RESULTS CONCERNING THE OUTPUT OF CERTAIN QUEUEING PROCESSES

By

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INTRODUCTION

This dissertation deals mainly with the investigation of the output of queueing processes in which the waiting room is assumed to be infinite. Both continuous and discrete time queueing processes are considered. Most of the queue analysis to be found in the literature assumes a continuous time process and deals with the number of customers in the system or the waiting times of these customers. Very little has been done concerning the number of customers served, the number of busy periods experienced by the server, or batch arrivals of customers. Although some authors feel that most important problems of interest concerning queues of infinite waiting room have been solved, little has been done concerning such basic problems as the derivation of total idle time available to the server. This dissertation will study single server queues with infinite waiting room and batch arrivals to obtain new results and some generalizations of known results.

The following summary is offered to acquaint the reader with some of the basic ideas and terminology in the theory of queues: In the basic queueing system units (customers) arrive at a service facility from some specified source, join a queue if necessary, and depart after receiving service. The source may be finite or infinite, and the service facility may have any specified number of servers (channels). The rule by which units are selected by the server from the queue is called the service discipline, and is, in practice, usually "first come-first served." Customers arrive at the queue at times $\tau_0 < \tau_1 < \ldots < \tau_k \quad \ldots$. If more than one customer is allowed to arrive at time τ_k, then the customers are said to arrive in batches. The random variables $t_k = \tau_k - \tau_{k-1}$ are called inter-arrival times, and are assumed to form a sequence of independent and identically distributed (i.i.d.) random variables. In this work, for the continuous time queue the common arrival distribution function (d.f.)

$$A(t) = \Pr[t_k \leq t]$$

is taken to be that of the so-called "completely random arrivals", or

$$A(t) = 1 - \exp(-\lambda t), \quad t \geq 0$$

$$= 0 \qquad\qquad , \quad t < 0,$$

where $\lambda > 0$ is known as the arrival intensity. The probability density function (p.d.f.) associated with this distribution is

$$\frac{d}{dt} A(t) = a(t) = \lambda\exp(-\lambda t), \quad t > 0.$$

The survivor function of $A(t)$ is

$$A_c(t) = 1 - A(t) = \exp(-\lambda t),$$

and the age specific failure rate, which is sometimes called the "first order probability of almost immediate failure" is, in conditional probability notation,

$$\phi(x) = \lim_{\Delta x \to 0^+} \frac{\Pr[x<X\leq x+\Delta x \mid x<X]}{\Delta x}$$

$$= \frac{a(x)}{A_c(x)}$$

$$= \lambda,$$

where X is a random variable whose distribution is A(x). The reader may refer to Appendix A for more specific information on probability distributions.

As is well known, the assumption of the exponential form of the arrival distribution implies that N(t), the random variable denoting the number of arrival instants in a time interval of length t, is Poisson distributed, i.e.

$$\Pr[N(t)=n] = \frac{(\lambda t)^n e^{-\lambda t}}{n!} \quad .$$

Service is performed on the k^{th} customer for a random service time θ_k, and the elements of the sequence $\{\theta_k\}$ of random variables thus defined are i.i.d. with distribution function

$$F(t) = \Pr[\theta_k \leq t].$$

Other functions associated with this distribution are defined as in the preceding paragraph.

It is appropriate at this time to mention a simplified notation for specifying the above information about a queue. A queue is called an $a|b|c$ queue if the distribution of time between arrivals is denoted a, the service time distribution is b, and c is the number of servers. Symbols that denote the various distributions include

M - exponential distribution (Markovian),

G - general distribution--no specified form,

E_κ - Erlang distribution with parameter κ.

For example, the $M|E_\kappa|s$ queue has completely random arrivals, κ-Erlang service (service in κ exponential stages), and s servers.

One type of distribution that is employed much in practice is the so-called steady-state or equilibrium distribution of a queue. Suppose, for a given time dependent queueing model, we consider the limit $t \to \infty$ of the probabilities under consideration. If this limit exists, the resulting limiting values still provide a probability distribution, and this distribution is called the <u>equilibrium probability distribution</u> of the given process. For $t >> 1$, little error would be committed if the time dependent probabilities at time t were assigned their equilibrium probabilities. One important property of the equilibrium distribution is that if, for the time dependent probabilities, we assign as initial values the corresponding equilibrium probabilities, then the instantaneous probability distribution is constant, or stationary, independent of time. Furthermore, these probabilities are equal to the equilibrium probabilities. Because of this property of the equilibrium distribution, it is sometimes called the <u>stationary</u> or <u>steady-state</u> distribution. Another important property of the equilibrium distribution is that, if we consider the different time dependent probabilities as "states" of the queueing model, and if we observe the model for a long period of time, the proportion of time which the system spends in any one "state is approximately the stationary probability of finding the system in that state.[1]

Now, some of the characteristics of a queueing process that are generally of interest include:

1. Queue length - the number of customers waiting or in service at time t,
2. Completions - the number of customers who completed service in (0,t],
3. Busy period - a time interval beginning with the arrival of a customer into a queue with zero queue length and ending with the next departure of a customer from a queue of length one,
4. Idle period - a time interval beginning with the departure of a customer from a queue of length one and ending with the next arrival of a customer,
5. Busy cycle - a busy period together with an adjoining idle period.

It is useful to notice that any queueing process is a stochastic process, i.e. a process whose characteristic states change with time,

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[1] For additional information on equilibrium probabilities, see Cox and Smith [5], Chapter 2.

and with each change of state is associated a probability. A stochastic process which has the property that a knowledge of the present state of the system is sufficient to know the probabilities associated with the next change of state is called a <u>Markov process</u>. Otherwise it is called non-Markovian.

It is sometimes possible to analyze a non-Markovian process by extracting a set of time points for which the Markov property holds. Such points, if they exist, are called regeneration points. A re-wording of the definition by D. G. Kendall [10] [2] is: A time point t_0 is said to be a <u>regeneration point for the stochastic process m(t)</u> iff for all $t > t_0$, the conditional distribution of m(t) given the state of the process at t_0 is identical to the conditional distribution of m(t) given the states at all time points of the set $\{t_1 | t_1 \leq t_0\}$. That is,

$$\text{distr}[m(t)|m(t_0)] = \text{distr}[m(t)|m(t_1) \text{ for all } t_1 \leq t_0].$$

Thus the development of the process during $t > t_0$ is independent of the history of the process during $(0,t_0]$. For a Markov process the whole range of t constitutes a set of regeneration points. Suppose there exists a denumerable set of regeneration points

$$\{t_n : n = 0,1,2,\ldots\}$$

such that $t_0 < t_1 < \ldots$, and put $X_n = X(t_n)$; then it follows from the

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[2] Numbers in square brackets refer to the bibliography.

above definition that the sequence of random variables $\{X_n\}$ forms a Markov chain, which is said to be embedded in the given process $X(t)$.

Once a set of regeneration points has been extracted, the general theory of Markov processes can often be used on that set to analyze the non-Markovian process. This method will be used to accomplish the following tasks:

In Chapter I we model the $M|M|1$ queue with unlimited waiting room, batch arrivals, and specified initial state. Theorems are proven concerning relationships between the equilibrium output distribution and the input distribution. In doing so, explicit expressions are obtained for the distribution of the number of arrivals in $(0,t]$ and the equilibrium distribution of queue size. Also, we obtain expressions for the Laplace transform of the g.f. associated with the joint time dependent (and equilibrium) probabilities of queue size, output, and number of busy periods completed in $(0,t]$, as well as the Laplace transform of the g.f. associated with the equilibrium output distribution. The conclusions of the theorems show that, although they agree in expected value, the two distributions under consideration are not the same.

Chapter II extends the results of Chapter I to a discrete queue with infinite waiting room, batch arrivals, specified initial state, and arrival and service distributions of the geometric type. Analogous expressions and theorems are obtained for each result of Chapter I.

Finally, Chapter III deals with the analysis of a queue with unlimited waiting room, specified initial state, exponential arrival distributions, and service dependent upon queue size. Explicit expressions are obtained for the joint p.d.f. and probability of the length of a busy period and the number of completions in that busy period. Also, the Laplace transform of the distribution of the queue length at time t, number of completions in (0,t], and the number of completed busy periods in (0,t] is derived. In each instance, these general results are compared with known results for the special case in which service is independent of queue size.

CHAPTER I

RESULTS CONCERNING THE OUTPUT OF AN $M|M|1$

QUEUE WITH BATCH ARRIVALS

In 1956 P. J. Burke [2] considered an $M|M|c$ queue and proved the very important result that the equilibrium output distribution of the queue is the same as the queue input distribution. This had been long suspected of being true. Then, in 1957 E. Reich [13] generalized Burke's result to a queue with Poisson arrivals and service dependent on queue size (i.e. If at time t there are n, ($n \geq 1$), customers in the system, the conditional probability that a customer completes service in $(t,t+\delta)$ is $\mu_n\delta + \sigma(\delta)$. Here $\sigma(\delta)$ is a function such that $\lim_{\delta\to 0} \frac{\sigma(\delta)}{\delta} = 0$.). Burke's result may be obtained from Reich's by setting

$$\mu_n = n\mu, \quad n < c,$$

$$\mu_n = c\mu, \quad n \geq c.$$

In this chapter we will consider the $M|M|1$ queue with batch arrivals and examine the output distribution of this queue. Contrary to expectation we discover that the equilibrium output distribution is not the same as the input distribution, except of course when batches are of size one. Thus Burke's type of result cannot be generalized in the direction of queueing models with batch arrivals. However, we did discover that the equilibrium output and input distributions have the same expected value. In obtaining the above results we found it necessary to determine several important additional results concerning the queue. These are, explicit expressions for the queue input and equilibrium queue size distributions, the Laplace transform of the g.f. of the joint distribution of queue size, number of customers served, and number of busy periods completed in $(0,t]$, as well as an explicit expression for the Laplace transform of the g.f. of the equilibrium output distribution.

We shall also see that, with the proper interpretations, the $M|M|1$ queue with batch arrivals is equivalent to a model given by Saaty.[1] Saaty has considered, for the model characterized by Poisson arrivals and service accomplished in phases (the distributions of each of which are exponential with identical distribution), the probabilities associated with the number of phases in the system at time t, assuming the probability a_k that an arriving customer will demand k, $(k \geq 1)$, phases of service. The significance of the discovery of the relationship lies in the fact that, since Saaty has determined only the time dependent distributions of the number of phases in the system, the results of this chapter can yield much additional information relative to Saaty's model.

To begin the analysis let customers arrive into the system according to a Poisson process with intensity λ, and let service

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[1] See Saaty [14], page 105.

time, which is exponentially distributed, have mean length $1/\mu$. Further, assume that at each arrival instant there is the probability a_k that k customers arrive simultaneously. Here $\sum_{k=1}^{\infty} a_k = 1$. The order in which customers are served will have no significance here. Define the random variables associated with the model as follows:

$M(t)$ is the number of customers in the system at time t,

$N(t)$ is the number of customers served in $(0,t]$,

$R(t)$ is the number of completed busy periods in $(0,t]$.

We further define

$$p(n,t) = \Pr[n \text{ customers arrive in } (0,t]],$$

$$g_i(m,t) = \Pr[M(t) = m/M(0) = i,\ i \geq 0],$$

$$p_i(m,n,r,t) = \Pr[M(t)=m,\ N(t)=n,\ R(t)=r/M(0)=i,\ N(0)=R(0)=0,\ i\geq 0].$$

We now define the generating functions (g.f.) associated with the above probabilities:

$$P(y,t) = \sum_{n=0}^{\infty} p(n,t)y^n,$$

$$G_i(x,t) = \sum_{m=0}^{\infty} g_i(m,t)x^m,$$

$$P_i(x,y,\cdot,t) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} p_i(m,n,0,t)x^m y^n,$$

$$P_i(x,y,z,t) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} p_i(m,n,r,t)x^m y^n z^r, \quad (i \geq 0),$$

$$P_i(\cdot,y,z,t) = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} p_i(0,n,r,t)y^n z^r, \quad (i \geq 0),$$

and

$$T_i(x,y,z,t) = P_i(\cdot,y,z,t) + P_i(x,y,z,t), \quad (i \geq 0).$$

We will also use the two generating functions

$$A(y) = \sum_{n=1}^{\infty} a_n y^n, \quad \text{and}$$

$$B(y) = \sum_{n=0}^{\infty} b_n y^n = \frac{1-A(y)}{1-y}, \quad |y| < 1, \quad \text{where } b_n = \sum_{k=n+1}^{\infty} a_k.$$

A proof of the relationship between $A(y)$ and $B(y)$ can be found in Feller [6] [2].

We shall now determine explicit expressions for the probabilities $p(n,t)$. For the $M|M|1$ queue with single arrivals, the g.f. associated with $p(n,t)$ is given by the expression $e^{-(1-y)\lambda t}$. For batch arrivals, using a method similar to the one employed to derive this latter expression, we can show that

(1) $$P(y,t) = e^{-[1-A(y)]\lambda t},$$

or alternately,

(2) $$P(y,t) = e^{-(1-y)B(y)\lambda t}.$$

Expanding (1) and collecting coefficients of y^n, $(n \geq 0)$, gives $p(n,t)$. Thus we have

(3) $$p(0,t) = e^{-\lambda t},$$

(4) $$p(1,t) = \lambda a_1 t e^{-\lambda t},$$

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[2] See Feller [6], Chapter XI.

and for $n \geq 2$,

(5) $$p(n,t) = e^{-\lambda t} \sum_{k=1}^{n} \frac{(\lambda t)^k}{k!} a_n^{(k)} .$$

Here $c_j^{(i)}$ denotes the j^{th} term of the i-fold convolution of the sequence $\{c_n\}_{n=0}^{\infty}$ with itself, where the i-fold convolution of $\{c_n\}_{n=0}^{\infty}$ with itself, denoted by $\{c_n\}^{(i)}$, is defined as follows:

Let $\{b_k\}_{k=0}^{\infty}$ be the sequence whose k^{th} term is given by $b_k = \sum_{j=0}^{k} c_j c_{k-j}$. Then we define

$$\{c_n\}^{(2)} = \{c_n\}*\{c_n\} = \{b_n\},$$

$$\{c_n\}^{(i)} = \{c_n\}^{(i-1)}*\{c_n\}, \quad i \geq 3.$$

Also, $\{c_n\}^{(1)} = \{c_n\}$, and

$$\{c_n\}^{(0)} = \{1,0,0,0,\ldots\}.$$

Thus (3) - (5) give the probabilities of the input into the queue in the interval $(0,t]$.

We now turn our attention to the probabilities $g_i(m,t)$. We shall determine the Laplace transform of $G_i(x,t)$, and then use this expression to obtain an explicit expression for the equilibrium distribution of queue size. This latter expression will then be used to prove that the equilibrium distribution of queue output is not the same as the distribution of queue input.

Considering the interval $(t,t+\Delta)$, $\Delta > 0$, we can write the time dependent equations associated with $g_i(m,t)$ as follows:

$$(6) \qquad \frac{d}{dt} g_i(0,t) + \lambda g_i(0,t) = \mu g_i(1,t),$$

$$(7) \qquad \frac{d}{dt} g_i(m,t) + (\lambda+\mu) g_i(m,t) = \mu g_i(m+1,t) + \lambda \sum_{j=1}^{m} a_j g_i(m-j,t), \quad m \geq 1.$$

Multiply (6) and (7) by the appropriate powers of x and sum. After some straightforward manipulations we have

$$(8) \qquad \frac{d}{dt} x G_i(x,t) + [(\lambda+\mu)x - \lambda x A(x) - \mu] G_i(x,t) = \mu(x-1) g_i(0,t).$$

Denote the Laplace transform of a function f(t) by a bar, i.e.

$$\overline{f}(s) = \int_0^\infty e^{-st} f(t)dt, \quad Re(s) > 0.$$

Using the initial condition that $G_i(x,0) = x^i$, the Laplace transform of (8) can be written as

$$(9) \qquad \overline{G}_i(x,s) = \frac{x^{i+1} + \mu(x-1)\overline{g}_i(0,s)}{(s+\lambda+\mu)x - \mu - \lambda x A(x)} \quad .$$

Equation (9) is then the transform of the g.f. of the probabilties associated with queue size. We remark that it was at this point in the analysis of our model that the discovery was made of the relationship between the model of this chapter and the model considered by Saaty. For, equation (9) is the equation derived by Saaty for the Laplace transform of the g.f. of the number of service phases in the system at time t. [3] If we then interpret number of customers as the number of service phases, and the probability a_k that a customer demands k phases of service as the probability a_k

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[3] See Saaty [14], page 106.

that k customers arrive simultaneously, we can obtain the probabilities $g_i(m,t)$ from the solution given by Saaty.

The g.f. of the equilibrium distribution of queue size is obtained from (9) by using the Tauberian theorem[4]

(10) $$\lim_{t\to\infty} G_i(x,t) = \lim_{s\to 0} s\overline{G}_i(x,s),$$ assuming the limits exist.

Denote by G(x) the g.f. of the equilibrium distribution of queue size, i.e.

(11) $$G(x) = \sum_{j=0}^{\infty} g(j)x^j,$$

where g(j) is the equilibrium probability that j customers are in the system.

Applying the limit given by (10) to (9) yields

(12) $$G(x) = \lim_{s\to 0} \frac{s[x^{i+1}+\mu(x-1)\overline{g}_i(0,s)]}{(s+\lambda+\mu)x-\mu-\lambda xA(x)}$$

$$= \frac{\mu(x-1)g(x)}{(\lambda+\mu)x-\mu-\lambda xA(x)} .$$

Using the fact that $\lim_{x\to 1} G(x) = 1$ and applying L'Hospital's rule to (12) yields

(13) $$g(0) = 1 - \rho, \quad \text{where } \rho = \frac{\lambda\overline{a}}{\mu} < 1 \quad \text{and} \quad \overline{a} = A'(1).$$

Substituting (13) into (12) and rearranging yields

(14) $$G(x) = (1-\rho)\sum_{j=0}^{\infty} \left[\frac{\lambda xB(x)}{\mu}\right]^j .$$

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[4] See Widder [19], Chapter V.

The infinite series above converges provided $\left|\frac{\lambda x B(x)}{\mu}\right| < 1$. For $|x| \leq 1$, we have $\left|\frac{\lambda x B(x)}{\mu}\right| \leq \frac{\lambda |B(1)|}{\mu} = \frac{\lambda \bar{a}}{\mu} = \rho < 1$. Thus the series converges. Expanding the series in (14) and collecting coefficients of x^j gives the value of $g(j)$:

$$(15) \qquad g(0) = 1 - \rho$$

$$(16) \qquad g(1) = (1-\rho)\,\frac{\lambda}{\mu}\,,$$

and for $j \geq 2$,

$$(17) \qquad g(j) = (1-\rho) \sum_{k=1}^{j} \left(\frac{\lambda}{\mu}\right)^k b_{j-k}^{(k)}.$$

Equations (15) - (17) are the explicit expressions for the equilibrium probabilities of queue size.

We now prove the following theorem:

THEOREM I: Let $p^{eq}(m,t)$, $m = 0,1,2,\ldots,$ denote the equilibrium probability that m customers are served in $(0,t]$. Then

$$p^{eq}(m,t) \neq p(m,t).$$

Thus the equilibrium output distribution is not the same as the queue input distribution.

PROOF.

To prove the conjecture it is sufficient to show that equality fails to hold for $m = 0$. Proceeding in this manner, let $p^{eq}(n,0,t)$ denote the joint equilibrium probability that n customers are in the system at time t and no service has been completed in $(0,t]$,

$p_i(n,0,t)$ denote the joint probability that n customers are in the system at time t and no service has been completed in (0,t], given $M(0) = i$, $i \geq 0$. Now

$$(18) \qquad p^{eq}(0,t) = \sum_{n=0}^{\infty} p^{eq}(n,0,t), \quad \text{and}$$

$$(19) \qquad p^{eq}(n,0,t) = \sum_{i=0}^{n} g(i)p_i(n,0,t).$$

Consider the probability $p_i(n,0,t)$. For $i = n = 0$, this is the probability of no arrivals in (0,t]. Thus

$$(20) \qquad p_0(0,0,t) = e^{-\lambda t}.$$

For $i \geq 1$ and $n \geq i$, $p_i(n,0,t)$ is the joint probability of n-i arrivals and no service in (0,t], starting initially with i customers. For this case we can write

$$(21) \qquad p_i(n,0,t) = p(n-i,t)\cdot e^{-\mu t}, \quad (i \geq 1,\ n \geq i).$$

The case $i = 0$, $n \geq 1$ remains. This is the joint probability of an arrival at time x, $(x > 0)$, consisting of k customers, $(k = 1,2,\ldots,n)$, n-k customers arriving in (x,t], and no service in (x,t]. Since $0 < x < t$, this is equal to

$$(22) \qquad p_0(n,0,t) = \sum_{k=1}^{n} \int_0^t a_k \lambda e^{-\lambda x} p(n-k,t-x) e^{-\mu(t-x)} dx, \quad n \geq 1.$$

Substituting (19) - (22) into (18) yields

$$(23) \qquad p^{eq}(0,t) = g(0)e^{-\lambda t} + \sum_{n=1}^{\infty} [g(0)\lambda \sum_{k=1}^{n} \int_0^t a_k e^{-\lambda x} p(n-k,t-x)e^{-\mu(t-x)}dx]$$

$$+ \sum_{n=1}^{\infty} e^{-\mu t} \sum_{k=1}^{n} g(k)p(n-k,t).$$

Performing the indicated sums and integration yields, upon simplification

$$(24) \qquad p^{eq}(0,t) = \mu(1-\rho)\left[\frac{e^{-\lambda t} - e^{-\mu t}}{\mu - \lambda}\right] + e^{-\mu t} .$$

From (3) we have $p(0,t) = e^{-\lambda t}$. Thus $p^{eq}(0,t) \neq p(0,t)$ and the theorem is proved.

We observe here that if we make the substitution $a_1 = 1$, $a_i = 0$, $i \neq 1$, equation (24) reduces to $e^{-\lambda t}$, which is the result given by Burke for the $M|M|c$ queue with $c = 1$. This is as it should be, for in this particular case, our model reduces to an $M|M|1$ queue.

Having proved in Theorem I that the two distributions are not the same, the question now arises as to whether or not they have the same expected value. We shall see that the answer is yes, but in order to do so, we first develop the g.f. associated with the probability $p_i(m,n,r,t)$.

Consider then $p_i(m,n,r,t)$. Restricting ourselves to the case $i \geq 1$, define the joint probability and p.d.f. of the number of customers served in the first busy period and the length of that busy period (which started with i customers present) to be $h_i(n,t)$. Also, define the g.f.

$$(25) \qquad H_i(y,t) = \sum_{n=1}^{\infty} h_i(n,t)y^n.$$

For $p_i(m,n,0,t)$, the time dependent equations associated with this probability are

$$(26) \qquad \frac{d}{dt} p_i(1,n,0,t) + (\lambda+\mu)p_i(1,n,0,t) = \mu p_i(2,n-1,0,t),$$

$$(27) \qquad \frac{d}{dt} p_i(m,n,0,t) + (\lambda+\mu)p_i(m,n,0,t)$$

$$= \mu p_i(m+1,n-1,0,t) + \lambda \sum_{k=1}^{m-1} a_k p_k(m-k,n,0,t),$$

$$m = 2,3,\dots .$$

Multiplying (26) and (27) by the appropriate powers of $x^m y^n$ and summing yields

$$(28) \qquad \frac{d}{dt} P_i(x,y,\cdot,t) + (\lambda+\mu)P_i(x,y,\cdot,t)$$

$$= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} p_i(m+1,n-1,0,t)x^m y^n$$

$$+ \lambda \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} [\sum_{k=1}^{m-1} a_k p_i(m-k,n,0,t)]x^m y^n .$$

After some simplification (28) reduces to

$$(29) \qquad x \frac{d}{dt} P_i(x,y,\cdot,t) + [(\lambda+\mu)x-\mu y-\lambda x A(x)]P_i(x,y,\cdot,t) = -xH_i(y,t).$$

Here we have used the fact that $h_i(n,t) = \mu p_i(1,n-1,0,t)$, $t \geq 0$, $n \geq 1$.[5]

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[5] See Appendix B, Remark A.

Using the initial condition $P_i(x,y,\cdot,0) = x^i$, the transform of $P_i(x,y,\cdot,t)$ can be written as

$$(30) \qquad \overline{P}_i(x,y,\cdot,s) = \frac{x^{i+1} - x\overline{H}_i(y,s)}{(s+\lambda+\mu)x - \mu y - \lambda x A(x)} \quad .$$

Equation (30) converges for $\mathrm{Re}(s) > 0$ and $|x| \leq 1$. Hence the zeros of the denominator and numerator of the RHS of (30) must coincide. By an application of Rouche's Theorem,[6] it can be shown that the denominator of the RHS of (30) has a single zero inside the unit circle, this zero being $\overline{H}_1(y,s)$. This latter fact will be used in the determination of the transform of the g.f. of the equilibrium output distribution. (30) gives the Laplace transform of the g.f., for the first busy period, of the busy system of the queue.

Now

$$(31) \qquad P_i(x,y,z,t) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} p_i(m,n,0,t)x^m y^n + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \sum_{r=1}^{\infty} p_i(m,n,r,t)x^m y^n z^r \ .$$

But for $i \geq 1$ and $r \geq 1$ we must have $n \geq 1$. Hence

$$(32) \qquad P_i(x,y,z,t) = P_i(x,y,\cdot,t) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} p_i(m,n,r,t)x^m y^n z^r .$$

From the Law of Total Probability (L.T.P.) and the regenerative property of the instants of commencement of the busy periods, we have, for $m \geq 1$, $n \geq 1$, $r \geq 1$, (See Figure 1).

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[6] See Appendix B, Theorem B.

$$(33)\qquad p_i(m,n,r,t) = \sum_{j=1}^{\infty}\sum_{k=1}^{n}\int_0^t\int_0^{t_2}[h_i(k,t_1)\cdot\lambda e^{-\lambda(t_2-t_1)}$$

$$\cdot\ a_j p_j(m,n-k,r-1,t-t_2)]\,dt_1 dt_2\ .$$

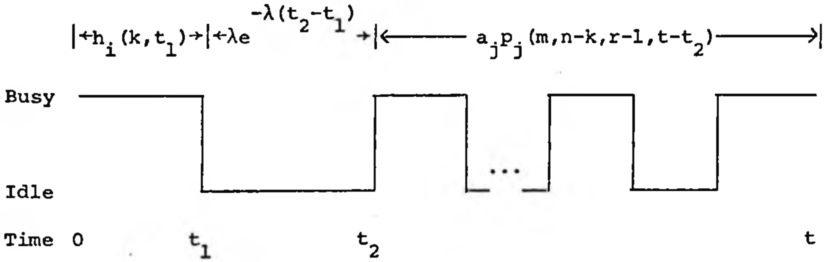

Figure 1

In terms of Laplace transforms (33) is

$$(34)\qquad \bar{p}_i(m,n,r,s) = \frac{\lambda}{s+\lambda}\sum_{j=1}^{\infty}\sum_{k=1}^{n}\bar{h}_i(k,s)\ \cdot\ a_j\bar{p}_j(m,n-k,r-1,s),$$

$$(m \geq 1,\ n \geq 1,\ r \geq 1).$$

Multiply by $x^m y^n z^r$ and sum:

$$(35)\qquad \bar{P}_i(x,y,z,s) - \bar{P}_i(x,y,\cdot,s) = \frac{\lambda z\bar{H}_i(y,s)}{s+\lambda}\sum_{j=1}^{\infty}a_j\bar{P}_j(x,y,z,s).$$

Since (35) is true for all $i \geq 1$, we have after some manipulations

$$(36)\qquad \bar{P}_i(x,y,z,s) = \bar{P}_i(x,y,\cdot,s) + \lambda z\bar{H}_i(y,s)\left[\frac{\sum_{j=1}^{\infty} a_j\bar{P}_j(x,y,\cdot,s)}{s+\lambda-\lambda z\sum_{j=1}^{\infty} a_j\bar{H}_j(y,s)}\right].$$

Equation (36) is the Laplace transform of the g.f. for the probabilities associated with a busy queue which started with $i \geq 1$ customers at time zero.

We now turn to the idle case. If $i \geq 1$, we have

$$(37)\qquad p_i(0,n,1,t) = \int_0^t h_i(n,t_1)e^{-\lambda(t-t_1)}dt_1,$$

a direct consequence of L.T.P. If $r \geq 2$

$$(38)\qquad p_i(0,n,r,t) = \sum_{j=1}^{\infty}\sum_{k=i}^{n}\int_0^t\int_0^{t_2}[h_i(k,t)\lambda e^{-\lambda(t_2-t_1)}$$

$$\cdot\ a_jp_j(0,n-k,r-1,t-t_2)]dt_1dt_2;$$

we have used the same regeneration point as before. Transforming (37) and (38) yields

$$(39)\qquad \bar{p}_i(0,n,1,s) = \frac{1}{s+\lambda}\bar{h}_i(n,s), \qquad \text{and if } r \geq 2,$$

$$(40)\qquad \bar{p}_i(0,n,r,s) = \frac{\lambda}{s+\lambda}\sum_{j=1}^{\infty}\sum_{k=i}^{n}\bar{h}_i(k,s)\cdot a_j\bar{p}_j(0,n-k,r-1,s).$$

Multiply (39) and (40) by the appropriate powers of z and y and sum. After some manipulation utilizing the facts

$$\overline{h}_i(k,s) = 0 \qquad \text{if } k < i \text{ and}$$

$$\overline{p}_j(m,n,r,s) = 0 \qquad \text{if } r > n+j-1$$

we arrive at the following expression:

$$(41) \qquad \overline{P}_i(\cdot,y,z,s) = \frac{z}{s+\lambda}\,\overline{H}_i(y,s) + \frac{\lambda z}{s+\lambda}\sum_{j=1}^{\infty}\overline{H}_i(y,s)a_j\overline{P}_j(\cdot,y,z,s).$$

Since (41) is true for all $i \geq 1$, we have after further manipulations

$$(42) \qquad \overline{P}_i(\cdot,y,z,s) = \frac{z\overline{H}_i(y,s)}{s+\lambda-\lambda z\sum_{j=1}^{\infty}a_j\overline{H}_j(y,s)} .$$

The sum of the g.f.s (36) and (42) yields the total Laplace transform of the g.f. for the probabilities associated with the queue for the case $i \geq 1$. For this case we then have

$$(43) \qquad \overline{T}_i(x,y,z,s) = \overline{P}_i(x,y,\cdot,s) + z\overline{H}_i(y,s)\,\frac{1+\lambda\sum_{j=1}^{\infty}a_j\overline{P}_j(x,y,\cdot,s)}{s+\lambda-\lambda z\sum_{j=1}^{\infty}a_j\overline{H}_j(y,s)},$$

$$(i \geq 1).$$

The case $i = 0$ remains. By the L.T.P. we have

$$(44) \qquad p_0(m,n,r,t) = \sum_{j=1}^{\infty}\int_0^t \lambda e^{-\lambda t_1}\cdot a_j p_j(m,n,r,t-t_1)dt_1, \quad (m \geq 1).$$

(See Figure 2.)

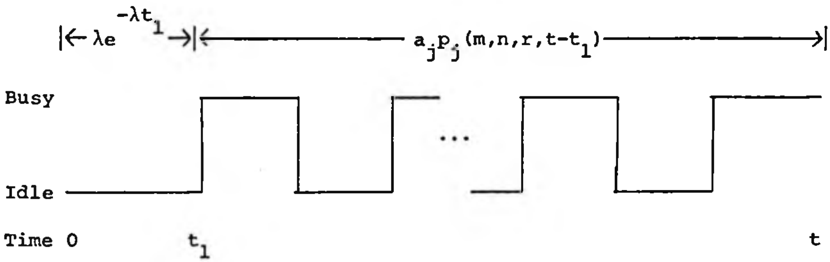

Figure 2

Applying the transform and forming the g.f. in the usual way yields

$$(45) \qquad \overline{P}_0(x,y,z,s) = \frac{\lambda}{s+\lambda} \sum_{j=1}^{\infty} a_j \overline{P}_j(x,y,z,s), \quad (m \geq 1).$$

The idle case (m = 0) is derived in a similar fashion: Appealing to the hypothesis of Poisson arrivals, we have

$$(46) \qquad p_0(0,0,0,t) = e^{-\lambda t}.$$

If $n \geq 1$ (so that $r \geq 1$) we have

$$(47) \qquad p_0(0,n,r,t) = \sum_{j=1}^{\infty} \int_0^t \lambda e^{-\lambda t_1} a_j p_j(0,n,r-1,t-t_1)\,dt_1.$$

Apply the transform, multiply by the appropriate powers of y and z, and sum. This process yields

$$(48) \qquad \overline{P}_0(\cdot,y,z,s) = \frac{1}{s+\lambda} + \frac{\lambda}{s+\lambda} \sum_{j=1}^{\infty} a_j \overline{P}_j(\cdot,y,z,s) .$$

The sum of the g.f.s (45) and (48) yields the Laplace transform of the total g.f. for the probabilities associated with the queue for the case $i = 0$. For this case we have after some simplification

$$(49)\qquad \overline{T}_0(x,y,z,s) = \frac{1+\lambda \sum_{j=1}^{\infty} a_j \overline{P}_j(x,y,\cdot,s)}{\lambda+s-\lambda z \sum_{j=1}^{\infty} a_j \overline{H}_j(y,s)} .$$

We will now derive an expression for the Laplace transform of the g.f. associated with the joint equilibrium probability of the number of customers in the system at time t, the number served in (0,t], and the number of busy periods completed in (0,t]. We begin by making the definition

$$p_{eq}(m,n,r,t) = pr[M(t)=m,\ N(t)=n,\ R(t)=r/Pr\{M(0)=j\}=g(j)].$$

Then

$$(50)\qquad p_{eq}(m,n,r,t) = \sum_{j=0}^{\infty} g(j)p_j(m,n,r,t).$$

Multiply by powers of $x^m y^n z^r$ and sum.

$$(51)\qquad P_{eq}(x,y,z,t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} p_{eq}(m,n,r,t)x^m y^n z^r$$

$$= \sum_{j=0}^{\infty} g(j)T_j(x,y,z,t).$$

Apply the transform to (51).

$$(52)\qquad \overline{P}_{eq}(x,y,z,s) = \sum_{j=0}^{\infty} g(j)\overline{T}_j(x,y,z,s).$$

This then is the transform to be found. Recall that $\overline{P}_0(x,y,z,s)$ is given by (49) and, for $j \geq 1$, $\overline{P}_j(x,y,z,s)$ is given by (43). Using (30), (43), and (49) in (52) yields

$$(53) \qquad \overline{P}_{eq}(x,y,z,s) = g(0)\left[\frac{1+\lambda \sum_{\ell=1}^{\infty} a_\ell \overline{P}_\ell(x,y,\cdot,s)}{s+\lambda-\lambda z \sum_{\ell=1}^{\infty} a_\ell \overline{H}_\ell(y,s)}\right]$$

$$+ \sum_{j=1}^{\infty} g(j)\left[\frac{x^{j+1}-x\overline{H}_j(y,s)}{(s+\lambda+\mu)x-\mu y-\lambda x A(x)} + z\overline{H}_j(y,s)\left\{\frac{1+\lambda \sum_{\ell=1}^{\infty} a_\ell \overline{P}_\ell(x,y,\cdot,s)}{s+\lambda-\lambda z \sum_{\ell=1}^{\infty} a_\ell \overline{H}_\ell(y,s)}\right\}\right].$$

Replacing $\overline{H}_j(y,s)$ by $\{\overline{H}_1(y,s)\}^j$ [7] and using the g.f. $G(\zeta)$, we can write, after some simplification,

$$(54) \qquad \overline{P}_{eq}(x,y,z,s) = \frac{x[G(x)-G(\overline{H}_1(y,s))]}{(s+\lambda+\mu)x-\mu y-\lambda x A(x)}$$

$$+ \left[\frac{1+\lambda \sum_{\ell=1}^{\infty} a_\ell \overline{P}_\ell(x,y,\cdot,s)}{s+\lambda-\lambda z \sum_{\ell=1}^{\infty} a_\ell \overline{H}_\ell(y,s)}\right] G(z\overline{H}_1(y,s)) \;.$$

Equation (54) is then the expression for the Laplace transform of the g.f. associated with the joint equilibrium distribution of queue length, queue output, and number of busy periods completed.

Letting $x \to 1$, $z \to 1$ in (54) yields $\overline{P}_{eq}(1,y,1,s)$, the g.f. of the equilibrium distribution of queue output in $(0,t]$.

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[7] For a proof that $h_i(n,t) = [h_1(n,t)]^{(i)}$, see Takács [17], page 32. That $\overline{H}_i(y,s) = [\overline{H}_1(y,s)]^i$ then follows from properties of g.f.

(55) $$\overline{P}_{eq}(1,y,1,s) = \frac{1-G(\overline{H}_1(y,s))}{s+\mu(1-y)} + \left[\frac{1+\lambda\sum_{\ell=1}^{\infty} a_\ell \overline{P}_\ell(1,y,\cdot,s)}{s+\lambda-\lambda\sum_{\ell=1}^{\infty} a_\ell \overline{H}_\ell(y,s)}\right] G(\overline{H}_1(y,s)).$$

Now, using the fact that $\overline{H}_1(y,s)$ is a solution, in x, of the denominator of the RHS of (30) [8], and the fact that $G(\overline{H}_1(y,s))$ also satisfies (12), equation (55) can be reduced to

(56) $$\overline{P}_{eq}(1,y,1,s) = \frac{1}{s+\mu(1-y)} + \frac{\mu(1-y)(1-\rho)[1-\overline{H}_1(y,s)]\overline{H}_1(y,s)}{[s+\mu(1-y)][y-\overline{H}_1(y,s)][\mu(1-y)+s\overline{H}_1(y,s)]}.$$

Equation (56) is then the Laplace transform of the g.f. of the equilibrium output probabilities over the interval (0,t].

We are now ready to prove that the equilibrium output distribution has the same expected value as the queue input distribution.

THEOREM II: Let $E(N(t))$ denote the expected number of arrivals in (0,t] and $E(N^{eq}(t))$ denote the expected number of departures (i.e. completions), under equilibrium conditions, in (0,t]. Then

$$E(N(t)) = E(N^{eq}(t)).$$

PROOF.

Recall that the g.f. of p(n,t) is given by

$$P(y,t) = e^{-[1-A(y)]\lambda t}.$$

Deriving this w.r.t. y and evaluating at y = 1 yields

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[8] See Appendix B, Theorem B.

$E(N(t)) = \lambda\bar{a}t.$

To evaluate $E(N^{eq}(t))$ we consider (56). $E(N^{eq}(t))$ is obtained by evaluating the derivative of the RHS of (56) w.r.t. y at y = 1, and then inverting the resulting expression. Now,

$$(57)\qquad \frac{d}{dy}\bar{P}_{eq}(1,y,1,s)\Big|_{y=1} = \frac{\mu}{s^2} + \frac{-\bar{H}_1(1,s)s\{1-\bar{H}_1(1,s)\}^2\mu(1-\rho)}{\{s(1-\bar{H}_1(1,s))\}^2 s\bar{H}_1(1,s)}$$

$$= \frac{\mu}{s^2} - \frac{\mu(1-\rho)}{s^2}$$

$$= \frac{\mu\rho}{s^2}$$

$$= \frac{\lambda\bar{a}}{s^2}\ .$$

Inverting we obtain

(58) $\quad E(N^{eq}(t)) = \lambda\bar{a}t$, and the theorem is proved.

If we assume $a_1 = 1$, $a_i = 0$ for $i \neq 1$, (58) becomes λt, the expected value of a Poisson distribution of parameter λ. This is as it should be for our model is now an $M|M|1$ queue.

In concluding this chapter, we remark that the generating functions (36), (42), (45), and (49) contain a lot of information. For example, if in (36) we let $x \to 1$ and $z \to 1$ we obtain the Laplace transform of the g.f. of the (joint) probability of there being n completions while the system is busy at time t. Letting $x \to 1$, $y \to 1$ in (36) gives a similar result for the (joint) probabilities associated with the number of busy periods and the busy system at time t. The Laplace transform of the probability that the system is busy at an arbitrary time is obtained by letting $x \to 1$, $y \to 1$, and $z \to 1$ in (36). Similar remarks apply to the other generating functions.

CHAPTER II

RESULTS CONCERNING THE OUTPUT OF A DISCRETE QUEUE WITH BATCH ARRIVALS

In this chapter we extend the results of Chapter I to a discrete time queue with batch arrivals characterized by geometric interarrival and service distributions. As the techniques involved are similar to those of the last chapter, we shall merely indicate methods of obtaining results and then state these results. Interestingly enough, very little has appeared in the literature concerning discrete queues. It has only been recently that interest in these types of queues has grown, this being due in part to the availability of computers to handle the models encountered. We shall obtain for the model of this chapter analogous results for each of the probabilities defined in Chapter I. We will also prove theorems which are the discrete analogy of Theorems I and II of Chapter I.

Employing the same definitions and associated notation as in Chapter I we begin the analysis as follows: Let customers arrive into the system with interarrival times given by the geometric distribution $\lambda_1\lambda_0^{k-1}$, $k = 1,2,3,\ldots$, and let service, which is also geometrically distributed, be characterized by the distribution $\mu_1\mu_0^{k-1}$, $k = 1,2,3,\ldots$. Further, assume that when an arrival instant occurs, there is the probability a_k that k customers arrive simultaneously. The order in which customers are served will have no significance here. Also, instead of a continuous time interval of length t, we shall now speak of a discrete time interval consisting of t time units. We consider probabilities and g.f.s in this chapter in the same order in which they are discussed in Chapter I.

First, for the probabilities p(n,t) we obtain the equation

(1) $$P(y,t+1) = [\lambda_0+\lambda_1 A(y)]P(y,t).$$

This equation can be solved recursively for P(y,t) using the initial condition

$$P(y,0) = \sum_{n=0}^{\infty} p(n,0)y^n = 1$$

to obtain

(2) $$P(y,t) = [\lambda_0+\lambda_1 A(y)]^t.$$

Compare this with equation (1) of Chapter I. Expanding the RHS of (2) and collecting coefficients of y^n then yields

(3) $$p(0,t) = \lambda_0^t ,$$

(4) $$p(1,t) = a_1 t\lambda_1\lambda_0^{t-1} ,$$

and for $n \geq 2$,

(5) $$p(n,t) = \sum_{k=1}^{n} \binom{t}{k} \lambda_0^{t-k}\lambda_1^k a_n^{(k)} .$$

Here $c_j^{(i)}$ is again the j^{th} term of the i-fold convolution of the sequence $\{c_n\}_{n=0}^{\infty}$ with itself. Equations (3)-(5) then give the probabilities associated with the number of arrivals into the queue in t units of time.

Before considering the probabilities associated with queue size we define the following g.f.:

$$\hat{F}(y,s) = \sum_{t=0}^{\infty} f(y,t)s^t, \quad |s| \leq 1.$$

This g.f. will play a role analogous to that of the Laplace transform for continuous time queueing models.

For the probabilities $g_i(m,t)$, corresponding to equations (8) and (9), respectively, of Chapter I we have

$$(6) \qquad xG_i(x,t+1) = [\mu_0 x+\mu_1][\lambda_0+\lambda_1 A(x)]G_i(x,t) + \mu_1(x-1)[\lambda_0+\lambda_1 A(x)]g_i(0,t),$$

and

$$(7) \qquad \hat{G}_i(x,s) = \frac{x^{i+1}+\mu_1 s(x-1)[\lambda_0+\lambda_1 A(x)]\hat{g}_i(0,s)}{x-s[\lambda_0+\lambda_1 A(x)][\mu_0 x+\mu_1]}.$$

Equation (7) is the g.f. of the g.f. of the probabilities associated with queue size. By a proof similar to the one used to show the denominator of $\bar{G}_i(x,s)$ has a single zero inside the unit circle, it can be shown that the denominator of the RHS of (7) also has a single zero inside the unit circle. This enables finding first $\hat{g}_i(0,s)$ and subsequently $g_i(0,t)$. From this values of $g_i(m,t)$, $m \geq 1$, can be obtained.

The equilibrium probabilities of queue size are obtained from (7) by using the Tauberian theorem [1]

(8) $\lim_{t\to\infty} G_i(x,t) = \lim_{s\to 1^-} (1-s)\hat{G}_i(x,s)$, assuming the limits exist.

Applying the limit to (7) yields

$$(9)\qquad G(x) = \frac{\mu_1(x-1)[\lambda_0+\lambda_1 A(x)]g(0)}{x-[\lambda_0+\lambda_1 A(x)][\mu_0 x+\mu_1]} .$$

Using the fact that $\lim_{x\to 1} G(x) = 1$ and applying L'Hospital's Rule to (9) yields

$$(10)\qquad g(0) = 1-\rho, \qquad \text{where } \rho = \frac{\lambda_1\bar{a}}{\mu_1} < 1 \text{ and } \bar{a} = A'(1).$$

Substituting (10) into (9) and simplifying gives

$$(11)\qquad G(x) = (1-\rho)[\lambda_0+\lambda_1 A(x)][\sum_{j=0}^{\infty} \{\frac{\lambda_1(\mu_0 x+\mu_1)B(x)}{\mu_1}\}^j] .$$

The series in (11) converges provided $\left|\frac{\lambda_1(\mu_0 x+\mu_1)B(x)}{\mu_1}\right| < 1$. For $|x| \le 1$, $\left|\frac{\lambda_0(\mu_0 x+\mu_1)B(x)}{\mu_1}\right| \le \left|\frac{\lambda_1(\mu_0+\mu_1)B(1)}{\mu_1}\right| = \frac{\lambda_1\bar{a}}{\mu_1} = \rho < 1.$

Thus the series converges. Expanding (11) and collecting coefficients of x^j gives the value of $g(j)$:

(12) $g(0) = 1 - \rho,$

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[1] See Takács [17], Appendix.

$$(13)\qquad g(1) = \frac{\mu_0\lambda_1}{\mu_1\lambda_0}(1-\rho),\quad \text{and for } j \geq 2$$

$$(14)\qquad g(j) = (1-\rho)\lambda_0\Big[\sum_{k=1}^{j}\Big(\frac{\mu_0}{\mu_1}\Big)^k \sum_{r=k}^{\infty}\binom{r}{r-k}\lambda_1^r\, b_{j-k}^{(r)} + \sum_{r=1}^{\infty}\lambda_1^r\, b_j^{(r)}\Big]$$

$$+ (1-\rho)\lambda_1\Big[\sum_{m=1}^{j-1} a_{j-m}\Big\{\sum_{k=1}^{m}\Big(\frac{\mu_0}{\mu_1}\Big)^k \sum_{r=k}^{\infty}\binom{r}{r-k}\lambda_1^r b_{m-k}^{(r)} + \sum_{r=1}^{\infty}\lambda_1^r b_m^{(r)}\Big\}$$

$$+ a_j\Big(\frac{\lambda_1}{\lambda_0}\Big)\Big] .$$

Equations (12) - (14) are the explicit expressions for the discrete equilibrium probabilities of queue size.

We now prove a theorem corresponding to Theorem I of Chapter I.

THEOREM I: Let $p^{eq}(m,t)$, $m = 0,1,2,\ldots,$ denote the equilibrium probability that m customers are served in t time units. Then

$$p^{eq}(m,t) \neq p(m,t).$$

Hence the equilibrium output distribution for the discrete queue is not the same as the queue input distribution.

PROOF.

Proceeding along the same lines as in the proof of Theorem I of Chapter I we may write

$$(15)\qquad p^{eq}(0,t) = \sum_{n=0}^{\infty} p^{eq}(n,0,t),\quad \text{and}$$

$$(16)\qquad p^{eq}(n,0,t) = \sum_{k=0}^{n} g(k)p_k(n,0,t).$$

Concerning the probability $p_k(n,0,t)$ we have the equations

(17) $$p_0(0,0,t) = \lambda_0^t,$$

(18) $$p_k(n,0,t) = p(n-k,t) \cdot \mu_0^t, \quad (k \geq 1,\ n \geq k),$$

(19) $$p_0(n,0,t) = \sum_{k=1}^{n} \sum_{\ell=0}^{t} a_k \lambda_1 \lambda_0^{\ell-1} \cdot p(n-k,t-\ell)\mu_0^{t-\ell}, \quad n \geq 1.$$

Substituting (16) – (19) into (15) yields

(20) $$p^{eq}(0,t) = g(0)\lambda_0^n + \sum_{n=1}^{\infty} [g(0)\lambda_1 \sum_{k=1}^{n} \sum_{\ell=0}^{t} a_k \lambda_0^{\ell-1} p(n-k,t-\ell)\mu_0^{t-\ell}]$$
$$+ \sum_{n=1}^{\infty} \mu_0^t \sum_{k=1}^{n} g(k)p(n-k,t).$$

Performing the indicated sums yields, upon simplification,

(21) $$p^{eq}(0,t) = \mu_1(1-\rho)[\frac{\lambda_0^t-\mu_0^t}{\mu_1-\lambda_1}] + \mu_0^t .$$

From (3) we have $p(0,t) = \lambda_0^t$. Thus $p^{eq}(0,t) \neq p(0,t)$ and the theorem is proved.

We observe here also that if we allow $a_1 = 1$, $a_i = 0$, $i \neq 1$, equation (21) reduces to λ_0^t. This is as it should be for our model is now a single arrival queue with geometric input and output distributions.

We now turn our attention to verifying, as in Theorem II of Chapter I, that the discrete equilibrium output distribution agrees in expected value with the discrete input distribution. To do so we again develop the probabilities $p_i(m,n,r,t)$ and associated g.f.s.

Restricting ourselves first to the case $i \geq 1$, we can obtain, for $m \geq 1$, the equation

$$(22) \qquad xP_i(x,y,\cdot,t) = [\mu_0 x+\mu_1 y][\lambda_0+\lambda_1 A(x)]P_i(x,y,\cdot,t) - xy\lambda_0\mu_1 \sum_{n=0}^{\infty} p_i(1,n,0,t)y^n.$$

Using the initial condition $P_i(x,y,\cdot,0) = x^i$, the g.f. of $P_i(x,y,\cdot,t)$ can be written as

$$(23) \qquad \hat{P}_i(x,y,\cdot,s) = \frac{x^{i+1}-x\hat{H}_i(y,s)}{x-s[\lambda_0+\lambda_1 A(x)][\mu_0 x+\mu_1 y]} \quad .$$

Equation (23) gives the g.f. of the g.f. of the probabilities associated with the first busy period of the queue. Compare it with (30) of Chapter I. As before, it can be shown that the denominator of the RHS of (23) has a single zero inside the unit circle $|x| = 1$, and we obtain a result analogous to Theorem B of Appendix B.

Now

$$(24) \qquad P_i(x,y,z,t) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} p_i(m,n,0,t)x^m y^n + \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\sum_{r=1}^{\infty} p_i(m,n,0,t)x^m y^n z^r \quad .$$

Corresponding to (34) of Chapter I we may write, for $m \geq 1$, $n \geq 1$, $r \geq 1$,

$$(25) \qquad \hat{p}_i(m,n,r,s) = \frac{\lambda_1}{(1-\lambda_0 s)} \sum_{j=1}^{\infty}\sum_{k=1}^{n} \hat{h}_i(k,s) \cdot a_j p_j(m,n-k,r-1,s).$$

Multiply by powers of x, y, z, and add. After simplification we have

$$(26)\qquad \hat{p}_i(x,y,z,s) = \hat{P}_i(x,y,\cdot,s) + \lambda_1 z\hat{H}_i(y,s)\left[\frac{\sum_{j=1}^{\infty} a_j\hat{P}_j(x,y,\cdot,s)}{1-\lambda_0 s-\lambda_1 zs\sum_{j=1}^{\infty} a_j\hat{H}_j(y,s)}\right].$$

Equation (26) is the g.f. of the g.f. for the probabilities associated with the busy queue which started with $i \geq 1$ customers. It is analogous to (36) of Chapter I.

Turning to the idle case ($m = 0$), for $i \geq 1$, we write

$$(27)\qquad p_i(0,n,1,t) = \sum_{k=0}^{t} h_i(n,k)\lambda_1\lambda_0^{t-k}, \quad \text{and for } n \geq 2$$

$$(28)\qquad p_i(0,n,r,t) = \sum_{j=1}^{\infty}\sum_{k=i}^{n}\sum_{t_2=0}^{t}\sum_{t_1=0}^{t_2} h_i(k,t_1)\cdot\lambda_1\lambda_0^{t_2-t_1} \cdot a_j p_j(0,n-k,r-1,t-t_2)\ .$$

These are obtained from L.T.P. and by using the concept of regeneration points as was done in Chapter I.

Forming g.f.s for (27) and (28) yields

$$(29)\qquad \hat{p}_i(0,n,1,s) = \frac{\lambda_1}{1-\lambda_0 s}\hat{h}_i(n,s), \quad \text{and for } r \geq 2,$$

$$(30)\qquad \hat{p}_i(0,n,r,s) = \frac{\lambda_1}{1-\lambda_0 s}\sum_{j=1}^{\infty}\sum_{k=i}^{n}\hat{h}_i(k,s)\cdot a_j p_j(0,n-k,r-1,s).$$

Now multiply (29) and (30) by powers of y and z and sum.

$$(31) \qquad \hat{P}_i(\cdot,y,z,s) = \frac{\lambda_1 z\hat{H}_i(y,s)}{1-\lambda_0 s} + \frac{\lambda_1 z\hat{H}_i(y,s)}{1-\lambda_0 s} \sum_{j=1}^{\infty} a_j \hat{P}_j(\cdot,y,z,s).$$

Further simplification yields

$$(32) \qquad \hat{P}_i(\cdot,y,z,s) = \frac{\lambda_1 z\hat{H}_i(y,s)}{1-\lambda_0 s-\lambda_1 zs \sum_{j=1}^{\infty} a_j \hat{H}_j(y,s)} .$$

The sum of (26) and (32) then yields the g.f. of the total g.f. for the probabilities associated with the queue for the case $i \geq 1$. This sum is

$$(33) \qquad \hat{T}_i(x,y,z,s) = \hat{P}_i(x,y,\cdot,s) + z\hat{H}_i(y,s)\left[\frac{1+\lambda_1 s \sum_{j=1}^{\infty} a_j \hat{P}_j(x,y,\cdot,s)}{1-\lambda_0 s-\lambda_1 zs \sum_{j=1}^{\infty} a_j \hat{H}_j(y,s)}\right] .$$

Compare it with (43) of Chapter I.

The case $i = 0$ remains. By the L.T.P. we have

$$(34) \qquad p_0(m,n,r,t) = \sum_{j=1}^{\infty} \sum_{k=0}^{t} \lambda_1 \lambda_0^k \, a_j p_j(m,n,r,t-k), \quad (m \geq 1).$$

Forming the g.f. for (20) yields

$$(35) \qquad \hat{P}_0(x,y,z,s) = \frac{\lambda_1}{1-\lambda_0 s} \sum_{j=1}^{\infty} a_j \hat{P}_j(x,y,z,s), \quad (m \geq 1).$$

The idle case $(m = 0)$ is derived in a similar fashion and we obtain the following equation analogous to (48) of Chapter I:

$$(36) \qquad \hat{P}_0(\cdot,y,z,s) = \frac{1}{1-\lambda_0 s} + \frac{\lambda_1}{1-\lambda_0 s} \sum_{j=1}^{\infty} a_j \hat{P}_j(\cdot,y,z,s) .$$

The sum of the g.f.s (35) and (36) yields the g.f. of the total g.f. for the probabilities associated with the queue for the case $i = 0$.

$$(37) \qquad T_0(x,y,z,s) = \frac{1+\lambda_1 s \sum_{j=1}^{\infty} a_j \hat{P}_j(x,y,\cdot,s)}{1-\lambda_0 s-\lambda_1 zs \sum_{j=1}^{\infty} a_j \hat{H}_j(y,s)} .$$

Our final task before proving the second theorem of this chapter will be to obtain expressions for the g.f. of the g.f. of the joint equilibrium distribution of queue size, queue output, and number of busy periods completed, as well as an expression for the g.f. of the g.f. of the equilibrium output distribution. Proceeding in a manner similar to that of Chapter I, we obtain, analogous to equations (54), (55), and (56), respectively, of that chapter the equations

$$(38) \qquad \hat{P}_{eq}(x,y,z,s) = \frac{x[G(x)-G(\hat{H}_1(y,s))]}{x-s[\lambda_0+\lambda_1 A(x)][\mu_0 x+\mu_1 y]} + [\frac{1+\lambda_1 s \sum_{\ell=1}^{\infty} a_\ell \hat{P}_\ell(x,y,\cdot,s)}{1-\lambda_0 s-\lambda_1 zs \sum_{\ell=1}^{\infty} a_\ell \hat{H}_\ell(y,s)}] G\{z\hat{H}_1(y,s)\},$$

$$(39) \qquad \hat{P}_{eq}(1,y,1,s) = \frac{1-G(\hat{H}_1(y,s))}{1-s(\mu_0+\mu_1 y)} + [\frac{1+\lambda_1 s \sum_{\ell=1}^{\infty} a_\ell \hat{P}_\ell(1,y,\cdot,s)}{1-\lambda_0 s-\lambda_1 s \sum_{\ell=1}^{\infty} a_\ell \hat{H}_\ell(y,s)}] G\{\hat{H}_1(y,s)\} ,$$

and

$$(40) \quad \hat{P}_{eq}(1,y,1,s) = \frac{1}{1-s(\mu_0+\mu_1 y)}$$
$$+ \frac{s\mu_1[\mu_0\hat{H}_1(y,s)+\mu_1 y][1-y][1-\rho][1-\hat{H}_1(y,s)]}{[y-\hat{H}_1(y,s)][1-s(\mu_0+\mu_1 y)][(1-s)\mu_0\hat{H}_1(y,s)+\mu_1(1-sy)]} .$$

In obtaining (40) from (39) we have used the fact that $\hat{H}_1(y,s)$ is a solution, in x, of the denominator of the RHS of (23) and the fact that $G(\hat{H}_1(y,s))$ also satisfies (9). Equation (40) is the g.f. of the g.f. of the equilibrium output distribution.

We are now ready to prove

THEOREM II: Let $E(N(t))$ denote the expected value of the number of arrivals in t time units and $E(N^{eq}(t))$ denote the expected value of the number of departures (i.e. completions), under equilibrium conditions, in t units of time. Then

$$E(N(t)) = E(N^{eq}(t)) .$$

PROOF.

Recall that the g.f. of p(n,t) is given by $[\lambda_0+\lambda_1 A(y)]^t$. Clearly then $E(N(t)) = \lambda_1 \bar{a} t$. To evaluate $E(N^{eq}(t))$ we consider equation (40). $E(N^{eq}(t))$ is obtained by evaluating the derivative of the RHS of (40) w.r.t. y at y = 1, and then inverting (i.e. extracting the coefficient of s^t) the resulting expression. Now,

$$(41)\qquad \frac{d}{dy}\hat{P}_{eq}(1,y,1,s)\Big|_{y=1} = \frac{s\mu_1}{(1-s)^2}$$

$$+\frac{-s\mu_1[1-\rho][\mu_0\hat{H}_1(1,s)+\mu_1]^2[1-s]^2[1-\hat{H}_1(1,s)]^2}{[(1-\hat{H}_1(1,s))(1-s)^2(\mu_0\hat{H}_1(1,s)+\mu_1)]^2}$$

$$= \frac{s\mu_1}{(1-s)^2} - \frac{s\mu_1(1-\rho)}{(1-s)^2}$$

$$= \frac{\mu_1\rho}{(1-s)^2}$$

$$= \frac{\lambda\bar{a}}{(1-s)^2}\ .$$

Extracting the coefficient of s^t then yields

$$(42)\qquad E\{N^{eq}(t)\} = \lambda_1\bar{a}t$$ and the theorem is proved.

As in Chapter I, if we make the assumption that $a_1 = 1$, $a_i = 0$ for $i \neq 1$, (42) reduces to $\lambda_1 t$, which is the expected value of a discrete queue with binomial input. This is as it should be, for in this particular case our model reduces to a model with single arrivals characterized by a binomial input and geometric service distribution. For this queue it is known that the expected value of the equilibrium output distribution is the same as the input distribution's expected value.

In concluding this chapter we remark that, as described in the final paragraph of Chapter I, by considering the limits $x \to 1$, $y \to 1$, or $z \to 1$ on the g.f. (26), (32), (35), and (37) we can obtain additional information about the queue.

CHAPTER III

RESULTS FOR A QUEUE WITH SERVICE DEPENDENT ON QUEUE SIZE

In many instances we encounter queueing models for which the rate of service at time t is dependent upon the size of the queue at that time, e.g. a long queue may cause the server to work at a faster rate whereas a short queue will cause the server to work at a more leisurely rate. For this type of queue only the equilibrium distribution of queue size [1] and the results of Reich stated in Chapter I have been determined. In this chapter we consider the model of Reich with a single server and obtain new results. Recall that this model is characterized by Poisson arrivals and service described as follows: If at time t there are n, $(n \geq 1)$, customers in the system, then the conditional probability that a customer will complete service in $(t,t+\delta)$ is $\mu_n \delta + \sigma(\delta)$. Here $\sigma(\delta)$ is a function such that $\lim_{\delta \to 0} \frac{\sigma(\delta)}{\delta} = 0$. We shall determine for this queue an explicit expression for the joint distribution of the number of customers served in a busy period and the length of that busy period. We then give a method of finding an expression for the Laplace transform of the joint distribution of the number of

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[1] See, for example, Cox and Smith [5], page 43.

customers in the system at time t, the number of customers served in $(0,t]$, and the busy period is still in progress at time t. Finally, we indicate a method of obtaining an expression for the Laplace transform of the joint distribution of the number of customers in the system at time t, the number of customers served in $(0,t]$, and the number of busy periods completed in $(0,t]$. In the special case $\mu_i = \mu$, $i = 1,2,3,\ldots$, we shall see that these distributions yield results known for an $M|M|1$ queue. Thus the $M|M|1$ queue, relative to the distributions discussed in this chapter, could be considered as a special case of the model of this chapter.

We begin the analysis by defining the following probabilities associated with the model:

$$p_k(m,n,0,t) = \Pr[M(t)=m,\ N(t)=n,\ R(t)=0|M(0)=k], \quad (m \geq 1),$$

and

$$p_k(m,n,r,t) = \Pr[M(t)=m,\ N(t)=n,\ R(t)=r|M(0)=k], \quad (r \geq 1).$$

Here the random variables $M(t)$, $N(t)$, and $R(t)$ are defined as in Chapter I.

Define $h_k(m,t)$ to be the joint probability and p.d.f. of the number of customers served in the first busy period and the length of that busy period (which started with k customers present).

THEOREM I: Let $h_k(m,t)$ be defined as above. Then

(i) $h_k(m,t) = 0$ for $m < k$,

(ii) $h_1(1,t) = e^{-\lambda t}\mu_1 e^{-\mu_1 t}$,

(iii) $h_k(k,t) = \{e^{-\lambda t}\mu_k e^{-\mu_k t}\} * h_{k-1}(k-1,t), \quad k = 2,3,\ldots,$

and for $n \geq 1$,

(iv) $$h_k(k+n,t) = \lambda^n \sum_{i_n=1}^{k} \sum_{i_{n-1}=1}^{i_n+1} \cdots \sum_{i_1=1}^{i_2+1} h_{i_1+1}(i_1+1,t)$$

$$* \{e^{-\lambda t} e^{-\mu_{i_1} t}\} * \mathop{\Omega}_{i_0=i_1+1}^{i_2+1} \{e^{-\lambda t}\mu_{i_0} e^{-\mu_{i_0} t}\}$$

$$* \ldots *\{e^{-\lambda t} e^{-\mu_{i_n} t}\}$$

$$* \mathop{\Omega}_{i_{n-1}=i_n+1}^{k} \{e^{-\lambda t}\mu_{i_{n-1}} e^{-\mu_{i_{n-1}} t}\} .$$

Here * is the convolution operation,

$$f(t) * g(t) = \int_0^t f(u)g(t-u)du, \quad \text{and}$$

$$\mathop{\Omega}_{j=1}^{s} \{e^{-\lambda t}\mu_j e^{-\mu_j t}\} = \{e^{-\lambda t}\mu_1 e^{-\mu_1 t}\} * \{e^{-\lambda t}\mu_2 e^{-\mu_2 t}\}$$

$$* \ldots * \{e^{-\lambda t}\mu_s e^{-\mu_s t}\} .$$

Also, for $m > k$, $h_k(m,t) = h_k(k+n,t)$ where $n = m-k$. Hence the notation in (iv).

PROOF.

(i) This is obvious since the number served in a busy period cannot be less than the number of customers that initiate the busy period.

(ii) Clearly. $h_1(1,t) = e^{-\lambda t}\mu_1 e^{-\mu_1 t}$, as this is the busy period consisting of no arrivals and one service in $(0,t]$.

(iii) First note that for $h_k(k,t)$ there are no arrivals in $(0,t]$. The service time of the first customer is completed at some time t_1, $(0 < t_1 < t)$, and then, in the interval $(t_1,t]$, a busy period consisting of k-1 services and no arrivals must be completed. As t_1 ranges from 0 to t, $h_k(k,t)$ can be written as

$$(2) \qquad h_k(k,t) = \{e^{-\lambda t}\mu_k e^{-\mu_k t}\} * h_{k-1}(k-1,t), \quad k = 2,3,\ldots .$$

(iv) The proof is by induction on n. We first note that, by considering arrival and service possibilities, for $n \geq 1$, $k \geq 1$, k fixed, we may write

$$(3) \qquad h_k(k+n,t) = \{\lambda e^{-\lambda t}e^{-\mu_k t}\} * h_{k+1}(k+n,t)$$

$$+ \{e^{-\lambda t}\mu_k e^{-\mu_k t}\} * h_{k-1}(k-1+n,t).$$

Now let $k \geq 1$ and fixed and assume $n = 1$. By (3)

$$(4) \qquad h_k(k+1,t) = \{\lambda e^{-\lambda t}e^{-\mu_k t}\} * h_{k+1}(k+1,t)$$

$$+ \{e^{-\lambda t}\mu_k e^{-\mu_k t}\} * h_{k-1}(k,t) .$$

Apply (3) to $h_{k-1}(k,t)$.

$$(5) \qquad h_{k-1}(k,t) = \{\lambda e^{-\lambda t} e^{-\mu_{k-1}t}\} * h_k(k,t) + \{e^{-\lambda t}\mu_{k-1}e^{-\mu_{k-1}t}\}$$

$$* \, h_{k-2}(k-1,t) \; .$$

Repeating the process on $h_{k-2}(k-1,t)$ we obtain

$$(6) \qquad h_{k-2}(k-1,t) = \{\lambda e^{-\lambda t} e^{-\mu_{k-2}t}\} * h_{k-1}(k-1,t) + \{e^{-\lambda t}\mu_{k-2}e^{-\mu_{k-2}t}\}$$

$$* \, h_{k-3}(k-2,t).$$

Continuation of this process on the last term of the second convolution of each equation obtained in this manner yields at the final step

$$(7) \qquad h_1(2,t) = \{\lambda e^{-\lambda t} e^{-\mu_1 t}\} * h_2(2,t).$$

Substituting into (4) the values of $h_{k-i}(k-i+1,t)$, $i = 1,2,\ldots,k-1$, obtained in this manner and simplifying yields

$$(8) \qquad h_k(k+1,t) = \lambda\{e^{-\lambda t} e^{-\mu_k t}\} * h_{k+1}(k+1,t) + \lambda\{e^{-\lambda t}\mu_k e^{-\mu_k t}\}$$

$$* \{e^{-\lambda t} e^{-\mu_{k-1}t}\} * h_k(k,t) + \lambda\{e^{-\lambda t}\mu_k e^{-\mu_k t}\}$$

$$* \, \{e^{-\lambda t}\mu_{k-1}e^{-\mu_{k-1}t}\} * \{e^{-\lambda t} e^{-\mu_{k-2}t}\}$$

$$* \, h_{k-1}(k-1,t) + \ldots + \lambda\{e^{-\lambda t}\mu_k e^{-\mu_k t}\} * \ldots$$

$$* \, \{e^{-\lambda t}\mu_2 e^{-\mu_2 t}\} * \{e^{-\lambda t} e^{-\mu_1 t}\} * h_2(2,t) \; .$$

Or, in the notation of (iv)

$$(9)\qquad h_k(k+1,t) = \lambda \sum_{i=1}^{k} h_{i+1}(i+1,t) * \{e^{-\lambda t}e^{-\mu_i t}\} * \prod_{j=i+1}^{k} \{e^{-\lambda t}\mu_j e^{-\mu_j t}\}.$$

Thus (iv) is true for n = 1. Now hypothesize that, for $n \geq 2$,

$$(10)\qquad h_k(k+n-1,t) = \lambda^{n-1} \sum_{i_{n-1}=1}^{k} \sum_{i_{n-2}=1}^{i_{n-1}+1} \cdots \sum_{i_1=1}^{i_2+1} h_{i_1+1}(i_1+1,t)$$

$$* \{e^{-\lambda t}e^{-\mu_{i_1} t}\} * \prod_{i_0=i_1+1}^{i_2+1} \{e^{-\lambda t}\mu_{i_0} e^{-\mu_{i_0} t}\}$$

$$* \{e^{-\lambda t}e^{-\mu_{i_2} t}\} * \prod_{i_1=i_2+1}^{i_3+1} \{e^{-\lambda t}\mu_{i_1} e^{-\mu_{i_1} t}\}$$

$$* \dots * \{e^{-\lambda t}e^{-\mu_{i_{n-1}} t}\}$$

$$* \prod_{i_{n-2}=i_{n-1}+1}^{k} \{e^{-\lambda t}\mu_{i_{n-2}} e^{-\mu_{i_{n-2}} t}\} .$$

Recall that

$$(3)\qquad h_k(k+n,t) = \{\lambda e^{-\lambda t}e^{-\mu_k t}\} * h_{k+1}(k+n,t) + \{e^{-\lambda t}\mu_k e^{-\mu_k t}\}$$

$$* h_{k-1}(k+n-1,t) .$$

Repeating on $h_k(k+n,t)$ the procedure employed in the proof of the case n = 1 yields, after simplification,

$$(11)\qquad h_k(k+n,t) = \lambda\{e^{-\lambda t}e^{-\mu_k t}\} * h_{k+1}(k+1+n-1,t) + \lambda\{e^{-\lambda t}\mu_k e^{-\mu_k t}\}$$

$$* \{e^{-\lambda t}e^{-\mu_{k-1} t}\} * h_k(k+n-1,t) + \ldots$$

$$+ \lambda\{e^{-\lambda t}\mu_k e^{-\mu_k t}\} * \{e^{-\lambda t}\mu_{k-1} e^{-\mu_{k-1} t}\} * \ldots$$

$$* \{e^{-\lambda t}\mu_2 e^{-\mu_2 t}\} * \{e^{-\lambda t}e^{-\mu_1 t}\} * h_2(2+n-1,t)\ .$$

Or,

$$(12)\qquad h_k(k+n,t) = \lambda \sum_{i_n=1}^{k} h_{i_n+1}(i_n+1+n-1) * \{e^{-\lambda t}e^{-\mu_{i_n} t}\}$$

$$* \mathop{\Omega}_{j=i_n+1}^{k} \{e^{-\lambda t}\mu_j e^{-\mu_j t}\}\ .$$

Hence, by the induction equation (9)

$$(13)\qquad h_k(k+n,t) = \sum_{i_n=1}^{k} \sum_{i_{n-1}=1}^{i_n+1} \ldots \sum_{i_1=1}^{i_2+1} h_{i_1+1}(i_1+1,t) * \{e^{-\lambda t}e^{-\mu_{i_1} t}\}$$

$$* \mathop{\Omega}_{i_0=i_1+1}^{i_2+1} \{e^{-\lambda t}\mu_{i_0} e^{-\mu_{i_0} t}\} * \ldots * \{e^{-\lambda t}e^{-\mu_{i_n} t}\}$$

$$* \mathop{\Omega}_{i_{n-1}=i_n+1}^{k} \{e^{-\lambda t}\mu_{i_{n-1}} e^{-\mu_{i_{n-1}} t}\}\ .$$

By the Principle of Mathematical deduction (iv) is established for all integers $n \geq 1$.

Theorem I gives then the explicit expressions for the joint time dependent probabilities of the number of customers served in the first busy period and the length of that busy period.

Because of the form of expressions (2) and (13), in practice it is easier to work with their Laplace transforms. These transforms are

$$(14) \qquad \bar{h}_k(k,s) = \frac{\mu_1\mu_2\cdots\mu_k}{(s+\lambda+\mu_1)(s+\lambda+\mu_2)\cdots(s+\lambda+\mu_k)}, \quad k = 1,2,3,\ldots,$$

and for $k = 1,2,3,\ldots;\ n = 1,2,3,\ldots,$

$$(15) \qquad \bar{h}_k(k+n,s) = \lambda^n \sum_{i_n=1}^{k} \sum_{i_{n-1}=1}^{i_n+1} \cdots \sum_{i_1=1}^{i_2+1} \bar{h}_{i_1+1}(i_1+1,s)$$

$$\cdot \frac{\mu_{i_1+1}\cdots\mu_{i_2}\mu_{i_2+1}}{(s+\lambda+\mu_{i_1})\cdots(s+\lambda+\mu_{i_2+1})} \times \cdots$$

$$\times \frac{\mu_{i_n+1}\mu_{i_n+2}\cdots\mu_k}{(s+\lambda+\mu_{i_n})(s+\lambda+\mu_{i_n+1})\cdots(s+\lambda+\mu_k)}.$$

If we make the assumption $\mu_i = \mu$, $i = 1,2,3,\ldots$, our model reduces to an $M|M|1$ queue and $h_k(m,t)$ becomes the probabilities given by Prabhu [12]. In this instance we have

$$(16) \qquad \bar{h}_k(k,s) = \frac{\mu^k}{(s+\lambda+\mu)^k}, \quad k = 1,2,3,\ldots, \text{ and}$$

$$(17) \qquad \bar{h}_k(k+n,s) = \frac{k(k+2n-1)!\,\lambda^n\mu^{k+n}}{n!\,(k+n)!\,(s+\lambda+\mu)^{k+2n}}, \quad k=1,2,3,\ldots,\ n=1,2,3,\ldots .$$

Equations (14) and (15) reduce to (16) and (17), respectively, if we put $\mu_i = \mu$, $i = 1,2,3,\ldots$, in our model. Thus, the results of Prabhu are obtainable as a special case of Theorem I.

The probabilities $h_k(m,t)$ will now be used to obtain expressions for the Laplace transform of $p_i(m,n,r,t)$. Restricting ourselves to the case $k \geq 1$ and $r = 0$, we obtain the following time dependent equations:

$$(18) \qquad \frac{d}{dt} p_k(1,n+1,0,t) + (\lambda+\mu_1)p_k(1,n+1,0,t) = \mu_2 p_k(2,n,0,t),$$

$$(19) \qquad \frac{d}{dt} p_k(m,n+1,0,t) + (\lambda+\mu_m)p_k(m,n+1,0,t) = \lambda p_k(m-1,n+1,0,t) + \mu_{m+1}p_k(m+1,n,0,t), \quad m = 2,3,\ldots .$$

We are assuming $m+n \geq k$ and $m \geq 1$. Using the initial condition

$$(20) \qquad \begin{aligned} &p_k(k,0,0,0) = 1 \\ &p_k(m,n,0,0) = 0 \quad \text{if } m \neq k \text{ and } n \geq 0, \end{aligned}$$

the Laplace transform of (18) and (19) can be written as

$$(21) \qquad (s+\lambda+\mu_1)\bar{p}_k(1,n+1,0,s) = \mu_2\bar{p}_k(2,n,0,s),$$

$$(22) \qquad (s+\lambda+\mu_m)\bar{p}_k(m,n+1,0,s) = \lambda\bar{p}_k(m-1,n+1,0,s) + \mu_{m+1}\bar{p}_k(m+1,n,0,s),$$

$$m = 2,3,\ldots .$$

Or

$$(23) \qquad \bar{p}_k(2,n,0,s) = \frac{(s+\lambda+\mu_1)}{\mu_2}\bar{p}_k(1,n+1,0,s), \quad \text{and for } m \geq 3,$$

$$(24)\qquad \bar{p}_k(m,n,0,s) = \frac{(s+\lambda+\mu_{m-1})}{\mu_m}\bar{p}_k(m-1,n+1,0,s) - \frac{\lambda}{\mu_m}\bar{p}_k(m-2,n+1,0,s).$$

Recall that

$$(25)\qquad h_k(n,t) = \mu_1 p_k(1,n-1,0,t)$$ [2], and hence

$$(26)\qquad \bar{h}_k(n,s) = \mu_1\bar{p}_k(1,n-1,0,s).$$

Thus

$$(27)\qquad \bar{p}_k(1,n,0,s) = \frac{\bar{h}_k(n+1,s)}{\mu_1}.$$

Equation (27) may be used in (23) to solve for $\bar{p}_k(2,n,0,s)$.

$$(28)\qquad \bar{p}_k(2,n,0,s) = \frac{(s+\lambda+\mu_1)}{\mu_1\mu_2}\bar{h}_k(n+2,s).$$

Equations (27) and (28) can be used now to obtain the expression for $\bar{p}_k(3,n,0,s)$ and, by recursion, we can obtain the expressions for $\bar{p}_k(m,n,0,s)$, $(m = 4,5,\ldots)$, in terms of $\bar{h}_k(r,s)$, $r = n+1,\ldots,n+m$. As an illustration we give expressions for $\bar{p}_k(3,n,0,s)$ and $\bar{p}_k(4,n,0,s)$:

$$(29)\qquad \bar{p}_k(3,n,0,s) = \frac{(s+\lambda+\mu_1)(s+\lambda+\mu_2)}{\mu_1\mu_2\mu_3}\bar{h}_k(n+3,s) - \frac{\lambda}{\mu_1\mu_3}\bar{h}_k(n+2,s),$$

$$(30)\qquad \bar{p}_k(4,n,0,s) = \frac{(s+\lambda+\mu_1)(s+\lambda+\mu_2)(s+\lambda+\mu_3)}{\mu_1\mu_2\mu_3\mu_4}\bar{h}_k(n+4,s) - \frac{\lambda\bar{h}_k(n+3,s)}{\mu_1\mu_4}\left[\frac{(s+\lambda+\mu_1)}{\mu_2} + \frac{(s+\lambda+\mu_3)}{\mu_3}\right].$$

Thus we have a method of obtaining expressions for the Laplace

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[2] See Appendix B, Remark A.

transform of the probabilities $p_k(m,n,0,t)$ associated with the first busy period.

For $k \geq 1$, $r \geq 1$, $m \geq 1$, and $n \geq r+k-1$, we have from the L.T.P. and the regenerative property of the instants of commencement of the busy periods,

$$(31) \qquad p_k(m,n,r,t) = \sum_{j=k}^{n} \int_0^t \int_0^{t_2} [h_k(j,t)\cdot\lambda e^{-\lambda(t_2-t_1)} \cdot p_1(m,n-j,r-1,t-t_2)]dt_1 dt_2.$$

Taking the transform yields

$$(32) \qquad \overline{p}_k(m,n,r,s) = \frac{\lambda}{s+\lambda} \sum_{j=k}^{n} \overline{h}_k(j,s)\cdot\overline{p}_1(m,n-j,r-1,s),$$

valid for $Re(s) > 0$, $k \geq 1$, $r \geq 1$, $m \geq 1$, $n \geq r+k-1$. The sum of (23), (24), (27), and (32) gives the expression for the Laplace transform of the total probabilities, associated with a busy queue for the case $k \geq 1$.

We turn now to the idle case. If $k \geq 1$, we have

$$(33) \qquad p_k(0,n,1,t) = \int_0^t h_k(n,t_1)e^{-\lambda(t-t_1)} dt_1,$$

a direct consequence of the L.T.P. If $r \geq 2$,

$$(34) \qquad p_k(0,n,r,t) = \sum_{j=k}^{n} \int_0^t \int_0^{t_2} [h_k(j,t)\cdot\lambda e^{-\lambda(t_2-t_1)} \cdot p_1(0,n-j,r-1,t-t_2)]dt_1 dt_2;$$

we have used the same regeneration point as before.

Transforming (33) and (34) yields

$$(35)\qquad \bar{p}_k(0,n,1,s) = \frac{1}{s+\lambda}\,\bar{h}_k(n,s), \quad \text{and if } r \geq 2,$$

$$(36)\qquad \bar{p}_k(0,n,r,s) = \frac{\lambda}{s+\lambda}\sum_{j=k}^{n} \bar{h}_k(j,s)\cdot\bar{p}_1(0,n-j,r-1,s).$$

By expressing n in the form $n = k+i$, $i \geq 0$, and using (14)-(15) we can determine an explicit expression for $\bar{p}_k(0,n,1,s)$. Equation (36) is then obtainable by recursion from (35).

The sum of (35) and (36) gives the expression for the Laplace transform of the probabilities associated with an idle queue for the case $k \geq 1$. Also, the sum of (23), (24), (27), (32), (35), and (36) gives the expression for the Laplace transform of the total probabilities associated with the queue for the case $k \geq 1$.

Before discussion the case $k = 0$, we note that by the structure of equations (32) and (36) we may write

$$(37)\qquad \bar{p}_k(m,n,r,s) = \frac{\lambda}{s+\lambda}\sum_{j=k}^{n} \bar{h}_k(j,s)\cdot\bar{p}_1(m,n-j,r-1,s),$$

valid for $m \geq 0$, $k \geq 1$, $r \geq 1$, $n \geq r+k-1$, $\mathrm{Re}(s) > 0$.

If we assume $\mu_i = \mu$, $i = 1,2,3,\ldots$, the model becomes an $M|M|1$ queue and the probabilities $p_k(m,n,r,t)$ become those determined by Scott [15]. For the values of m,n,r, and k given by (37), the Laplace transform of the probabilities given by Scott is

$$(38)\qquad \bar{p}_k(m,n,r,s) = \frac{(m+r+k-1)\,(m+2n-r-k)!\,\lambda^{m+n-k}\mu^{n}}{(m+n)!\,(n-r-k+1)!\,(s+\lambda)^{r}\,(s+\lambda+\mu)^{m+2n-r-k+1}},$$

$\mathrm{Re}(s) > 0$, $m \geq 0$, $k \geq 1$, $r \geq 1$, $n \geq k+r-1$.

Assuming then that $\mu_i = \mu$, $i = 1,2,3,\ldots$, for representative values of m,n,k, and r it has been shown that (37) reduces to (38). Thus it appears that the results of Scott are obtainable as a special case of the given model.

Turning to the case $k = 0$, the probabilities $p_0(m,n,r,t)$ can be obtained from the equation

$$(39) \qquad p_0(m,n,r,t) = \delta_{mnr} e^{-\lambda t} + (1-\delta_{mnr}) \int_0^t \lambda e^{-\lambda t_1} \cdot p_1(m,n,r,t-t_1)\,dt_1,$$

$$(m \geq 0,\ n \geq 0,\ r \geq 0).$$

Here $\delta_{mnr} = 1$ for $m = n = r = 0$,

$= 0$ otherwise.

Thus

$$(40) \qquad \bar{p}_0(m,n,r,s) = \frac{\delta_{mnr}}{\lambda+s} + \frac{\lambda(1-\delta_{mnr})}{s+\lambda} \cdot \bar{p}_1(m,n,r,s), \quad (m\geq 0,\ n\geq 0,\ r\geq 0).$$

Equation (40) is the expression of the Laplace transform of the probabilities associated with the queue for the case $k = 0$.

In concluding this chapter we remark that the discrete model of this type of queue can be analyzed in a manner similar to that employed here. Thus, as was done in Chapter II, results are obtainable for the discrete queue associated with the model of this chapter. However, we shall not consider this discrete model at this time.

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APPENDICES

APPENDIX A

Basic Notions of Probability and Random Variables

Definitions

Experiment. Any process of observations.

Outcome. A result of an experiment.

Sample Space. The set of all logical possibilities for the outcome of an experiment.

Event. Subset of a sample space.

Random Variable. A function from a sample space into the reals.

Probability Space. Let S be a sample space and P a set function from the power set of S into the closed unit interval with the following properties:

1) For every set A in the power set of S, $0 \leq P(A) \leq 1$.

2) $P(S) = 1$.

3) If $A_1, A_2, \ldots,$ is a sequence of mutually disjoint subsets of S, then $P(\bigcup_{i=1}^{\infty} A_i) = P(A_1) + P(A_2) + \ldots$.

Then P is called a probability measure on S and (S,P) is called a probability space.

Probability. Let (S,P) be a probability space and X a random variable defined on S. Then if K is a subset of the reals, the probability of K is

$$Pr[K] = P[X^{-1}(K)],$$

i.e. Pr[K] is the measure of the inverse image by the random variable of the set K.

The notation $Pr[X \leq x] = a$ means $Pr\{(-\infty,x]\} = a$, and similar definitions apply to $Pr[X < x] = a$, $Pr[x < X \leq y] = a$, etc. The function

$$F(x) = Pr[X \leq x]$$

is called the distribution function (d.f.) of the random variable X. It is well known that F has the following properties:

1) $\lim_{x\to-\infty} F(x) = 0,$

2) $\lim_{x\to\infty} F(x) = 1,$

3) F is a non-decreasing, bounded function from R into [0,1].

If F is differentiable we may define the probability density function (p.d.f.) of X by

$$f(x) = \frac{d}{dx} F(x)$$

so that $F(x) = \int_{-\infty}^{x} f(x)dx$. Another function associated with the random variable X is the survivor function

$$F_c(x) = 1 - F(x).$$

If X has a p.d.f. f(x), we have

$$F_c(x) = \int_{x}^{\infty} f(x)dx.$$

Similar remarks hold for a discrete random variable Y if the integrals above are replaced by sums (over the same interval), excepting the fact that the p.d.f. f(y) of Y is no longer determinable by differentiation.

APPENDIX B

Some Supporting Remarks

REMARK A. Let T denote the length of the first busy period of the $M|M|1$ queue with batch arrivals. If $h_i(n,t)$ is the joint p.d.f. and probability of T and N(t), then

$$h_i(n,t) = \mu p_i(1,n-1,0,t).$$

THEOREM B. The denominator of the RHS of (30) Chapter I, namely

$$(\lambda+\mu+s)x-\mu y-\lambda xA(x), \quad [\mathrm{Re}(s) > 0,\ \lambda > 0,\ \mu > 0,\ y \in (0,1]],$$

has one root inside the unit circle $|x| = 1$. This root is $\bar{H}_1(y,s)$.

PROOF.

The functions $f(x) = (\lambda+\mu+s)x$ and $g(x) = -\mu y-\lambda xA(x)$ are both analytic on and inside the circle $|x| = 1$. For $|x| = 1$, we have

$$|f(x)| = |\lambda+\mu+s|, \text{ and}$$

$$|g(x)| = |\mu y+\lambda xA(x)| \leq |\mu y+\lambda| \leq |\mu+\lambda|.$$

Since $\mathrm{Re}(s) > 0$, $|g(x)| < |f(x)|$ on $|x| = 1$. Hence by Rouche's Theorem, f and f+g have the same number (one) of zeros inside $|x| = 1$. Call it $\alpha_1(y,s)$. Since the zeros of the denominator and numerator of the RHS of (30) must coincide, $\alpha_1(y,s)$ is a root of

(1) $$x^{i+1} - x\bar{H}_i(y,s) = 0.$$

Equation (1) is equal to

(2) $x[x^i - \{\overline{H}_1(y,s)\}^i] = 0.$

Thus the numerator is zero when $x = 0$ or when $x = \overline{H}_1(y,s)$. Assume $\alpha_1(y,s) = 0$. Then we must have

$$(\lambda+\mu+s)\cdot 0 - \mu y - \lambda 0\cdot 0 = 0,$$

or $\mu y = 0$. Since $\mu > 0$, $y > 0$, this is a contradiction. Therefore

$$\alpha_1(y,s) = \overline{H}_1(y,s).$$

Rouche's Theorem. If $f(z)$ and $g(z)$ are regular inside and on a closed contour C, and $|g(z)| < |f(z)|$ on C, then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside C.