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*Research Article*

# On Integral Operators with Operator-Valued Kernels

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Here, we study the continuity of integral operators with operator-valued kernels. Particularly we get $L_q(S;X) \to L_p(T;Y)$ estimates under some natural conditions on the kernel $k : T \times S \to B(X,Y)$, where $X$ and $Y$ are Banach spaces, and $(T, \sum_T, \mu)$ and $(S, \sum_S, \nu)$ are positive measure spaces: Then, we apply these results to extend the well-known Fourier Multiplier theorems on Besov spaces.

## 1. Introduction

It is well known that solutions of inhomogeneous differential and integral equations are represented by integral operators. To investigate the stability of solutions, we often use the continuity of corresponding integral operators in the studied function spaces. For instance, the boundedness of Fourier multiplier operators plays a crucial role in the theory of linear PDE's, especially in the study of maximal regularity for elliptic and parabolic PDE's. For an exposition of the integral operators with scalar-valued kernels see [1] and for the application of multiplier theorems see [2].

Girardi and Weis [3] recently proved that the integral operator

$$(Kf)(\cdot) = \int_S k(\cdot, s) f(s) d\nu(s) \tag{1.1}$$

defines a bounded linear operator

$$K : L_p(S, X) \longrightarrow L_p(T, Y) \tag{1.2}$$

provided some measurability conditions and the following assumptions

$$\sup_{s\in S}\int_T \|k(t,s)x\|_Y d\mu(t) \le C_1\|x\|_X, \quad \forall x \in X,$$
$$\sup_{t\in T}\int_S \left\|k^*(t,s)y^*\right\|_{X^*} d\nu(s) \le C_2\left\|y^*\right\|_{Y^*}, \quad \forall y^* \in Y^* \tag{1.3}$$

are satisfied. Inspired from [3] we will show that (1.1) defines a bounded linear operator

$$K : L_q(S,X) \longrightarrow L_p(T,Y) \tag{1.4}$$

if the kernel $k : T \times S \to B(X,Y)$ satisfies the conditions

$$\sup_{s\in S}\left(\int_T \|k(t,s)x\|_Y^\theta dt\right)^{1/\theta} \le C_1\|x\|_X, \quad \forall x \in X,$$
$$\sup_{t\in T}\left(\int_S \left\|k^*(t,s)y^*\right\|_{X^*}^\theta ds\right)^{1/\theta} \le C_2\left\|y^*\right\|_{Y^*}, \quad \forall y^* \in Y^*, \tag{1.5}$$

where

$$\frac{1}{q} - \frac{1}{p} = 1 - \frac{1}{\theta} \tag{1.6}$$

for $1 \le q < \theta/(\theta-1) \le \infty$ and $\theta \in [1,\infty)$.

Here $X$ and $Y$ are Banach spaces over the field $C$ and $X^*$ is the dual space of $X$. The space $B(X,Y)$ of bounded linear operators from $X$ to $Y$ is endowed with the usual uniform operator topology.

Now let us state some important notations from [3]. A subspace $Y$ of $X^*$ $\tau$-norms $X$, where $\tau \ge 1$, provided

$$\|x\|_X \le \tau \sup_{x^*\in B(Y)} |x^*(x)| \quad \forall x \in X. \tag{1.7}$$

It is clear that if $Y$ $\tau$-norms $X$ then the canonical mapping

$$u : X \longrightarrow Y^* \quad \text{with } \langle y, ux\rangle = \langle x, y\rangle \tag{1.8}$$

is an isomorphic embedding with

$$\frac{1}{\tau}\|x\|_X \le \|u(x)\|_{Y^*} \le \|x\|_X. \tag{1.9}$$

Let $(T, \sum_T, \mu)$ and $(S, \sum_S, \nu)$ be $\sigma$-finite (positive) measure spaces and

$$\sum_S^{\text{finite}} = \left\{ A \in \sum_S : \nu(A) < \infty \right\}, \qquad \sum_S^{\text{full}} = \left\{ A \in \sum_S : \nu(S \setminus A) = 0 \right\}. \tag{1.10}$$

$\varepsilon(S, X)$ will denote the space of finitely valued and finitely supported measurable functions from $S$ into $X$, that is,

$$\varepsilon(S, X) = \left\{ \sum_{i=1}^{n} x_i 1_{A_i} : x_i \in X,\ A_i \in \sum_S^{\text{finite}},\ n \in N \right\}. \tag{1.11}$$

Note that $\varepsilon(S, X)$ is norm dense in $L_p(S, X)$ for $1 \le p < \infty$. Let $L_\infty^0(S, X)$ be the closure of $\varepsilon(S, X)$ in the $L_\infty(S, X)$ norm. In general $L_\infty^0(S, X) \neq L_\infty(S, X)$ (see [3, Proposition 2.2] and [3, Lemma 2.3]).

A vector-valued function $f : S \to X$ is measurable if there is a sequence $(f_n)_{n=1}^\infty \subset \varepsilon(S, X)$ converging (in the sense of $X$ topology) to $f$ and it is $\sigma(X, \Gamma)$-measurable provided $\langle f(\cdot), x^* \rangle : S \to K$ is measurable for each $x^* \in \Gamma \subset X^*$. Suppose $1 \le p \le \infty$ and $1/p + 1/p' = 1$. There is a natural isometric embedding of $L_{p'}(T, Y^*)$ into $[L_p(T, Y)]^*$ given by

$$\langle f, g \rangle = \int_T \langle f(t), g(t) \rangle d\mu(t) \quad \text{for } g \in L_{p'}(T, Y^*),\ f \in L_p(T, Y). \tag{1.12}$$

Now, let us note that if $X$ is reflexive or separable, then it has the Radon-Nikodym property, which implies that $[E(X)]^* = E^*(X^*)$.

## 2. $L_q \to L_p$ Estimates for Integral Operators

In this section, we identify conditions on operator-valued kernel $k : T \times S \to B(X, Y)$, extending theorems in [3] so that

$$\|K\|_{L_q(S,X) \to L_p(T,Y)} \le C \tag{2.1}$$

for $1 \le q \le p$. To prove our main result, we shall use some interpolation theorems of $L_p$ spaces. Therefore, we will study $L_1(S, X) \to L_\theta(T, Y)$ and $L_{\theta'}(S, X) \to L_\infty(T, Y)$ boundedness of integral operator (1.1). The following two conditions are natural measurability assumptions on $k : T \times S \to B(X, Y)$.

*Condition 1.* For any $A \in \sum_S^{\text{finite}}$ and each $x \in X$

(a) there is $T_{A,x} \in \sum_T^{\text{full}}$ so that if $t \in T_{A,x}$ then the Bochner integral

$$\int_A k(t,s)x\,d\nu(s) \quad \text{exists,} \tag{2.2}$$

(b) $T_{A,x} : t \longrightarrow \int_A k(t,s)x\,d\nu(s)$ defines a measurable function from $T$ into $Y$.

Note that if $k$ satisfies the above condition then for each $f \in \varepsilon(S,X)$, there is $T_f \in \sum_T^{\text{full}}$ so that the Bochner integral

$$\int_S k(t,s)f(s)\,d\nu(s) \quad \text{exists} \tag{2.3}$$

and (1.1) defines a linear mapping

$$K : \varepsilon(S,X) \longrightarrow L_0(T,Y), \tag{2.4}$$

where $L_0$ denotes the space of measurable functions.

*Condition 2.* The kernel $k : T \times S \longrightarrow B(X,Y)$ satisfies the following properties:

(a) a real-valued mapping $\|k(t,s)x\|_X^\theta$ is product measurable for all $x \in X$,

(b) there is $S_x \in \sum_S^{\text{full}}$ so that

$$\|k(t,s)x\|_{L_{\theta(T,Y)}} \leq C_1\|x\|_X \tag{2.5}$$

for $1 \leq \theta < \infty$ and $x \in X$.

**Theorem 2.1.** *Suppose* $1 \leq \theta < \infty$ *and the kernel* $k : T \times S \longrightarrow B(X,Y)$ *satisfies Conditions 1 and 2. Then the integral operator* (1.1) *acting on* $\varepsilon(S,X)$ *extends to a bounded linear operator*

$$K : L_1(S,X) \longrightarrow L_\theta(T,Y). \tag{2.6}$$

*Proof.* Let $f = \sum_{i=1}^n x_i 1_{A_i}(s) \in \varepsilon(S,X)$ be fixed. Taking into account the fact that $1 \leq \theta$ and using the general Minkowski-Jessen inequality with the assumptions of the theorem we obtain

$$\begin{aligned}
\left\|(Kf)(t)\right\|_{L_\theta(T,Y)} &\leq \left[\int_T\left(\int_S\left\|k(t,s)\sum_{i=1}^n x_i 1_{A_i}(s)\right\|_Y d\nu(s)\right)^\theta d\mu(t)\right]^{1/\theta} \\
&\leq \int_S\left(\int_T\left\|k(t,s)\sum_{i=1}^n x_i 1_{A_i}(s)\right\|_Y^\theta d\mu(t)\right)^{1/\theta} d\nu(s) \\
&\leq \int_S\left[\int_T\left(\sum_{i=1}^n 1_{A_i}(s)\|k(t,s)x_i\|_Y\right)^\theta d\mu(t)\right]^{1/\theta} d\nu(s) \\
&\leq \int_S\sum_{i=1}^n 1_{A_i}(s)\left(\int_T\|k(t,s)x_i\|_Y^\theta d\mu(t)\right)^{1/\theta} d\nu(s) \\
&\leq \int_S\sum_{i=1}^n 1_{A_i}(s)\|k(t,s)x_i\|_{L_\theta(T,Y)}\, d\nu(s) \leq C_1\sum_{i=1}^n\|x_i\|_X\int_S 1_{A_i}(s)d\nu(s) \\
&= C_1\sum_{i=1}^n\|x_i\|_X\nu(A_i) = C_1\left\|f\right\|_{L_1(S,X)}.
\end{aligned} \tag{2.7}$$

Hence, $\|K\|_{L_1\to L_\theta}\leq C_1$. □

*Condition 3.* For each $y^*\in Z$ there is $T_{y^*}\in\sum_T^{\text{full}}$ so that for all $t\in T_{y^*}$,

(a) a real-valued mapping $\|k^*(t,s)x^*\|_{X^*}^\theta$ is measurable for all $x^*\in X^*$,

(b) there is $S_x\in\sum_S^{\text{full}}$ so that

$$\left\|k^*(t,s)y^*\right\|_{L_{\theta(S,X^*)}}\leq C_2\left\|y^*\right\|_{Y^*} \tag{2.8}$$

for $1\leq\theta<\infty$ and $x\in X$.

**Theorem 2.2.** *Let $Z$ be a separable subspace of $Y^*$ that $\tau$-norms $Y$. Suppose $1\leq\theta<\infty$ and $k: T\times S\to B(X,Y)$ satisfies Conditions 1 and 3. Then integral operator* (1.1) *acting on $\varepsilon(S,X)$ extends to a bounded linear operator*

$$K: L_{\theta'}(S,X)\longrightarrow L_\infty(T,Y). \tag{2.9}$$

*Proof.* Suppose $f\in\varepsilon(S,X)$ and $y^*\in Z$ are fixed. Let $T_f,T_{y^*}\in\sum_T^{\text{full}}$ be corresponding sets due to Conditions 1 and 3. By separability of $Z$, we can choose a countable set of $T_{y^*}\in\sum_T^{\text{full}}$ satisfying the above condition (note that since $\sum_T^{\text{full}}$ is a sigma algebra, the union of these countable sets still belongs to $\sum_T^{\text{full}}$ and the intersection of these sets should be nonempty). If $t \in T_f \cap T_{y^*}$ then, by using Hölder's inequality and assumptions of the theorem, we get

$$\begin{aligned} \left|\langle y^*, (Kf)(t)\rangle_Y\right| &= \left|\left\langle y^*, \int_S k(t,s) f(s) d\nu(s)\right\rangle\right| \\ &\le \int_S \left|\left[k^*(t,s) y^*\right] f(s)\right| d\nu(s) \\ &\le \|k^*(t,s) y^*\|_{L_\theta(S,X^*)} \|f(s)\|_{L_{\theta'}(S,X)} \\ &\le C_2 \|y^*\| \|f\|_{L_{\theta'}(S,X)}. \end{aligned} \tag{2.10}$$

Since, $T_f \cap T_{y^*} \in \sum_T^{\text{full}}$ and $Z$ $\tau$-norms $Y$

$$\|Kf\|_{L_\infty(T,Y)} \le C_2 \tau \|f\|_{L_{\theta'}(S,X)}. \tag{2.11}$$

Hence, $\|K\|_{L_{\theta'} \to L_\infty} \le \tau C_2$. □

In [3, Lemma 3.9], the authors slightly improved interpolation theorem [4, Theorem 5.1.2]. The next lemma is a more general form of [3, Lemma 3.9].

**Lemma 2.3.** *Suppose a linear operator*

$$K : \varepsilon(S,X) \longrightarrow L_\theta(T,Y) + L_\infty(T,Y) \tag{2.12}$$

*satisfies*

$$\|Kf\|_{L_\theta(T,Y)} \le C_1 \|f\|_{L_1(S,X)}, \qquad \|Kf\|_{L_\infty(T,Y)} \le C_2 \|f\|_{L_{\theta'}(S,X)}. \tag{2.13}$$

*Then, for* $1/q - 1/p = 1 - 1/\theta$ *and* $1 \le q < \theta/(\theta-1) \le \infty$ *the mapping* $K$ *extends to a bounded linear operator*

$$K : L_q(S,X) \longrightarrow L_p(T,Y) \tag{2.14}$$

*with*

$$\|K\|_{L_q \to L_p} \le (C_1)^{\theta/p} (C_2)^{1-\theta/p}. \tag{2.15}$$

*Proof.* Let us first consider the conditional expectation operator

$$(K_0 f) = E\Big((Kf)_{1_B} |_\Sigma\Big), \tag{2.16}$$

where $\sum$ is a $\sigma$-algebra of subsets of $B \in \sum_T^{\text{finite}}$. From (2.13) it follows that

$$\|K_0 f\|_{L_\theta(T,Y)} \le C_1 \|f\|_{L_1(S,X)} < \infty,$$
$$\|K_0 f\|_{L_\infty(T,Y)} \le C_2 \|f\|_{L_{\theta'}(S,X)} < \infty. \tag{2.17}$$

Hence, by Riesz-Thorin theorem [4, Theorem 5.1.2], we have

$$\|K_0 f\|_{L_p(T,Y)} \le (C_1)^{\theta/p} (C_2)^{1-\theta/p} \|f\|_{L_q(S,X)}. \tag{2.18}$$

Now, taking into account (2.18) and using the same reasoning as in the proof of [3, Lemma 3.9], one can easily show the assertion of this lemma. □

**Theorem 2.4** (operator-valued Schur's test). *Let $Z$ be a subspace of $Y^*$ that $\tau$-norms $Y$ and $1/q - 1/p = 1 - 1/\theta$ for $1 \le q < \theta/(\theta-1) \le \infty$. Suppose $k : T \times S \to B(X,Y)$ satisfies Conditions 1, 2, and 3 with respect to $Z$. Then integral operator* (1.1) *extends to a bounded linear operator*

$$K : L_q(S,X) \longrightarrow L_p(T,Y) \tag{2.19}$$

*with*

$$\|K\|_{L_q \to L_p} \le (C_1)^{\theta/p} (\tau C_2)^{1-\theta/p}. \tag{2.20}$$

*Proof.* Combining Theorems 2.1 and 2.2, and Lemma 2.3, we obtain the assertion of the theorem. □

*Remark 2.5.* Note that choosing $\theta = 1$ we get the original results in [3].

For $L_\infty$ estimates (it is more delicate and based on ideas from the geometry Banach spaces) and weak continuity and duality results see [3]. The next corollary plays important role in the Fourier Multiplier theorems.

**Corollary 2.6.** *Let $Z$ be a subspace of $Y^*$ that $\tau$-norms $Y$ and $1/q - 1/p = 1 - (1/\theta)$ for $1 \le q < \theta/\theta - 1 \le \infty$. Suppose $k : R^n \to B(X,Y)$ is strongly measurable on $X$, $k^* : R^n \to B(Y^*, X^*)$ is strongly measurable on $Z$ and*

$$\|kx\|_{L_\theta(R^n,Y)} \le C_1 \|x\|_X, \quad \forall x \in X,$$
$$\|k^* y^*\|_{L_\theta(R^n,X^*)} \le C_2 \|y^*\|_{Y^*}, \quad \forall y^* \in Y^*. \tag{2.21}$$

*Then the convolution operator defined by*

$$(Kf)(t) = \int_{R^n} k(t-s) f(s) ds \quad \text{for } t \in R^n \tag{2.22}$$

*satisfies* $\|K\|_{L_q \to L_p} \le (C_1)^{\theta/p} (C_2)^{1-\theta/p}$.

It is easy to see that $k : R^n \to B(X, Y)$ satisfies Conditions 1, 2, and 3 with respect to $Z$. Thus, assertion of the corollary follows from Theorem 2.4.

## 3. Fourier Multipliers of Besov Spaces

In this section we shall indicate the importance of Corollary 2.6 in the theory of Fourier multipliers (FMs). Thus we give definition and some basic properties of operator valued FM and Besov spaces.

Consider some subsets $\{J_k\}_{k=0}^{\infty}$ and $\{I_k\}_{k=0}^{\infty}$ of $R^n$ given by

$$\begin{gathered} J_0 = \{t \in R^n : |t| \le 1\}, \qquad J_k = \left\{t \in R^n : 2^{k-1} < |t| \le 2^k\right\} \quad \text{for } k \in N, \\ I_0 = \{t \in R^n : |t| \le 2\}, \qquad I_k = \left\{t \in R^n : 2^{k-1} < |t| \le 2^{k+1}\right\} \quad \text{for } k \in N. \end{gathered} \tag{3.1}$$

Let us define the partition of unity $\{\varphi_k\}_{k \in N_0}$ of functions from $S(R^n, R)$. Suppose $\psi \in S(R, R)$ is a nonnegative function with support in $[2^{-1}, 2]$, which satisfies

$$\begin{gathered} \sum_{k=-\infty}^{\infty} \psi\left(2^{-k}s\right) = 1 \quad \text{for } s \in R \setminus \{0\}, \\ \varphi_k(t) = \psi\left(2^{-k}|t|\right), \quad \varphi_0(t) = 1 - \sum_{k=1}^{\infty} \varphi_k(t) \quad \text{for } t \in R^n. \end{gathered} \tag{3.2}$$

Note that

$$\operatorname{supp} \varphi_k \subset \bar{I}_k, \qquad \operatorname{supp} \varphi_k \subset I_k. \tag{3.3}$$

Let $1 \le q \le r \le \infty$ and $s \in R$. The Besov space is the set of all functions $f \in S'(R^n, X)$ for which

$$\begin{aligned} \|f\|_{B_{q,r}^s(R^n,X)} &:= \left\|2^{ks}\{(\check{\varphi}_k * f)\}_{k=0}^{\infty}\right\|_{l_r(L_q(R^n,X))} \\ &\equiv \begin{cases} \left[\displaystyle\sum_{k=0}^{\infty} 2^{ksr} \|\check{\varphi}_k * f\|_{L_q(R^n,X)}^r\right]^{1/r} & \text{if } r \ne \infty \\ \displaystyle\sup_{k \in N_0}\left[2^{ks}\|\check{\varphi}_k * f\|_{L_q(R^n,X)}\right] & \text{if } r = \infty \end{cases} \end{aligned} \tag{3.4}$$

is finite; here $q$ and $s$ are main and smoothness indexes respectively. The Besov space has significant interpolation and embedding properties:

$$\begin{gathered} B^s_{q,r}(R^n;X) = \left(L_q(R^n;X), W^m_q(R^d;X)\right)_{s/m,r}, \\ W^{l+1}_q(X) \hookrightarrow B^s_{q,r}(X) \hookrightarrow W^l_q(X) \hookrightarrow L_q(X), \quad \text{where } l < s < l+1, \\ B^s_{\infty,1}(X) \hookrightarrow C^s(X) \hookrightarrow B^s_{\infty,\infty}(X) \quad \text{for } s \in \mathbf{Z}, \\ B^{d/p}_{p,1}\left(R^d, X\right) \hookrightarrow L_\infty\left(R^d, X\right) \quad \text{for } s \in \mathbf{Z}, \end{gathered} \tag{3.5}$$

where $m \in N$ and $C^s(X)$ denotes the Holder-Zygmund spaces.

*Definition 3.1.* Let X be a Banach space and $1 \le u \le 2$. We say X has Fourier type $u$ if

$$\|\mathcal{F} f\|_{L_{u'}(R^n,X)} \le C\|f\|_{L_u(R^n,X)} \quad \text{for each } f \in S\left(R^N, X\right), \tag{3.6}$$

where $1/u + 1/u' = 1, \mathcal{F}_{u,n}(X)$ is the smallest $C \in [0,\infty]$. Let us list some important facts:

(i) any Banach space has a Fourier type 1,

(ii) $B$-convex Banach spaces have a nontrivial Fourier type,

(iii) spaces having Fourier type 2 should be isomorphic to a Hilbert spaces.

The following corollary follows from [5, Theorem 3.1].

**Corollary 3.2.** *Let X be a Banach space having Fourier type $u \in [1,2]$ and $1 \le \theta \le u'$. Then the inverse Fourier transform defines a bounded operator*

$$\mathcal{F}^{-1} : B^{n(1/\theta - 1/u')}_{u,1}(R^n, X) \longrightarrow L_\theta(R^n, X). \tag{3.7}$$

*Definition 3.3.* Let $(E_1(R^n, X), E_2(R^n, Y))$ be one of the following systems, where $1 \le q \le p \le \infty$:

$$(L_q(X), L_p(Y)) \quad \text{or} \quad \left(B^s_{q,r}(X), B^s_{p,r}(Y)\right). \tag{3.8}$$

A bounded measurable function $m : R^n \to B(X,Y)$ is called a Fourier multiplier from $E_1(X)$ to $E_2(Y)$ if there is a bounded linear operator

$$T_m : E_1(X) \longrightarrow E_2(Y) \tag{3.9}$$

such that

$$T_m(f) = \mathcal{F}^{-1}[m(\cdot)(\mathcal{F}f)(\cdot)] \quad \text{for each } f \in S(X), \tag{3.10}$$

$$T_m \text{ is } \sigma(E_1(X), E_1^*(X^*)) \text{ to } \sigma(E_2(Y), E_2^*(Y^*)) \text{ continuous.} \tag{3.11}$$

The uniquely determined operator $T_m$ is the FM operator induced by $m$. Note that if $T_m \in B(E_1(X), E_2(Y))$ and $T_m^*$ maps $E_2^*(Y^*)$ into $E_1^*(X^*)$ then $T_m$ satisfies the weak continuity condition (3.11).

For the definition of Besov spaces and their basic properties we refer to [5].

Since (3.10) can be written in the convolution form

$$T_m(f)(t) = \int_{R^n} \check{m}(t-s) f(s) ds, \tag{3.12}$$

Corollaries 2.6 and 3.2 can be applied to obtain $L_q(R^n, X) \to L_p(R^n, Y)$ regularity for (3.10).

**Theorem 3.4.** *Let $X$ and $Y$ be Banach spaces having Fourier type $u \in [1,2]$ and $p, q \in [1, \infty]$ so that $0 \le 1/q - 1/p \le 1/u$. Then there is a constant $C$ depending only on $\mathcal{F}_{u,n}(X)$ and $\mathcal{F}_{u,n}(Y)$ so that if*

$$m \in B_{u,1}^{n(1/u+1/p-1/q)}(R^n, B(X,Y)) \tag{3.13}$$

*then $m$ is a FM from $L_q(R^n, X)$ to $L_p(R^n, Y)$ with*

$$\|T_m\|_{L_q(R^n,X) \to L_p(R^n,Y)} \le C M_u(m), \tag{3.14}$$

*where*

$$M_u^{p,q}(m) = \inf\left\{ a^{n(1/q-1/p)} \|m(a\cdot)\|_{B_{u,1}^{n(1/u+1/p-1/q)}(R^n,B(X,Y))} : a > 0 \right\}. \tag{3.15}$$

*Proof.* Let $1/q - 1/p = 1 - 1/\theta$ and $1 \le q < \theta/(\theta-1) \le \infty$. Assume that $m \in S(B(X,Y))$. Then $\check{m} \in S(B(X,Y))$. Since $\mathcal{F}^{-1}[m(a\cdot)x](s) = a^{-n}\check{m}(s/a)x$, choosing an appropriate $a$ and using (3.7) we obtain

$$\begin{aligned} \|\check{m}x\|_{L_\theta(Y)} &= a^{n-n/\theta} \left\| [m(a\cdot)x]^{\vee} \right\|_{L_\theta(Y)} \\ &\le C_1 a^{n/\theta'} \|m(a\cdot)\|_{B_{u,1}^{n(1/\theta-1/u')}} \|x\|_X \\ &\le 2C_1 M_u^{p,q}(m) \|x\|_X, \end{aligned} \tag{3.16}$$

where $C_1$ depends only on $\mathcal{F}_{u,n}(Y)$. Since $m \in S(B(X,Y))$ we have $[m^*]^{\vee} = [\check{m}]^* \in S(B(Y^*, X^*))$ and $M_u^{p,q}(m) = M_u^{p,q}(m^*)$. Thus, in a similar manner as above, we get

$$\left\| [\check{m}(\cdot)]^* y^* \right\|_{L_\theta(Y)} \le 2C_2 M_u^{p,q}(m) \left\| y^* \right\|_{Y^*} \tag{3.17}$$

for some constant $C_2$ depending on $F_{u,n}(X^*)$. Hence by (3.16)-(3.17) and Corollary 2.6

$$(T_m f)(t) = \int_{R^n} \check{m}(t-s) f(s) ds \tag{3.18}$$

satisfies

$$\|T_m f\|_{L_p(R^n,Y)} \leq C M_u^{p,q}(m) \|f\|_{L_q(R^n,X)} \tag{3.19}$$

for all $p,\ q \in [1,\infty]$ so that $0 \leq 1/q - 1/p \leq 1/u$. Now, taking into account the fact that $S(B(X,Y))$ is continuously embedded in $B_{u,1}^{n(1/u+1/p-1/q)}(B(X,Y))$ and using the same reasoning as [5, Theorem 4.3] one can easily prove the general case $m \in B_{u,1}^{n(1/u+1/p-1/q)}$ and the weak continuity of $T_m$. □

**Theorem 3.5.** *Let X and Y be Banach spaces having Fourier type $u \in [1,2]$ and $p,q \in [1,\infty]$ be so that $0 \leq 1/q - 1/p \leq 1/u$. Then, there exist a constant C depending only on $\mathcal{F}_{u,n}(X)$ and $\mathcal{F}_{u,n}(Y)$ so that if $m : R^n \to B(X,Y)$ satisfy*

$$\varphi_k \cdot m \in B_{u,1}^{n(1/u+1/p-1/q)}(R^n, B(X,Y)), \qquad M_u^{p,q}(\varphi_k \cdot m) \leq A \tag{3.20}$$

*then m is a FM from $B_{q,r}^s(R^n,X)$ to $B_{p,r}^s(R^n,Y)$ and $\|T_m\|_{B_{q,r}^s \to B_{p,r}^s} \leq CA$ for each $s \in R$ and $r \in [1,\infty]$.*

Taking into consideration Theorem 3.4 one can easily prove the above theorem in a similar manner as [5, Theorem 4.3].

The following corollary provides a practical sufficient condition to check (3.20).

**Lemma 3.6.** *Let $n(1/u + 1/p - 1/q) < l \in N$ and $\theta \in [u,\infty]$. If $m \in C^l(R^n, B(X,Y))$ and*

$$\begin{gathered} \|D^\alpha m\|_{L_\theta(I_0)} \leq A, \\ \left(2^{k-1}\right)^{n(1/q-1/p)} \|D^\alpha m_k\|_{L_\theta(I_1)} \leq A, \quad m_k(\cdot) = m\left(2^{k-1}\cdot\right), \end{gathered} \tag{3.21}$$

*for each $\alpha \in N^n$, $|\alpha| \leq l$ and $k \in N$, then m satisfies* (3.20)*.*

Using the fact that $W_u^l(R^n, B(X,Y)) \subset B_{u,1}^{n(1/u+1/p-1/q)}(R^n, B(X,Y))$, the above lemma can be proven in a similar fashion as [5, Lemma 4.10].

Choosing $\theta = \infty$ in Lemma 3.6 we get the following corollary.

**Corollary 3.7** (Mikhlin's condition). *Let X and Y be Banach spaces having Fourier type $u \in [1,2]$ and $0 \leq 1/q - 1/p \leq 1/u$. If $m \in C^l(R^n, B(X,Y))$ satisfies*

$$(1+|t|)^{|\alpha|+n(1/q-1/p)} \|D^\alpha m\|_{L_\infty(R^n, B(X,Y))} \leq A \tag{3.22}$$

*for each multi-index α with $|\alpha| \leq l = [n(1/u + 1/p - 1/q)] + 1$, then m is a FM from $B_{q,r}^s(R^n,X)$ to $B_{p,r}^s(R^n,Y)$ for each $s \in R$ and $r \in [1,\infty]$.*

*Proof.* It is clear that for $t \in I_0$

$$\|D^\alpha m\|_{L_\infty(I_0)} \leq (1+|t|)^{|\alpha|+n(1/q-1/p)} \|D^\alpha m\|_{L_\infty(R^n)}. \tag{3.23}$$

Moreover, for $t \in I_1$ we have

$$\begin{aligned}\left(2^{k-1}\right)^{n(1/q-1/p)}\|D^\alpha m_k(t)\|_{B(X,Y)} &= \left(2^{k-1}\right)^{|\alpha|+n(1/q-1/p)}\left\|m(2^{k-1}t)\right\|_{B(X,Y)} \\ &\le \left|2^{k-1}t\right|^{|\alpha|+n(1/q-1/p)}\left\|m(2^{k-1}t)\right\|_{B(X,Y)},\end{aligned} \tag{3.24}$$

which implies

$$\left(2^{k-1}\right)^{n(1/q-1/p)}\|D^\alpha m_k\|_{L_\infty(I_1)} \le (1+|t|)^{|\alpha|+n(1/q-1/p)}\|D^\alpha m(t)\|_{L_\infty(R^n)}. \tag{3.25}$$

Hence by Lemma 3.6, (3.22) implies assumption (3.20) of Theorem 3.5. □

*Remark 3.8.* Corollary 3.7 particularly implies the following facts.

(a) if $X$ and $Y$ are arbitrary Banach spaces then $l = [n(1/p + 1/q')] + 1$,

(b) if $X$ and $Y$ be Banach spaces having Fourier type $u \in [1, 2]$ and $1/q - 1/p = 1/u$ then $l = 1$, suffices for a function to be a FM in $(B^s_{q,r}(R^n, X), B^s_{p,r}(R^n, Y))$.

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## References

[1] G. B. Folland, *Real Analysis: Modern Techniques and Their Applications*, Pure and Applied Mathematics, John Wiley & Sons, New York, NY, USA, 1984.

[2] R. Denk, M. Hieber, and J. Prüss, "*R*-boundedness, Fourier multipliers and problems of elliptic and parabolic type," *Memoirs of the American Mathematical Society*, vol. 166, no. 788, pp. 1–106, 2003.

[3] M. Girardi and L. Weis, "Integral operators with operator-valued kernels," *Journal of Mathematical Analysis and Applications*, vol. 290, no. 1, pp. 190–212, 2004.

[4] J. Bergh and J. Löfström, *Interpolation spaces. An Introduction*, vol. 223 of *Grundlehren der Mathematischen Wissenschaften*, Springer, Berlin, Germany, 1976.

[5] M. Girardi and L. Weis, "Operator-valued Fourier multiplier theorems on Besov spaces," *Mathematische Nachrichten*, vol. 251, pp. 34–51, 2003.