THE EXISTENCE OF MATRICES WITH PRESCRIBED CHARACTERISTIC POLYNOMIALS

by

DAVID KELLER BAXTER

A DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in
the Department of Mathematics
in the Graduate School of
The University of Alabama

UNIVERSITY, ALABAMA

1978

ACKNOWLEDGMENTS

I wish to express my gratitude to my committee for the supervision I received during the writing of this dissertation. I particularly wish to express my sincere appreciation to my advisor, Professor Peter M. Gibson of The University of Alabama in Huntsville, for his guidance in the development of this dissertation, for the many helpful discussions, and for his encouragement.

TABLE OF CONTENTS

CHAPTER I

Introduction

It is well known that every monic polynomial of degree n with coefficients in a field F is the characteristic polynomial of some $n \times n$ matrix A with entries in F. This result is actually quite elementary. Efforts have been made to impose additional restrictions on such a matrix A. A survey of much of the work that has been done in this area of research is given in a paper by L. Mirsky [11]. Although much of the literature treats the existence of matrices with prescribed characteristic polynomials and prescribed entries, a number of interesting questions have been raised concerning the existence of matrices having various other properties. Of these, several were selected as being of sufficient import to warrant further investigation. It is the primary purpose of this dissertation to present results on two problems pertaining to the existence of matrices with prescribed characteristic polynomials.

In Chapter II we shall consider the existence of matrices and linear operators with prescribed characteristic polynomials and prescribed image vectors. The primary motivation for this consideration was a paper by H. Wielandt [14]. In his paper, Wielandt first noted that if x and y are two linearly independent vectors of n complex components, then there is no restriction imposed upon the characteristic values of an $n \times n$ matrix A by postulating that $Ax = y$. However, the following more general question was raised and is apparently still open.

> Given an $n \times n$ complex matrix A, n-vectors $x_1, x_2, \ldots, x_s$ and $Ax_1, Ax_2, \ldots, Ax_s$, what can be said about the spectrum of A?

Chapter II includes conditions on the prescribed image vectors which ensure that there is no restriction imposed upon the characteristic polynomial of the matrix A.

Perhaps a more natural setting for Wielandt's question is in terms of linear operators on a finite-dimensional vector space. We may consider the following problem.

> Let V be an n-dimensional vector space over a field F, $\{\xi_1, \xi_2, \ldots, \xi_s\}$ a linearly independent subset of V, $U = \langle\xi_1, \xi_2, \ldots, \xi_s\rangle$, and $\sigma: U \to V$ a fixed linear transformation. What conditions on σ are necessary and sufficient such that for each monic polynomial $p(\lambda)$ over F of degree n, there exists a linear operator $\tau: V \to V$ with the properties that the characteristic polynomial of τ is $p(\lambda)$ and the restriction of τ to U is σ?

We shall present a solution to this problem in Chapter II.

The existence of matrices with prescribed characteristic polynomials and prescribed permanental polynomials is discussed in Chapter III. Recall that the permanent function is a scalar-valued matrix function whose value for an $n \times n$ matrix $A = (a_{ij})$ is defined to be

$$\mathrm{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)} ,$$

where S_n is the symmetric group of degree n.

Numerous properties of the characteristic polynomial $\det(\lambda I - A)$ of a matrix A over a field F can be found in the literature. Recently, a number of results have been obtained which involve the permanental polynomial $\text{per}(\lambda I - A)$ (for example, see R. Merris [9] and G. N. de Oliveira [13]). In addition, new questions concerning this polynomial have been raised. It should be noted that efforts to investigate the permanent function by relating it in some simple way to the determinant function have met with little success, although inequalities relating the two functions have been established in special circumstances. Thus, the following problem posed by G. N. de Oliveira [13] was chosen for investigation:

> Find a necessary and sufficient condition for scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $z_1, z_2, \ldots, z_n$ in a field F to be, respectively, the characteristic roots of an $n \times n$ matrix A over F and the roots of the equation $\text{per}(zI - A) = 0$.

Chapter III contains results on the following existence problem.

> Let $d(\lambda)$ and $p(\lambda)$ be monic polynomials of degree $n \geq 2$ with coefficients in a field F. What conditions are necessary and sufficient for the existence of an $n \times n$ matrix A over F such that $\det(\lambda I - A) = d(\lambda)$ and $\text{per}(\lambda I - A) = p(\lambda)$?

This problem is solved over algebraically closed fields and also over the field of all real numbers by obtaining the desired conditions in terms of simple relations on the coefficients of $d(\lambda)$ and $p(\lambda)$.

Chapter IV contains some remarks concerning related topics for future consideration.

CHAPTER II

The Existence of Linear Operators (Matrices) with Prescribed Characteristic Polynomials and Prescribed Image Vectors

Let V be an n-dimensional vector space over a field F, $\{\xi_1, \xi_2, \ldots, \xi_s\}$ a linearly independent subset of V, and $\{\alpha_1, \alpha_2, \ldots, \alpha_s\}$ an arbitrary subset of V. It is easy to show that there exists a linear operator $\tau: V \to V$ such that $\tau(\xi_i) = \alpha_i$ for each $i = 1, 2, \ldots, s$. (Moreover, if $s = n$, τ is uniquely determined.) The question by H. Wielandt [14] mentioned in the preceding chapter asks for any restrictions imposed upon the characteristic polynomial of such a linear operator τ. We consider a slightly different question, asking not for restrictions on the polynomial but instead seeking conditions which ensure that there is no restriction upon the characteristic polynomial. In particular we shall present a solution to the following problem.

> Let V be an n-dimensional vector space over a field F, $\{\xi_1, \xi_2, \ldots, \xi_s\}$ a linearly independent subset of V, $U = \langle\xi_1, \xi_2, \ldots, \xi_s\rangle$. and $\sigma: U \to V$ a fixed linear transformation. What conditions on σ are necessary and sufficient such that for each monic polynomial $p(\lambda)$ over F of degree n, there exists a linear operator $\tau: V \to V$ with the properties that the characteristic polynomial of τ is $p(\lambda)$ and the restriction of τ to U is σ?

If U is a nonzero subspace of a vector space V and W is a complementary subspace of U in V such that $V = U \oplus W$, let $\pi_u: V \to U$ and $\pi_w: V \to W$ be the projection operators from V into U and W, respectively. Thus if $\xi \in V$ and $\xi = \xi_u + \xi_w$, where $\xi_u \in U$ and $\xi_w \in W$, then $\pi_u(\xi) = \xi_u$ and $\pi_w(\xi) = \xi_w$.

We shall begin with a result establishing a sufficient condition for the existence of a linear operator which has a prescribed characteristic polynomial and prescribed image vectors.

Theorem 2.1 Let V be an n-dimensional vector space over an arbitrary field F, $\{\xi_1, \xi_2, \ldots, \xi_s\}$ a linearly independent subset of V, $U = \langle\xi_1, \xi_2, \ldots, \xi_s\rangle$, $\sigma: U \to V$ a fixed linear transformation, and W a complementary subspace of U in V. If

(a) $$\bigcap_{i=0}^{\infty} \ker(\pi_w\sigma(\pi_u\sigma)^i) = \{0\},$$

then

(b) for each monic polynomial $p(\lambda)$ over F of degree n, there exists a linear operator $\tau: V \to V$ with the properties that the characteristic polynomial of τ is $p(\lambda)$ and the restriction of τ to U is σ.

We shall establish this theorem by using a result that is generally known in the case that F is the field of all complex numbers. Several existing proofs could possibly have been adapted to obtain the result for arbitrary fields. The proof that is presented is perhaps the nicest and is contained in an unpublished communication received from H. K. Wimmer.

Lemma 2.1 Let A_1 and A_2 be $s \times s$ and $s \times (n-s)$ matrices, respectively, over an arbitrary field F. If

(a) rank $[A_2, A_1A_2, A_1^2A_2, \ldots, A_1^{s-1}A_2] = s$,

then

(b) for each monic polynomial $p(\lambda)$ over F of degree n, there exist appropriately-sized matrices A_3 and A_4 over F such that the characteristic polynomial of the $n \times n$ matrix

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

is $p(\lambda)$.

Proof. (Wimmer) It will be convenient to establish the lemma by proceeding as follows. Let

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}$$

be a fixed $n \times n$ matrix, and let

$$B = \begin{bmatrix} 0 \\ I_{n-s} \end{bmatrix}$$

be the $n \times (n-s)$ matrix where I_{n-s} is the identity matrix of order n-s. We shall show that if (a) holds, then

(c) for each monic polynomial $p(\lambda)$ over F of degree n, there exists a matrix $M \in F^{(n-s)\times n}$ such that the characteristic polynomial of $A + BM$ is $p(\lambda)$.

Assume (a) holds. It follows that

$$\text{rank } [B, AB, A^2B, \ldots, A^{s-1}B] = n. \qquad (1)$$

Let Φ be an algebraically closed extension field of F. According to (1), if x is an n-dimensional row vector over Φ and $xA^iB = 0$ for each $i = 0, 1, \ldots, s-1$, then $x = 0$. Moreover, for each scalar $\mu \in \Phi$ and n-dimensional row vector x over Φ such that $xA = \mu x$ and $xB = 0$, we must have that $xA^iB = \mu^i xB = 0$ for each $i = 0, 1, \ldots, s-1$, and hence $x = 0$. It follows that

$$\operatorname{rank}[A - \mu I_n, B] = n \text{ for every } \mu \in \Phi. \tag{2}$$

(The fact that (1) implies (2) was first noticed by M. Hautus [5].)

We shall proceed by taking advantage of the observation that the polynomial matrix (2) is indeed a singular pencil of matrices. Much is known about the general theory of pencils of matrices, including a criterion for strict equivalence of two pencils of matrices and a canonical form for an arbitrary pencil of matrices under strict equivalence. We shall use the results and terminology of F. Gantmacher [4]. According to (2), the pencil $[A - \mu I_n, B]$ has a nonzero constant minor or order n, and therefore it has no elementary divisors in the field F. It also follows from (2) that there are no minimal row indices for the pencil. Thus, the strictly canonical form of $[A - \mu I_n, B]$ consists only of blocks of the form

$$\begin{bmatrix} \mu & 1 & & & \\ & \mu & 1 & & \\ & & \ddots & \ddots & \\ & & & \mu & 1 \end{bmatrix},$$

where the number of columns in each block exceeds the number of rows by one. That is, there exist nonsingular matrices $P \in F^{n\times n}$ and $R \in F^{(2n-s)\times(2n-s)}$ such that

$$P[A - \mu I_n, B]R = \text{block diag}(N_1, N_2, \ldots, N_r) \tag{3}$$

where

$$N_i = \begin{bmatrix} \mu & 1 & & & & \\ & \mu & 1 & & & \\ & & \mu & 1 & & \\ & & & \ddots & \ddots & \\ & & & & \mu & 1 \end{bmatrix} \varepsilon\ F^{j_i \times (j_i+1)}$$

for each $i = 1, 2, \ldots, r$. Following a suitable permutation of the columns of the canonical form, we have established that there exist nonsingular matrices $P \ \varepsilon\ F^{n\times n}$ and $Q \ \varepsilon\ F^{(2n-s)\times(2n-s)}$ such that

$$P[A - \mu I_n . B]Q = \text{block diag}(H_1 + \mu I_{j_1}, \ldots, H_r + \mu I_{j_r}, E_1, \ldots, E_r) \quad (4)$$

where

$$H_i = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{bmatrix} \varepsilon\ F^{j_i \times j_i}$$

and

$$E_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \varepsilon\ F^{j_i \times 1}$$

for each $i = 1, 2, \ldots, r$. We now examine the form of the nonsingular matrix Q. Suppose

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$$

where $Q_{11} \varepsilon F^{n\times n}$. Whereas

$$P[A - \mu I_n, B]Q = [C + \mu I_n, D],$$

we obtain the matrix equation

$$[PAQ_{11} - \mu PQ_{11} + PBQ_{21}, P(A - \mu I_n)Q_{12} + PBQ_{22}] = [C + \mu I_n, D].$$

Thus, $PQ_{11} = -I_n$, $PQ_{12} = 0$, and it follows that the nonsingular matrix Q has the form

$$Q = \begin{bmatrix} -P^{-1} & 0 \\ Q_{21} & Q_{22} \end{bmatrix}. \qquad (5)$$

We see from (4) and (5) that there exist matrices $P \varepsilon F^{n\times n}$, $Q_{21} \varepsilon F^{(n-s)\times n}$, and $Q_{22} \varepsilon F^{(n-s)\times(n-s)}$ such that

$$-PAP^{-1} + PBQ_{21} = \text{block diag}(H_1, H_2, \ldots, H_r) \qquad (6)$$

and

$$PBQ_{22} = \text{block diag}(E_1, E_2, \ldots, E_r) \qquad (7)$$

where the matrices H_i and E_i are as described in (4). Let M_1 be the $(n-s)\times n$ matrix having $(i, 1 + \sum_{k=1}^{i} j_k)$-th entry equal to 1 for each $i = 1, 2, \ldots, r-1$ and all other entries equal to 0, and let M_2 be the $(n-s)\times 1$ matrix having $(r, 1)$-th entry equal to 1 and all other entries equal to 0. It is easily verified that

$$\hat{A} = -PAP^{-1} + PBQ_{21} + PBQ_{22}M_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 1 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 0 & 1 & . & . & . & 0 & 0 \\ 0 & 0 & 0 & 0 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & 0 & 1 \\ 0 & 0 & 0 & 0 & . & . & . & 0 & 0 \end{bmatrix} \varepsilon F^{n\times n} \quad (8)$$

and

$$\hat{B} = PBQ_{22}M_1 = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ 1 \end{bmatrix} \varepsilon F^{n\times 1} \quad (9)$$

We now establish that (c) holds for the pair of matrices $\hat{A}$ and $\hat{B}$, and hence for A and B. Let $p(\lambda) = \lambda^n + c_{n-1}\lambda^{n-1} + c_{n-2}\lambda^{n-2} + \ldots + c_0$ be any monic polynomial over F of degree n. Choose $\hat{M} = (-c_0, -c_1, \ldots, -c_{n-1}) \varepsilon F^{1\times n}$. Then

$$\hat{A} + \hat{B}\hat{M} = \begin{bmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & . & . & . & 0 \\ 0 & 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 1 \\ -c_0 & -c_1 & -c_2 & . & . & . & -c_{n-1} \end{bmatrix} \quad (10)$$

and clearly the characteristic polynomial of $\hat{A} + \hat{B}\hat{M}$ is $p(\lambda)$. From (8) and (9), it is easily verified that

$$\hat{A} + \hat{B}\hat{M} = (-P)[A + B(Q_{21} + Q_{22}M_1 + Q_{22}M_2\hat{M})P](-P)^{-1}. \quad (11)$$

Choosing $M = (Q_{21} + Q_{22}M_1 + Q_{22}M_2\hat{M})P \varepsilon F^{(n-s)\times n}$, it follows that the characteristic polynomial of A + BM is $p(\lambda)$. The proof of the lemma is complete.

<u>Proof of Theorem 2.1</u> Extend the linearly independent set of vectors $S = \{\xi_1, \xi_2, \ldots, \xi_s\}$, if necessary, to a basis $S' = \{\xi_1, \xi_2, \ldots, \xi_s, \ldots, \xi_n\}$

for V, where $\{\xi_{s+1}, \ldots, \xi_n\}$ is a basis for W. The matrix representing $\sigma : U \to V$ with respect to the bases S and S' is an $n \times s$ matrix of the form

$$\begin{bmatrix} A_u \\ A_w \end{bmatrix}$$

where A_u is an $s \times s$ matrix. We note that the matrix A_u represents the linear operator $\pi_u\sigma: U \to U$, A_w represents $\pi_w\sigma: U \to W$, and $A_wA_u^i$ represents $\pi_w\sigma(\pi_u\sigma)^i : U \to W$ for each i = 1, 2, ..., s-1. A matrix-theoretic interpretation of statement (a) in Theorem 2.1 is easily seen to be the following. If x is an s-vector over F and $A_wA_u^ix = 0$ for each i = 0, 1, ..., s-1, then x = 0. This statement, in turn, is equivalent to the following.

(c) The rank of the matrix

$$\begin{bmatrix} A_w \\ A_wA_u \\ A_wA_u^2 \\ \vdots \\ A_wA_u^{s-1} \end{bmatrix}$$

is s.

Statement (b) in Theorem 2.1 is equivalent to the following.

(d) For each monic polynomial $p(\lambda)$ over F of degree n, there exist appropriately-sized matrices P and Q over F such that the characteristic polynomial of the $n \times n$ matrix

$$\begin{bmatrix} A_u & P \\ A_w & Q \end{bmatrix}$$

is $p(\lambda)$.

The fact that (c) implies (d) follows from Lemma 2.1. The proof of the theorem is complete.

We will extend Theorem 2.1 and obtain the main result of this chapter with the aid of the following observation.

Theorem 2.2 Let V be a vector space over an arbitrary field F, U a nonzero subspace of V, σ: U $\to$ V a fixed linear transformation, and W a complementary subspace of U in V. The following two statements are equivalent.

(a) For each nonzero subspace X of U, $\sigma(X) \not\subseteq X$.

(b) $\bigcap_{i=0}^{\infty} \ker(\pi_w \sigma(\pi_u \sigma)^i) = \{0\}$.

Proof: Suppose (a) holds. Let

$$K = \bigcap_{i=0}^{\infty} \ker(\pi_w \sigma(\pi_u \sigma)^i).$$

Clearly, K is a subspace of U. Assume that $K \neq \{0\}$. Then by (a), $\sigma(K) \not\subseteq K$. Thus, there exists a nonzero vector $\xi \in K$ such that $\sigma(\xi) \notin K$. But since $\xi \in K$, we must have that

$$\pi_w \sigma(\pi_u \sigma)^0(\xi) = \pi_w \sigma(\xi) = 0.$$

It follows that $\sigma(\xi) \in U - K$, and hence $\pi_u \sigma(\xi) \in U - K$. Thus for some positive integer j,

$$\pi_w \sigma(\pi_u \sigma)^{j+1}(\xi) = \pi_w \sigma(\pi_u \sigma)^j(\pi_u \sigma(\xi)) \neq 0,$$

a contradiction to the assumption that $\xi \in K$. Since the assumption that $K \neq \{0\}$ leads to a contradiction, we must have that $K = \{0\}$. Hence (a) $\to$ (b).

Suppose (b) holds. Let X be any nonzero subspace of U. If $\sigma(X) \not\subseteq X$, choose any nonzero vector $\xi \in X$. Necessarily,

$$(\pi_u \sigma)^i(\xi) \in X \subseteq U$$

and

$$\sigma(\pi_u \sigma)^i(\xi) \in X \subseteq U$$

for each nonnegative integer i. Hence,

$$\pi_w \sigma(\pi_u \sigma)^i(\xi) = 0$$

for each nonnegative integer i, and so

$$0 \neq \xi \in \bigcap_{i=0}^{\infty} \ker(\pi_W\sigma(\pi_U\sigma)^i),$$

a contradiction to the assumption that (b) is true. It follows that for each nonzero subspace X of U, $\sigma(X) \nsubseteq X$. Hence, (b) → (a). The proof of the theorem is complete.

We may now establish the main result.

Theorem 2.3 Let V be an n-dimensional vector space over an arbitrary field F, $\{\xi_1, \xi_2, \ldots, \xi_s\}$ a linearly independent subset of V, $U = \langle\xi_1, \xi_2, \ldots, \xi_s\rangle$, $\sigma: U \to V$ a fixed linear transformation, and W a complementary subspace of U in V. The following statements are equivalent.

(a) For each monic polynomial $p(\lambda)$ over F of degree n, there exists a linear operator $\tau: V \to V$ such that the characteristic polynomial of τ is $p(\lambda)$ and the restriction of τ to U is σ.

(b) For each nonzero subspace X of U, $\sigma(X) \nsubseteq X$.

(c) $\bigcap_{i=0}^{s-1} \ker(\pi_W\sigma(\pi_U\sigma)^i) = \{0\}$.

Proof. The equivalence of statements (b) and (c) is established by Theorem 2.2. Suppose ξ is a nonzero vector in U such that

$$\pi_W\sigma(\pi_U\sigma)^i(\xi) = 0$$

for each $i = 0, 1, \ldots, s-1$. Since $\dim U = s$, $1 \leq s < \infty$, there is an integer p, $0 \leq p \leq s$ such that the vectors

$$\xi,\ (\pi_U\sigma)(\xi),\ \ldots,\ (\pi_U\sigma)^{p-1}(\xi)$$

are linearly independent, while $(\pi_U\sigma)^p(\xi)$ is a linear combination of these vectors with coefficients in F. It follows that for each positive integer $j \geq s$,

$$(\pi_u\sigma)^j(\xi) = \sum_{i=0}^{s-1} a_i(\pi_u\sigma)^i(\xi)$$

for some scalars $a_i \in F$, and

$$\pi_w\sigma(\pi_u\sigma)^j(\xi) = \sum_{i=0}^{s-1} a_i\pi_w\sigma(\pi_u\sigma)^i(\xi) = 0.$$

Thus, we see that

$$\bigcap_{i=0}^{s-1} \ker(\pi_w\sigma(\pi_u\sigma)^i) = \bigcap_{i=0}^{\infty} \ker(\pi_w\sigma(\pi_u\sigma)^i),$$

and the equivalence of (b) and (c) follows from Theorem 2.2.

The implication (c) $\rightarrow$ (a) has been established by Theorem 2.1. It remains to be shown that (a) implies (c).

Suppose (a) holds. Let $p(\lambda)$ be any monic polynomial over F of degree n which is relatively prime to the characteristic polynomial $\Delta(\lambda)$ of the linear operator $\pi_u\sigma: U \rightarrow U$. Then there exists a linear operator $\tau: V \rightarrow V$ such that the characteristic polynomial of τ is $p(\lambda)$ and the restriction of τ to U is σ. Assume (c) fails to hold. Then there exists a nonzero vector $\xi \in U$ such that

$$\pi_w (\pi_u\sigma)^i(\xi) = 0$$

for each $i = 0, 1, \ldots, s-1$. It is easily seen that $\sigma(\xi) \in U$, and hence $\pi_u\sigma(\xi) = \sigma(\xi) = \tau(\xi)$. Let $\phi(\lambda)$ be the minimal annihilating polynomial of ξ with respect to the operator τ. Clearly, $\phi(\lambda)$ is also the minimal annihilating polynomial of ξ with respect to the operator $\pi_u\sigma$. Since every annihilating polynomial of ξ with respect to τ or $\pi_u\sigma$ must be divisible by $\phi(\lambda)$, it follows that both $p(\lambda)$ and $\Delta(\lambda)$ are divisible by $\phi(\lambda)$. Moreover, $\phi(\lambda) \not\equiv 1$ since $\xi \neq 0$. Thus, the assumption that (c) fails to hold leads to a contradiction of the fact that

$p(\lambda)$ and $\Delta(\lambda)$ are relatively prime. Hence, (a) $\rightarrow$ (c). The proof of the theorem is complete.

CHAPTER III

The Existence of Matrices with Prescribed Characteristic and Permanental Polynomials

In this chapter, we shall present results on the following existence problem suggested by G. N. de Oliveira [13].

> Let $M_n(F)$ denote the set of all $n \times n$ matrices over a field F where $n \geq 2$. Let $d(\lambda) = \lambda^n + d_1\lambda^{n-1} + d_2\lambda^{n-2} + \ldots + d_n$ and $p(\lambda) = \lambda^n + p_1\lambda^{n-1} + p_2\lambda^{n-2} + \ldots + p_n$ be monic polynomials over F of degree n. What conditions are necessary and sufficient for the existence of a matrix $A \in M_n(F)$ such that $\det(\lambda I - A) = d(\lambda)$ and $\operatorname{per}(\lambda I - A) = p(\lambda)$?

Since the permanent is equal to the determinant whenever the characteristic of F is 2, throughout this chapter F will denote a field of characteristic different from 2. Under this restriction, it is first shown that if F is an algebraically closed field and if $\operatorname{char}(F) \neq 3$ when $n = 3$, then there exists a matrix $A \in M_n(F)$ such that $\det(\lambda I - A) = d(\lambda)$ and $\operatorname{per}(\lambda I - A) = p(\lambda)$ if and only if $d_1 = p_1$. Next, the case in which $n = 3 = \operatorname{char}(F)$ is settled for algebraically closed fields by proving that in addition to the requirement that $d_1 = p_1$, at least one of an additional three requirements must be satisfied. Then for $F = R$, the field of all real numbers, it is shown that there exists a matrix $A \in M_n(F)$ such that $\det(\lambda I - A) = d(\lambda)$ and $\operatorname{per}(\lambda I - A) = p(\lambda)$ if and only if $d_1 = p_1$ and $d_1^2 \geq \frac{n}{n-1}(d_2 + p_2)$, with $d_3 - p_3 = \frac{n-2}{n} d_1(d_2 - p_2)$ if equality holds and $n > 2$.

1. ALGEBRAICALLY CLOSED FIELDS

We first consider the existence of a matrix $A \in M_n(F)$ having prescribed characteristic and permanental polynomials over an algebraically closed field F.

Theroem 3.1 Let $d(\lambda) = \lambda^n + d_1\lambda^{n-1} + d_2\lambda^{n-2} + \ldots + d_n$ and $p(\lambda) = \lambda^n + p_1\lambda^{n-1} + p_2\lambda^{n-2} + \ldots + p_n$ be monic polynomials of degree $n \geq 2$ with coefficients in an algebraically closed field F, where $\mathrm{char}(F) \neq 3$ if $n = 3$. There exists a matrix $A \in M_n(F)$ such that $\det(\lambda I - A) = d(\lambda)$ and $\mathrm{per}(\lambda I - A) = p(\lambda)$ if and only if $d_1 = p_1$.

Proof. If there exists a matrix $A \in M_n(F)$ such that $\det(\lambda I - A) = d(\lambda)$ and $\mathrm{per}(\lambda I - A) = p(\lambda)$, then clearly $d_1 = -\mathrm{tr}(A) = p_1$.

Conversely, suppose that $d_1 = p_1$. First, let $n = 2$. In this case, it can easily be verified that if

$$A = \begin{bmatrix} \dfrac{-d_1 + \sqrt{d_1^2 - 2(d_2 + p_2)}}{2} & \dfrac{p_2 - d_2}{2} \\ 1 & \dfrac{-d_1 - \sqrt{d_1^2 - 2(d_2 + p_2)}}{2} \end{bmatrix}$$

then $\det(\lambda I - A) = d(\lambda)$ and $\mathrm{per}(\lambda I - A) = p(\lambda)$.

Next, let $n = 3$. Choosing any scalar $s \in F$ such that $t = (d_1 - 3s)(d_1 + s) - 2(d_2 + p_2) \neq 0$, it can easily be verified that if

$$A = \begin{bmatrix} \dfrac{-(d_1+s)+\sqrt{t}}{2} & \dfrac{2(d_3-p_3)+(d_2-p_2)(\sqrt{t}-(d_1+s))}{-4\sqrt{t}} & \dfrac{st-s(d_1+s)^2-2(d_3+p_3)}{4} \\ 1 & s & \dfrac{2(d_3-p_3)-(d_2-p_2)(\sqrt{t}+d_1+s))}{4\sqrt{t}} \\ 0 & 1 & \dfrac{-(d_1+s)-\sqrt{t}}{2} \end{bmatrix}$$

then $\det(\lambda I - A) = d(\lambda)$ and $\mathrm{per}(\lambda I - A) = p(\lambda)$.

Finally, let $n \geq 4$. Let $x_0, y_0, x_1, y_1 \in F$ such that $x_1 \neq y_1$, $x_1 + x_0 \neq y_1 + y_0$, $x_1 + x_0 + y_1 + y_0 = -d_1$ and $x_1(x_0 + y_0 + y_1) + x_0(y_0 + y_1) + y_0 y_1 = \frac{1}{2}(d_2 + p_2)$. These scalars can be chosen in the algebraically closed field F in the following way. If $d_1 \neq 0$ or $d_2 + p_2 \neq 0$, choose $x_0 \in F$ such that $t = (d_1 - 2x_0)^2 - 3(2x_0)^2 - 2(d_2 + p_2) \neq 0$, then let $y_0 = x_0$, $x_1 = -\frac{1}{2}(d_1 + 2x_0 - \sqrt{t})$, and $y_1 = -\frac{1}{2}(d_1 + 2x_0 + \sqrt{t})$. If $d_1 = 0 = d_2 + p_2$ and $\mathrm{char}(F) \neq 3$, let $x_0 = 1$, $y_0 = 0$, $x_1 = -\frac{1}{2}(1 - \sqrt{-3})$, and $y_1 = \frac{1}{2}(1 + \sqrt{-3})$. If $d_1 = 0 = d_2 + p_2$ and $\mathrm{char}(F) = 3$, let $x_0 = y_0 = x_1 = 1$, and $y_1 = 0$.

For $i = 1, 2, 3, 4$, let σ_i denote the i-th elementary symmetric function of x_0, y_0, x_1 and y_1 and let $\sigma_i = 0$ if $4 < i \leq n$. Since $x_1 \neq y_1$ and $x_1 + x_0 \neq y_1 + y_0$, it is easy to show that we can select $x_2, x_3, \ldots, x_n, y_2, y_3, \ldots, y_{n-1} \in F$ in the following way. If $n = 4$. select $x_2, x_3, x_4, y_2, y_3 \in F$ such that

$$\begin{bmatrix} 1 & 1 \\ y_1 + y_0 & x_1 + x_0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(p_2 - d_2) \\ -\frac{1}{2}(p_3 - d_3) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ y_1 & x_1 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}(p_3 + d_3) - \sigma_3 \\ \frac{1}{2}(p_4 + d_4) - \sigma_4 - x_2 y_2 \end{bmatrix}$$

$$x_4 = \tfrac{1}{2}(p_4 - d_4) - x_0 x_1 y_2 - y_0 y_1 x_2.$$

If $n > 4$, select $x_2, x_3, \ldots, x_n, y_2, y_3, \ldots, y_{n-1} \in F$ such that

$$\begin{bmatrix} 1 & 1 \\ y_1 + y_0 & x_1 + x_0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(p_2 - d_2) \\ -\frac{1}{2}(p_3 - d_3) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ y_1 + y_0 & x_1 + x_0 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}(p_3 + d_3) - \sigma_3 \\ \frac{1}{2}(p_4 + d_4) - \sigma_4 - x_2 y_2 \end{bmatrix} \tag{1.1}$$

$$\begin{bmatrix} 1 & 1 \\ y_1 + y_0 & x_1 + x_0 \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix} = \begin{bmatrix} t_k \\ v_k \end{bmatrix} \text{ if } 4 \le k \le n-2,$$

$$\begin{bmatrix} 1 & 1 \\ y_1 & x_1 \end{bmatrix} \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} t_{n-1} \\ v_{n-1} \end{bmatrix}$$

$$x_n = t_n$$

where for $k = 4,5,\ldots,n$,

$$t_k = -[x_0x_1y_{k-2} + y_0y_1x_{k-2}] + \begin{cases} \tfrac{1}{2}(p_k - d_k) & \text{, if } k \text{ is even} \\ -\tfrac{1}{2}(p_k + d_k) - \sigma_k & \text{, if } k \text{ is odd} \end{cases}$$

$$v_k = -\sum_{i=2}^{k-1} x_i y_{k+1-i} + \begin{cases} -\tfrac{1}{2}(p_{k+1} - d_{k+1}) & \text{, if } k \text{ is even} \\ \tfrac{1}{2}(p_{k+1} + d_{k+1}) - \sigma_{k+1} & \text{, if } k \text{ is odd.} \end{cases}$$

We now show that if

$$A = \begin{bmatrix} x_1 & x_2 & x_3 & \cdot & \cdot & \cdot & x_{n-2} & x_{n-1} & x_n \\ 1 & x_0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & y_{n-1} \\ 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 & y_{n-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & y_3 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 & y_0 & y_2 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 & y_1 \end{bmatrix}$$

then $\det(\lambda I - A) = d(\lambda)$ and $\operatorname{per}(\lambda I - A) = p(\lambda)$. For each $k = 1,2,\ldots, n-2$, let A_k denote the matrix obtained from A be deleting the first k rows and columns. An expansion of $\det(\lambda I - A)$ in terms of the first row gives

$$\det(\lambda I - A) = (\lambda - x_1)\det(\lambda I - A_1) - \sum_{k=2}^{n-2} x_k \det(\lambda I - A_k)$$

$$-x_{n-1}(\lambda - y_1) - x_n$$

$$= [\lambda^2 - (x_0 + x_1)\lambda + (x_0x_1 - x_2)][\lambda^{n-2} - (y_0 + y_1)\lambda^{n-3} +$$

$$(y_0y_1 - y_2)\lambda^{n-4} - y_3\lambda^{n-5} - \dots - y_{n-2}]$$

$$- \sum_{k=3}^{n-2} x_k[\lambda^{n-k} - (y_0 + y_1)\lambda^{n-k-1} + (y_0y_1 - y_2)\lambda^{n-k-2} -$$

$$y_3\lambda^{n-k-3} - \dots - y_{n-k}]$$

$$- x_{n-1}(\lambda - y_1) - y_{n-1}(\lambda - x_1) - x_n,$$

which, after rearrangement, can be written as

$$\det(\lambda I - A) = \lambda^n - \sigma_1\lambda^{n-1} + [\sigma_2 - (x_2 + y_2)]\lambda^{n-2}$$

$$+ [(y_0 + y_1)x_2 + (x_0 + x_1)y_2 - \sigma_3 - (x_3 + y_3)]\lambda^{n-3}$$

$$+ \sum_{k=4}^{n-1} [(y_0 + y_1)x_{k-1} + (x_0 + x_1)y_{k-1} + \sum_{i=2}^{k-2} x_iy_{k-i} +$$

$$(-1)^k\sigma_k - (x_k + y_k + x_0x_1y_{k-2} + y_0y_1x_{k-2})]\lambda^{n-k}$$

$$+ [y_1x_{n-1} + x_1y_{n-1} + \sum_{i=2}^{n-2} x_iy_{n-i} + (-1)^n\sigma_n -$$

$$(x_n + x_0x_1y_{n-2} + y_0y_1x_{n-2})]. \qquad (1.2)$$

Using the relationships among the scalars $x_0, x_1, \dots, x_n, y_0, y_1, \dots, y_{n-1}$ given by (1.1) it is not difficult to show that (1.2) can be written as

$$\det(\lambda I - A) = \lambda^n + d_1\lambda^{n-1} + [\tfrac{1}{2}(p_2 + d_2) - \tfrac{1}{2}(p_2 - d_2)]\lambda^{n-2} + \dots$$

$$+ [\tfrac{1}{2}(p_k + d_k) - \tfrac{1}{2}(p_k - d_k)]\lambda^{n-k} + \dots + [\tfrac{1}{2}(p_n + d_n) - \tfrac{1}{2}(p_n - d_n)]$$

$$= \lambda^n + d_1\lambda^{n-1} + d_2\lambda^{n-2} + \dots + d_k\lambda^{n-k} + \dots + d_n = d(\lambda).$$

A similar expansion of per(λI - A) gives

$$\mathrm{per}(\lambda I - A) = \lambda^n - \sigma_1\lambda^{n-k} + [\sigma_2 + (x_2 + y_2)]\lambda^{n-2}$$

$$- [(y_0 + y_1)x_2 + (x_0 + x_1)y_2 + \sigma_3 + (x_3 + y_3)]\lambda^{n-3}$$

$$+ \sum_{k=4}^{n-1} (-1)^k [(y_0 + y_1)x_{k-1} + (x_0 + x_1)y_{k-1} + \sum_{i=2}^{k-2} x_i y_{k-i} +$$

$$\sigma_k + (x_k + y_k + x_0x_1y_{k-2} + y_0y_1x_{k-2})]\lambda^{n-k}$$

$$+ (-1)^n [y_1x_{n-1} + x_1y_{n-1} + \sum_{i=2}^{n-2} x_iy_{n-i} + \sigma_n +$$

$$(x_n + x_0x_1y_{n-2} + y_0y_1x_{n-2})]. \qquad (1.3)$$

Again from (1.1), it is not difficult to show that (1.3) can be written as

$$\mathrm{per}(\lambda I - A) = \lambda^n + p_1\lambda^{n-1} + [\tfrac{1}{2}(p_2 + d_2) + \tfrac{1}{2}(p_2 - d_2)]\lambda^{n-2} + \ldots$$

$$+ [\tfrac{1}{2}(p_k + d_k) + \tfrac{1}{2}(p_k - d_k)]\lambda^{n-k} + \ldots + [\tfrac{1}{2}(p_n + d_n) + \tfrac{1}{2}(p_n - d_n)]$$

$$= \lambda^n + p_1\lambda^{n-1} + p_2\lambda^{n-2} + \ldots + p_k\lambda^{n-k} + \ldots + p_n = p(\lambda).$$

The proof of the theorem is complete.

For completeness, we consider the case where F is an algebraically closed field and n = 3 = char(F).

<u>Proposition 3.1</u> Let $d(\lambda) = \lambda^3 + d_1\lambda^2 + d_2\lambda + d_3$ and $p(\lambda) = \lambda^3 + p_1\lambda^2 + p_2\lambda + p_3$ be monic polynomials over an algebraically closed field F of characteristic 3. There exists a matrix $A \in M_3(F)$ such that $\det(\lambda I - A) = d(\lambda)$ and $\mathrm{per}(\lambda I - A) = p(\lambda)$ if and only if $d_1 = p_1$ and at least one of the following holds:

(a) $d_1 \neq 0$,

(b) $d_2 + p_2 \neq 0$,

(c) $d_1 = d_2 + p_2 = 0$ and $d_3 - p_3 = a(d_2 - p_2)$ for some $a \in F$.

Proof. Suppose there exists a matrix $A = (a_{ij}) \in M_3(F)$ such that $\det(\lambda I - A) = d(\lambda)$ and $\operatorname{per}(\lambda I - A) = p(\lambda)$. Clearly, $d_1 = -\operatorname{tr}(A) = p_1$. It suffices to assume that (a) and (b) fail to hold. Then $d_1 = d_2 + p_2 = 0$. From the polynomial equations $\det(\lambda I - A) = d(\lambda)$ and $\operatorname{per}(\lambda I - A) = p(\lambda)$, we can obtain the following system of equations:

$$a_{11} + a_{22} + a_{33} = 0 \qquad (1.4)$$

$$a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} = 0 \qquad (1.5)$$

$$a_{12}a_{21} + a_{13}a_{31} + a_{23}a_{32} = \tfrac{1}{2}(p_2 - d_2) \qquad (1.6)$$

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21} = -\tfrac{1}{2}(p_3 + d_3) \qquad (1.7)$$

$$a_{23}a_{32}a_{11} + a_{13}a_{31}a_{22} + a_{12}a_{21}a_{33} = -\tfrac{1}{2}(p_3 - d_3) \qquad (1.8)$$

Using (1.4), (1.5) and the fact that $\operatorname{char}(F) = 3$, it is not difficult to show that $a_{11} = a_{22} = a_{33}$. Then from (1.6) and (1.8), we obtain

$$-\tfrac{1}{2}(p_3 - d_3) = a_{11}(a_{23}a_{32} + a_{13}a_{31} + a_{12}a_{21}) = \tfrac{1}{2}a_{11}(p_2 - d_2),$$

or equivalently,

$$d_3 - p_3 = -a_{11}(d_2 - p_2).$$

Hence, $d_1 = p_1$ and at least one of the properties (a), (b), and (c) hold.

It remains to be shown that the stated conditions are sufficient conditions. The proof of Theorem 3.1 for the case $n = 3$ clearly shows that if $d_1 = p_1$ and either (a) or (b) holds, then there exists a matrix $A \in M_3(F)$ such that $\det(\lambda I - A) = d(\lambda)$ and $\operatorname{per}(\lambda I - A) = p(\lambda)$.

Suppose $d_1 = p_1$ and (c) holds. Then by our assumption, we have that $d_1 = p_1 = 0$, $p_2 = -d_2$ and $d_3 - p_3 = a(d_2 - p_2) = 2ad_2$ for some $a \in F$. Let

$$A = \begin{bmatrix} -a & -d_2 & a^3-\frac{1}{2}(p_3 + d_3) \\ 1 & -a & 0 \\ 0 & 1 & -a \end{bmatrix}.$$

It follows that $\det(\lambda I - A) = d(\lambda)$ and $\mathrm{per}(\lambda I - A) = p(\lambda)$. The proof is complete.

2. THE REAL FIELD

We now consider the existence of real matrices having prescribed characteristic and permanental polynomials with real coefficients. We shall use the following in this consideration.

Lemma 3.1 Let $d(\lambda) = \lambda^n + d_2\lambda^{n-2} + d_3\lambda^{n-3} + \ldots + d_n$ and $p(\lambda) = \lambda^n + p_2\lambda^{n-2} + p_3\lambda^{n-3} + \ldots + p_n$ be monic polynomials of degree $n \geq 2$ with real coefficients. There exists a matrix $A \in M_n(R)$ such that $\det(\lambda I - A) = d(\lambda)$ and $\mathrm{per}(\lambda I - A) = p(\lambda)$ if one of the following holds:

(a) $d_2 + p_2 < 0$,

(b) $d_2 + p_2 = 0$ and, if $n > 2$, $d_3 = p_3$.

Proof. Rewriting the polynomials $d(\lambda)$ and $p(\lambda)$ as

$$d(\lambda) = \lambda^n + (r_2 - s_2)\lambda^{n-2} + \ldots + (r_k - s_k)\lambda^{n-k} + \ldots + (r_n - s_n)$$

and $$p(\lambda) = \lambda^n + (r_2 + s_2)\lambda^{n-2} + \ldots + (r_k + s_k)\lambda^{n-k} + \ldots + (r_n + s_n)$$

where $r_k = \frac{1}{2}(p_k + d_k)$ and $s_k = \frac{1}{2}(p_k - d_k)$ for $k = 2, 3, \ldots, n$, we see that the assumption that (a) or (b) holds is equivalent to the assumption that

(c) $r_2 < 0$,

or

(d) $r_2 = 0$ and, if $n > 2$, $s_3 = 0$.

Suppose (c) holds. Then an argument similar to that in the proof of Theorem 3.1 establishes that a real $n \times n$ matrix with the properties $\det(\lambda I - A) = d(\lambda)$ and $\operatorname{per}(\lambda I - A) = p(\lambda)$ is

$$A = \begin{bmatrix} x_1 & x_2 & x_3 & . & . & . & x_{n-2} & x_{n-1} & x_n \\ 1 & 0 & 0 & . & . & . & 0 & 0 & y_{n-1} \\ 0 & 1 & 0 & . & . & . & 0 & 0 & y_{n-2} \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 0 & 0 & y_3 \\ 0 & 0 & 0 & . & . & . & 1 & 0 & y_2 \\ 0 & 0 & 0 & . & . & . & 0 & 1 & y_1 \end{bmatrix} \qquad (2.1)$$

where $x_1 = \sqrt{-r_2}$, $y_1 = -\sqrt{-r_2}$, and the other x_i and y_i are determined as in the proof of Theorem 3.1.

Suppose (d) holds. First, let $n = 2$. Clearly, if

$$A = \begin{bmatrix} 0 & -d_2 \\ 1 & 0 \end{bmatrix}$$

then $\det(\lambda I - A) = \lambda^2 + d_2 = d(\lambda)$ and $\operatorname{per}(\lambda I - A) = \lambda^2 - d_2 = p(\lambda)$.

Next, let $n \geq 3$. We now show that a real $n \times n$ matrix A with the desired properties can be found, where A has the form

$$A = \begin{bmatrix} 0 & x_1 & x_2 & x_3 & x_4 & \cdots & x_{n-2} & x_{n-1} \\ 1 & 0 & y_1 & y_2 & y_3 & \cdots & y_{n-3} & y_{n-2} \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \qquad (2.2)$$

We first establish by induction on n that if A has the form given by (2.2), then

$$\begin{aligned} \det(\lambda I - A) = {} & \lambda^n + 0\lambda^{n-1} - [x_1 + y_1 + (n-3)]\lambda^{n-2} - [x_2 + y_2]\lambda^{n-3} \\ & + [(n-3)x_1 + (n-4)y_1 + P_1 - (x_3 + y_3 + Q_1)]\lambda^{n-4} + \cdots \\ & + [(n-k+1)x_{k-3} + (n-k)y_{k-3} + P_{k-3} - \\ & \qquad (x_{k-1} + y_{k-1} + Q_{k-3})]\lambda^{n-k} + \cdots \\ & + [2x_{n-4} + y_{n-4} + P_{n-4} - (x_{n-2} + y_{n-2} + Q_{n-4})]\lambda \\ & + [x_{n-3} + P_{n-3} - (x_{n-1} + Q_{n-3})] \end{aligned} \qquad (2.3)$$

and

$$\begin{aligned} \operatorname{per}(\lambda I - A) = {} & \lambda^n + 0\lambda^{n-1} + [x_1 + y_1 + (n-3)]\lambda^{n-2} - [x_2 + y_2]\lambda^{n-3} \\ & + [(n-3)x_1 + (n-4)y_1 + P_1 + (x_3 + y_3 + Q_1)]\lambda^{n-4} + \cdots \\ & + (-1)^k[(n-k+1)x_{k-3} + (n-k)y_{k-3} + P_{k-3} + \\ & \qquad (x_{k-1} + y_{k-1} + Q_{k-3})]\lambda^{n-k} + \cdots \\ & + (-1)^{n-1}[2x_{n-4} + y_{n-4} + P_{n-4} + (x_{n-2} + y_{n-2} + Q_{n-4})]\lambda \\ & + (-1)^n[x_{n-3} + P_{n-3} + (x_{n-1} + Q_{n-3})] \end{aligned} \qquad (2.4)$$

where for each $i = 1, 2, \ldots, n-3$, P_i is a polynomial in

$x_1, y_1, x_2, y_2, \ldots, x_{i-4}$, and y_{i-4} of degree at most one, Q_i is a polynomial in $x_1, y_1, x_2, y_2, \ldots, x_{i-2}$, and y_{i-2} of degree at most one, and $x_i = y_i = P_i = Q_i = 0$ if $i \leq 0$. This assertion is easily verified for $n = 3$ and $n = 4$. Assume the assertion is true for matrices of order t, $4 \leq t < n$, having the form given by (2.2). Then if an $n \times n$ matrix has the form given by (2.2), expansions of $\det(\lambda I - A)$ and $\operatorname{per}(\lambda I - A)$ in terms of the last column give

$$\det(\lambda I - A) = \lambda\det(\lambda I - A') - \det(\lambda I - A'') - y_{n-2}\lambda - x_{n-1} \tag{2.5}$$

and

$$\operatorname{per}(\lambda I - A) = \lambda\operatorname{per}(\lambda I - A') + \operatorname{per}(\lambda I - A'') + (-1)^{n-1}y_{n-2}\lambda + (-1)^n x_{n-1} \tag{2.6}$$

where A' is the matrix obtained from A by deleting the last row and column, and A'' is the matrix obtained from A by deleting the last two rows and columns. By our induction assumption,

$$\begin{aligned}\det(\lambda I - A') = {} & \lambda^{n-1} + 0\lambda^{n-2} - [x_1 + y_1 + (n-4)]\lambda^{n-3} - [x_2 + y_2]\lambda^{n-4} \\ & + [(n-4)x_1 + (n-5)y_1 + P_1' - (x_3 + y_3 + Q_1')]\lambda^{n-5} \\ & + \sum_{k=5}^{n-2}[(n-k)x_{k-3} + (n-k-1)y_{k-3} + P_{k-3}' - \\ & \qquad (x_{k-1} + y_{k-1} + Q_{k-3}')]\lambda^{n-k-1} \\ & + [x_{n-4} + P_{n-4}' - (x_{n-2} + Q_{n-4}')]\end{aligned} \tag{2.7}$$

$$\begin{aligned}\det(\lambda I - A'') = {} & \lambda^{n-2} + 0\lambda^{n-3} - [x_1 + y_1 + (n-5)]\lambda^{n-4} - [x_2 + y_2]\lambda^{n-5} \\ & + [(n-5)x_1 + (n-6)y_1 + P_1'' - (x_3 + y_3 + Q_1'')]\lambda^{n-6} \\ & + \sum_{k=5}^{n-3}[(n-k-1)x_{k-3} + (n-k-2)y_{k-3} + P_{k-3}'' - \\ & \qquad (x_{k-1} + y_{k-1} + Q_{k-3}'')]\lambda^{n-k-2} \\ & + [x_{n-5} + P_{n-5}'' - (x_{n-3} + Q_{n-5}'')]\end{aligned} \tag{2.8}$$

$$\begin{aligned}\mathrm{per}(\lambda I - A') = {} & \lambda^{n-1} + 0\lambda^{n-2} + [x_1 + y_1 + (n-4)]\lambda^{n-3} - [x_2 + y_2]\lambda^{n-4} \\ & + [(n-4)x_1 + (n-5)y_1 + P_1' + (x_3 + y_3 + Q_1')]\lambda^{n-5} \\ & + \sum_{k=5}^{n-2} (-1)^k [(n-k)x_{k-3} + (n-k-1)y_{k-3} + P_{k-3}' + \\ & \qquad (x_{k-1} + y_{k-1} + Q_{k-3}')]\lambda^{n-k-1} \\ & + (-1)^{n-1}[x_{n-4} + P_{n-4}' + (x_{n-2} + Q_{n-4}')] \qquad (2.9)\end{aligned}$$

$$\begin{aligned}\mathrm{per}(\lambda I - A'') = {} & \lambda^{n-2} + 0\lambda^{n-3} + [x_1 + y_1 + (n-5)]\lambda^{n-4} - [x_2 + y_2]\lambda^{n-5} \\ & + [(n-5)x_1 + (n-6)y_1 + P_1'' + (x_2 + y_2 + Q_1'')]\lambda^{n-6} \\ & + \sum_{k=5}^{n-3} (-1)^k [(n-k-1)x_{k-3} + (n-k-2)y_{k-3} + P_{k-3}'' + \\ & \qquad (x_{k-1} + y_{k-1} + Q_{k-3}'')]\lambda^{n-k-2} \\ & + (-1)^{n-2}[x_{n-5} + P_{n-5}'' + (x_{n-3} + Q_{n-5}'')] \qquad (2.10)\end{aligned}$$

where P_i' and P_i'' are polynomials in $x_1, y_1, x_2, y_2, \ldots, x_{i-4}$, and y_{i-4} of degree at most one and Q_i' and Q_i'' are polynomials in $x_1, y_1, x_2, y_2, \ldots, x_{i-2}$, and y_{i-2} of degree at most one. Substituting (2.7) and (2.8) into (2.5) we get

$$\begin{aligned}\det(\lambda I - A) = {} & \lambda^n + 0\lambda^{n-1} - [x_1 + y_1 + (n-3)]\lambda^{n-2} - [x_2 + y_2]\lambda^{n-3} \\ & + [(n-3)x_1 + (n-4)y_1 + P_1' + (n-5) - (x_3 + y_3 + Q_1')]\lambda^{n-4} \\ & + [(n-4)x_2 + (n-5)y_2 + P_2' - (x_4 + y_4 + Q_2')]\lambda^{n-5} \\ & + \sum_{k=6}^{n-1} [(n-k+1)x_{k-3} + (n-k)y_{k-3} + P_{k-3}' + Q_{k-5}'' - (x_{k-1} + \\ & \qquad y_{k-1} + Q_{k-3}' + P_{k-5}'' + (n-k+1)x_{k-5} + (n-k)y_{k-5})]\lambda^{n-k} \\ & + [x_{n-3} + Q_{n-5}'' - (x_{n-1} + P_{n-5}'' + x_{n-5})].\end{aligned}$$

Letting $P_1 = P_1' + (n - 5)$, $P_2 = P_2', \ldots, P_k = P'_{k-3} + Q''_{k-5}, \ldots, P_{n-3} = Q''_{n-5}$ and $Q_1 = Q_1'$, $Q_2 = Q_2', \ldots, Q_k = Q_k' + P''_{k-5} + (n - k + 1)x_{k-5} + (n - k)y_{k-5}, \ldots,$ $Q_{n-3} = P''_{n-5} + x_{n-5}$, we see that (2.3) is valid. Similarly, the substitution of (2.9) and (2.10) into (2.6) establishes (2.4).

The polynomial equations

$$\det(\lambda I - A) = \lambda^n + 0\lambda^{n-1} - s_2\lambda^{n-2} + r_3\lambda^{n-3} + (r_4 - s_4)\lambda^{n-4} + \ldots + (r_n - s_n)$$

$$\operatorname{per}(\lambda I - A) = \lambda^n + 0\lambda^{n-1} + s_2\lambda^{n-2} + r_3\lambda^{n-3} + (r_4 + s_4)\lambda^{n-4} + \ldots + (r_n + s_n)$$

can be replaced by the following system of equations:

$$\begin{bmatrix} 1 & 1 \\ (n - 3) & (n - 4) \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} s_2 - (n - 3) \\ r_4 - P_1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ (n - 4) & (n - 5) \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} -r_3 \\ -s_5 - P_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ (n - k - 2) & (n - k - 3) \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix} = \begin{bmatrix} t_k \\ v_k \end{bmatrix} \text{ if } 3 \leq k \leq n-3,$$

$$x_{n-2} + y_{n-2} = t_{n-2}$$

$$x_{n-1} = t_{n-1}$$

where for $k = 3,4,\ldots,n-1$,

$$t_k = \begin{cases} -r_{k+1} - Q_{k-2}, & \text{if } k \text{ is even} \\ s_{k+1} - Q_{k-2}, & \text{if } k \text{ is odd} \end{cases}$$

$$v_k = \begin{cases} -s_{k+3} - p_k, & \text{if } k \text{ is even} \\ r_{k+3} - p_k, & \text{if } k \text{ is odd.} \end{cases}$$

The system is clearly consistent over R, and the proof of the lemma is complete.

With the aid of Lemma 3.1, we now establish the following result.

Theorem 3.2. Let $d(\lambda) = \lambda^n + d_1\lambda^{n-1} + d_2\lambda^{n-2} + \ldots + d_n$ and $p(\lambda) = \lambda^n + p_1\lambda^{n-1} + p_2\lambda^{n-2} + \ldots + p_n$ be monic polynomials of degree $n \geq 2$ with real coefficients. There exists a matrix $A \in M_n(R)$ such that $\det(\lambda I - A) = d(\lambda)$ and $\operatorname{per}(\lambda I - A) = p(\lambda)$ if and only if $d_1 = p_1$ and one of the following holds:

(a) $d_1^2 > \frac{n}{n-1}(d_2 + p_2)$,

(b) $d_1^2 = \frac{n}{n-1}(d_2 + p_2)$ and, if $n > 2$, $d_3 - p_3 = \frac{n-2}{n} d_1 (d_2 - p_2)$.

Proof. Suppose there exists a matrix $A = (a_{ij}) \in M_n(R)$ such that

$$\det(\lambda I - A) = d(\lambda) = \prod_{i=1}^{n} (\lambda - \lambda_i)$$

and

$$\operatorname{per}(\lambda I - A) = p(\lambda) = \prod_{i=1}^{n} (\lambda - \mu_i).$$

Then

$$d_1 = -\sum_{i=1}^{n} \lambda_i = -\sum_{i=1}^{n} a_{ii} = -\sum_{i=1}^{n} \mu_i = p_1, \tag{2.11}$$

$$d_2 = \sum_{1 \leq i < j \leq n} \lambda_i\lambda_j = \sum_{1 \leq i < j \leq n} a_{ii}a_{jj} - \sum_{1 \leq i < j \leq n} a_{ij}a_{ji}, \tag{2.12}$$

and

$$p_2 = \sum_{1 \leq i < j \leq n} \mu_i\mu_j = \sum_{1 \leq i < j \leq n} a_{ii}a_{jj} + \sum_{1 \leq i < j \leq n} a_{ij}a_{ji}. \tag{2.13}$$

Hence,

$$d_2 + p_2 = 2 \sum_{1\leq i<j\leq n} a_{ii}a_{jj} = (\sum_{j=1}^{n} a_{jj})^2 - \sum_{i=1} a_{ii}^2. \qquad (2.14)$$

It follows from the Cauchy-Schwarz inequality that

$$-\sum_{i=1}^{n} a_{ii}^2 \leq -(\sum_{i=1}^{n} a_{ii})^2/n$$

with equality if and only if $a_{ii} = a_{jj}$ for $i,j = 1,2, \ldots, n$. Therefore, we see from (2.14) that

$$d_2 + p_2 \leq \frac{n-1}{n}(\sum_{i=1}^{n} a_{ii})^2 = \frac{n-1}{n}d_1^2$$

or equivalently,

$$d_1^2 \geq \frac{n}{n-1}(d_2 + p_2) \qquad (2.15)$$

with equality if and only if $a_{ii} = -\frac{1}{n}d_1$ for $i = 1, 2, \ldots, n$. Suppose that $n > 2$ and $d_1^2 = \frac{n}{n-1}(d_2 + p_2)$. Then $a_{ii} = -\frac{1}{n}d_1$ for $i = 1, 2, \ldots, n$. Since $-d_3$ is equal to the sum of the principal minors of A of order 3 and $-p_3$ is equal to the sum of the principal permanental minors of A of order 3, we see that

$$d_3 - p_3 = 2(n-2)(-\frac{1}{n}d_1) \sum_{1\leq i<j\leq n} a_{ij}a_{ji} = \frac{n-2}{n} d_1(d_2 - p_2).$$

Hence, $d_1 = p_1$ and (a) or (b) holds.

Now suppose that $d_1 = p_1$ and either (a) or (b) holds. Note that if

$$d(\lambda) = \lambda^n + d_1\lambda^{n-1} + d_2\lambda^{n-2} + \ldots + d_n = \prod_{i=1}^{n} (\lambda - \lambda_i) ,$$

$$p(\lambda) = \lambda^n + p_1\lambda^{n-1} + p_2\lambda^{n-2} + \ldots + p_n = \prod_{i=1}^{n} (\lambda - \mu_i),$$

$$\tilde{d}(\lambda) = \lambda^n + 0\lambda^{n-1} + \tilde{d}_2\lambda^{n-2} + \ldots + \tilde{d}_n = \prod_{i=1}^{n} (\lambda - (\lambda_i + \tfrac{1}{n}d_1)),$$

and $$\tilde{p}(\lambda) = \lambda^n + 0\lambda^{n-1} + \tilde{p}_2\lambda^{n-2} + \ldots + \tilde{p}_n = \prod_{i=1}^{n} (\lambda - (\mu_i + \tfrac{1}{n}d_1)),$$

then for each $A \in M_n(R)$,

$$\det(\lambda I - (A - \tfrac{1}{n}d_1 I)) = d(\lambda) \text{ and } \operatorname{per}(\lambda I - (A - \tfrac{1}{n}d_1 I)) = p(\lambda)$$

if and only if

$$\det(\lambda I - A) = \tilde{d}(\lambda) \text{ and } \operatorname{per}(\lambda I - A) = \tilde{p}(\lambda).$$

Thus it is sufficient to prove the existence of a matrix $A \in M_n(R)$ such that $\det(\lambda I - A) = d(\lambda)$ and $\operatorname{per}(\lambda I - A) = p(\lambda)$ in the case that $d_1 = p_1 = 0$. In this case, the assumption that $d_1 = p_1$ and either (a) or (b) holds is equivalent to the assumption that $d_1 = p_1 = 0$ and either (c) $d_2 + p_2 < 0$ or (d) $d_2 + p_2 = 0$ and, if $n > 2$, $d_3 = p_3$. Under these conditions, Lemma 3.1 establishes the existence of a matrix $A \in M_n(R)$ such that $\det(\lambda I - A) = d(\lambda)$ and $\operatorname{per}(\lambda I - A) = p(\lambda)$, and the proof of the theorem is complete.

It should be noted that in the case that the prescribed polynomials $d(\lambda)$ and $p(\lambda)$ are identical, then Theorem 3.2 becomes the following.

Corollary. Let $p(\lambda) = \lambda^n + p_1\lambda^{n-1} + p_2\lambda^{n-2} + \ldots + p_n$ be a monic polynomial of degree $n \geq 2$ with real coefficients. There exists a matrix $A \in M_n(R)$ such that $\det(\lambda I - A) = p(\lambda) = \operatorname{per}(\lambda I - A)$ if and only if $p_1^2 \geq \frac{2n}{n-1} p_2$.

We conclude by noting that Proposition 3.1 and Theorems 3.1 and 3.2 imply the following.

Theorem 3.3 Let $p(\lambda)$ and $q(\lambda)$ be monic polynomials of degree $n \geq 2$ with coefficients in F, an algebraically closed field or the field of all real numbers. There exists a matrix $A \in M_n(F)$ such that $\det(\lambda I - A) = p(\lambda)$ and $\operatorname{per}(\lambda I - A) = q(\lambda)$ if and only if there exists a matrix $B \in M_n(F)$ such that $\det(\lambda I - B) = q(\lambda)$ and $\operatorname{per}(\lambda I - B) = p(\lambda)$.

CHAPTER IV

Related Topics for Further Consideration

In Chapter II we considered the following question.

> Let V be an n-dimensional vector space over a field F, $\{\xi_1, \xi_2, \ldots, \xi_s\}$ a linearly independent subset of V, and $\{\alpha_1, \alpha_2, \ldots, \alpha_s\}$ an arbitrary subset of V. What restrictions are imposed upon the characteristic polynomial of a linear operator $\tau: V \to V$ by postulating that $\tau(\xi_i) = \alpha_i$ for each $i = 1, 2, \ldots, s$?

Although conditions on the image vectors were obtained which ensure that there is no restriction on the characteristic polynomial, the question is not yet completely solved.

A second matter for further consideration would be the extension of the results in Chapter III to results on the relationship between the characteristic polynomial and permanental polynomial of a positive semidefinite hermitian matrix A (denoted by $A \geq 0$). It is well known that if $A \geq 0$, then $\text{per}(A) \geq \det(A)$. This inequality perhaps indicates that the relationship between the characteristic polynomial and the permanental polynomial of a positive semidefinite matrix is much more complex than the results of Chapter III would suggest. However, with additional knowledge of this relationship, it may be possible to use known results on the characteristic values of positive semidefinite hermitian matrices in answering some of the following questions recently raised by R. Merris [9].

1. If $A \geq 0$, are the real parts of the zeros of $\text{per}(\lambda I - A)$ nonnegative?

2. Can $\{A \geq 0: \text{per}(\lambda I - A) \text{ has nonnegative zeros}\}$ be characterized?

3. Can any significance be attached to the convex hull of the zeros of $\text{per}(\lambda I - A)$?

4. Do the zeros of $\text{per}(\lambda I - A)$ have any interlacing properties?

We conclude the discussion with some observations on a result by S. Friedland. An initial objective of my investigation was to obtain new and improved results concerning the existence of matrices with prescribed characteristic polynomials and prescribed entries. A review of the literature revealed that this subject has already received considerable attention. For example, L. Mirsky [10] first obtained the following result.

> If $p(x)$ is any monic polynomial of degree n with coefficients in the field C of complex numbers and if $a_1, a_2, \ldots, a_{n-1}$ are any $n-1$ complex numbers, then there exists an $n \times n$ complex matrix $A = (a_{ij})$ having $a_1, a_2, \ldots, a_{n-1}$ in any prescribed positions on its main diagonal and having characteristic polynomial $p(x)$.

H. K. Farahat and W. Ledermann [2] showed that Mirsky's result is valid when the ground field is arbitrary and further improved the result by obtaining the following result.

> If $B = (b_{ij})$ is any $(n-1)\times(n-1)$ nonderogatory matrix over a field F and $p(x)$ is any monic polynomial over F

of degree n, then there exists an $n \times n$ matrix $A = (a_{ij})$ over F having the matrix B in the top left-hand corner and having characteristic polynomial p(x).

Several variations of this result concerning the existence of matrices with prescribed characteristic polynomial and prescribed submatrix have also appeared in the literature. G. N. de Oliveira [12] obtained necessary and sufficient conditions for the existence of an $n \times n$ complex matrix $A = (a_{ij})$ having prescribed principal minor contained in rows whose indices are $i_1, i_2, \ldots, i_{n-k}$ $(1 \leq i_1 < i_2 < \ldots < i_{n-k} \leq n)$ and prescribed characteristic polynomial p(x) of degree n. K. Fan and G. Pall [1] had resolved this question in the case that the matrix A and the prescribed submatrix are both real symmetric or both hermitian. Along this line, it should also be noted that the results of Chapter II in this dissertation can be used to obtain necessary and sufficient conditions for the existence of matrices having prescribed rows (or columns) of entries and prescribed characteristic polynomial.

Recently, S. Friedland [3] determined the existence of matrices having all of the characteristic values and off-diagonal entries prescribed. Let n be a positive integer, and consider for the moment the following definition. A field F is said to have property P(n) if for each set of n^2 elements of F, a_{ij}, $i \neq j$, $i, j = 1, 2, \ldots, n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$, there exists an $n \times n$ matrix $A = (a_{ij})$ over F with characteristic values $\lambda_1, \lambda_2, \ldots, \lambda_n$ and (i, j)-th entry a_{ij}, $i \neq j$, $i, j = 1, 2, \ldots, n$. (Moreover, the number of such matrices is finite.) Then Friedland's remarkable result can be rephrased as follows.

Theorem (Friedland) If a field F is algebraically closed, then F has property P(n) for every positive integer n.

Our next theorem shows that the converse to Friedland's theorem is also true.

Theorem 4.1 A field F is algebraically closed if F has property P(n) for each positive integer n.

Proof. Under the assumption that the field F has property P(n) for each positive integer n, it suffices to establish that each monic polynomial $f(\lambda)$ over F of degree $d \geq 1$ factors completely into a product of linear factors over F. We shall accomplish this by induction on d. The assertion is clearly true for d = 1. Assume the assertion is true for monic polynomials over F of degree k, $1 \leq k < d$. Let $f(\lambda) = \lambda^d + c_{d-1}\lambda^{d-1} + \ldots + c_1\lambda + c_0$ be an arbitrary monic polynomial over F of degree d. Let $f(\lambda) = \lambda\hat{f}(\lambda) + c_0$, where $\hat{f}(\lambda) = \lambda^{d-1} + c_{d-1}\lambda^{d-2} + \ldots + c_1$. By our induction assumption there exist scalars $\lambda_1, \lambda_2, \ldots, \lambda_{d-1} \in F$ such that

$$\hat{f}(\lambda) = \prod_{i=1}^{d-1} (\lambda - \lambda_i).$$

Since F has property P(d), there exist scalars $a_1, a_2, \ldots, a_d \in F$ such that the $d \times d$ matrix

$$\begin{bmatrix} a_1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & 1 & \cdots & 0 & 0 \\ 0 & 0 & a_3 & & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & a_{d-1} & 1 \\ c_0 & 0 & 0 & \cdots & 0 & a_d \end{bmatrix}$$

has characteristic values $0, \lambda_1, \lambda_2, \ldots, \lambda_{d-1}$. It follows that

$$\det(\lambda I - A) = -c_0 + \prod_{i=1}^{d} (\lambda - a_i) = \lambda \prod_{i=1}^{d-1} (\lambda - \lambda_i)$$

and hence

$$\prod_{i=1}^{d} (\lambda - a_i) = \lambda \hat{f}(\lambda) + c_0 = f(\lambda)$$

where $a_1, a_2, \ldots, a_d \in F$. The induction proof of the theorem is complete.

In addition to his result, Friedland's approach to the problem is noteworthy. He gave a simple proof of the theorem by applying the well-known Hilbert Nullstellensatz in determining the solvability of a certain system of polynomial equations. Such an approach may be suitable for obtaining existence theorems in cases where constructing matrices having prescribed properties is difficult. Our efforts to exploit Friedland's technique in the solution of several existence problems failed, perhaps due to the absence of the special systems of polynomial equations required for application of the technique. However, with careful consideration given to the restrictions upon the desired matrices, further efforts may yield worthwhile results on several open problems of current interest. Examples of such problems are given in the following paragraphs.

Theorem 4.1 establishes that Friedland's result does not hold for fields that are not algebraically closed. Thus the following problem remains to be solved.

Suppose the field F is not algebraically closed.

What conditions are necessary and sufficient for the

> elements a_{ij}, $i \neq j$, $i, j = 1, 2, \ldots, n$ and monic polynomial $p(\lambda)$ of degree n over F to be the off-diagonal entries and characteristic polynomial, respectively, of a matrix $A = (a_{ij}) \in M_n(F)$?

Results in the case that F is the field of real numbers R would be of practical importance. For example, H. Zimmer and J. Van Ness [16] have considered the problem of assigning the characteristic values of a real n × n matrix constrained by the specification of certain zero entries, noting that the problem arises in the design of dynamic systems with specified configurations. In their paper, they obtain necessary and sufficient conditions for a third- or fourth-order real matrix of the form

$$A = \begin{bmatrix} a_1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & 1 & \cdots & 0 & 0 \\ 0 & 0 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & 1 \\ b & 0 & 0 & \cdots & 0 & a_n \end{bmatrix}$$

to exist when A is required to have a prescribed set of n characteristic values consisting of real values and complex conjugate pairs. It may be possible to solve this problem for single-loop matrices of all orders, and it is hoped that necessary and sufficient conditions can be obtained for the real numbers a_{ij}, $i \neq j$, $i, j = 1, 2, \ldots, n$ and complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ consisting of real values and conjugate pairs to be the off-diagonal entries and the characteristic values, respectively, of a real n × n matrix $A = (a_{ij})$. One approach to this problem would be to use Friedland's technique of considering the solvability of certain systems of polynomial equations. The key to obtaining the result might be to determine when a certain system of real polynomial equations in n unknowns has a solution in R^n.

L. Mirsky [10] obtained a necessary and sufficient condition for the real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $a_1, a_2, \ldots, a_n$ to be the characteristic values and the diagonal entries, respectively, of a real symmetric $n \times n$ matrix. One might also ask if there is a necessary and sufficient condition for the real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ and a_{ij}, $i \neq j$, $i, j = 1, 2, \ldots, n$ to be the characteristic values and the off-diagonal entries, respectively, of a real symmetric $n \times n$ matrix $A = (a_{ij})$. Mirsky noted that the following generalization deserves attention.

> If f(x) is an irreducible monic polynomial over a field F, what are the conditions for the existence of a symmetric matrix over F which has f(x) as its characteristic polynomial?

In seeking to impose further restrictions on a matrix having prescribed characteristic polynomial and prescribed off-diagonal entries, one may seek necessary and sufficient conditions for the complex numbers a_{ij}, $i \neq j$, $i, j = 1, 2, \ldots, n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ to be the off-diagonal entries and characteristic values, respectively, of a normal (or hermitian) $n \times n$ complex matrix $A = (a_{ij})$. Several well-known necessary conditions are found in [8]. For example, if $A = (a_{ij}) \in M_n(C)$ is normal and has characteristic values $\lambda_1, \lambda_2, \ldots, \lambda_n$, then it must be true that

$$\max_{i,j} |\lambda_i - \lambda_j| \geq \sqrt{3} \max_{i \neq j} |a_{ij}|.$$

Moreover, if $A = (a_{ij}) \in M_n(C)$ is hermitian and has characteristic values $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the inequality

$$\max_{i,j} |\lambda_i - \lambda_j| \geq 2 \max_{i \neq j} |a_{ij}|$$

must hold. These necessary conditions are clearly not sufficient.

It may be that results concerning the existence of normal and hermitian matrices with prescribed off-diagonal entries and characteristic values, together with an artistic choice of off-diagonal entries, would lead to answers to the following unsolved problems mentioned by L. Mirsky [11]. Firstly, given real numbers a_k, b_k, and c_k, $1 \leq k \leq n$, what conditions are necessary and sufficient for the existence of $n \times n$ hermitian matrices A and B such that $\{a_k\}$, $\{b_k\}$, and $\{c_k\}$ are the characteristic values of A, B, and A + B, respectively? A. Horn [7] has been able to obtain a definitive answer only for $n \leq 4$. The analogous question for normal matrices is a second problem. H. Wielandt [15] solved the following variation of this problem. Given complex numbers $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ and c, what conditions are necessary and sufficient for the existence of $n \times n$ normal matrices A and B with characteristic values $\{a_k\}$ and $\{b_k\}$, respectively, such that c is a characteristic value of A + B? Mirsky noted, however, that "Wielandt's solution is given not by means of a system of inequalities but in geometric terms, and it does not obviously lead to an effective procedure for deciding whether a given set of numbers possesses the requisite properties."

Next, recall that if A is an $n \times n$ complex matrix, then the nonnegative square roots of the characteristic values of the hermitian matrix A^*A are called the singular values of A. A. Horn [6] obtained the following result.

> Given complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ and nonnegative real numbers $s_1, s_2, \ldots, s_n$, there exists an $n \times n$ complex matrix A having characteristic values $\lambda_1, \lambda_2, \ldots, \lambda_n$ and singular values $s_1, s_2, \ldots, s_n$ if and only if the relations $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$ and $s_1 \geq s_2 \geq \ldots \geq s_n$ imply that
>
> $$|\lambda_1 \lambda_2 \ldots \lambda_k| \leq s_1 s_2 \ldots s_k \quad (1 \leq k < n)$$
>
> and
>
> $$|\lambda_1 \lambda_2 \ldots \lambda_n| = s_1 s_2 \ldots s_n .$$

In view of Friedland's result, the following problem may be considered.

> Given complex numbers a_{ij}, $i \neq j$, $i, j = 1, 2, \ldots, n$ and nonnegative real numbers $s_1, s_2, \ldots, s_n$, when does there exist an $n \times n$ matrix $A = (a_{ij})$ having off-diagonal entries a_{ij}, $i \neq j$, $i, j = 1, 2, \ldots, n$ and singular values $s_1, s_2, \ldots, s_n$?

A stronger result would be the solution of the following problem.

> If a_{ij}, $i \neq j$, $i, j = 1, 2, \ldots, n$ are any given complex numbers, and if $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $s_1, s_2, \ldots, s_n$ are complex numbers and nonnegative real numbers, respectively, such that whenever $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$ and $s_1 \geq s_2 \geq \ldots \geq s_n$ then $|\lambda_1 \lambda_2 \ldots \lambda_k| \leq s_1 s_2 \ldots s_n$ for $1 \leq k < n$ and $|\lambda_1 \lambda_2 \ldots \lambda_n| = s_1 s_2 \ldots s_n$, when does there exist

an $n \times n$ complex matrix $A = (a_{ij})$ having off-diagonal entries a_{ij}, $i \neq j$, $i, j = 1, 2, \ldots, n$, characteristic values $\lambda_1, \lambda_2, \ldots, \lambda_n$, and singular values $s_1, s_2, \ldots, s_n$?

The above are just a few of the open problems for which worthwhile results might be obtained by following the lead of Friedland. In seeking additional problems, it is suggested again that one review the survey by L. Mirsky [11].

BIBLIOGRAPHY

1. Fan, K. and Pall, G., "Imbedding conditions for hermitian and normal matrices", Can. J. Math. 9 (1957), pp. 298-304.

2. Farahat, H. K. and Ledermann, W., "Matrices with prescribed characteristic polynomials", Proc. Edinburgh, Math. Soc. 11 (1959), pp. 143-146.

3. Friedland, S., "Matrices with prescribed off-diagonal elements", Israel Jnl. Math. 11 (1972), pp. 184-189.

4. Gantmacher, F. R., The Theory of Matrices, Vol. II, Chelsea Pub. Co., New York, 1960.

5. Hautus, M. L. J., "Controllability and observability conditions of linear autonomous systems", Indagationes Mathematicae, Vol. XXXI (1969), pp. 443-448.

6. Horn, A., "On the eigenvalues of a matrix with prescribed singular values", Proc. Amer. Math. Soc. 5(1954), pp. 4-7.

7. ________, "Eigenvalues of sums of hermitian matrices", Pacific J. Math., 12 (1962), pp. 225-241.

8. Marcus, M. and Minc, H., A Survey of Matrix Theory and Matrix Inequalities, Prindle Weber and Schmidt, Inc., Boston, 1964.

9. Merris, R., Two problems involving Schur functions", Linear Algebra and Its Applications 10(1975), pp. 155-162.

10. Mirsky, L, "Matrices with prescribed characteristic roots and diagonal elements", J. London Math. Soc. 33(1958), pp. 14-21.

11. ________, "Inequalities and existence theorems in the theory of matrices", J. Math. Anal. and Appl. 9 (1964), pp. 99-118.

12. de Oliveira, G. N., "Matrices with prescribed characteristic polynomial and a prescribed submatrix - II", Pacific J. Math. 29 (1969), pp. 663-667.

13. ________, "A conjecture and some problems on permanents", Pacific J. Math. 32(1970), pp. 495-499.

14. Taussky, O. and Paige, L., Simultaneous Linear Equations and the Determination of Eigenvalues, U. S. National Bureau of Standards Applied Mathematics Series 29 (1953), pp. 75-78.

15. Wielandt, H., "On eigenvalues of sums of normal matrices", Pacific J. Math. 5 (1955), pp. 633-638.

16. Zimmer, H. and Van Ness, J., "Eigenvalue specification for system matrices", Poc. 1973 Joint Automat. Control Conf., pp. 465-472.