ON THE THEORY OF STRUCTURES IN SETS

by

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I. INTRODUCTION

In many branches of mathematics the property of being a subspace receives considerable attention. The subspaces of a given space might be regarded as values of <u>a structure in a set</u>, that is, a function P on the subsets of a set V such that $P(X) \subseteq V$ for all $X \subseteq V$. Thus, structures in sets form a basis for an abstract treatment of the property of being a subspace.

The following properties of a structure P in a set V form a basis for an abstract treatment of the notion of linear independence over a vector space. (Cf. e. g. Bleicher-Preston [3], Bleicher-Marczewski [2] and Zariski-Samuel [10], pp.50-52.)

Monotonicity: If $X \subseteq Y \subseteq V$, then $P(X) \subseteq P(Y)$.

Extensiveness: If $X \subseteq V$, then $X \subseteq P(X)$.

Idempotence: If $X \subseteq V$, then $P(P(X)) = P(X)$.

Exchange property: If $X \subseteq V$, $y \in V$, $x \in P(X \cup \{y\})$ and $x \notin P(X)$, then $y \in P(X \cup \{x\})$.

Finite character: If $X \subseteq V$ and $x \in P(X)$, then $x \in P(Y)$ for some finite subset Y of X.

(If P has these five properties, then P is called <u>a span in V</u>.) Such a treatment studies the notions of independence and equivalence defined as follows:

A subset X of V is P-independent if and only if $x \notin P(X-\{x\})$ for all $x \in X$, where $S-\{s\} = \{t \in S: t \neq s\}$ for all sets S and all $s \in S$. Subsets X and Y of V are P-equivalent [or Y is a P-generator of P(X)] if and only if $P(Y) = P(X)$.

Bleicher and Preston in [3] used these five properties and proved the following propositions:

(1) Every subset of a P-independent subset of V is P-independent.

(2) If X is a P-independent subset of V and $y \in [V-P(X)]$, then $X \cup \{y\}$ is P-independent.

(3) Any two maximal (relative to inclusion) P-independent subsets of V are P-equivalent.

(4) If X and Y are P-independent P-equivalent subsets of V, then for each $x \in [X-(X \cap Y)]$ there is a $y \in [Y-(X \cap Y)]$ such that $(X-\{x\}) \cup \{y\}$ is P-independent and P-equivalent to X.

(5) If X and Y are P-independent P-equivalent subsets of V, then X and Y have the same cardinal number.

Zariski and Samuel in [10] proved (2), (3) and (5) in the case of finite subsets of V. Bleicher and Marczewski in [2] showed that if P is a closure structure, that is, a structure having monotonicity, extensiveness and idempotence, then (5) may be decomposed as follows:

(5a) If P has the exchange property, then any two P-independent P-equivalent subsets of V, one of which is finite, have the same cardinal number.

(5b) If P has finite character, then any two P-independent P-equivalent subsets of V, one of which is infinite, have the same cardinal number.

In [5] Hammer considered monotonicity, extensiveness and idempotence along with additivity and α-character (with α being a cardinal number not exceeding the first infinite cardinal number) as defined below:

Additivity: If $X \subseteq V$ and $Y \subseteq V$, then $P(X \cup Y) = P(X) \cup P(Y)$.

α-character: If $X \subseteq V$ and $x \in P(X)$, then $x \in P(Y)$ for some $Y \subseteq X$ such that $|Y| < \alpha$, where $|A|$ denotes the cardinal number of a set A.

Hammer studied the operations union, intersection and finite composition on families of structures. If G is a family of structures in a set V, then the union of G is the structure sending each subset X of V onto the union of all $Q(X)$ such that $Q \in G$. If G is a non-empty family of structures in a set V, then the intersection of G is the structure sending each subset X of V onto the intersection of all $Q(X)$ such that $Q \in G$. He observed that monotonicity, extensiveness, additivity and finite character are preserved under union and finite composition, that monotonicity and extensiveness are preserved under intersection, that finite character is preserved under finite intersection and that being a closure structure is preserved under intersection. With the understanding that "a structure R_1 in a set V contains a structure R_2 in V" (or R_2 is contained in R_1) means that $R_2(X) \subseteq R_1(X)$ for all subsets X of V, Hammer studied an arbitrary structure R in a set and deduced the existence of the following:

a unique minimal monotone structure containing R,

a unique maximal monotone structure contained in R,

a unique minimal extensive structure containing R,

a unique maximal additive structure contained in R, a unique maximal structure having finite character and

being contained in R.

He considered the iterated sequence $\{R^n\}$ [with $R^1 = R$ and $R^{n+1} = RR^n$ for all positive integers n] of a monotone and extensive structure R having finite character, and deduced that the union of the structures R^n is idempotent. The existence of a unique minimal closure structure having finite character and containing R followed.

In order to merely observe another effort at generalizing an important notion under serious investigation, I remark that Hammer's work in [5] was an additional contribution to his treatment of "Extended Topology" in [6].

Any treatment of abstract linear dependence based on properties considered thus far necessarily has certain limitations. For example, we consider the infinite dimensional real inner product space V of all real-valued sequences of functions which are square-summable. The structure P sending each subset X of V onto the closed linear manifold generated by X satisfies the conclusion in (5b). (See [9], pp.106-117.) However, P does not have finite character. Some relief is provided upon replacing finite character in (5b) by the following property:

S5: If X and Y are P-equivalent subsets of V and $x \in P(X)$,

then $x \in P(Z)$ for some subset Z of Y such that $|Z| \leq |X|$.

(See Theorem II-3L of Chapter II.)

One might observe that the conclusions in (5a) and (5b) involve comparison of cardinal numbers of pairs of equivalent sets under the structure P, and that S5 is stated in such a way that pairs of equivalent sets receive attention. However, the statement of finite character does not refer to pairs of equivalent sets.

Another "normality condition" which assists in providing some relief is a property to which we shall refer as S0. We shall say that "a structure P in a set V satisfies S0" and will mean that P satisfies the following condition:

S0: If G is an increasing (inclusion) chain of subsets of V, then $P(\cup G) = \cup\{P(X)\}_{X \in G}$.

A study of structures satisfying S0 can lead to a clearer understanding of the role played by finite character. Indeed, every monotone structure having finite character satisfies S0. (See Corollary 2 to Theorem II-2C, Section 2 of Chapter II.)

It is the purpose of this dissertation to provide a basic theory of structures in sets (Chapter II) based on monotonicity, extensiveness, idempotence, the exchange property, α-character (with α being an arbitrary cardinal number), S0 and S5. Many results from earlier treatments of abstract linear dependence with the notion of finite character will follow. Certain notions introduced and treated by Hammer are shown to have applicability in a larger class of structures. In fact, much of the theory enlarges one's awareness of the size of the class of structures to which many well-known notions common to several branches of mathematics apply.

In Section 1 of Chapter II some of the results by Hammer in [5] are developed. Hammer's treatment of idempotence is improved. (See Proposition II-1E.) Explicit definitions of certain maximal and minimal structures relative to a given structure are provided, and preservation of properties under finite composition and in passing to minimal and maximal structures is discussed in Propositions II-1B, II-1D, II-1G, II-1I and II-1K along with Theorems II-1F, II-1H, II-1J and II-1L. Proposition II-1M deals with the problem of commutativity of pairs of operations in passing to maximal and minimal structures relative to a given structure. Theorem II-1N provides two equivalent formulations of the Moore-family characterization of closure structures as appears in [1], p.111.

In Section 2 of Chapter II the independence of monotonicity, extensiveness, idempotence, the exchange property, α-character, S5 and the property of sending ϕ onto ϕ is established. (See Theorem II-2D.) The independence of the axioms for spans follows as an immediate consequence. (See the corollary to Theorem II-2D.) Other results in Section 2 provide sufficient or necessary conditions for the property S0 to hold.

Section 3 of Chapter II provides a basic theory of independence and equivalence relative to a given structure. The main results are Theorems II-3G, II-3H, II-3I and II-3L. Each of these theorems is a generalization of some result from earlier treatments of abstract linear dependence. The importance of S0 is demonstrated in Theorems II-3G and II-3H. The importance of S5 is demonstrated in Theorem II-3L where it is shown that the hypothesis of (5b) can be replaced by S5.

Section 4 of Chapter II provides an abstract treatment of the notions of homomorphism and continuity. The theory in this section might be regarded as an abstract treatment of some of the properties of open and closed maps. The treatment is enhanced by a study of factor spaces and factor structures induced by maps whose range space has a structure defined in it. Propositions II-4A, II-4B, II-4C and Theorem II-4J are results which appear in several areas in mathematics. Proposition II-4D provides results on commutativity of passing to factor structures with passing to maximal and minimal structures defined by a given structure. Theorems II-4E, II-4F and II-4G deal with inheritance of properties. Theorems II-4I and II-4K are generalizations of well-known results from topology. Theorem II-4H explains a relation between independence with respect to a domain structure and the factor structure defined by a map. Theorem II-4J is a generalization of a well-known result concerning linear spaces which states the relation between the dimension of the factor space, the null space and the domain space of a linear homomorphism. The notion of point-wise continuity as considered in topology (one of the definitions of continuity) is given an abstract treatment, and it is shown in Theorem II-4K that the notion of point-wise continuity coincides with the previously studied notion of concinuity in the case of closure structures. Theorem II-4L provides criteria for extendability of maps relative to homomorphism. Theorem II-4M gives a sufficient condition for extendability of maps relative to homomorphism. In the development of the theory in this section the definition of the factor structure induced by a given map having a structure defined in its range space is most crucial.

In Section 5 of Chapter II an abstract treatment of Cartesian products is provided. The definition of the Cartesian product structure in the Cartesian product of a family of sets in which structures are defined is most crucial. Proposition II-5A shows that the defined Cartesian product structure behaves in a familiar and appropriate fashion by producing "box-like" images. Propositions II-5B and II-5D study commutativity relations with respect to Cartesian product structures. Proposition II-5C considers Cartesian products of extensive structures, and inheritance of extensiveness by Cartesian product structures immediately follows. Proposition II-5E provides information useful in studying inheritance of idempotence by Cartesian product structures. Theorem II-5F deals with inheritance of properties by Cartesian product structures, and one might note from this theorem that almost every property with which we are concerned is inherited by such structures. Theorem II-5G deals with the construction of independent sets and generators under Cartesian product structures from independent sets and generators under structures in the coordinate spaces. Theorem II-5H provides formulae for calculating images and inverse images using coordinate projections, and its two corollaries are generalizations of well-known characterizations of continuity of maps to Cartesian products and the homomorphic nature of maps to such products. Finally, Theorem II-5J provides formulae for calculating images and inverse images with respect to evaluation maps, and its two corollaries give criteria for evaluation maps to be continuous and criteria for such maps to be homomorphisms.

Chapter III includes applications of some of the theory developed in Chapter II. It is shown how the structure sending a subset X of an algebra onto the intersection of all subalgebras containing X is related to certain structures defined by the operations in the algebra. (See Propositions III-1A and III-1D.) Proposition III-1C gives a characterization of subalgebra in terms of structures defined by the operations in the algebra. The notion of homomorphism with respect to structures defined by the operations in an algebra is studied in relation to the notion of algebra morphism (or algebra homomorphism). (See Propositions III-1E and III-1G.) Explicit definitions of the operations for Cartesian product algebras are given, and it turns out that such operations are also Cartesian products. Propositions III-1H, III-1I and III-1J give attention to some properties of the induced Cartesian product structures for algebras. Finally, in Section 1 we consider a ring J with a multiplicative identity, the class of all non-trivial, faithful and unitary J-modules and the class $\{P_V\}_{V\epsilon G}$ [with G being the class of all such J-modules] of all structures such that if $V\epsilon G$, then $P_V(\phi) = \phi$ and $P_V(X)$ is the set of all finite linear combinations of elements of X (if X is a non-empty subset of V). In Proposition III-1K we give a necessary and sufficient condition that J be a division ring in terms of the existence of a $V\epsilon G$ such that P_V has the exchange property.

Section 2 of Chapter III deals with additivity. It proceeds to study the notion of c-additivity, a generalization of the notion of additivity which requires that all families H of subsets of V such that $|H| \leq c$ satisfy the following condition: $P(\cup H) = \cup\{P(X)\}_{X\epsilon H}$. Proposition III-2B observes that c-additivity implies monotonicity. Propositions III-2C, III-2D, III-2F and III-2J deal with inheritance of c-additivity. Proposition III-2E observes that the smallest extensive structure containing a c-additive structure can be obtained in two ways. Proposition III-2G gives a characterization of c-additivity for monotone structures while Proposition III-2H gives one of c-additivity for monotone structures having α-character. Theorem III-2I pinpoints the largest c-additive structure contained in a given structure. Proposition III-1K establishes the equivalence of c-additivity and finite additivity.

The remaining portion of Section 2 of Chapter III deals with universal additivity. The trivial result that universally additive structures are additive and satisfy S0 appears as Proposition III-2L. The definition of the structure A_P determined by a structure P in a set V is given as follows:

$$\text{If } X \subseteq V, \text{ then } A_P(X) = \cup\{P(Y)\}_{Y \subseteq X, |Y|=1}.$$

Proposition III-2M deals with some properties of the structure A_P, showing that $A_P(\phi) = \phi$, giving the relation between A_P and the structure sending each subset X of V onto $\cup\{P(Y)\}_{Y \subseteq X, |Y| < 2}$, giving a necessary and sufficient condition for A_P to have the exchange property and dealing with the problem of passing to A_P with iteration. As a consequence, Corollary 1 to Proposition III-2M gives a characterization of universal additivity of a structure P in terms of its coincidence with A_P, and Corollary 3 to Proposition III-2M gives a sufficient condition (the exchange property) that a universally additive closure structure satisfy S5. Proposition III-2N and III-2O provide information which might be used to study decompositions of subspaces under universally additive closure structures, and the latter proposition shows that the exchange property is necessary and sufficient for every subspace under a given universally additive closure structure to be decomposable into one-dimensional subspaces.[1]

[1] The theory in Chapter II was developed to its present arrangement in sequence of topics before the writer decided to treat additivity. It was decided, therefore, to include a study of additivity as an application of the theory in Chapter II. However, additivity might have been considered as a part of Chapter II, and rightfully so.

Section 3 of Chapter III provides a treatment of "generalized compactness". All results in this section are generalizations of well-known results on compactness usually found in treatments of elementary topology. A characterization of such a notion of compactness is given in Proposition III-3A. A generalization of the proposition in elementary topology that closed subspaces of compact spaces are compact is given in Proposition III-3B. A generalization of Tychonoff's theorem is given in Proposition III-3C. It is interesting to note the extreme weakness in the hypothesis of Proposition III-3C, and that the availability of Proposition II-5A in Chapter II made the construction of a simple proof possible. Proposition III-3D is a generalization of the theorem on the preservation of compactness under continuous maps. Proposition III-3E is related to the theorem in functional analysis which characterizes the compactness of the unit sphere in a normed linear space in terms of finiteness of the dimension of the space.

Finally, a large variety of classes of examples of structures in sets is given in Chapter IV. These examples appear under two main headings referred to as Closure Structures (Section 1) and Miscellaneous Structures (Section 2).

II. STRUCTURES IN SETS

1. Some Maximal and Minimal Structures

Throughout this chapter it is assumed that V is a set. Unless the contrary is specified, all structures referred to are to be considered as structures in V. The symbol α will be used as a variable for cardinal numbers not exceeded by 1.

If G is a non-empty family of structures, then $\cup G$ denotes the structure which sends each subset X of V onto $\cup\{P(X)\}_{P \in G}$ and $\cap G$ denotes the structure which sends each subset X of V onto $\cap\{P(X)\}_{P \in G}$.

Excluding the cases of α-character (for $\alpha \geq \infty$), SO and the exchange property, the following results are known. (See [1], pp.111-112 and [5], pp. 25-26.)

Proposition II-1A

Suppose that G is a non-empty family of structures.

a) If each member of G has either monotonicity or extensiveness, then $\cap G$ has the corresponding property.

b) If each member of G has either monotonicity, extensiveness, the exchange property, the property of satisfying SO or α-character, then $\cup G$ has the corresponding property.

c) If each member of G is a closure structure, then $\cap G$ is a closure structure.

<u>Proof:</u> If G is a family of monotone structures and $X \subseteq Y \subseteq V$, then $P(X) \subseteq P(Y)$ for all $P \in G$; hence, $(\cap G)(X) \subseteq (\cap G)(Y)$ and $(\cup G)(X) \subseteq (\cup G)(Y)$. If G is a family of extensive structures and $X \subseteq V$, then $X \subseteq P(X)$ for all $P \in G$; hence, $X \subseteq (\cap G)(X)$ and $X \subseteq (\cup G)(X)$. Part a) follows. Part b) follows in the cases of monotonicity and extensiveness. If G is a family of closure structures and $X \subseteq V$ while $P \in G$, then it follows that $(\cap G)(X) \subseteq (\cap G)([\cap G](X)) \subseteq P([\cap G](X)) \subseteq P(P(X)) = P(X)$. Part c) follows. If G is a family of structures which have the exchange property while $X \subseteq V$, $y \in V$ $x \in (\cup G)(X \cup \{y\})$ and $x \notin (\cup G)(X)$, then one chooses $P \in G$ such that $x \in P(X \cup \{y\})$ and notes that $x \notin P(X)$ [since $P(Z) \subseteq (\cup G)(Z)$ for all $Z \subseteq V$ and $x \notin (\cup G)(X)$]; hence, $y \in P(X \cup \{x\}) \subseteq (\cup G)(X \cup \{x\})$ [since P has the exchange property]. Part b) in the case of the exchange property follows. If G if a family of structures which have α-character while $X \subseteq V$ and $x \in (\cup G)(X)$, then one chooses $Y \subseteq X$ such that $|Y| < \alpha$ and $x \in P(Y) \subseteq (\cup G)(Y)$. Part b) in the case of α-character follows. If G is a family of structures which satisfy SO while F is an increasing chain of subsets of V, then it follows that

$$(\cup G)(\cup F) = \cup\{P(\cup F)\}_{P \in G} \subseteq \cup\{\cup\{P(X)\}_{X \in F}\}_{P \in G} = \cup\{\cup\{P(X)\}_{P \in G}\}_{X \in F}$$

while the latter set is $\cup\{(\cup G)(X)\}_{X \in F}$. Part b) in the case of SO follows. This completes a proof of part b). The proposition follows.

One might observe that the class of all structures in V is closed under composition of pairs of structures. If P and Q are structures, then the symbol PQ denotes the application of Q followed by the application of P.

Part a) of the following proposition may be found in [5], p. 26.

Proposition II-1B

a) If P and Q are either monotone or extensive structures, then PQ has the corresponding property.

b) If Q is a monotone structure and both P and Q satisfy SO, then PQ satisfies SO.

c) If P and Q are monotone structures, P has α-character and Q has β-character, then PQ has $\alpha\beta$-character.

Proof: If P and Q are monotone structures while $X \subseteq Y \subseteq V$, then $(PQ)(X) = P(Q(X)) \subseteq P(Q(Y)) = (PQ)(Y)$. If P and Q are extensive structures while $X \subseteq V$, then $X \subseteq Q(X) \subseteq P(Q(X)) = (PQ)(X)$. Part a) follows. If Q is a monotone structure and both P and Q are structures satisfying SO while G is an increasing chain of subsets of V, then

$$P(Q(\cup G)) \subseteq P(\cup\{Q(X)\}_{X\in G}) \subseteq \cup\{P(Q(X))\}_{X\in G};$$

hence, $(PQ)(\cup G) \subseteq \cup\{(PQ)(X)\}_{X\in G}$. Part b) follows. We assume that P and Q are monotone structures, that P has α-character and that Q has β-character while $X \subseteq V$ and $x\in(PQ)(X)$. We choose a subset Z of $Q(X)$ such that $|Z| < \alpha$ and $x\in P(Z)$; then we choose a family $\{X_z\}_{z\in Z}$ of subsets of X such that $|X_z| < \beta$ and $z\in Q(X_z)$ for all $z\in Z$. It follows from the monotonicity of P and Q that $P(X) \subseteq P(\cup\{Q(X_z)\}_{z\in Z}) \subseteq (PQ)(\cup\{X_z\}_{z\in Z})$ while, also, $|\cup\{X_z\}_{z\in Z}| \leq \sum_{z\in Z}|X_z| \leq \sum_{z\in Z}\beta < \alpha\beta$. Part c) follows. The proposition is proved.

We shall use the symbol N to denote the set of positive integers. If P is a structure, then the sequence $\{P^n\}_{n\in N}$ and P^∞ are defined as follows:

$$P^1 = P,\ P^{n+1} = PP^n \text{ for all } n\in N \text{ and } P^\infty = \cup\{P^n\}_{n\in N}.$$

Proposition II-1C

Suppose that P is a structure, $n \in N$ and $m \in N$.

a) $P^n P^m = P^{n+m} = P^m P^n$.

b) $(P^n)^m = P^{nm} = (P^m)^n$.

Proof: Assuming that P is a structure, it is clear that the equalities are true for $n = 1 = m$. If $P^k P^1 = P^{k+1}$ for some $k \in N$, then it follows that $P^{k+1}P^1 = (P^1P^k)P^1 = P^1(P^kP^1) = P^1P^{k+1} = P^{(k+1)+1}$. We suppose that some element k of N is such that $P^nP^k = P^{n+k}$ for all $n \in N$. It follows that if $n \in N$, then $P^nP^{k+1} = P^n(P^1P^k) = P^n(P^kP^1) = (P^nP^k)P^1 = P^{n+k}P^1$ while $P^{n+k}P^1 = P^{(n+k)+1} = P^{n+(k+1)}$ and $P^{n+k}P^1 = (P^nP^k)P^1 = P^n(P^kP^1) = P^nP^{k+1}$. Part a) follows by induction. It is clear that $(P^n)^1 = P^{n1}$ for all $n \in N$. If $k \in N$ such that $(P^n)^k = P^{nk}$ for all $n \in N$, then it follows that $(P^n)^{k+1} = P^n(P^n)^k = P^nP^{nk} = P^{n+nk} = P^{n(k+1)}$ for all $n \in N$. Part b) follows by induction. The proposition is proved.

Proposition II-1D

Suppose that P is a structure and that $n \in N$.

a) P^n and P^∞ inherit monotonicity, extensiveness and the property of satisfying SO from P; if $X \subseteq V$ such that $P(X) = X$, then $P^n(X) = X$ and $P^\infty(X) = X$.

b) If P is monotone, then P^n inherits α^n-character from α-character of P.

Proof: The first assertion in part a) follows by induction upon application of part a) of Proposition II-1B and part b) of Proposition II-1A. The second assertion in part a) follows by simple induction. Part b) follows from part c) of Proposition II-1B upon simple induction. The proposition follows.

Proposition II-1E

Suppose that P is a structure.

a) The following are equivalent:

i) P^n commutes with P^∞ for all $n \in N$;

ii) $P \cup (P^\infty)^2 = P^\infty$.[1]

b) If P is monotone and Q is an idempotent structure containing P, then $P^\infty \subseteq Q$.

Proof: If $n \in N$, then it follows from the definition of P^∞ and part a) of Proposition II-1C that $P^\infty P^n = \cup\{P^{m+n}\}_{m \in N}$ for all $n \in N$. Therefore,

$$P \cup [\cup\{P^\infty P^n\}_{n \in N}] = P \cup [\cup\{\cup\{P^{m+n}\}_{m \in N}\}_{n \in N}] = P \cup [\cup\{P^n\}_{n \in N, n \geq 2}] = P^\infty.$$

It follows that P^n commutes with P^∞ for all $n \in N$ if and only if

$$P^\infty = P \cup [\cup\{P^\infty P^n\}_{n \in N}] = P \cup [\cup\{P^n P^\infty\}_{n \in N}] = P \cup (P^\infty)^2.$$

The proposition follows.

An example of a structure P such that $P \not\subseteq (P^\infty)^2$ is as follows. Let V be the ring of integers. If $X \subseteq V$, let $P(X) = \emptyset$ if $X = \emptyset$ and let $P(X) = \{2x : x \in X\}$ if $X \neq \emptyset$. Then $P(\{3\}) \not\subseteq (P^\infty)^2(\{3\})$.

The reader might recall from the introduction the promise to improve some results in Hammer's treatment of idempotence in [5]. Hammer proved that if P is a monotone and extensive structure having finite character, then P^∞ is a closure structure. Hammer's result is an immediate consequence of Theorem II-1F. (See Corollary 4 to Theorem II-1F and Corollary 2 to Theorem II-2C.)

[1] The form $P \cup (P^\infty)^2 = P^\infty$ is the best that can be obtained under the assumption that P is an arbitrary structure.

Theorem II-1F

Suppose that P is a structure.

a) If P is monotone and extensive, then the intersection of all idempotent structures containing P is the smallest closure structure containing P.

b) P^∞ is the smallest idempotent structure containing P if and only if $P \subseteq (P^\infty)^2$ and P^n commutes with P^∞ for all $n \in N$.

Proof: Note that $Q(X) = V$ for all $X \subseteq V$ defines a structure which is idempotent and contains P. Part a) follows from Proposition II-1A. Part b) follows from Proposition II-1E. The theorem follows.

Corollary 1

For all monotone structures P, P^∞ is idempotent if and only if $P \subseteq (P^\infty)^2$ and $P^n P^\infty \subseteq P^\infty P^n$ for all $n \in N$.

Proof: If P is a monotone structure while $n \in N$, then it follows that $P^\infty P^n = \cup\{P^m P^n\}_{m\in N} = \cup\{P^n P^m\}_{m\in N} \subseteq P^n(\cup\{P^m\}_{m\in N}) = P^n P^\infty$. The corollary follows.

Corollary 2

For all monotone and extensive structures P, the following are equivalent:

i) P^∞ is idempotent;

ii) $P^n P^\infty \subseteq P^\infty P^n$ for all $n \in N$;

iii) P^∞ is the intersection of all idempotent structures which contain P.

Proof: Assuming that P is a monotone and extensive structure, it follows that $P \subseteq P^\infty \subseteq (P^\infty)^2$; hence, the equivalence of i) and ii) follows from the theorem and Corollary 1 above. The equivalence of i) and iii) follows from Proposition II-1E. The corollary is proved.

Corollary 3

For all structures P such that P satisfies SO and $P^n \subseteq P^{n+1}$ for all $n\in N$, P^∞ is idempotent if and only if $P^\infty P^n \subseteq P^n P^\infty$ for all $n\in N$.

Proof: For such structures P, it follows that $P \subseteq P^2 \subseteq (P^\infty)^2$ and that

$$P^n P^\infty = P^n(\cup\{P^m\}_{m\in N}) \subseteq \cup\{P^n P^m\}_{m\in N} = \cup\{P^m P^n\}_{m\in N} = [\cup\{P^m\}_{m\in N}]P^n = P^\infty P^n$$

for all $n\in N$. The corollary follows from the theorem.

Corollary 4

For all monotone structures P such that P satisfies SO and $P^n \subseteq P^{n+1}$ for all $n\in N$, P^∞ is the smallest idempotent structure containing P.

Proof: Apply Corollaries 2 and 3 above.

If P is a structure, then the structure M_P is the structure which sends each subset X of V onto $\cup\{P(Z)\}_{Z\subseteq X}$.

Proposition II-1G

If P is a structure, then M_P inherits extensiveness, α-character, the property of satisfying SO and the exchange property from P.

Proof: Assuming that P is a structure, if P is extensive while $X \subseteq V$, then $Z \subseteq P(Z) \subseteq M_P(X)$ for all $Z \subseteq X$. If P has α-character while $X \subseteq V$ and $x\in M_P(X)$, then one chooses $Y \subseteq X$ such that $x\in P(Y)$; then one chooses $Z \subseteq Y$ such that $|Z| < \alpha$ and $x\in P(Z) \subseteq M_P(X)$. If P satisfies SO while G is an increasing chain of subsets of V and $x\in M_P(\cup G)$, then one chooses $Y \subseteq \cup G$ such that $x\in P(Y)$ and notes that

$$P(Y) = P(\cup\{Y\cap X\}_{X\in G}) \subseteq \cup\{P(Y\cap X)\}_{X\in G} \subseteq \cup\{M_P(X)\}_{X\in G};$$

hence, $x\in \cup\{M_P(X)\}_{X\in G}$. It follows that M_P inherits the property of satisfying SO from P. Finally, we suppose that P has the exchange property while $X \subseteq V$, $y \in V$, $x \in M_P(X \cup \{y\})$ and $x \notin M_P(X)$. Then we choose a subset Z of $X \cup \{y\}$ such that $x \in P(Z)$ and notes that $y \in Z$ [since $x \notin P(U)$ for all $U \subseteq X$]. It follows that $x \in P([Z-\{y\}] \cup \{y\})$ while $x \notin P(Z-\{y\})$, so that $[Z-\{y\}] \cup \{x\} \subseteq X \cup \{x\}$. It follows that $y \in M_P(X \cup \{x\})$. Therefore, M_P inherits the exchange property from P. The proposition follows.

Theorem II-1H

Suppose that P is a structure.

a) M_P is the smallest monotone structure containing P.

b) The union of all monotone structures contained in P is the largest monotone structure contained in P.

Proof: Since $X \subseteq X$ for all $X \subseteq V$, then $P(X) \subseteq M_P(X)$ for all $X \subseteq V$. It follows that $P \subseteq M_P$. If $X \subseteq Y \subseteq V$, then $Z \subseteq Y$ for all $Z \subseteq X$; hence, it follows that $M_P(X) \subseteq M_P(Y)$. Therefore, M_P is a monotone structure containing P. Furthermore, if Q is a monotone structure containing P while $X \subseteq V$, then it follows that $P(Z) \subseteq Q(Z) \subseteq Q(X)$ for all $Z \subseteq X$; hence, $M_P(X) \subseteq Q(X)$. Part a) follows. Part b) follows from Proposition II-1A. The theorem is proved.

If P is a structure, then the structure E_P is defined to be the structure which sends each subset X of V onto $X \cup P(X)$. The results in Proposition II-1I and Theorem II-1J are obvious and already known (excepting, possibly, the assertions concerning property SO and α-character).

Proposition II-1I

If P is a structure, then E_P inherits monotonicity, α-character and the property of satisfying SO from P.

Proposition II-1J

If P is a structure, then E_P is the smallest extensive structure containing P.

If P is a structure, then P_α denotes the structure which sends each subset X of V onto $\cup\{P(Z)\}_{Z\subseteq X, |Z|<\alpha}$.

Proposition II-1K

If P is a structure, then P_α inherits monotonicity, extensiveness, the property of satisfying SO and the exchange property from P.

Proof: The assertion concerning monotonicity is obvious. The remainder of the proof of the proposition parallels the proof of Proposition II-1H.

Theorem II-1L

If P is a monotone structure, then P_α is the largest structure having α-character and being contained in P.

Proof: Assuming that P is a monotone structure while $X \subseteq V$, it follows that $P(Z) \subseteq P(X)$ for all $Z \subseteq X$ such that $|Z| < \alpha$; hence, P_α is contained in P. It is clear from the definition that P_α has α-character. Therefore, P_α is a structure having α-character and being contained in P. If Q is a structure having α-character and being contained in P, then it follows that

$$\begin{aligned} Q(X) &\subseteq \cup\{Q(Z)\}_{Z\subseteq X, |Z|<\alpha} \\ &\subseteq \cup\{P(Z)\}_{Z\subseteq X, |Z|<\alpha} \\ &= P_\alpha(X) \end{aligned}$$

for all $X \subseteq V$; hence, $Q \subseteq P_\alpha$. It follows that P_α is the largest structure having α-character and being contained in P. The theorem follows.

Proposition II-1M

Suppose that P is a structure.

a) $M_{P^n} \subseteq (M_P)^n$ for all $n\in N$.

b) If P is monotone, then $E_{P^n} \subseteq (E_P)^n$ for all $n\in N$.

c) $(P^n)_\alpha \subseteq (P_\alpha)^n$ for all $n\in N$.

d) $M_{P_\alpha} = (M_P)_\alpha$.

e) $M_{E_P} = E_{M_P}$.

f) $E_{P_\alpha} = (E_P)_\alpha$.

Proof: Parts a), b) and c) are obvious for $n = 1$.

We suppose that $k\in N$ such that $M_{P^k} \subseteq (M_P)^k$ while $X \subseteq V$. Since $P^k \subseteq M_{P^k}$, then it follows that

$$\begin{aligned} M_{P^{k+1}}(X) &= \cup\{P^{k+1}(Z)\}_{Z\subseteq X} \\ &= \cup\{P(P^k(Z))\}_{Z\subseteq X} \\ &\subseteq \cup\{M_P(M_{P^k}(Z))\}_{Z\subseteq X} \\ &\subseteq (M_P)^{k+1}(X) \end{aligned}$$

for all $X \subseteq V$. Part a) follows by induction.

We suppose that $k\in N$ such that $E_{P^k} \subseteq (E_P)^k$ while $X \subseteq V$. It follows that

$$\begin{aligned} (E_P)^{k+1}(X) &= [E_P(E_P)^k](X) \\ &\supseteq X \cup [P(E_P)^k](X) \\ &\supseteq X \cup [P(E_{P^k})](X) \\ &\supseteq X \cup [P(P^k)](X) \end{aligned}$$

Part b) follows by induction.

A proof of part c) parallels the proof of part a)

If $X \subseteq V$, then it follows that

$$\begin{aligned}(M_P)_\alpha(X) &= \cup\{M_P(Y)\}_{Y\subseteq X,\,|Y|<\alpha} \\ &= \cup\{\cup\{P(Z)\}_{Z\subseteq Y}\}_{Y\subseteq X,\,|Y|<\alpha} \\ &= \cup\{\cup\{P(Z)\}_{Z\subseteq Y,\,|Z|<\alpha}\}_{Y\subseteq X} \\ &= \cup\{P_\alpha(Y)\}_{Y\subseteq X} \\ &= M_{P_\alpha}(X).\end{aligned}$$

Part d) follows.

If $X \subseteq V$, then it follows that

$$\begin{aligned}E_{M_P}(X) &= X \cup M_P(X) \\ &= X \cup [\cup\{P(Z)\}_{Z\subseteq X}] \\ &= \cup\{Z \cup P(Z)\}_{Z\subseteq X} \\ &= \cup\{E_P(Z)\}_{Z\subseteq X} \\ &= M_{E_P}(X).\end{aligned}$$

Part e) follows.

If $X \subseteq V$, then it follows that

$$\begin{aligned}(E_P)_\alpha(X) &= \cup\{E_P(Y)\}_{Y\subseteq X,\,|Y|<\alpha} \\ &= \cup\{Y \cup P(Y)\}_{Y\subseteq X,\,|Y|<\alpha} \\ &= X \cup [\cup\{P(Y)\}_{Y\subseteq X,\,|Y|<\alpha}] \\ &= X \cup P_\alpha(X) \\ &= E_{P_\alpha}(X).\end{aligned}$$

Part f) follows. This completes a proof of the proposition.

Theorem II-1N below includes two versions of the Moore-family characterization of closure structures. (See [1], p.111.) Condition iii) is used to construct examples in the proof of Theorem II-2D on the independence of axioms studied in Chapter II.

Theorem II-1N

The following are equivalent:

i) P is a closure structure;

ii) There is a non-empty family F of subsets of V such that the following conditions are satisfied:

1) If $X \subseteq V$, then $X \subseteq Y$ for some $Y \in F$,

2) If $X \subseteq V$, then $P(X) = \cap\{Y\}_{Y \in F, X \subseteq Y}$.

iii) There is a pair H and G of non-empty families of subsets of V such that the following conditions are satisfied:

1) If $X \subseteq V$ such that $Y \not\subseteq X$ for all $Y \in H$, then $X \subseteq Y$ for some $Y \in G$,

2) If $X \subseteq V$, then $P(X) = V$ if $Y \subseteq X$ for some $Y \in H$; $P(X) = \cap\{Y\}_{Y \in G, X \subseteq Y}$ if $Y \not\subseteq X$ for all $Y \in H$.

Proof: We assume that i) is true and let F be the family of all $P(X)$ such that $X \subseteq V$. Since P is extensive, if $X \subseteq V$, then $P(X)$ is an element Y of F such that $X \subseteq Y$. We assume that $X \subseteq V$. Since P is monotone and idempotent, then $P(X) \subseteq P(Y) = Y$ for all $Y \in F$ such that $X \subseteq Y$; hence, it follows that $P(X) \subseteq \cap\{Y\}_{Y \in F, X \subseteq Y}$. On the other hand, since $P(X) \in F$ and P is extensive, it follows that $\cap\{Y\}_{Y \in F, X \subseteq Y} \subseteq P(X)$. Therefore, i) implies ii).

We assume that ii) is true while $X \subseteq V$, $Z \subseteq V$ and $X \subseteq Z$. It follows that $X \subseteq Y$ for all $Y \in F$ such that $Z \subseteq Y$ and, hence, it follows that

$P(X) = \cup\{Y\}_{Y\in F, X\subseteq Y} \subseteq \cap\{Y\}_{Y\in F, Z\subseteq Y} = P(Z)$. Therefore, P is monotone. It is clear that $X \subseteq \cap\{Y\}_{Y\in F, X\subseteq Y} = P(X)$ and that $P(Y) = Y$ for all $Y\in F$. It follows that P is extensive and idempotent. Therefore, ii) implies i).

We assume that i) is true. We let H be the family of all subsets of V which are P-equivalent to V and G be the family of all P(X) such that $X \subseteq V$ and X is not P-equivalent to V. Then H and G are families of subsets of V which meet the conditions in iii) above. It follows that i) implies iii).

We assume that iii) is true. It is clear that P is extensive. Since V is P-equivalent to V, then $P(V) = V$ and, therefore, if $X \subseteq V$ such that $Y \subseteq X$ for some $Y\in H$, then $P(P(X)) = P(V) = V$. As in the proof of the equivalence of i) and ii) above, one shows that if $Y \not\subseteq X$ for all $Y\in H$, then $P(P(X)) = P(X)$. It follows that P is idempotent. We assume that $X \subseteq Z \subseteq V$. If $Y \not\subseteq Z$ for all $Y\in H$, then $Y \subseteq X$ for all $Y\in H$ and, therefore, it follows that $P(X) = \cap\{Y\}_{Y\in G, X\subseteq Y} \subseteq \cap\{Y\}_{Y\in G, Z\subseteq Y} = P(Z)$. Also, if $Y \subseteq Z$ for some $Y\in H$, then $P(X) \subseteq V = P(Z)$. It follows that P is monotone; hence, P is a closure structure. Therefore, iii) implies i). The theorem follows.

<u>Corollary</u>

If P is a structure and F is a non-empty family of subsets of V such that the following condition is satisfied: If $X \subseteq V$, then $X \subseteq Y$ for some $Y\in F$ and $P(X) = \cap\{Y\}_{Y\in F, X\subseteq Y}$, then

a) $P(\phi) = \cap F$,

b) $P(V) = V$.

2. The Axioms: Contingency

We have used the phrase 'the axioms' to refer to the following properties of structures: monotonicity, extensiveness, idempotence, the exchange property, α-character, the property of satisfying S0, the property of satisfying S5 and the property of sending ϕ onto ϕ. (See pages 1, 2 and 4 of Chapter I.) This section of Chapter II is a study of a relationship between α-character and S5, some relationships between S0 and α-character and the independence of the axioms other than the property S0.

Proposition II-2A

If P is a structure having α-character, then the following condition is satisfied by all P-equivalent subsets X and Y of V such that $\alpha \leq |X|$:

If $x \epsilon P(X)$, then $x \epsilon P(Z)$ for some $Z \subseteq Y$ such that $|Z| < |X|$.

Proof: Assuming that P is a structure having α-character while X and Y are P-equivalent subsets of V and $x \epsilon P(X)$, one chooses $Z \subseteq Y$ such that $x \epsilon P(Z)$ and $|Z| < \alpha$. The proposition follows.

Proposition II-2B

If P is a structure having α-character and G is a family of subsets of V, then i) and ii) are equivalent:

i) $\cup\{P(Z)\}_{Z \subseteq \cup G, |Z| < \alpha,\ Z \notin G} \subseteq \cup\{P(Z)\}_{Z \epsilon G, |Z| < \alpha}$;

ii) $P(\cup G) = \cup\{P(Z)\}_{Z \epsilon G, |Z| < \alpha}$.

Proof: We note that a subset Z of V satisfies the relations $Z \subseteq \cup G$ and $|Z| < \alpha$ if and only if Z satisfies one of the following: $Z \epsilon G$ and $|Z| < \alpha$, $Z \subseteq \cup G$ and $|Z| < \alpha$ and $Z \notin G$. It follows that

$$P(\cup G) = \cup\{P(Z)\}_{Z \subseteq \cup G, |Z| < \alpha}$$

$$= [\cup\{P(Z)\}_{Z \in G, |Z| < \alpha}] \cup [\cup\{P(Z)\}_{Z \subseteq \cup G, |Z| < \alpha, Z \notin G}]$$

The proposition follows.

Theorem II-2C

a) Every structure P having α-character and satisfying the following conditions:

If G is an increasing chain of subsets of V, then

$$\cup\{P(Z)\}_{Z \subseteq \cup G, |Z| < \alpha, Z \notin G} \subseteq \cup\{P(Z)\}_{Z \in G, |Z| < \alpha}.$$

satisfies SO.

b) A monotone structure P having α-character satisfies SO if and only if the following condition is satisfies:

If G is an increasing chain of subsets of V and H is a subchain of G such that $|H| < \alpha$ [where $|H|$ denotes the cardinal number of the range of H], then $P(\cup H) \subseteq \cup\{P(Z)\}_{Z \in G}$.

Proof: Assuming that P is a structure having α-character and satisfying the condition in part a) while G is an increasing chain of subsets of V, it follows from Proposition II-2B that $P(\cup G) \subseteq \cup\{P(Z)\}_{Z \in G}$. Part a) follows. We assume that P is a monotone structure having α-character. If P satisfies SO and G is an increasing chain of subsets of V while H is a subchain of G such that $|H| < \alpha$, then it follows from the monotonicity of P that $P(\cup H) \subseteq \cup\{P(Z)\}_{Z \in G}$. Conversely, we assume that the condition in part b) is satisfied while G is an increasing chain of subsets of V and $x \in P(\cup G)$. Then we choose $Y \subseteq \cup G$ such that $|Y| < \alpha$ and $x \in P(Y)$; then we choose a family $H = \{X_y\}_{y \in Y}$ of elements of G such that $y \in X_y$ for all $y \in Y$ We observe that $|H| < \alpha$ and use the monotonicity of P along with the condition to deduce that $x \in P(Y) \subseteq P(\cup H) \subseteq \cup\{P(Z)\}_{Z\in G}$. It follows that $P(\cup G) \subseteq \cup\{P(Z)\}_{Z\in G}$. Therefore, P satisfies S0. This completes a proof of part b). The theorem follows.

Corollary

If P is a monotone structure having α-character while the following condition is satisfied:

If G is an increasing chain of subsets of V and H is a subchain of G such that $|H| < \alpha$, then $(\cup H) \subseteq Y$ for some $Y \in G$.

then P satisfies S0.

Corollary 2

Every monotone structure having finite character satisfies S0.

Theorem II-2D below is proved by many examples and requires as its proof general methods of construction of different types of structures. Conditions (1), (2) and (3) in the proof of the theorem are examples of such methods of construction. The condition $\alpha \geq 3$ is used in proving the independence of monotonicity and the independenc of idempotence. For closure structures P having the exchange property, any two P-independent P-equivalent subsets of V, one of which is finite, have the same cardinal number. [See part b) of Theorem II-3L or (5a) in the introduction.] It follows that if P is a closure structure such that for all $X \subseteq V$ there is a finite P-independent subset of V which is P-equivalent to X, then P satisfies S5. It follows that the condition that α exceed the first infinite cardinal number (in the case of S5) is necessary.

Proposition II-2D

The following axioms are independent:

i) Monotonicity

ii) Idempotence

iii) Extensiveness

iv) The exchange property

v) α-character (for $\alpha \geq 3$)

vi) S5 (for $\alpha > \infty$)

vii) The property of sending ϕ onto ϕ

Proof: It follows from Theorem II-1N that if F is a non-empty family of non-empty subsets of V and $A \subseteq V$, then the following defines a closure structure which sends ϕ onto A:

$$\text{If } X \subseteq V, \text{ then } P(X) = \begin{cases} V \text{ if } Y \subseteq V-A \text{ for some } Y \in F \\ X \cup A \text{ otherwise} \end{cases}$$

To see that such is the case one takes H = F and G as the family of all $Z \subseteq V$ such that $Z \notin H$; then one applies the equivalence of i) and iii) of Theorem II-1N, thereupon verifying that P is a closure structure. One notes that $Y \not\subseteq \phi - A = \{x \in \phi : x \notin A\}$ for all $Y \in H$; hence, $P(\phi) = A$.

The following assertions follow quite easily (as will be shown), with μ being a cardinal number less than $|V|$:

(1) If F is the family of all subsets Y of V such that $|Y| = \mu$, then P has the exchange property.

(2) If F is any family of subsets Y of V such that $|Y| = \mu$, then P satisfies S5.

(3) If F is any family of subsets Y of V such that $|Y| = \mu < \alpha$, then P has α-character.

We assume that F is the family of all subsets Y of V such that $|Y| = \mu$ while $X \subseteq V$, $y \epsilon V$ and $x \epsilon [P(X \cup \{y\}) - P(X)]$. Either $Y \subseteq [X \cup \{y\}] - A$ for some $Y \epsilon F$ or not. We consider the case that $Y \subseteq [X \cup \{y\}] - A$ for some $Y \epsilon F$. Since P is extensive and $x \epsilon [P(X \cup \{y\}) - P(X)]$, then $x \notin X$ and $y \notin X$. Therefore, if $Y \subseteq X$, then $Y \subseteq [X \cup \{x\}] - A$; hence, $P(X \cup \{x\}) = V$ and $y \epsilon P(X \cup \{x\})$. If $Y \nsubseteq X$, then $y \epsilon Y$; hence, $[Y - \{y\}] \cup \{x\} \subseteq [X \cup \{x\}] - A$ while, also, $|[Y - \{y\}] \cup \{x\}| = \mu$, so that $P(X \cup \{x\}) = V$ and $y \epsilon P(X \cup \{x\})$. In case $Y \nsubseteq [X \cup \{y\}] - A$ for all $Y \epsilon F$, one has the relation $P(X \cup \{y\}) - P(X) = \{y\}$. It follows that P has the exchange property. This proves (1).

We assume that X and Y are P-equivalent subsets of V and that F is a family of subsets of V which have cardinal number μ. It follows that $Z_1 \subseteq X - A$ for some $Z_1 \epsilon F$ if and only if $Z_2 \subseteq Y - A$ for some $Z_2 \epsilon F$ and, therefore, P satisfies S5. This proves (2).

We assume that F is a family of subsets of V which have cardinal number μ. If $Y \subseteq X - A$ for some $Y \epsilon F$ while $X \subseteq V$, then $Y \subseteq X$ and $Y \subseteq Y - A$, so that $P(Y) = V$ while $|Y| = \mu$. Therefore, if $\mu < \alpha$ and $Y \nsubseteq X - A$ for all $Y \epsilon F$ while $X \subseteq V$ [so that $\mu \geq 1$] and $x \epsilon P(X)$, then $x \epsilon (X \cup A)$; hence, it follows that $x \epsilon (\{x\} \cup A) = P(\{x\})$ or $x \epsilon A = P(\phi)$. It follows that P has α-character. This proves (3).

<u>Independence of the property of sending ϕ onto ϕ</u>: We take $A = \phi$ and let F be the family of all $Y \subseteq V$ such that $|Y| = \mu < \alpha$.

<u>Independence of monotonicity</u>: We let $A = \phi$ and F be the family of all $Y \subseteq V$ such that $|Y| = \mu \geq 3$ while $\mu < \alpha$. We define Q as follows:

$$\text{If } X \subseteq V, \text{ then } Q(X) = \begin{cases} \{a,b\} & \text{if } X \subseteq \{a,b\} \text{ and } X \neq \phi \\ P(X) & \text{otherwise} \end{cases}$$

where a and b are distinct elements of V. We let c be a third element of V. Then $Q(\{a\}) = \{a,b\} \not\subseteq \{a,c\} = Q(\{a,c\})$ while $\{a\} \subseteq \{a,c\}$. It follows that Q is not monotone. We suppose that X is a non-empty subset of $\{a,b\}$. It follows that $X \subseteq \{a,b\} = Q(X) = Q(Q(X))$. It is obvious that $Q(\phi) = \phi$. We assume that $\phi \neq X \not\subseteq \{a,b\}$ while $X \subseteq V$. Then $Q(X) = P(X)$ while $P(X)$ is either X or V; hence, $X \subseteq Q(X) = Q(Q(X))$. It follows that Q is extensive and idempotent. We assume, also, that $y \in V$ and $x \in Q(X \cup \{y\})$ while $x \notin Q(X)$. If $X \cup \{y\}$ is a non-empty subset of $\{a,b\}$, then it follows that $Q(X \cup \{y\}) - Q(X) = \{a,b\}$ and, hence, x is one of a and b, so that $y \in Q(X \cup \{x\}) = \{a,b\}$. Assuming that $X \cup \{y\}$ is non-empty and $X \cup \{y\} \not\subseteq \{a,b\}$, it follows that $Q(X \cup \{y\}) - Q(X) = P(X \cup \{y\}) - Q(X)$. Either $Q(X) = P(X)$ or $Q(X) = \{a,b\}$. If $Q(X) = P(X)$, then one applies the exchange property to deduce that $y \in P(X \cup \{x\})$. If $P(X \cup \{x\}) = \{a,b\}$, then $X \cup \{x\} \subseteq \{a,b\}$; hence, it follows that $X \cup \{y\} \subseteq \{a,b\}$. But $X \cup \{y\} \not\subseteq \{a,b\}$; hence, it follows that $P(X \cup \{x\}) \neq \{a,b\}$. Therefore, $Q(X \cup \{x\}) = P(X \cup \{x\})$ if $P(X) = Q(X)$. If $Q(X) = \{a,b\}$, then $X \subseteq \{a,b\}$ while $x \notin Q(X) = \{a,b\}$; hence, $X \cup \{x\} \not\subseteq \{a,b\}$ and it follows that $Q(X \cup \{x\}) = P(X \cup \{x\})$. It follows that $y \in Q(X \cup \{x\})$. Therefore, Q has the exchange property. The sets which are Q-equivalent to $\{a,b\}$ are $\{a,b\}$, $\{a\}$ and $\{b\}$ while P satisfies S5. It follows that Q satisfies S5. Since P already has α-character, it follows that Q has α-character. The independence of monotonicity follows.

Independence of idempotence: We let V be a vector space over a division ring F and assume that V has dimension at least 2. We let a and b be linearly independent elements of V while $z = 0$ is the identity in the Abelian group V. If $x \in V$ and $y \in V$, then we let $L(x,y) = \{rx+sy\colon\ r+s=1\}$. We let $Q(X) = \cup\{L(x,y)\}_{x \in X, y \in X}$ for all $X \subseteq V$. Then $Q(Q(\{a,b,z\})$ is the subspace spanned by $\{a,b\}$ while $Q(\{a,b,z\}) = L(a,b) \cup L(a,z) \cup L(b,z)$, a proper subset of $Q(Q(\{a,b,z\}))$. It follows that P is not idempotent. We assume that $X \subseteq V$. Since $\{x\} = L(x,x) \subseteq Q(X)$ for all $x \in X$, it follows that Q is monotone and extensive. It is clear that $Q(\phi) = \phi$. Assuming, also, that $y \in V$ and $x \in [Q(X \cup \{y\}) - Q(X)]$, we choose elements c and d of $X \cup \{y\}$ such that $x \in L(c,d)$ and observe that one of c and d, say d, is y [since $x \notin Q(X)$]. Then we choose $r \in F$ and $s \in F$ such that $r+s=1$ and $x = rc + sy$ and observe that $s \neq 0$ [since $x \notin Q(X)$]. We solve for y to obtain the relation $y = s^{-1}x + (-s^{-1}r)c$ and notice that $s^{-1} + (-s^{-1}r) = 1$. It follows that $y \in Q(X \cup \{x\})$. Therefore, Q has the exchange property. If X is a non-empty subset of V and $x \in Q(X)$, then $x \in L(y,w)$ for some $\{y,w\} \subseteq X$ while $|\{y,w\}| < \alpha$ [since $\alpha \geq 3$]. It follows that Q has α-character. If X and Y are Q-equivalent subsets of V and $x \in Q(X)$, then $x \in L(y,w)$ for some $\{y,w\} \subseteq Y$ while $|\{y,w\}| \leq |X|$ [since $L(y,w) = L(a,b)$ for some $\{a,b\} \subseteq X$ such that $|\{a,b\}| = |\{y,w\}|$]. It follows that Q satisfies S5. The independence of idempotence follows.

Independence of extensiveness: We let μ be a cardinal number less than α and F be the family of all subsets Y of V such that $|Y| = \mu$. We let $Q(X) = \phi$ if $Y \not\subseteq X \subseteq V$ for all $Y \in F$ and we let $Q(X) = V$ if $Y \subseteq X \subseteq V$ for some $Y \in F$. It is easy to see that $Q(\phi) = \phi$, that Q is monotone and idempotent, that Q has the exchange property and α-character, and that Q satisfies S5 if $\mu = 2$. However, if $\mu = 2$, then $\{x\} \not\subseteq Q(\{x\})$ for all $x \in V$. Therefore, Q is not extensive. The independence of extensiveness follows.

Independence of the exchange property: We let B be a non-empty subset of V such that $|B| < \alpha$. We let $F = \{B\}$ and $A = \phi$. Since B is non-empty, we let $b \in B$. Since $V-B \neq \phi$, we let $a \in (V-B)$. It follows that $a \in [V-(B-\{b\})]$

while $V-(V-\{b\}) = P([B-\{b\}] \cup \{b\}) - P(B-\{b\})$, $b \notin [(B-\{b\}) \cup \{a\}$ and $P([B-\{b\}] \cup \{a\}) = (B-\{b\}) \cup \{a\}$. It follows that P does not have the exchange property. It is clear that P is a closure structure having α-character, satisfying S5 and sending ϕ onto ϕ. The independence of the exchange property follows.

Independence of α-character: We assume that $\alpha < |V|$. We let F be the family of all $Y \subseteq V$ such that $|Y| = \alpha$ and $A = \phi$. If P has α-character, then it follows that every subset X of V is such that

$$P(X) = \cup\{P(Y)\}_{Y \subseteq X, |Y| < \alpha} = \cup\{Y\}_{Y \subseteq X, |Y| < \alpha} = X.$$

It is obvious that no element X of F is such that $P(X) = X$. Therefore, P does not have α-character. It is clear that P is a closure structure having the exchange property, satisfying S5 and sending ϕ onto ϕ. The independence of α-character follows.

Independence of S5: We assume that V contains disjoint subsets A_1 and A_2 such that $|A_1| < |A_2| < \alpha$ and $|A_1| \geq \infty$. We let $A = \phi$ and F be the family of all $X \subseteq V$ such that $|A_i - X| < \infty$ and $|X - A_i| \geq |A_i - X|$ for some $i \in \{1,2\}$. Since $A_i - A_i = \phi$, then it follows that $A_i \in F$ (for $i = 1, 2$). Therefore, $P(A_1) = V = P(A_2)$. If $Y \subseteq A_2$ such that $|Y| \leq |A_1|$, then $|A_1 - Y| = |A_1| \nless \infty$ and $|Y - A_2| = 0 \ngeq |A_2 - Y|$. It follows that P does not satisfy S5. It is clear that P is a closure structure having α-character and sending ϕ onto ϕ. To show that P has the exchange property, we assume that $X \subseteq V$, $y \in V$, $x \in P(X \cup \{y\})$ and $x \notin P(X)$. Either $(X \cup \{y\}) \in F$ or not. If $(X \cup \{y\}) \notin F$, then $P(X \cup \{y\}) - P(X) = \{y\}$; hence, $x = y \in P(X \cup \{x\})$. We suppose that $(X \cup \{y\}) \in F$. We choose $k \in \{1,2\}$ such that $|A_k - (X \cup \{y\})| < \infty$ while $|(X \cup \{y\}) - A_k| \geq |A_k - (X \cup \{y\})|$. It follows that $X \cup \{y\}$ is infinite; hence,

$X \cup \{x\}$ is infinite. It follows that $|(X \cup \{x\}) - A_k| \geq |A_k - (X \cup \{x\})|$ and $|A_k - (X \cup \{x\})| < \infty$. Therefore, $P(X \cup \{x\}) = V$; hence, $y \in P(X \cup \{x\})$. It follows that P has the exchange property. The independence of S5 follows.

Corollary

The following are independent:

i) The property of sending ϕ onto ϕ

ii) Monotonicity

iii) Extensiveness

iv) Idempotence

v) The exchange property

vi) Finite character

Proof: One observes that the structures used in the proofs of the independence of the property of sending ϕ onto ϕ, monotonicity and extensiveness will have finite character if μ is finite. If B (as used in the proof of the independence of the exchange property) is finite, then the structure in the proof of the independence of the exchange property will have finite character. It is clear that the structure used in proving the independence of idempotence has finite character. The structure used in proving the independence of S5 suffices to prove the independence of finite character.

3. Independence and Equivalence

The main concern of this section of Chapter II is to treat certain cases of inheritance of independence and equivalence, maximization relative to independence and the notion of <u>dimension with respect to a given structure</u> (that is, equality of cardinal numbers of pairs of independent and equivalent sets under a structure). As has been pointed out in Chapter I, the main results in this section are Theorems II-3G, II-3H, II-3I and II-3L. The roles of the exchange property, S0 and S5 are vital in the development of the main results.

<u>Proposition II-3A</u>

Suppose that P is a structure.

a) $P(\phi) = \phi$ if and only if $\{x\}$ is P-independent for all $x \in V$.

b) If P is monotone, then every subset of a P-independent subset of V is P-independent.

c) If P is monotone, then every subset of V which P sends onto ϕ is P-independent.

d) If P has the exchange property, then every P-independent subset X of V satisfies the following condition:

If $x \in [V - P(X)]$, then $X \cup \{x\}$ is P-independent.

e) ϕ is a P-independent subset of V.

<u>Proof:</u> Part a) follows from the fact that if $x \in V$, then $P(\{x\} - \{y\}) = P(\phi)$ for all $y \in \{x\}$. If P is monotone and X is a P-independent subset of V while $Y \subseteq X$ and $y \in Y$, then $y \notin P(X - \{y\}) \supseteq P(Y - \{y\})$ and, hence, Y is P-independent. Part b) follows. Part c) is obvious. Assuming that P has the exchange property while X is a P-independent subset of V and $x \in [V - P(X)]$, we suppose that some member y of $X \cup \{x\}$ is in $P([X \cup \{x\}] - \{y\})$. Since $x \notin P(X)$, then $y \neq x$ and, therefore, $y \in X$. It follows that $y \in P([X - \{y\}] \cup \{x\})$ while $y \notin P(X - \{y\})$ [since X is P-independent]. Since P

has the exchange property, it follows that $x \in P([X-\{y\}] \cup \{x\}) = P(X)$. But $x \notin P(X)$. It follows that no such member y of $X \cup \{x\}$ exists; hence, $X \cup \{x\}$ is P-independent. Part d) follows. It is obvious that part e) is true. The proposition follows.

Proposition II-3B

a) If G is any family of structures, then a subset X of V is $(\cup G)$-independent if and only if X is P-independent for all $P \in G$.

b) If P is an extensive structure while Q is any structure, then the (PQ)-independent subsets of V are among the Q-independent subsets of V.

Proof: Assuming that G is a family of structures and that $X \subseteq V$, it follows that X is $(\cup G)$-independent if and only if $x \notin \cup\{P(X-\{x\})\}_{P \in G}$ for all $x \in X$. Part a) follows. Assuming that P is an extensive structure, that Q is any structure and that X is a (PQ)-independent subset of V, it follows that $x \notin (PQ)(X-\{x\})$ for all $x \in X$; hence, $x \notin Q(X-\{x\})$ for all $x \in X$ [since P is extensive]. Part b) follows. The proposition is proved.

Proposition II-3C

If P is a structure, then the class of P-independent subsets of V contains the class of M_P-independent subset of V and coincides with the class of E_P-independent subsets of V.

Proof: Assuming that P is a structure and $X \subseteq V$, it follows that X is P-independent if and only if $x \notin [(X-\{x\}) \cup P(X-\{x\})]$ for all $x \in X$; hence, it follows that X is P-independent if and only if X is E_P-independent. This proves the proposition in the case of the structure E_P. The remaining assertion follows from the fact that $P \subseteq M_P$.

Proposition II-3D

If P is a monotone structure, then subsets X and Y of V such that $X \subseteq P(Y)$ and $Y \subseteq P(X)$ are P^{∞}-equivalent.

Proof: Assuming that P is a monotone structure while X and Y are subsets of V such that $X \subseteq P(Y)$ and $Y \subseteq P(X)$, it follows by induction that $P^n(X) \subseteq P^{n+1}(Y)$ and $P^n(Y) \subseteq P^{n+1}(X)$ for all $n \in N$. Therefore, $P^{\infty}(X) = P^{\infty}(Y)$. The proposition follows.

Corollary

If P is a monotone and idempotent structure, then subsets X and Y of V are P-equivalent if and only if $X \subseteq P(Y)$ and $Y \subseteq P(X)$.

Proposition II-3E

Suppose that P is a structure and that $n \in N$.

a) Any two P^n-equivalent subsets of V are P^{n+1}-equivalent subsets of V.

b) The P^{∞}-independent subsets of V are P^n-independent subsets of V.

Proof: If X and Y are P^n-equivalent subsets of V, then it follows that $P^{n+1}(X) = P(P^n(X)) = P(P^n(Y)) = P^{n+1}(Y)$. If X is P^{∞}-independent, then $x \notin \cup\{P^m(X-\{x\})\}_{m \in N}$ for all $x \in X$ and, hence, $x \notin P^n(X-\{x\})$ for all $x \in X$. The proposition follows.

Proposition II-3F

If P is a monotone structure having the exchange property and $U \subseteq V$, then every maximal P-independent subset of U is P^{∞}-equivalent to U.

Proof: Assuming that P is a monotone structure having the exchange property while $U \subseteq V$ and X is a maximal P-independent subset of U, we use the exchange property along with part d) of Proposition II-3A to deduce that $U \subseteq P(X)$. Then we use the monotonicity of P to deduce that $P(X) \subseteq P(U)$.

It follows that $P^n(X) \subseteq P^n(U) \subseteq P^{n+1}(X)$ for all $n\in N$. Therefore, $P^\infty(X) = P^\infty(U)$. The proposition follows.

Corollary

If P is a closure structure having the exchange property and $U \subseteq V$, then a subset X of U is a maximal P-independent subset of U if and only if X is a P-independent subset of U and X is P-equivalent to U.

Theorem II-3G

Suppose that P is a structure satisfying SO.

a) Every subset of V contains a maximal P-independent subset.

b) Each P-independent subset of V is contained in a maximal P-independent subset of V.

Proof: Assuming that Y is a subset of V, it follows from part e) of Proposition II-3A that the family of all P-independent subsets of Y contains ϕ. We suppose that A is a P-independent subset of Y and let F be the family of all $Z \subseteq Y$ such that Z is P-independent and $Z \subseteq Z$. Then F is non-empty. We assume that G is an increasing chain in F. If $x\in(\cup G)$, then the family $\{X-\{x\}\}_{X\in G}$ is an increasing chain of subsets of V while X is P-independent for all $X\in G$ and P satisfies SO. It follows that

$$x \notin \cup\{P(X-\{x\})\}_{X\in G} \supseteq P(\cup\{X-\{x\}\}_{X\in G}) = P([\cup G]-\{x\})$$

It follows that $\cup G$ is P-independent. We use Zorn's lemma to assert the existence of maximal elements of F. The theorem follows.

The following corollaries to Theorem II-3G follow from Theorem II-2C and the corollaries to Theorem II-2C.

Corollary 1

Suppose that P is a structure having α-character and satisfying the following condition:

If G is an increasing chain of subsets of V, then

$$\cup\{P(Z)\}_{Z\subseteq \cup G,\, |Z|<\alpha,\, Z\notin G} \subseteq \cup\{P(Z)\}_{Z\in G,\, |Z|<\alpha}.$$

a) Every subset of V contains a maximal P-independent subset.

b) Each P-independent subset of V is included in a maximal P-independent subset of V.

Corollary 2

Suppose that P is a monotone structure having α-character and satisfying the following condition:

If G is an increasing chain of subsets of V and H is a subchain of G such that $|H|<\alpha$, then $P(\cup H) \subseteq \cup\{P(Z)\}_{Z\in G}$.

a) Every subset of V includes a maximal P-independent subset.

b) Each P-independent subset of V is included in a maximal P-independent subset of V.

Corollary 3

Suppose that P is a monotone structure having α-character and satisfying the following condition:

If G is an increasing chain of subsets of V and H is a subchain of G such that $|H| < \alpha$, then $(\cup H) \subseteq Y$ for some $Y\in G$.

a) Every subset of V includes a maximal P-independent subset.

b) Each P-independent subset of V is included in a maximal P-independent subset of V.

Corollary 4

Suppose that P is a monotone structure having finite character.

a) Every subset of V includes a maximal P-independent subset.

b) Each P-independent subset of V is included in a maximal P-independent subset of V.

Theorem II-3H

Every monotone and extensive structure having the exchange property and satisfying SO satisfies the following condition:

Proof: We assume that P is a monotone and extensive structure having the exchange property and satisfying SO. We suppose that $U \subseteq V$, X and Y are maximal P-independent subsets of U and $A \subseteq X$. The family F of all $B \subseteq Y$ such that $(X-A) \cup B$ is P-independent contains ϕ. We suppose, further, that G is an increasing chain in F. Then $\{(X-A) \cup B\}_{B \in G}$ is an increasing chain of subsets of V while P satisfies SO and $(X-A) \cup B$ is P-independent for all $B \in G$. Therefore, if $x \in \cup\{(X-A) \cup B\}_{B \in G}$, then $x \notin \cup\{P([(X-A) \cup B]-\{x\})\}_{B \in G}$ while $\cup\{P([(X-A) \cup B]-\{x\})\}_{B \in G} \supseteq P(\cup\{[(X-A) \cup B]-\{x\}\}_{B \in G}) = P([X-A] \cup [\cup G])$. It follows that $(X-A) \cup (\cup G)$ is P-independent while $\cup G \subseteq Y$. Therefore, it follows upon application of Zorn's lemma that F contains a maximal element. We let B be a maximal element of F. It follows from part d) of Theorem II-3A that $Y-[(X-A) \cup B] \subseteq P([X-A] \cup B)$; hence, $Y \subseteq P([X-A] \cup B)$ [since P is extensive]. We use the monotonicity of P to deduce the relations $P^{\infty}(Y) \subseteq P^{\infty}([X-A] \cup B) \subseteq P^{\infty}(U)$. Since Y is a maximal P-independent subset of U, it follows from part d) of Theorem II-3A that $U \subseteq P(Y)$; hence, it follows that $P^{\infty}(U) \subseteq P^{\infty}(Y)$. Therefore, $P^{\infty}(Y) = P^{\infty}([X-A] \cup B) = P^{\infty}(X)$. The theorem follows.

Theorem II-3I

Every closure structure P having the exchange property satisfies the following condition:

If $U \subseteq V$, Z and W are maximal P-independent subsets of U and A is a finite subset of Z, then there is a subset B of W such that $|B| = |A|$ and $(Z-A) \cup B$ is a maximal P-independent subset of U.

Proof: We assume that P is a closure structure having the exchange property while $U \subseteq V$. It follows that

(I) If Z and W are maximal P-independent subsets of U and $C \subseteq Z$ such that $W-(Z-C) \subseteq P(Z-C)$, then $C = \phi$.

Indeed, for such subsets Z and W of U and such a subset C of Z, it follows from the monotonicity and extensiveness of P that $W \subseteq P(Z-C) \subseteq P(Z)$; hence, since $P(W) = P(Z)$ [as follows from the corollary to Theorem II-3F] while P is monotone and idempotent, it follows that $P(Z-C) = P(Z)$. Since the subset Z-C of the P-independent subset Z of V is P-equivalent to Z while P is monotone, it follows from part b) of Theorem II-3A that $C = \phi$. This proves (I) above.

Continuing with the proof of the theorem, we suppose that X and Y are maximal P-independent subsets of U. If $A = \phi$, we let $B = \phi$. We suppose that $x \in X$. We use (I) to choose an element y of $Y-(X-\{x\})$ such that $y \notin P(X-\{x\})$. Then we use part d) of Theorem II-3A to deduce that $(X-\{x\}) \cup \{y\}$ is P-independent. It follows that $y \notin (X-\{x\})$. It follows that $x \in P([X-\{x\}] \cup \{y\})$; otherwise, we use part d) of Theorem II-3A to deduce that $X \cup \{y\} = ([X-\{x\}] \cup \{y\}) \cup \{x\}$ is P-independent while $y \notin (X-\{x\})$ and X is a maximal P-independent subset of U, so that $y = x$ and, hence,

$x \notin P(X)$. The assertion $x \notin P(X)$ is contrary to the extensiveness of P. It follows from the extensiveness of P that $X \subseteq P([X-\{x\}] \cup \{y\})$ and, therefore, since P is a closure structure, it follows that $P(X) = P([X-\{x\}] \cup \{y\})$. Since P is a closure structure having the exchange property, it follows from the corollary to Proposition II-3F that $(X-\{x\}) \cup \{y\}$ is a maximal P-independent subset of U. Therefore, the following proposition has been proved:

(II) If Z and W are maximal P-independent subsets of U and $x \in Z$, then there is a $y \in [W-(Z-\{x\})]$ such that $(Z-\{x\}) \cup \{y\}$ is a maximal P-independent subset of U and, moreover, such is the case for all elements y of $W-(Z-\{x\})$.

To complete the proof of the theorem, we make the inductive assumption that if Z and W are maximal P-independent subsets of U and $A \subseteq Z$ such that $|A| = n$, then there is a $B \subseteq W$ such that $|B| = n$ and $(Z-A) \cup B$ is a maximal P-independent subset of U. We suppose that X and Y are maximal P-independent subsets of U, that $A \subseteq X$ such that $|A| = n+1$ and that $x \in A$. We use the inductive assumption to obtain $B \subseteq Y$ such that $|B| = n = |A-\{x\}|$ and $[X-(A-\{x\})] \cup B$ is a maximal P-independent subset of U. We apply (II) above with $Z = [X-(A-\{x\})] \cup B$ and $W = Y$ to obtain an element y of $Y-(Z-\{x\})$ such that $(Z-\{x\}) \cup \{y\}$ is a maximal P-independent subset of U. We note that $(Z-\{x\}) \cup \{y\} = (X-A) \cup (B \cup \{y\})$ while $y \in (Y-B)$. It follows that $|B \cup \{y\}| = |B| + 1 = |A|$. The theorem follows by induction.

Corollary

Every closure structure P having the exchange property satisfies the following condition:

If $U \subseteq V$, X and Y are maximal P-independent subsets of U and $x \in X$, then $[Y-(X-\{x\})] - P(X-\{x\}) \neq \phi$ and $(X-\{x\}) \cup \{y\}$ is a maximal P-independent subset of U for all $y \in [Y-(X-\{x\})]$ such that $y \notin P(X-\{x\})$.

Theorem II-3J

Every monotone and extensive structure P satisfying S5 satisfies the following condition:

If X and Y are P-equivalent subsets of V and $A \subseteq P(X)$, then there is a sequence $\{G_n\}_{n \in M}$ $[M = N \cup \{0\}]$ of families of subsets of V such that i), ii) and iii) are true, with P^0 being the identity structure:

i) $Z \subseteq P^n(Y)$ for all $n \in M$ and all $Z \in G_n$.

ii) $|Z| \leq |P^n(A)|\,|P^n(X)|$ for all $n \in M$ and all $Z \in G_n$

iii) $P^n(A) \subseteq \cap\{P(Z)\}_{Z \in G_n} \subseteq P^{n+1}(A)$ for all $n \in M$.

Proof: We assume that P is a monotone and extensive structure satisfying S5, that X and Y are P-equivalent subsets of V and that $A \subseteq P(X)$. It follows that $P^n(X)$ and $P^n(Y)$ are P-equivalent and $P^n(A) \subseteq P^{n+1}(X)$ for all $n \in M$. Therefore, since P satisfies S5, we obtain a family $\{Z_{n,x}\}_{x \in P^n(A)}$ of subsets of V such that $Z_{n,x} \subseteq P^n(Y)$, $|Z_{n,x}| \leq |P^n(X)|$ and $x \in P(Z_{n,x})$ for all $x \in P^n(A)$ and all $n \in M$. It follows from the monotonicity of P that if $n \in M$, then

$$P^n(A) = \cup\{x\}_{x \in P^n(A)} \subseteq \cup\{P(Z_{n,x})\}_{x \in P^n(A)} \subseteq P(\cup\{Z_{n,x}\}_{x \in P^n(A)})$$

while $|\cup\{Z_{n,x}\}_{x\in P^n(A)}| \leq \sum_{x\in P^n(A)} |Z_{n,x}| \leq |P^n(A)||P^n(X)|$. We let G_n be the family of all $Z \subseteq V$ such that $Z \subseteq P^n(Y)$, $|Z| \leq |P^n(A)||P^n(X)|$ and $P^n(A) \subseteq P(Z)$. Then $P^n(A) \subseteq \cap\{P(Z)\}_{Z\in G_n}$. Since $P^n(A) \subseteq P^n(Y)$ and $|P^n(A)| \leq |P^n(A)||P^n(X)|$ and $P^n(A) \subseteq P(P^n(A)) = P^{n+1}(A)$ [extensiveness of P], it follows that $P^n(A) \in G_n$ and, hence, it follows that $P^n(A) \subseteq \cap\{P(Z)\}_{Z\in G_n} \subseteq P^{n+1}(A)$. The theorem follows.

Corollary 1

Every monotone and extensive structure P satisfying S5 satisfies the following condition:

If X and Y are P-equivalent subsets of V and $A \subseteq P(X)$, then there is a sequence $\{G_n\}_{n\in M}$ of families of subsets of V such that $P^\infty(A) = \cup\{\cap\{P(Z)\}_{Z\in G_n}\}_{n\in M}$.

Proof: The corollary is an immediate consequence of iii) in the theorem.

Corollary 2

Every monotone and extensive structure satysfying S5 satisfies the following condition:

If X and Y are P-equivalent subsets of V and Y is P^∞-independent, then $|Y| \leq |X|^2$.

Proof: One observes that extensiveness allows one to replace A in the theorem by X. As in the proof of the theorem, one obtains the relations $X = P^0(X) \subseteq P(\cup\{Z_{0,x}\}_{x\in P^0(X)}) \subseteq P(Y)$ along with the relation $|\cup\{Z_{0,x}\}_{x\in P^0(X)}| \leq |P^0(X)||P^0(X)| = |X|^2$. It fill follow that $P^\infty(X) = P^\infty(\cup\{Z_{0,x}\}_{x\in P^0(X)}) = P^\infty(Y)$ while $Y = \cup\{Z_{0,x}\}_{x\in P^0(X)}$ [since Y is P^∞-independent, $\cup\{Z_{0,x}\}_{x\in P^0(X)} \subseteq Y$ and Y is P^∞-equivalent to $\cup\{Z_{0,x}\}_{x\in P^0(X)}$]. The corollary follows.

Proposition II-3K

Suppose that P is a structure satisfying the following conditions:

i) If $Z \subseteq V$, then Z contains a P-independent subset which is P-equivalent to Z.

ii) Any two P-independent P-equivalent subsets of V have the same cardinal number.

Then P satisfies S5.

Proof: We assume that P fails to satisfy S5. We choose P-equivalent subsets X and Y of V and $x \in P(X)$ such that if $Z \subseteq Y$ such that $|Z| \leq |X|$, then $x \notin P(Z)$. We use i) to obtain P-independent subsets Z_1 of X and Z_2 of Y such that Z_1 is P-equivalent to X and Z_2 is P-equivalent to Y. Then Z_1 and Z_2 are P-independent P-equivalent subsets of V while $x \in P(Z_2)$. It follows that $|Z_2| > |X| \geq |Z_1|$. The latter result is contrary to ii). Therefore, P does not fail to satisfy S5. The proposition follows.

Theorem II-3L

a) If P is a monotone and extensive structure satisfying S5, then any two P-independent P^∞-equivalent subsets of V, one of which is infinite, have the same cardinal number.

b) If P is a closure structure having the exchange property, then any two P-independent P-equivalent subsets of V, one of which is finite, have the same cardinal number.

Proof: We assume that P is a monotone and extensive structure satisfying S5 while X and Y are P-independent P^∞-equivalent subsets of V. It follows from Corollary 2 to Theorem II-3J that $|Y| \leq |X|^2$ and $|X| \leq |Y|^2$ while $c^2 = c$ for all infinite cardinal numbers c. Part a) follows.

To prove part b) we assume that P is a closure structure having the exchange property while X and Y are P-independent P-equivalent [and, therefore, are maximal P-independent subsets of $X \cup Y$]. We apply Theorem III-3I with $Z = X$ being finite and $W = Y$ to obtain $B \subseteq Y$ such that $|B| = |X|$ and $(X-X) \cup B$ is a maximal P-independent subset of $X \cup Y$. It follows that $B = Y$ and, hence, that $|Y| = |X|$. Part b) follows. The theorem follows.

Corollary 1

If P is a closure structure having the exchange property and satisfying S5, then any two P-independent P-equivalent subsets of V have the same cardinal number.

Corollary 2

If P is a closure structure having the exchange property and finite character, then any two P-independent P-equivalent subsets of V have the same cardinal number.

Proofs: Corollary 1 is an immediate consequence of the theorem. We assume that P is a closure structure having the exchange property and finite character while X and Y are P-equivalent subsets of V and $x \in P(X)$. It follows from Corollary 2 to Theorem II-2C that P satisfies S0. We choose maximal P-independent subsets Z_1 of X and Z_2 of Y (Theorem II-3G) and deduce from the corollary to Proposition II-3F that $P(Z_1) = P(Z_2)$. If Z_1 is finite, then it follows from part b) of Theorem II-3L that Z_1 and Z_2 have the same cardinal number; hence, $|Z_2| \leq |X|$ while $x \in P(Z_2)$ and $Z_2 \subseteq Y$. If Z_1 is infinite, we choose a finite subset Z of Y such that $x \in P(Z)$, so that $|Z| < |Z_1| \leq |X|$. It follows that P satisfies S5. Therefore, we apply Corollary 1 to Theorem II-3L and deduce that any two P-independent P-equivalent subsets of V have the same cardinal number. The corollaries follow.

4. Homomorphism and Continuity

Throughout this section it is assumed that P is a structure in a set V and that Q is a structure in a set W. The word 'map' will mean function. A symbol $f: S \longrightarrow T$ will be used to indicate a map f from a set S to a set T.

One can regard a structure P_1 in a set S as a unary operation on the class 2^S of all subsets of S; therefore, the pair $(2^S, P_1)$ can be regarded as an algebra. Thus, our use of the term 'homomorphism' in the discussion which follows is standard. The concept of continuity as defined below was motivated by the definition of continuity occurring in most treatments of elementary topology.

We assume that $f: V \longrightarrow W$ is a map. f is <u>a (P,Q)-homomorphism</u> if and only if $f(P(X)) = Q(f(X))$ for all $X \subseteq V$. f is <u>(P,Q)-continuous</u> if and only if $f^{-1}(Q(Y))$ is P-closed [that is, is $P(X)$ for some $X \subseteq V$] for all $Y \subseteq W$. A map $g: X \longrightarrow W$ from a subset X of V <u>has a (P,Q)-homomorphic extension to V</u> if and only if there is a (P,Q)-homomorphism $h: V \longrightarrow W$ such that the restriction $h|_X$ of h to X is g. A map $g: X \longrightarrow W$ from a subset X of V <u>has a (P,Q)-continuous extension to V</u> if and only if there is a (P,Q)-continuous map $h: V \longrightarrow W$ such that $h|_X = g$.

If $f: S \longrightarrow T$ is a map, then the symbol $f^{-1}(y)$ will be used for the symbol $f^{-1}(\{y\})$ for all $y \in T$.

Assuming that $g: V \longrightarrow W$ is a map, then <u>the factor space modulo g</u>, V_g, is defined as follows:

$$V_g = \{g^{-1}(y)\}_{y \in g(V)}.$$

<u>The factor space structure modulo g</u>, Q_g, is defined as follows:

$$\text{If } \beta \subseteq V_g, \text{ then } Q_g(\beta) = \{g^{-1}(y)\}_{y \in Q(g(\cup\beta))}.$$

<u>The canonical projection modulo g</u>, τ_g, is defined as follows:

$$\text{If } x \in V, \text{ then } \tau_g(x) = g^{-1}(g(x)).$$

<u>The canonical injection modulo g</u>, i_g, is defined as follows:

$$\text{If } y \in g(V), \text{ then } i_g(g^{-1}(y)) = y.$$

<u>Proposition II.4A</u>

Suppose that R is a structure in a set U.

a) If $f: V \longrightarrow W$ is a (P,Q)-homomorphism and $g: W \longrightarrow U$ is a (Q,R)-homomorphism, then gf is a (P,R)-homomorphism.

b) If $f: V \longrightarrow W$ is (P,Q)-continuous and $g: W \longrightarrow U$ is (Q,R)-continuous, then gf is (P,R)-continuous.

<u>Proof:</u> Under the hypothesis of part a), if $X \subseteq V$, then it follows that $g(f(P(X))) = g(Q(f(X))) = R(g(f(X)))$. Under the hypothesis of part b), if $Y \subseteq U$, then $(gf)^{-1}(R(Y)) = f^{-1}(g^{-1}(R(Y))) = f^{-1}(Q(Z)) = P(X)$ for some $Z \subseteq W$ and some $X \subseteq V$. The proposition follows.

If $Y \subseteq g(V)$, then we will use the symbol β_Y as notation for the class of all $g^{-1}(y)$ such that $y \in Y$. It follows that $g(\cup\beta_Y) = Y$ for all subsets Y of $g(V)$.

Proposition II-4B

Suppose that $g: V \longrightarrow W$ is a map.

a) The following are equivalent:

i) g is a (P,Q)-homomorphism;

ii) If $X \subseteq V$, then $Q_g(\beta_{g(X)}) = \{g^{-1}(y)\}_{y \in g(P(X))}$;

iii) τ_g is a (P,Q_g)-homomorphism.

b) The following are equivalent:

i) g is (P,Q)-continuous;

ii) If $Z \subseteq W$, then $\cup Q_g(\{g^{-1}(y)\}_{y \in Z}) = g^{-1}(Q(Z))$;

iii) τ_g is (P,Q_g)-continuous.

Proof: We assume that $X \subseteq V$. It follows that

$$\tau_g(P(X)) = \{g^{-1}(g(x))\}_{x \in P(X)} = \{g^{-1}(y)\}_{y \in g(P(X))},$$

and that

$$Q_g(\beta_{g(X)}) = \{g^{-1}(y)\}_{y \in Q(g(\cup \beta_{g(X)}))} = \{g^{-1}(y)\}_{y \in Q(g(X))}.$$

Part a) follows. Assuming that $\beta \subseteq V_g$, the following statements are equivalent:

(1) $x \in \tau_g^{-1}(Q_g(\beta))$;

(2) $g^{-1}(g(x)) \in Q_g(\beta)$;

(3) $g(x) \in Q(g(\cup\beta))$;

(4) $x \in g^{-1}(Q(g(\cup\beta)))$.

It follows that $\tau_g^{-1}(Q_g(\beta)) = g^{-1}(Q(g(\cup\beta)))$ for all $\beta \subseteq V_g$. Also, if $Y \subseteq W$, then $\{g^{-1}(y)\}_{y \in Y} = \beta$ for some $\beta \subseteq V_g$. Furthermore, if $\beta \subseteq V_g$, then $\cup Q_g(\beta) = \cup\{g^{-1}(y)\}_{y \in Q(g(\cup\beta))} = g^{-1}(Q(g(\cup\beta))) = g^{-1}(Q(Z))$ for $Z = \cup\beta$. Part b) follows. The proposition is proved.

Proposition II-4C

If $g: V \longrightarrow W$ is a map, then the diagram to the right is commutative, i_g is an injective (Q_g, Q)-homomorphism and i_g^{-1} is a (Q, Q_g)-homomorphism.

$$\begin{tikzcd} V \arrow[rr, "g"] \arrow[dr, "\tau_g"'] & & W \\ & V_g \arrow[ur, "i_g"'] & \end{tikzcd}$$

Proof: We assume that $g: V \longrightarrow W$ is a map. If $x \in V$, then it follows that $i_g(\tau_g(x)) = i_g(g^{-1}(g(x))) = g(x)$. Therefore, the diagram is commutative. It is clear from the definition that i_g is injective. If $\beta \subseteq V_g$, then it follows that

$$\begin{aligned} i_g(Q_g(\beta)) &= i_g(\{g^{-1}(y)\}_{y \in Q(g(\cup\beta))}) \\ &= \{i_g(g^{-1}(y))\}_{y \in Q(g(\cup\beta))} \\ &= \{y\}_{y \in Q(g(\cup\beta))} \\ &= Q(g(\cup\beta)). \end{aligned}$$

Therefore, i_g is a (Q_g, Q)-homomorphism. Finally, if $Y \subseteq g(V)$, then it follows that

$$\begin{aligned} i_g^{-1}(Q(Y)) &= \{i_g^{-1}(y)\}_{y \in Q(Y)} \\ &= \{i_g^{-1}(i_g(g^{-1}(y)))\}_{y \in Q(Y)} \\ &= \{g^{-1}(y)\}_{y \in Q(Y)} \\ &= \{g^{-1}(y)\}_{y \in Q(g(\cup\beta_Y))} \\ &= Q_g(\beta_Y) \end{aligned}$$

It follows that i_g^{-1} is a (Q, Q_g)-homomorphism. The proposition is proved.

One might observe that no mention of τ_g was made in the statement of Proposition II-4C. Under the hypothesis of the proposition, the only statement that can be made about τ_g is that it is surjective.

Proposition II-4D

Suppose that $g: V \longrightarrow W$ is a map.

a) $Q_g(\phi) = \{g^{-1}(y)\}_{y \in Q(\phi)}$.

b) $(Q_g)^n = (Q^n)_g$ for all $n \in N$; hence, $(Q_g)^\infty = (Q^\infty)_g$.

c) $(M_Q)_g = M_{Q_g}$.

d) $(E_Q)_g = E_{Q_g}$.

e) If G is a family of subsets of V_g and $Z = \cup\{g(\cup\beta)\}_{\beta \in G}$, then $Q_g(\cup G) = \{g^{-1}(y)\}_{y \in Q(Z)}$.

Proof: Part a) follows from the fact that $Q(g(\cup\phi) = Q(\phi)$. If $\beta \subseteq V_g$, then it follows that

$$(Q_g)^2(\beta) = Q_g(Q_g(\beta)) = \{g^{-1}(y)\}_{y \in Q(g(\cup Q_g(\beta)))}.$$

Also, $\cup Q_g(\beta) = g^{-1}(Q(g(\cup\beta)))$. Therefore, $g(\cup Q_g(\beta)) = Q(g(\cup\beta))$; hence, $Q(g(\cup Q_g(\beta))) = Q^2(g(\cup\beta))$. It follows that $(Q_g)^2(\beta) = \{g^{-1}(y)\}_{y \in Q^2(g(\cup\beta))}$. Therefore, $(Q_g)^2 = (Q^2)_g$. We assume that $n \in N$ such that $(Q_g)^n = (Q^n)_g$ while $\beta \subseteq V_g$. It follows that

$$\begin{aligned} \cup(Q_g)^n(\beta) &= \cup(Q^n)_g(\beta) \\ &= \cup\{g^{-1}(y)\}_{y \in Q^n(g(\cup\beta))} \\ &= g^{-1}(Q^n(g(\cup\beta))). \end{aligned}$$

Therefore, $g(\cup[Q_g]^n(\beta)) = Q^n(g(\cup\beta))$; hence, $Q(g(\cup[Q_g]^n(\beta))) = Q^{n+1}(g(\cup\beta))$. It follows that

$$\begin{aligned} (Q_g)^{n+1}(\beta) = Q_g([Q_g]^n(\beta)) &= \{g^{-1}(y)\}_{y \in Q(g(\cup[Q_g]^n(\beta)))} \\ &= \{g^{-1}(y)\}_{y \in Q^{n+1}(g(\cup\beta))} \end{aligned}$$

The latter set is $(Q^{n+1})_g(\beta)$. It follows that $(Q_g)^{n+1} = (Q^{n+1})_g$. Part b) follows by induction.

We assume that $\beta \subseteq V_g$ and let $g(U\mu) = X_\mu$ for all $\mu \subseteq \beta$. It follows that $M_Q(g(U\beta)) = U\{Q(g(U\mu))\}_{\mu \subseteq \beta} = U\{Q(X_\mu)\}_{\mu \subseteq \beta}$. Therefore, it follows that

$$\begin{aligned}(M_Q)_g(\beta) &= \{g^{-1}(y)\}_{y \in M_Q(g(U\beta))} \\ &= \{g^{-1}(y)\}_{y \in [U\{Q(X_\mu)\}_{\mu \subseteq \beta}]} \\ &= U\{\{g^{-1}(y)\}_{y \in Q(X_\mu)}\}_{\mu \subseteq \beta} \\ &= M_{Q_g}(\beta).\end{aligned}$$

It follows that $(M_Q)_g = M_{Q_g}$. Part c) follows.

We assume that $\beta \subseteq V_g$. Then $E_Q(g(U\beta)) = g(U\beta) \cup Q(g(U\beta))$ and $\beta = \{g^{-1}(y)\}_{y \in g(U\beta)}$. It follows that

$$\begin{aligned}(E_Q)_g(\beta) &= \{g^{-1}(y)\}_{y \in E_Q(g(U\beta))} \\ &= \{g^{-1}(y)\}_{y \in ([g(U\beta)] \cup [Q(g(U\beta))])} \\ &= [\{g^{-1}(y)\}_{y \in g(U\beta)}] \cup [\{g^{-1}(y)\}_{y \in Q(g(U\beta))}] \\ &= \beta \cup Q_g(\beta).\end{aligned}$$

Therefore, $(E_Q)_g(\beta) = E_{Q_g}(\beta)$. Part d) follows.

One can prove part e) using an argument paralleling the argument used in proving part c).

We assume that G is a family of subsets of V_g. It follows that $Q(g(U[UG])) = Q(g(U[U\{\beta\}_{\beta \in G}])) = Q(U\{g(U\beta)\}_{\beta \in G})$. Therefore, $Q_g(UG) = \{g^{-1}(y)\}_{y \in Q(g(U[UG]))} = \{g^{-1}(y)\}_{y \in Q(U\{g(U\beta)\}_{\beta \in G})}$. Part f) follows. This completes a proof of the proposition.

Theorem II-4E

Suppose that $g: V \longrightarrow W$ is a (P,Q)-homomorphism.

a) g is an (M_P, M_Q)-homomorphism.

b) g is an (E_P, E_Q)-homomorphism.

c) g is a (P_α, Q_α)-homomorphism.

d) g is a (P^n, Q^n)-homomorphism for all $n \in N$.

e) g is a (P^∞, Q^∞)-homomorphism.

Proof: We assume that $X \subseteq V$. It follows that

$$\begin{aligned} g(\cup\{P(Z)\}_{Z \subseteq X}) &= \cup\{g(P(Z))\}_{Z \subseteq X} \\ &= \cup\{Q(g(Z))\}_{Z \subseteq X} \\ &= \cup\{Q(Y)\}_{Y \subseteq g(X)} \\ &= M_Q(g(X)). \end{aligned}$$

Therefore, g is an (M_P, M_Q)-homomorphism. Similarly, g is a (P_α, Q_α)-homomorphism. Since $g(X \cup P(X) = g(X) \cup g(P(X)) = g(X) \cup Q(g(X))$, it follows that $g(E_P(X)) = E_Q(g(X))$. Therefore, g is an (E_P, E_Q)-homomorphism. Assuming that $n \in N$ such that g is a (P^n, Q^n)-homomorphism, it follows that $g(P^{n+1}(X)) = g(P(P^n(X))) = Q(g(P^n(X))) = Q(Q^n(g(X))) = Q^{n+1}(g(X))$. It follows that g is a (P^{n+1}, Q^{n+1})-homomorphism. Part d) follows; hence, part e) follows. The theorem is proved.

Theorem II-4F

Suppose that $g: V \longrightarrow W$ is a map.

a) Q_g satisfies a given axiom (or SO) if and only if Q satisfies the same axiom (or SO).

b) Subsets β and μ of V_g are Q_g-equivalent if and only if $g(\cup\beta)$ and $g(\cup\mu)$ are Q-equivalent.

c) A subset β of V_g is Q_g-independent if and only if $g(\cup\beta)$ is Q-independent.

Proof: It follows from Section 1 of this chapter that the assertions $Q = M_Q$, $Q = E_Q$ and $Q = Q_\alpha$ mean that Q is monotone, Q is extensive and Q has α-character (respectively). Therefore, part a) follows from Proposition II-4D in the case of monotonicity, the case of extensiveness and the case of α-character. Also, it follows from the same proposition that $Q = Q^2$ if and only if $Q_g = (Q^2)_g = (Q_g)^2$, and that $Q(\phi) = \phi$ if and only if $Q_g(\phi) = \phi$, and that Q satisfies S0 if and only if the following condition is satisfied by all increasing chains G of subsets of V_g:

$$\begin{aligned} Q_g(\cup G) &= \{g^{-1}(y)\}_{y\in Q(\cup\{g(\cup\beta)\}_{\beta\in G}} \\ &\subseteq \{g^{-1}(y)\}_{y\in[\cup\{Q(g(\cup\beta))} \\ &= \cup\{Q_g(\beta)\}_{\beta\in G}. \end{aligned}$$

This proves part a) except for the case of S5 and the exchange property. Assertion a) in the case of the exchange property follows from the relation $Q_g(\beta\cup\{g^{-1}(y)\}) - Q_g(\beta) = \{g^{-1}(z)\}_{z\in[Q(g(\cup\beta)\cup\{y\}) - Q(g(\cup\beta))]}$. Part b) follows from the fact that subsets β and μ of V_g are Q_g-equivalent if and only if the associated index sets $Q(g(\cup\beta))$ and $Q(g(\cup\mu))$ coincide; hence, if β and μ are subsets of V_g, then $g^{-1}(y)\in Q_g(\beta) = Q_g(\mu)$ if and only if $y\in Q(g(\cup\beta)) = Q(g(\cup\mu))$. Also, $|g(\cup\beta_1)| = |\beta_1|$ for all $\beta_1 \subseteq V_g$. Part a) in the case of S5 follows. If $\beta\subseteq V_g$ and $g^{-1}(b)\in\beta$, then the following are equivalent:

(1) $g^{-1}(b)\notin Q_g(\beta-\{g^{-1}(b)\})$; (2) $b\neq y$ for all $y\in Q(g(\cup[\beta-\{g^{-1}(b)\}]))$;

(3) $b\notin Q(g(\cup\beta)-\{b\})$.

Part c) follows. The theorem is proved.

Theorem II-4G

Suppose that $g:V \longrightarrow W$ is a surjective (P,Q)-homomorphism.

a) Q inherits monotonicity, extensiveness, idempotence, α-character the property of satisfying SO and the property of sending ϕ onto ϕ from P.

b) If $X \subseteq V$, then the class of all $g(Y)$ such that $Y \subseteq V$ and Y is P-equivalent to X is contained in the class of all $Z \subseteq W$ such that Z is Q-equivalent to $g(X)$.

Proof: One can use Theorem II-4E along with some results from Section 1 of this chapter to prove part a) [except in the cases of SO and the property of sending ϕ onto ϕ]. For example, if P is monotone, then $P = M_P$; hence, it follows that $Q(g(X)) = g(P(X)) = g(M_P(X)) = M_Q(g(X))$, that is, Q is monotone. If $P(\phi) = \phi$, then $\phi = g(\phi) = g(P(\phi)) = Q(g(\phi)) = Q(\phi)$. If P satisfies SO while G is an increasing chain $\{g(X)\}_{X\in H}$ of subsets of W, then it follows that $Q(\cup G) = Q(\cup\{g(X)\}) = Q(g(\cup H)) = g(P(\cup H))$ while $g(P(\cup H)) \subseteq g(\cup\{P(X)\}_{X\in H}) = \cup\{g(P(X))\}_{X\in H} = \cup\{Q(g(X))\}_{X\in H} = \cup\{Q(Y)\}_{Y\in G}$; hence, Q satisfies SO. Part a) follows. If B and C are P-equivalent subsets of V, then it follows that $Q(g(B)) = Q(g(C))$; hence, $g(C)$ is Q-equivalent to $g(B)$. Part b) follows. The theorem is proved.

Theorem II-4H

Suppose that X is a P-independent subset of V while P is monotone and extensive and $g:V \longrightarrow W$ is a (P,Q)-homomorphism.

a) $g(X-g^{-1}(Q(\phi)))$ is Q^{∞}-equivalent to $g(X)$.

b) $\{g^{-1}(g(x))\}_{x\in[X-g^{-1}(Q(\phi))]}$ is Q_g-independent and $(Q_g)^{\infty}$-equivalent to $\{g^{-1}(g(x))\}_{x\in X}$.

Proof: We suppose that some member y of $g(X)$ fails to be in $Q(g(X-Y))$, where $Y = g^{-1}(Q(\phi))$. Then $y \notin g(P(X-Y))$. We let $y = g(x)$ with $x \in X$. Then $x \notin P(X-Y)$; hence, since P is extensive and $x \in X$, it follows that $x \in Y$. But then, since P is monotone, it follows that $y \in g(Y) = g(g^{-1}(Q(\phi)) = Q(\phi)$ while $Q(\phi) \subseteq Q(g(X-Y))$. This latter result is contrary to our supposition. It follows that $g(X) \subseteq Q(g(X-Y))$; hence, $Q(g(X)) \subseteq Q^2(g(X-Y))$ while $Q^2(g(X-Y)) \subseteq Q^2(g(X))$. Therefore, $g(X-Y)$ is Q^{∞}-equivalent to $g(X)$. It follows, therefore, from Theorem II-4F that $g(\cup\{g^{-1}(y)\}_{y \in g(X-Y)}$ is Q-equivalent to $g(\cup\{g^{-1}(y)\}_{y \in g(X)}$; hence, $\{g^{-1}(y)\}_{y \in g(X-Y)}$ is $(Q_g)^{\infty}$-equivalent to $\{g^{-1}(y)\}_{y \in g(X)}$ [as follows from Theorem II-4F and part b) of Proposition II-4D]. It follows that if $x \in (X-Y)$, then $x \notin P([X-Y]-\{x\})$ [since X is P-independent] and $Q(g(\cup\{g^{-1}(g(z))\}_{z \in [(X-Y)-\{x\}]}) = g(P([X-Y]-\{x\}))$; hence, $Q_g(\beta) = \{g^{-1}(y)\}_{y \in g(P([X-Y]-\{x\})}$ and $g^{-1}(g(x)) \notin Q_g(\beta)$, where $\beta = \{g^{-1}(g(z))\}_{z \in [(X-Y)-\{x\}]}$. Part b) follows. The theorem is proved.

Theorem II-4I

If P is extensive and Q is idempotent, then every surjective (P,Q)-homomorphism $g: V \longrightarrow W$ is (P,Q)-continuous.

Proof: We assume that P is extensive and Q is idempotent while $g: V \longrightarrow W$ is a surjective (P,Q)-homomorphism. We suppose that $X \subseteq V$. It follows that $g(P(g^{-1}(Q(X)))) = Q(g(g^{-1}(Q(X)))) = Q(Q(X)) = Q(X))$; hence, it follows that $g^{-1}(Q(X)) = g^{-1}(g(P(g^{-1}(X)))) \supseteq P(g^{-1}(Q(X))) \supseteq g^{-1}(Q(X))$. Therefore, $P(g^{-1}(Q(X))) = g^{-1}(Q(X))$. It follows that g is (P,Q)-continuous. The theorem follows.

Corollary

If P is extensive, Q is idempotent and $g: V \longrightarrow W$ is a surjective (P,Q)-homomorphism, then $g^{-1}(Q(X))$ is left fixed by P for all $X \subseteq V$.

Assuming that P_1 is a structure in a set V_1 and that $U \subseteq V_1$, we shall write '$[P_1\text{-dim}](P_1(U))$ exists' and will mean that any two P-independent subsets of V_1 which are P-equivalent to U have the same cardinal number. We shall write '$[P_1\text{-dim}](U)$' for $[P_1\text{-dim}](P_1(U))$ with the understanding that $P_1(U) = U$.

Theorem II-4J

Suppose that $g: V \longrightarrow W$ is a map.

a) If β is a Q_g-generator of V_g and one of $[Q\text{-dim}](Q(g(V)))$ and $[Q_g\text{-dim}](Q_g(\beta))$ exists, then

$$[Q_g\text{-dim}](Q_g(\beta)) = [Q\text{-dim}](Q(g(V))).$$

b) If P is monotone and extensive, Q is idempotent, g is a surjective (P,Q)-homomorphism, $[Q\text{-dim}](W)$ exists and the following condition is satisfied:

If $X \subseteq V$ such that X is P^{∞}-equivalent to V and $A \subseteq V$, then $X \cap P^{\infty}(A)$ is P^{∞}-equivalent to A and $[P^{\infty}\text{-dim}](P^{\infty}(A))$ exists.

then

$$[P^{\infty}\text{-dim}[(V) = [P^{\infty}\text{-dim}](g^{-1}(Q(\phi))) + [Q_g\text{-dim}](V_g).$$

Proof: We suppose that β is a Q_g-generator of V_g. We consider the case that $[Q\text{-dim}](Q(g(V)))$ exists. We assume that β_1 and β_2 are Q_g-independent subsets of V_g which are Q_g-equivalent to β. It follows from parts b) and c) of Theorem II-4F that $g(\cup\beta_1)$ and $g(\cup\beta_2)$ are Q-independent subsets of W which are Q-equivalent to $g(\cup\beta)$. Therefore, $g(\cup\beta_1)$ and $g(\cup\beta_2)$ have the same cardinal number. But β_1 and $g(\cup\beta_1)$ have the same cardinal number for each $i\in\{1,2\}$. Therefore, $|\beta_1| = |\beta_2|$. It follows that $[Q_g\text{-dim}](Q_g(\beta))$ exists and coincides with $[Q\text{-dim}](Q(g(V)))$. Similarly, if $[Q_g\text{-dim}](Q_g(\beta))$ exists, then $[Q\text{-dim}](Q(g(V)) = [Q_g\text{-dim}](Q_g(\beta))$. Part a) follows.

We assume the hypothesis of part b) and suppose that $X \subseteq V$ such that X is P-independent and P-equivalent to V. It follows from part b) of Theorem II,4H and part a) of the corollary to Proposition II-3B that $\{g^{-1}(g(x))\}_{x\in(X-Y)}$ is $(Q_g)^{\infty}$-independent and $(Q_g)^{\infty}$-equivalent to $\{g^{-1}(g(x))\}_{x\in X}$, where $Y = g^{-1}(Q(\phi))$. It follows from Theorem II-4E that g is a (P^{∞},Q^{∞})-homomorphism. Since P-equivalent subsets of V are among the P^{∞}-equivalent subsets of V [Proposition II-3B], then X is P^{∞}-equivalent to V and, therefore, $Q^{\infty}(g(X)) = g(P^{\infty}(X)) = g(P^{\infty}(V)) = Q^{\infty}(g(V))$. But Q is idempotent; hence, $Q(g(X)) = Q(g(V))$. Also, it follows that $Q(g(\cup\{g^{-1}(g(x))\}_{x\in X}) = Q(g(X))$. Therefore, $Q_g(\{g^{-1}(g(x))\}_{x\in X}) = V_g$ [since Q_g is extensive and $\{g^{-1}(y)\}_{y\in Q(g(V))} = V_g$]. It follows that $\{g^{-1}(g(x))\}_{x\in(X-Y)}$ is a Q_g-independent Q_g-generator of V_g. Therefore, $[Q_g\text{-dim}](V_g) = |\{g^{-1}(g(x))\}_{x\in(X-Y)}| = |X-Y|$. It follows from the corollary to Theorem II-4I that $P^{\infty}(g^{-1}(Q(\phi))) = g^{-1}(Q(\phi))$; it follows from the hypothesis of part b) that $X\cap P^{\infty}(g^{-1}(Q(\phi)))$ is P^{∞}-equivalent to $g^{-1}(Q(\phi))$, and that $[P^{\infty}\text{-dim}](P^{\infty}(g^{-1}(Q(\phi))))$ exists. Therefore, it follows that $[P^{\infty}\text{-dim}](g^{-1}(Q(\phi))) = |X\cap g^{-1}(Q(\phi))|$ while $[P^{\infty}\text{-dim}](V) = |X|$. We note that $|X| = |X-g^{-1}(Q(\phi))| + |X\cap g^{-1}(Q(\phi))|$. The equality in part b) follows. This completes a proof of the theorem.

A map $f:V\longrightarrow W$ is point-wise (P,Q)-continuous if and only if the following condition is satisfied:

If $x \in V$ and $Z \subseteq W$ such that $f(x) \in [W-Q(Z)]$, then there is a subset X of V such that $x \in [V-P(X)]$ and $f(V-P(X)) \subseteq W-Q(Z)$.

The following theorem reveals that the notion of point-wise (P,Q)-continuity coincides with the notion of (P,Q)-continuity if P is a closure structure.

Theorem II-4K

If $f: V \longrightarrow W$ is a map, then the following are equivalent:

i) f is point-wise (P,Q)-continuous;

ii) For each subset Z of W such that $M = V - f^{-1}(Q(Z)) \neq \phi$, there is a family $\{B_x\}_{x \in M}$ of subsets of V subject to the following conditions:

1) $x \in [V-P(B_x)]$ for all $x \in M$;

2) $f^{-1}(Q(Z)) = \cap\{P(B_x)\}_{x \in M}$.

Proof: We assume that $f: V \longrightarrow W$ is a map. We suppose that i) is true while $Z \subseteq W$ such that $M = V - f^{-1}(Q(Z)) \neq \phi$. We use the point-wise (P,Q)-continuity of f to obtain a family $\{B_x\}_{x \in M}$ of subsets of V such that if $x \in M$, then $x \in [V-P(B_x)]$ and $f(V-P(B_x)) \subseteq W-Q(Z)$. We observe that if $x \in M$, then $V-P(B_x) \subseteq f^{-1}(f(V-P(B_x))) \subseteq f^{-1}(W-Q(Z)) = V-f^{-1}(Q(Z)) = M$. It follows that $M = \cup\{x\}_{x \in M} \subseteq \cup\{V-P(B_x)\}_{x \in M} \subseteq M$. Therefore, it follows that $f^{-1}(Q(Z)) = \cap\{P(B_x)\}_{x \in M}$. Consequently, i) implies ii). Conversely, we suppose that ii) is true while $x \in V$, $Z \subseteq W$ and $f(x) \in [W-Q(Z)]$. Then $x \in f^{-1}(W-Q(Z)) = V-f^{-1}(Q(Z)) = M$; hence, $M \neq \phi$. We use ii) to obtain a family $\{B_x\}_{x \in M}$ of subsets of V such that $y \in [V-P(B_y)]$ for all $y \in M$ and $f^{-1}(Q(Z)) = \cap\{P(B_y)\}_{y \in M}$. Since $x \in M$, then $x \in [V-P(B_x)]$ while, also, $V-P(B_x) \subseteq \cup\{V-P(B_y)\}_{y \in M} = V-f^{-1}(Q(Z))$. It follows that $W-Q(Z)$ contains

$f(V-P(B_x))$. Therefore, f is point-wise (P,Q)-continuous; hence, ii) implies i). This completes a proof of the theorem.

Corollary

If P is a closure structure, then a map $f: V \longrightarrow W$ is point-wise (P,Q)-continuous if and only if f is (P,Q)-continuous.

Theorem II-4L

Suppose that P is monotone and extensive, that P satisfies SO, that $U \subseteq V$ such that U is P-equivalent to V and that $g: U \longrightarrow W$ is a map. The following are equivalent:

i) g has a (P,Q)-homomorphic extension to V;

ii) There is an increasing chain G of subsets of U whose union is U and a family $\{f_X\}_{X \in G}$ of (P,Q)-homomorphisms $f_X: P(X) \longrightarrow W$ such that

1) If $X \in G$, then $f_X|_X = g|_X$,

2) If $X \in G$, $Z \in G$ and $X \subseteq Z$, then $f_Z|_{P(X)} = f_X$.

Proof: If i) is true, we let $G = \{X\}_{X=U}$ and $f_X = f$ for all $X \in G$, with f being a (P,Q)-homomorphic extension of g to all of V. We assume that ii) is true and choose such a chain G and such a family $\{f_X\}_{X \in G}$. Since P is monotone and satisfies SO, it follows that $P(\cup G) = \cup\{P(X)\}_{X \in G}$. We define $f: P(\cup G) \longrightarrow W$ as follows:

If $x \in P(\cup G)$, then $f(x) = f_X(x)$ if $x \in P(X)$ with $X \in G$.

Since G is increasing and P satisfies SO, it follows that f is well-defined [as follows from condition 2) above]. We suppose that $A \subseteq P(\cup G)$. Then $A = \cup\{A \cap P(X)\}_{X \in G}$ and $\{A \cap P(X)\}_{X \in G}$ is an increasing chain of subsets of V. It follows that

$$\begin{aligned} f(P(A)) &= f(P(\cup\{A \cap P(X)\}_{X\in G})) \\ &= f(\cup\{P(A \cap P(X))\}_{X\in G}) \\ &= \cup\{f(P(A \cap P(X)))\}_{X\in G} \\ &= \cup\{f_X(P(A \cap P(X)))\}_{X\in G} \\ &= \cup\{Q(f_X(A \cap P(X)))\}_{X\in G} \\ &= \cup\{Q(f(A \cap P(X)))\}_{X\in G} \\ &= Q(\cup\{f(A \cap P(X))\}_{X\in G}) \\ &= Q(f(\cup\{A \cap P(X)\}_{X\in G})) \\ &= Q(f(A)). \end{aligned}$$

It is clear that $f|_{\cup G} = g$. It follows that f is a (P,Q)-homomorphic extension of g to $P(\cup G) = P(U) = V$. This completes a proof of the theorem.

Theorem II-4M

Suppose that P is a closure structure satisfying SO, that $U \subseteq V$ such that U is P-equivalent to V, that $g{:}U \longrightarrow W$ is a map, that there is a map $f{:}P(\phi) \longrightarrow W$ such that $f(P(\phi)) \subseteq Q(f(\phi))$, and that the following condition is satisfied:

If $X \subseteq U$ and $x\in[U-P(X)]$, then $g|_{X\cup\{x\}}$ has an extension $f{:}P(X\cup\{x\}) \longrightarrow W$ such that $f(P(A)) \subseteq Q(f(A))$ for all $A \subseteq X\cup\{x\}$.

Then g has an extension $f{:}V \longrightarrow W$ such that $f(P(A)) \subseteq Q(f(A))$ for all $A \subseteq U$.

Proof: We let F be the family of all $X \subseteq U$ such that $g|_X$ has an extension $f{:}P(X) \longrightarrow W$ for which $f(P(A)) \subseteq Q(f(A))$ for all $A \subseteq X$. It follows from the hypothesis that $\phi\in F$. We define the family $\{F_X\}_{X\in F}$ by letting F_X be the family of all extensions $f{:}P(X) \longrightarrow W$ of $g|_X$ such that if $A \subseteq X$, then $f(P(A)) \subseteq Q(f(A))$. We define a partial order relation $<$ on $\cup\{F_X\}_{X\in F}$

as follows:

If $X \in F$, $Z \in F$, $f \in F_X$ and $h \in F_Z$, then $f < h$ if and only if $X \subseteq Z$ and $h|_{P(X)} = f$.

We assume that G is a totally ordered chain (relative to $<$) in $\cup\{F_X\}_{X \in F}$ and let H be the family of all $X \in F$ such that F_X contains a member of G. Since P is monotone and satisfies SO, it follows that $P(\cup H) = \cup\{P(Z)\}_{Z \in H}$. If $x \in \cup\{P(Z)\}_{Z \in H}$, we let G_x be the family of all $X \in H$ such that $x \in P(X)$. Since G is totally ordered, it follows that if $x \in \cup\{P(Z)\}_{Z \in H}$, $A \in G_x$, $f_1 \in F_A$, $B \in G_x$ and $f_2 \in F_B$, then $f_1(x) = f_2(x)$. Therefore, we may define a map f from $P(\cup H)$ to W as follows:

If $x \in P(\cup H)$, $A \in G_x$ and $h \in F_A$, then $f(x) = h(x)$.

If $Z \subseteq \cup H$, then it follows that

$$\begin{aligned}
f(P(Z)) &= f(P(\cup\{Z \cap C\}_{C \in H})) \\
&= f(\cup\{P(Z \cap C)\}_{C \in H}) \\
&= \cup\{f(P(Z \cap C))\}_{C \in H} \\
&= \cup\{f_C(P(Z \quad C))\}_{C \in H}, \text{ where } f_C \in F_C \text{ for all } C \in H \\
&\subseteq \cup\{Q(f_C(Z \cap C))\}_{C \in H} \\
&= \cup\{Q(f(Z \cap C))\}_{C \in H} \\
&= Q(\cup\{f(Z \cap C)\}_{C \in H}) \\
&= Q(f(\cup\{Z \cap C\}_{C \in H})) \\
&= Q(f(Z))
\end{aligned}$$

Also, $f(\cup H) = \cup\{f(C)\}_{C \in H} = \cup\{f_C(C)\}_{C \in H} = \cup\{g(C)\}_{C \in H} = g(\cup H)$. Therefore, f is an extension of $g|_{\cup H}$ such that if $A \subseteq \cup H$, then $f(P(A)) \subseteq Q(f(A))$. It follows that f is an upper bound for G. We are in a position to apply Zorn's lemma to obtain a maximal element h of $\cup\{F_X\}_{X \in F}$. Then in view of the condition in the statement of the theorem, no enlargement $X \cup \{x\}$ of of X with $x \in [U - P(X)]$ is permitted, where X is an element of F such that $h \in F_X$. It follows that $U \subseteq P(X)$ while $X \subseteq P(U)$ and P is a closure structure. Therefore, $P(X) = P(U) = V$. It follows that h is defined on V. The theorem follows.

Corollary

Suppose that P is a closure structure satisfying SO, that $U \subseteq V$ such that U is P-equivalent to V, that $g: U \longrightarrow W$ is a map, that there is a map $f: P(\phi) \longrightarrow W$ such that $f(P(\phi)) = Q(f(\phi))$, and that the following condition is satisfied:

If $X \subseteq V$ and $x \in [U - P(X)]$, then there is a map $f: P(X \cup \{x\}) \longrightarrow W$ such that $f(P(A)) = Q(f(A))$ for all $A \subseteq X \cup \{x\})$.

Then there is an extension $f: V \longrightarrow W$ of g such that $f(P(A)) = Q(f(A))$ for all $A \subseteq U$.

5. Cartesian Products

Throughout this section it is assumed that M is a non-empty set and that $\{(V_m, P_m)\}_{m \in M}$ is a non-empty family such that if $m \in M$, then V_m is a non-empty set and P_m is a structure in V_m.

The Cartesian product, $\prod\{V_m\}_{m \in M}$, of $\{V_m\}_{m \in M}$ is the class of all functions $f: M \longrightarrow \cup\{V_m\}_{m \in M}$ such that $f(m) \in V_m$ for all $m \in M$. The Cartesian product structure, $\prod\{P_m\}_{m \in M}$, in $\prod\{V_m\}_{m \in M}$ is defined as follows:

If $\beta \subseteq \prod\{V_m\}_{m \in M}$, then $[\prod\{P_m\}_{m \in M}](\beta)$ is the class of all $f \in \prod\{V_m\}_{m \in M}$ such that $f(m) \in P_m(\cup\{h(m)\}_{h \in \beta})$ for all $m \in M$.

In the following discussion we will use the symbols W and Q as notation for $\prod\{V_m\}_{m\in M}$ and $\prod\{P_m\}_{m\in M}$, respectively. If $\beta \subseteq W$, then $X_{m,\beta} = \cup\{h(m)\}_{h\in\beta}$.

Proposition II-5A

If $\beta \subseteq W$, then $Q(\beta) = \prod\{P_m(X_{m,\beta})\}_{m\in M}$.

Proof: If $\beta \subseteq W$ and $f\in W$, then $f\in Q(\beta)$ if and only if $f(m)\in P_m(X_{m,\beta})$ for all $m\in M$; hence, $f\in Q(\beta)$ if and only if $f\in \prod\{P_m(X_{m,\beta})\}_{m\in M}$. The proposition follows.

Proposition II-5B

If G is any family of subsets of W, then

$$\prod\{\cup\{P_m(X_{m,\beta})\}_{\beta\in G}\}_{m\in M} = \cup\{\prod\{P_m(X_{m,\beta})\}_{m\in M}\}_{\beta\in G}.$$

Proof: We assume that G is a family of subsets of W. If some $\beta\in G$ is such that $f\in \prod\{P_m(X_{m,\beta})\}_{m\in M}$, then $f(m)\in P_m(X_{m,\beta})$ for all $m\in M$; hence, $f(m)\in \cup\{P_m(X_{m,\mu})\}_{\mu\in G}$ for all $m\in M$ and, therefore, $f\in \prod\{\cup\{P_m(X_{m,\beta})\}_{\beta\in G}\}_{m\in M}$. This shows that the left member of the equality in the proposition is contained in the right member. Also, if $f\in \prod\{\cup\{P_m(X_{m,\beta})\}_{\beta\in G}\}_{m\in M}$, then $f(m)\in \cup\{P_m(X_{m,\beta})\}_{\beta\in G}$ for all $m\in M$; hence, if $m\in M$, then $f(m)\in P_m(X_{m,\beta})$ for some $\beta\in G$. It follows that the right member of the equality in the proposition contains the left member. The proposition follows.

Proposition II-5C

If P_m is extensive for all $m\in M$ and $\beta \subseteq W$, then $\beta \subseteq \prod\{X_{m,\beta}\}_{m\in M} \subseteq Q(\beta)$.

Proof: Assuming that P_m is extensive for all $m\in M$ and that $\beta \subseteq W$, it follows that $h(m)\in X_{m,\beta} \subseteq P_m(X_{m,\beta})$ for all $h\in\beta$ and all $m\in M$. Therefore, if $h\in\beta$, then $h\in \prod\{X_{m,\beta}\}_{m\in M} \subseteq \prod\{P_m(X_{m,\beta})\}_{m\in M} = Q(\beta)$. The proposition follows.

Corollary

Suppose P_m is extensive for all $m\in M$.

a) β is Q^{∞}-equivalent to $\prod\{X_{m,\beta}\}_{m\in M}$ for all $\beta \subseteq W$.

b) $Q^{\infty}(\beta) = \prod\{X_{m,Q^{\infty}(\beta)}\}_{m\in M}$ for all $\beta \subseteq W$.

Proof: Assuming that $\beta \subseteq W$, it follows from the proposition that if $n\in N$, then $Q^n(\beta) \subseteq \prod\{X_{m,Q^n(\beta)}\}_{m\in M} \subseteq Q^{n+1}(\beta)$ while $\beta \subseteq \prod\{X_{m,\beta}\}_{m\in M} \subseteq Q(\beta)$. Therefore,

$$\begin{aligned} Q^{\infty}(\beta) &= \cup\{\prod\{X_{m,Q^n(\beta)}\}_{m\in M}\}_{n\in N} \\ &= \prod\{\cup\{X_{m,Q^n(\beta)}\}_{n\in N}\}_{m\in M} \\ &= \prod\{X_{m,Q^{\infty}(\beta)}\}_{m\in M}. \end{aligned}$$

If follows, also, that $Q^{\infty}(\beta) = Q^{\infty}(\prod\{X_{m,\beta}\}_{m\in M})$. The corollary follows.

Proposition II-5D

a) $M_Q = \prod\{M_{P_m}\}_{m\in M}$.

b) $E_Q \subseteq \prod\{E_{P_m}\}_{m\in M}$.

c) $Q_{\alpha} = \prod\{(P_m)_{\alpha}\}_{m\in M}$.

d) If P_m is monotone for all $m\in M$, then $Q^{n+1} \subseteq \prod\{(P_m)^{n+1}\}_{m\in M}$ for all $n\in N$.

Proof: We assume that $\beta \subseteq W$.

To prove part a) is suffices to show that $M_Q(\beta) = (\prod\{M_{P_m}\}_{m\in M})(\beta)$. Indeed, (1), (2), (3), (4) and (5) are equivalent:

(1) $f\in M_Q(\beta)$;

(2) There is a $\mu \subseteq \beta$ such that $f\in Q(\mu)$;

(3) There is a $\mu \subseteq \beta$ such that $f(m)\in P_m(X_{m,\mu})$ for all $m\in M$;

(4) If $m\in M$, then $f(m)\in M_{P_m}(X_{m,\beta})$;

(5) $f \in (\prod\{M_{P_m}\}_{m\in M})(\beta)$.

Part a) follows.

To prove part b) it suffices to show that $E_Q(\beta) \subseteq (\prod\{E_{P_m}\}_{m\in M})(\beta)$. Indeed, (6) implies (7) while (7), (8) and (9) are equivalent:

(6) $f\in E_Q(\beta)$;

(7) If $m\in M$, then $f(m)\in[X_{m,\beta} \cup P_m(X_{m,\beta})]$;

(8) If $m\in M$, then $f(m)\in E_{P_m}(X_{m,\beta})$;

(9) $f\in (\prod\{E_{P_m}\}_{m\in M})(\beta)$.

Part b) follows.

To prove part c) one modifies (2) and (3) above by replacing the phrase '$\mu \subseteq \beta$' with '$\mu \subseteq \beta$ and $|\mu| < \alpha$'.

To prove part d) it suffices to show that if P_m is monotone for all $m\in M$, then $Q^n(\beta) \subseteq (\prod\{(P_m)^n\}_{m\in M})(\beta)$ for all $n\in N$ such that $n > 1$. We assume that P_m is monotone for all $m\in M$ while $n\in N$ such that $Q^n \subseteq \prod\{(P_m)^n\}_{m\in M}$. It follows that (10) and (11) are equivalent while (11) implies (12):

(10) $f\in Q^{n+1}(\beta)$;

(11) If $m\in M$, then $f(m)\in P_m(X_{m,Q^n(\beta)})$;

(12) If $m\in M$, then $f(m)\in P_m(X_{m,R(\beta)})$, $R = \prod\{(P_m)^n\}_{m\in M}$.

Also, $h\in R(\beta)$ if and only if $h(m)\in(P_m)^n(X_{m,\beta})$ for all $m\in M$. It follows that if $m\in M$, then $f(m)\in P_m([P_m]^n(X_{m,\beta})) = (P_m)^{n+1}(X_{m,\beta})$ if (12) holds. Therefore, if $f\in Q^{n+1}(\beta)$, then $f\in (\prod\{(P_m)^{n+1}\}_{m\in M})(\beta)$. Part d) follows. The proposition follows.

Corollary

If P_m is monotone for all $m\in M$, then $Q^\infty \subseteq \prod\{(P_m)^\infty\}_{m\in M}$.

Proof: If P_m is monotone for all $m\in M$, then it follows from part d) of the proposition, the definition of Q and Proposition II-5B that the inclusion in the corollary holds.

Proposition II-5E

Suppose that $(P_m)^n$ commutes with $(P_m)^\infty$ for all $n\in N$ and all $m\in M$.

a) $Q \cup [\sqcap\sqcap\{[(P_m)^\infty]^2\}_{m\in M}] \subseteq \sqcap\sqcap\{(P_m)^\infty\}_{m\in M}$.

b) If P_m is monotone for all $m\in M$, then

$$[\sqcap\sqcap\{(P_m)^\infty\}_{m\in M}]^{n+1} \subseteq [\sqcap\sqcap\{(P_m)^\infty\}_{m\in M}]^n \text{ for all } n\in N.$$

Proof: It follows from Proposition II-1E that $P_m \cup [(P_m)^\infty] = (P_m)^\infty$ for all $m\in M$. Therefore, it follows that

$$\sqcap\sqcap\{(P_m)^\infty\}_{m\in M} = \sqcap\sqcap\{P_m \quad [(P_m)^\infty]^2\}_{m\in M}$$

$$\supseteq [\sqcap\sqcap\{P_m\}_{m\in M}] \cup \sqcap\sqcap\{[(P_m)^\infty]^2\}_{m\in M}.$$

It follows from part d) of Proposition II-5D that $\sqcap\sqcap\{[(P_m)^\infty]^2\}_{m\in M}$ contains $[\sqcap\sqcap\{(P_m)^\infty\}_{m\in M}]^2$ if P_m is monotone for all $m\in M$ [since $(P_m)^\infty$ will be monotone for all $m\in M$]. The proposition follows.

Corollary

If $(P_m)^n$ commutes with $(P_m)^\infty$ for all $n\in N$ and all $m\in M$ while P_m is monotone for all $m\in M$, then $[\sqcap\sqcap\{(P_m)^\infty\}_{m\in M}]^\infty = \sqcap\sqcap\{(P_m)^\infty\}_{m\in M}$.

Proof: The corollary is an immediate consequence of part b) of the proposition.

Theorem II-5F

a) Q inherits monotonicity, extensiveness, α-character, the property of satisfying S0 and the property of satisfying S5 from the members of $\{P_m\}_{m\in M}$.

b) If P_m is monotone for all $m\in M$, then Q inherits the exchange property from the members of $\{P_m\}_{m\in M}$.

c) If P_m is monotone for all $m\in M$, then $Q^2 \subseteq Q$.

Proof: Excluding the cases of the properties S0 and S5, part a) follows from Proposition II-5C and parts a) and c) of Proposition II-5D. We assume that P_m satisfies S0 for all $m\epsilon M$ while G is an increasing chain of subsets of W. Then $X_{m,\cup G} = \cup\{X_{m,\beta}\}_{\beta\epsilon G}$ for all $m\epsilon M$ and $\{X_{m,\beta}\}_{\beta\epsilon G}$ is an increasing chain of subsets of V_m for all $m\epsilon M$. Since P_m satisfies S0 for all $m\epsilon M$, it follows from Proposition II-5A and Proposition II-5B that

$$
\begin{aligned}
Q(\cup G) &= \prod\{P_m(X_{m,\cup G})\}_{m\epsilon M} \\
&= \prod\{P_m(\cup\{X_{m,\beta}\}_{\beta\epsilon G})\}_{m\epsilon M} \\
&\subseteq \prod\{\cup\{P_m(X_{m,\beta})\}_{m\epsilon M}\}_{\beta\epsilon G} \\
&= \cup\{\prod\{P_m(X_{m,\beta})\}_{m\epsilon M}\}_{\beta\epsilon G} \\
&= \cup\{Q(\beta)\}_{\beta\epsilon G}.
\end{aligned}
$$

It follows that Q satisfies S0. Furthermore, we assume that P_m satisfies S5 for all $m\epsilon M$ while β and μ are Q-equivalent subsets of W and $f\epsilon Q(\beta)$. Then $f\epsilon Q(\mu)$; hence, $f(m)\epsilon P_m(X_{m,\mu})$ for all $m\epsilon M$. If there is an element m of M such that $P_m(X_{m,\beta}) - P_m(X_{m,\mu})$ contains an element x, one can choose $g\epsilon W$ such that $g(m) = x$ and deduce that $g\epsilon[Q(\beta)-Q(\mu)]$; hence, β and μ are not Q-equivalent. It follows that $P_m(X_{m,\beta}) = P_m(X_{m,\mu})$ for all $m\epsilon M$. Since P_m satisfies S5 for all $m\epsilon M$, we choose $\mu_m \subseteq \mu$ such that $|\mu_m| \leq |\beta|$ and $f(m)\epsilon P_m(X_{m,\mu_m})$ for all $m\epsilon M$. Then $f(m)\epsilon \cup\{P_m(X_{m,\beta_1})\}_{\beta_1\subseteq\mu, |\beta_1|\leq|\beta|}$ for all $m\epsilon M$. It follows that

$$
\begin{aligned}
f &\epsilon \prod\{\cup\{P_m(X_{m,\beta_1})\}_{\beta_1\subseteq\mu,|\beta_1|\leq|\beta|}\}_{m\epsilon M} \\
&= \cup\{\prod\{P_m(X_{m,\beta_1})\}_{m\epsilon M}\}_{\beta_1\subseteq\mu,|\beta_1|\leq|\beta|} \quad \text{[Proposition II-5B]} \\
&= \cup\{Q(\beta_1)\}_{\beta_1\subseteq\mu,|\beta_1|\leq|\beta|} \quad \text{[Proposition II-5A]}
\end{aligned}
$$

It follows that $f \in Q(\beta_1)$ for some $\beta_1 \subseteq \mu$ such that $|\beta_1| \leq |\beta|$. Therefore, Q satisfies S5. This proves part a).

We assume that P_m is monotone for all $m \in M$. We suppose that P_m has the exchange property for all $m \in M$ while $\beta \subseteq W$, $g \in W$ and $f \in Q(\beta \cup \{g\})$. We let $M_1 = \{m \in M: f(m) \notin P_m(X_{m,\beta})\}$. Then $M_1 \neq \phi$. Moreover, $f(m) \notin P_m(X_{m,\beta})$ for all $m \in M_1$ while $f(m) \in P_m(X_{m,\beta} \cup \{g(m)\})$ for all $m \in M$. Therefore, since P_m has the exchange property for all $m \in M$, it follows that $g(m) \in P_m(X_{m,\beta} \cup \{f(m)\})$ for all $m \in [M - M_1]$. It follows that $g(m) \in P_m(X_{m,\beta} \cup \{f(m)\})$ for all $m \in M$. Therefore, $g \in Q(\beta \cup \{f\})$. It follows that Q has the exchange property. Part b) follows. Also, if P_m is idempotent for all $m \in M$, then it follows from part d) of Proposition II-5D that $Q^2 \subseteq \prod\{(P_m)^2\}_{m \in M} = \prod\{P_m\}_{m \in M} = Q$. Part c) follows. The theorem is proved.

<u>Corollary 1</u>

If $\{P_m\}_{m \in M}$ is a family of closure structures, then Q is a closure structure.

<u>Proof:</u> If $\{P_m\}_{m \in M}$ is a family of closure structures, then it follows from part a) of the theorem that Q is monotone and extensive; it follows from part c) of the theorem that $Q^2 \subseteq Q$ while Q is extensive. The corollary follows.

<u>Corollary 2</u>

If $\{P_m\}_{m \in M}$ is a family of spans, then Q is a span.

<u>Proof:</u> One applies Corollary 1 above and notices from the theorem that Q inherits ω-character (that is, finite character) and the exchange property from the members of $\{P_m\}_{m \in M}$. The corollary follows.

Theorem II-50

Suppose that 1) - 5) are true:

1) $\cap\{P_r(\phi)\}_{r\epsilon M} \neq \phi$.

2) P_m is monotone for all $m\epsilon M$.

3) $B_m \subseteq V_m$ for all $m\epsilon M$.

4) $X \cup Y$ is P_m-equivalent to X for all $m\epsilon M$, all $X \subseteq V_m$ and all $Y \subseteq P_m(\phi)$.

5) $\beta_m = \{f\epsilon W:\ f(m)\epsilon B_m$ and $f(M-\{m\}) \subseteq \cap\{P_r(\phi)\}_{r\epsilon M}\}$ for all $m\epsilon M$.

a) If B_m is P_m-equivalent to V_m for all $m\epsilon M$, then $\cup\{\beta_m\}_{m\epsilon M}$ is a Q-generator of $\prod\{P_m(V_m)\}_{m\epsilon M}$.

b) If B_m is P_m-independent for all $m\epsilon M$, then $\cup\{\beta_m\}_{m\epsilon M}$ is Q-independent.

Proof: We assume that B_m is P_m-equivalent to V_m for all $m\epsilon M$. We let β denote $\cup\{\beta_m\}_{m\epsilon M}$. It follows that

$$h(m)\,\epsilon \begin{cases} B_m \text{ if } h\epsilon\beta_m \\ \cap\{P_r(\phi)\}_{r\epsilon M} \text{ if } h\notin\beta_m \end{cases} \qquad \text{for all } m\epsilon M.$$

Also, if $y\epsilon B_m$, then any function sending m onto y and $M-\{m\}$ onto some subset of non-empty $\cap\{P_r(\phi)\}_{r\epsilon M}$ is in β_m; hence, $B_m \subseteq X_{m,\beta} \subseteq B_m \cup [\cap\{P_r(\phi)\}_{r\epsilon M}]$. Therefore, since $B_m \cup [\cap\{P_r(\phi)\}_{r\epsilon M}]$ is P_m-equivalent to B_m, it follows that $X_{m,\beta}$ is P_m-equivalent to B_m. Therefore, we have the relations $Q(\beta) = \prod\{P_m(B_m)\}_{m\epsilon M} = \prod\{P_m(V_m)\}_{m\epsilon M}$. Part a) follows. To prove part b), we assume that B_m is P_m-independent for all $m\epsilon M$ while $g\epsilon\beta$. Then $Q(\beta-\{g\}) = \{f\epsilon W:\ f(n)\epsilon P_n(X_{n,\beta-\{g\}})\}$ for all $n\epsilon M$. Furthermore, if $n\epsilon M$, then $P_n(X_{n,\beta-\{g\}}) = P_n([B_n-\{g(n)\}] \cup [\cap\{P_r(\phi)\}_{r\epsilon M}] = P_n(B_n-\{g(n)\})$; hence, $g(n)\notin P_n(B_n-\{g(n)\})$ [since B_n is P_n-independent]. It follows that $g\notin Q(\beta-\{g\})$. Therefore, β is Q-independent. Part b) follows. The theorem is proved.

If $n \in N$, then <u>the nth projection</u>, $\tau_n: W \longrightarrow V_n$, is defined to be the map which sends $f \in W$ onto $f(n)$.

<u>Theorem II-5H</u>

a) If $V_m \neq \emptyset$ for all $m \in M$, then each projection τ_n $(n \in M)$ is surjective.

b) If $\beta \subseteq W$ such that $P_m(X_{m,\beta}) \neq \emptyset$ for all $m \in M$, then $\tau_m(Q(\beta)) = P_m(X_{m,\beta})$ for all $m \in M$.

c) If $n \in M$ and $X \subseteq V_n$, then $\tau_n^{-1}(P_n(X)) = Q(\{g \in W: f(n) \in X\})$.

<u>Proof:</u> We assume that $m \in M$. If $y \in V_m$, then there is an element f of W which sends m onto y; hence, $\tau_m(f) = y$. It follows that τ_m is surjective. Part a) follows. To prove part b), we assume, further, that $\beta \subseteq W$ such that $P_n(X_{n,\beta}) \neq \emptyset$ for all $n \in N$. It is clear that $\tau_m(Q(\beta)) \subseteq P_m(X_{m,\beta})$. If $y \in P_m(X_{m,\beta})$, then since $P_n(X_{n,\beta}) \neq \emptyset$ for all $n \in M$, there is an element f of W such that $f(n) \in P_n(X_{n,\beta})$ for all $n \in M$ while $f(m) = y$; hence, $\tau_m(f) = y$ and $f \in Q(\beta)$. It follows that $P_m(X_{m,\beta}) \subseteq \tau_m(Q(\beta))$; hence, $\tau_m(Q(\beta)) = P_m(X_{m,\beta})$. Part b) follows. To prove part c), we assume that $X \subseteq V_m$ and that $f \in \tau_m^{-1}(P_m(X))$. If $n \in N$ and β is the set of all $g \in W$ such that $g(n) \in X$, then $f(n) = \tau_n(f) \in P_n(X_{n,\beta})$. It follows that $f \in Q(\beta)$. Therefore, we have the relation $\tau_m^{-1}(P_m(X)) \subseteq Q(\beta)$. Now, we assume that $f \in Q(\beta)$. Then $f(n) \in P_n(X_{n,\beta})$ for all $n \in M$; hence, it follows that $f(m) \in P_m(X_{m,\beta})$. It follows that $f \in \tau_m^{-1}(P_m(X))$ [since $P_m(X_{m,\beta}) = P_m(X)$]. Therefore $Q(\beta) \subseteq \tau_m^{-1}(P_m(X))$. Part c) follow. The theorem is proved.

<u>Corollary 1</u>

Each projection $\tau_m: W \longrightarrow V_m$ is (Q, P_m)-continuous.

<u>Corollary 2</u>

If $\emptyset \neq P_m(\emptyset) \subseteq P_m(X)$ for all $m \in M$ and all $X \subseteq V_m$, then each projection $\tau_m: W \longrightarrow V_m$ is a (Q, P_m)-continuous (Q, P_m)-homomorphism.

Theorem II-5I

Suppose that $g: V \longrightarrow W$ is a map.

a) If $\phi \neq P_m(\phi) \subseteq P_m(X)$ for all $m\in M$ and all $X \subseteq V_m$, then the following are equivalent:

i) g is a (P,Q)-homomorphism;

ii) $\tau_m g$ is a (P,P_m)-homomorphism for all $m\in M$.

b) If $\cap\{P(B)\}_{B\in G}$ is P-closed for each family G of subsets of V, then the following are equivalent:

i) g is (P,Q)-continuous;

ii) $\tau_m g$ is (P,P_m)continuous for all $m\in M$.

Proof: We assume the hypothesis of part a). We note that if $X \subseteq V$ and $m\in M$, then $X_{m,g(X)} = \cup\{h(m)\}_{y\in g(X)} = \cup\{\tau_m(h)\}_{h\in g(X)} = (\tau_m g)(X)$. It follows from Corollary 2 to Theorem II-5H that $\tau_m(Q(g(X))) = P_m(X_{m,g(X)})$ for all $m\in M$ and all $X \subseteq V$. Therefore, $\tau_m(Q(g(X))) = P_m([\tau_m g](X))$ for all $m\in M$ and all $X \subseteq V$. The equivalence of i) and ii) under part a) follows. To prove part b), we assume the hypothesis of part b). It follows from Corollary 1 to Theorem II-5H and Proposition II-4A that i) under part b) implies ii) under part b). We suppose that ii) under part b) is true while $m\in M$ and $\beta \subseteq W$. It follows from Theorem II-5H that $\tau_m(Q(\beta)) = P_m(X_{m,\beta})$. Therefore, $Q(\beta) \subseteq \tau_m^{-1}(\tau_m(Q(\beta))) = \tau_m^{-1}(P_m(X_{m,\beta}))$. It follows that $g^{-1}(Q(\beta)) \subseteq g^{-1}(\tau_m^{-1}(P_m(X_{m,\beta}))) = (\tau_m g)^{-1}(P_m(X_{m,\beta}))$. Taking intersection over the index set M, we obtain the relation $g^{-1}(Q(\beta)) \subseteq \cap\{(\tau_m g)^{-1}(Y_m)\}_{m\in M}$, where $Y_m = P_m(X_{m,\beta})$ for all $m\in M$. Also, assuming that $x \in \cap\{(\tau_m g)^{-1}(Y_m)\}_{m\in M}$, it follows that $(\tau_m g)(x) \in P_m(X_{m,\beta})$ for all $m\in M$; hence, $[g(x)](m) \in P_m(X_{m,\beta})$ for all $m\in M$. It follows that $g(x) \in \prod\{P_m(X_{m,\beta}) = Q(\beta)$; hence, it follows that $x \in g^{-1}(Q(\beta))$. Therefore, $\cap\{(\tau_m g)^{-1}(P_m(X_{m,\beta}))\}_{m\in M} \subseteq g^{-1}(Q(\beta))$. Part b) follows. This completes a proof of the theorem.

Corollary 1

If P_m is monotone and $P_m(\phi) \neq \phi$ for all $m\in M$, then a map $g: V \longrightarrow W$ is a (P,Q)-homomorphism if and only if $\tau_m g$ is a (P,P_m)-homomorphism for all $m\in M$.

Corollary 2

If P is a closure structure, then a map $g: V \longrightarrow W$ is (P,Q)-continuous if and only if $\tau_m g$ is (P,P_m)-continuous for all $m\in M$.

Proofs: Corollary 1 is an immediate consequence of part a) of the theorem. To prove Corollary 2, it suffices to show that if G is any family of subsets of V, then $\cap\{P(B)\}_{B\in G}$ is P-closed for each closure structure P. We assume that P is a closure structure while G is any family of subsets of V. We let F be a family of subsets of V such that if $X \subseteq V$, then $X \subseteq Y$ for some $Y\in F$ and $P(X) = \cap\{Y\}_{Y\in F, X\subseteq Y}$. (See Theorem II-1N.) It follows that $T = \cap\{P(B)\}_{B\in G} = \cap\{P(P(B))\}_{B\in G} \supseteq P(\cap\{P(B)\}_{B\in G}) = \cap\{Y\}_{Y\in F, T\subseteq Y} \supseteq T$. Therefore, $\cap\{P(B)\}_{B\in G}$ is P-closed. The corollaries follow.

We assume that the family $\{(P_m,V_m)\}_{m\in M}$ satisfies the following condition: If $m\in M$, then m is a function from V to V_m. The evaluation map, $e: V \longrightarrow W$, [relative to $\{V_m\}_{m\in M}$] is defined as follows:

$$\text{If } x\in V, \text{ then } [e(x)](m) = m(x) \text{ for all } m\in M.$$

The following theorem includes formulae for calculating images of closed sets under e^{-1} and formulae for use in obtaining characterizations of continuity of e and the condition that e be a homomorphism. (See the corollaries to the theorem.)

Theorem II-5J

a) If $\beta \subseteq W$, then $e^{-1}(Q(\beta)) = \cap\{m^{-1}(P_m(X_{m,\beta}))\}_{m\in M}$.

b) If $m\in M$, $m(V) \subseteq P_m(V_m)$ and $A \subseteq V_m$, then

$$e^{-1}(Q(\{f\in W\colon f(n)\in A\})) = m^{-1}(P_m(A)).$$

c) If $X \subseteq V$ and $P_n(\phi) \neq \phi$ for all $n\in N$ and $m\in M$, then

i) $\tau_m(Q(e(X))) = P_m(m(X))$,

ii) $\tau_m(e(P(X))) = m(P(X))$.

Proof: If $\beta \subseteq W$, then

$$\begin{aligned} e^{-1}(Q(\beta)) &= \{x\in V\colon e(x)\in Q(\beta)\} \\ &= \{x\in V\colon [e(x)](m)\in P_m(X_{m,\beta}) \text{ for all } m\in M\} \\ &= \{x\in V\colon m(x)\in P_m(X_{m,\beta}) \text{ for all } m\in M\} \\ &= \{x\in V\colon x\in m^{-1}(P_m(X_{m,\beta})) \text{ for all } m\in M\}. \end{aligned}$$

Part a) follows. To prove part b), we assume that $n\in M$ and $A \subseteq V_n$ while $\beta = \{f\in W\colon f(n)\in A\}$. It follows from part a) that $e^{-1}(Q(\beta))$ is $\cap\{m^{-1}(P_m(X_{m,\beta}))\}_{m\in M}$. Also, $X_{n,\beta} = \cup\{h(n)\}_{h\in W, h(n)\in A}$; hence,

$$X_{n,\beta} = \begin{cases} V_m \text{ if } m \neq n \\ A \text{ if } m = n \end{cases}$$

Therefore, $\cap\{m^{-1}(P_m(X_{m,\beta}))\}_{m\in M} = n^{-1}(P_n(A)) \cap [\cap\{m^{-1}(P_m(V_m))\}_{m\in(M-\{n\})}]$. The latter set is $n^{-1}(P_n(A)) \cap V$ or $n^{-1}(P_n(A))$. Part b) follows. To prove part c), we assume that $X \subseteq V$, $m\in M$ and $\beta = \{e(x)\colon x\in X\}$ It follows that $m(X) = \cup\{m(x)\}_{x\in X} = \cup\{[e(x)](m)\}_{m\in M} = X_{m,\beta}$; it follows, also, that $(\tau_m e)(X) = \cup\{\tau_m(e(x))\}_{x\in X} = \cup\{[e(x)](m)\}_{x\in X} = \cup\{m(x)\}_{x\in X} = m(X)$. Therefore, $P_m(m(X)) = P_m(X_{m,\beta}) = \tau_m(Q(\{e(x)\colon x\in X\})) = \tau_m(Q(e(X)))$ and, also, $\tau_m(e(P(X))) = m(P(X))$. Part c) follows. The theorem is proved.

Corollary 1

If P is a closure structure, then the evaluation map, $e: V \longrightarrow W$, relative to $\{V_m\}_{m \in M}$ is (P,Q)-continuous if and only if m is (P,P_m)-continuous for all $m \in M$.

Corollary 2

If $\emptyset \neq P_m(\emptyset) \subseteq P_m(X)$ for all $m \in M$ and all $X \subseteq V_m$, then the evaluation map, $e: V \longrightarrow W$, relative to $\{V_m\}_{m \in M}$ is a (P,Q)-homomorphism if and only if m is a (P,P_m)-homomorphism for all $m \in M$.

Corollary 3

If P_m is monotone and $P_m(\emptyset) \neq \emptyset$ for all $m \in M$, then the evaluation map, $e: V \longrightarrow W$, relative to $\{V_m\}_{m \in M}$ is a (P,Q)-homomorphism if and only if m is a (P,P_m)-homomorphism for all $m \in M$.

Proofs: Corollary 1 follows from part a) of the theorem. Under the hypothesis of Corollary 2, it follows from part c) of the theorem that $Q(e(X)) = e(P(X))$ for all $X \subseteq V$ if and only if $\tau_m(Q(e(X))) = \tau_m(e(P(X))$ for all $m \in M$ and all $X \subseteq V$, while $\tau_m(Q(e(X))) = P_m(m(X))$ and $\tau_m(e(P(X))) = m(P(X))$. Corollary 2 follows. Corollary 3 follows from Corollary 2. The proof of the corollaries is complete.

III. APPLICATIONS

1. Universal Algebra

In [1], p.132 and [8], p.16 the notion of <u>algebra</u> is defined as follows:

> $(V,\{F_j\}_{j\in J})$ is an algebra of similarity type μ if and only if V and J are non-empty sets, $\mu\in(\mathrm{Ord})^J$ [with Ord being the class of all ordinal numbers] and F_j is a map $V^{\mu(j)}\longrightarrow V$ for all $j\in J$.

In [1], p.133 and [8], pp.28-29 <u>the subalgebras of an algebra</u> $(V,\{F_j\}_{j\in J})$ of similarity type μ are defined to be precisely the class of all algebras $(X,\{F_j|_{X^{\mu(j)}}\}_{j\in J})$ of similarity type μ with $X\subseteq V$.

Henceforth, it is assumed that $(V,\{F_j\}_{j\in J})$ is an algebra of similarity type μ. We shall say that "X is a subalgebra of V" and will mean that $(X,\{F_j|_{X^{\mu(j)}}\}_{j\in J})$ is a subalgebra of $(V,\{F_j\}_{j\in J})$. We agree that <u>the subalgebra structures in V</u> are precisely those structures Q in V such that $F_j([Q(X)]^{\mu(j)})\subseteq Q(X)$ for all $j\in J$ and all $X\subseteq V$.

We let S denote the structure in V which sends each subset X of V onto the intersection of all $Y\subseteq V$ such that Y is a subalgebra of V and $X\subseteq Y$. We let P_j denote the structure in V which sends each subset X of V onto $\{F_j(x)\colon x\in X^{\mu(j)}\}$ for all $j\in J$. It follows that P_j is monotone for all $j\in J$. We let $P=\cup\{P_j\}_{j\in J}$, so that P is monotone. [See part b) of Proposition II-1A.] According to [8], pp.29-30, S is both a closure structure and a subalgebra structure. Therefore, it follows that

$$P(X) = \cup\{P_j(X)\}_{j\in J} \subseteq \cup\{P_j(S(X))\}_{j\in J} = \cup\{F_j([S(X)]^{\mu(j)})\}_{j\in J} \subseteq S(X).$$

Therefore, $P \subseteq S$; hence, since S is extensive and idempotent, it follows that $(E_P)^\infty \subseteq S$. Assuming that $j\in J$, $X \subseteq V$ and $x\in P_j(X)$, then $x\in F_j(X^{\mu(j)})$. We choose $y\in X^{\mu(j)}$ and let Y be the image of y. It follows that $F_j(y)\in P_j(Y)$ while $|Y| \leq |\mu(j)|$ and $Y \subseteq X$. This proves the following propositions:

<u>Proposition III-1A</u>

$(E_P)^\infty \subseteq S$.

<u>Proposition III-1B</u>

a) If $j\in J$ and $|\mu(j)| < \alpha$, then P_j has α-character.

b) If $|\mu(j)| < \alpha$ for all $j\in J$, then P has α-character.

Since $F_j(X^{\mu(j)}) = P_j(X)$ for all $j\in J$ and all $X \subseteq V$, we have the following characterization of subalgebra:

<u>Proposition III-1C</u>

If $X \subseteq V$, then X is a subalgebra of V if and only if $P(X) \subseteq X$.

<u>Corollary 1</u>

If $j\in J$, then P_j is a subalgebra structure if and only $PP_j \subseteq P_j$.

<u>Proof:</u> If $j\in J$, then P_j is a subalgebra structure if and only if $P_i(P_j(X)) \subseteq P_j(X)$ for each $i\in J$ and each $X \subseteq V$, while $P_i(P_j(X)) \subseteq P_j(X)$ for each $i\in J$ and $X \subseteq V$ if and only if $P(P_j(X)) = \cup\{P_i(P_j(X))\}_{i\in J} \subseteq P_j(X)$ for each $X \subseteq V$. The corollary follows.

<u>Corollary 2</u>

P^∞ is a subalgebra structure if and only if $(P^\infty)^2 \subseteq P^\infty$.

<u>Proof:</u> Since $(P^\infty)^2 = \cup\{P^n P^\infty\}_{n\in N}$, it follows that $(P^\infty)^2 \subseteq P^\infty$ if and

only if $P^n P^\infty \subseteq P^\infty$ for each $n \in N$. If P^∞ is a subalgebra structure, then $P^n P^\infty \subseteq P^\infty$ for all $n \in N$; hence, $(P^\infty)^2 \subseteq P^\infty$. The corollary follows.

Theorem III-1D

$(E_P)^\infty$ is a subalgebra structure if and only if $(E_P)^\infty = S$.

Proof: Since S is a subalgebra structure, then $(E_P)^\infty$ is a subalgebra structure if $(E_P)^\infty = S$. We assume that $(E_P)^\infty$ is a subalgebra structure. Since E_P is extensive, then $(E_P)^\infty(X)$ is a subalgebra of V which contains X; hence, it follows that $S(X) \subseteq (E_P)^\infty(X)$ while, already, $(E_P)^\infty(X) \subseteq S(X)$ (for all $X \subseteq V$). It follows that $(E_P)^\infty = S$. The theorem follows.

Proposition III-1E

If P satisfies SO, then $(E_P)^\infty$ is a subalgebra structure.

Proof: We assume that P satisfies SO while $X \subseteq V$. Since E_P is extensive and monotone, it follows that $\{(E_P)^n(X)\}_{n \in (N \cup \{0\})}$ is an increasing chain of subsets of V. Therefore, since $P \subseteq E_P$ and P satisfies SO, it follows that

$$\begin{aligned} P([E_P]^\infty(X)) &\subseteq E_P(\cup\{(E_P)^n(X)\}_{n \in (N \cup \{0\})}) \\ &\subseteq \cup\{(E_P)^n(X)\}_{n \in N} \\ &= (E_P)^\infty(X). \end{aligned}$$

The proposition follows.

Corollary 1

If $|\mu(j)| < \alpha$ for all $j \in J$ while the following condition is satisfied:

If G is an increasing chain of subsets of V and H is a subchain of G such that $|H| < \alpha$, then $P(\cup H) \subseteq \cup\{P(Z)\}_{Z \in G}$.

then $(E_P)^\infty$ is a subalgebra structure.

Corollary 2

If $|\mu(j)|$ is finite for all $j\varepsilon J$, then $(E_p)^{\infty}$ is a subalgebra structure.

Proofs: Apply part b) of Theorem II-2C and Corollary 2 to Theorem II-2C.

The notion of algebra morphism (or algebra homomorphism) is defined in [1], p.134 and [8], p.20 as follows:

Assuming that $(W,\{G_j\}_{j\varepsilon J})$ is an algebra of similarity type μ, a map $g:V\longrightarrow W$ is an algebra morphism if and only if $g(F_j(x) = G_j(gx)$ for all $j\varepsilon J$ and all $x\varepsilon V^{\mu(j)}$.

We begin with a set S and a map $g:S\longrightarrow V$ sending S into $F_j(V^{\mu(j)})$ for all $j\varepsilon J$. We construct an algebra $(S,\{G_j\}_{j\varepsilon J})$ of similarity type μ such that g is an algebra morphism. It suffices to require that $\{G_j\}_{j\varepsilon J}$ be a family of maps $S^{\mu(j)}\longrightarrow S$ such that $G_j(x)\varepsilon g^{-1}(F_j(gx))$ for all $j\varepsilon J$ and all $x\varepsilon S^{\mu(j)}$. We let $Q_j(X) = \{G_j(x)\colon x\varepsilon X^{\mu(j)}\}$ for all $X\subseteq S$ and all $j\varepsilon J$. We let $Q = \cup\{Q_j\}_{j\varepsilon J}$. If $j\varepsilon J$ and $X\subseteq S$, then it follows that $g(Q_j(X)) = g(G_j(X^{\mu(j)})) = F_j([g(X)]^{\mu(j)}) = P_j(g(X))$; hence, it follows that $g(Q(X)) = P(g(X))$. This proves the following proposition:

Proposition III-1F

Every map $g:S\longrightarrow V$ such that $g(S)\subseteq F_j(V^{\mu(j)})$ for all $j\varepsilon J$ induces an algebra $(S,\{G_j\}_{j\varepsilon J})$ of similarity type μ such that g is an algebra morphism, a (Q_j,P_j)-homomorphism and a (Q,P)-homomorphism, where $Q_j(X) = \{G_j(x)\colon x\varepsilon X^{\mu(j)}\}$ for all $X\subseteq S$ and all $j\varepsilon J$ while $Q = \cup\{Q_j\}_{j\varepsilon J}$.

We shall refer to $[(S,\{G_j\}_{j\varepsilon J}),\{Q_j\}_{j\varepsilon J}]$ as "an algebra system induced by the map $g:S\longrightarrow V$".

Continuing with the map $g:S\longrightarrow W$, we consider the map i_g from the factor space S_g modulo g to V defined by the relation $i_g(g^{-1}(y)) = y$. We obtain an algebra system $[(S_g,\{H_j\}_{j\in J}),\{T_j\}_{j\in J}]$ incuded by i_g. If $\beta \subseteq S_g$, then $i_g(T_j(\beta)) = P_j(i_g(\beta)) = i_g([P_j]_g(\beta))$ [since i_g is a $([P_j]_g,P_j)$-homomorphism, according to Proposition II-4C]. Therefore, it follows that $T_j(\beta) = [P_j]_g(\beta)$ [since i_g is injective]. It follows that $T_j = [P_j]_g$.

We consider the canonical projection $\tau_g: S \longrightarrow S_g$ and obtain an algebra system $[(S,\{\bar{G}_j\}_{j\in J}),\{\bar{Q}_j\}_{j\in J}]$ induced by τ_g. It follows that if $X \subseteq S$, then $\tau_g(\bar{Q}_j(X)) = [P_j]_g(\tau_g(X)) = \tau_g(Q_j(X))$ for all $j\in J$. Upon application of Proposition III-1F, we obtain the following proposition:

Proposition III-1G

Suppose that S is a set and $g: S \longrightarrow V$ is a map.

$$\begin{tikzcd} S \arrow[rr, "g"] \arrow[dr, "\tau_g"'] & & V \\ & S_g \arrow[ur, "i_g"'] & \end{tikzcd}$$

a) Every algebra system induced by the map $i_g: S_g \longrightarrow V$ is the factor algebra system $[(S,\{G_j\}_{j\in J}),\{(P_j)_g\}_{j\in J}]$ modulo g.

b) If $[(S,\{G_j\}_{j\in J}),\{Q_j\}_{j\in J}]$ is an algebra system induced by g, then the diagram above is a commutative diagram of algebra morphisms and one of continuous structure homomorphisms.

c) If $[(S,\{G_j\}_{j\in J}),\{Q_j\}_{j\in J}]$ and $[(S,\{\bar{G}_j\}_{j\in J}),\{\bar{Q}_j\}_{j\in J}]$ are algebra systems induced by g, then $\tau_g(Q_j(X)) = \tau_g(\bar{Q}_j(X))$ for all $j\in J$ and all $X \subseteq S$.

We assume that $\{(V_m,\{F_{j,m}\}_{j\in J})\}_{m\in M}$ is a non-empty family of algebras of similarity type μ. We let W denote the Cartesian product of $\{V_m\}_{m\in M}$. We suppose that $j\in J$ and $x\in W^{\mu(j)}$. Then $x(i)\in W$ for all i such that $1 \leq i \leq |\mu(j)|$; hence, $[x(i)](m)\in V_m$ for all i such that $1 \leq i$ and

$i \leq |\mu(j)|$ and all $m \in M$. We let $y_{m,x}$ be the member of $(V_m)^{\mu(j)}$ which sends each i such that $1 \leq i \leq |\mu(j)|$ onto $[x(i)](m)$ if $m \in M$. Then $F_{j,m}(y_{m,x}) \in V_m$ for all $m \in M$. Therefore, we define $(\prod\{F_{j,m}\}_{m \in M})(x)$ to be the element of W which sends each $m \in M$ onto $F_{j,m}(y_{m,x})$. Application of Proposition II-4B leads to the following proposition:

Proposition III-1H

The family of functions $\prod\{F_{j,m}\}_{m \in M}$ $(j \in J)$ defines an algebra of similarity type μ on $\prod\{V_m\}_{m \in M}$ with the structure defined by $\prod\{F_{j,m}\}_{m \in M}$ being $\prod\{P_{j,m}\}_{m \in M}$, where $P_{j,m}$ is the structure defined by $F_{j,m}$ for all $j \in J$ and all $m \in M$. Moreover,

$$\prod\{\cup\{P_{j,m}\}_{j \in J}\}_{m \in M} = \cup\{\prod\{P_{j,m}\}_{m \in M}\}_{j \in J}.$$

It follows from Theorem II-5F and Proposition III-1B that

Proposition III-1I

a) If $j \in J$ and $|\mu(j)| < \alpha$, then $\prod\{P_{j,m}\}_{m \in M}$ has α-character.

b) If $|\mu(j)| < \alpha$ for all $j \in J$, then $\prod\{\cup\{P_{j,m}\}_{j \in J}\}_{m \in M}$ has α-character.

It follows from Corollary 2 to Proposition III-1C that

Proposition III-1J

If $\cup\{P_{j,m}\}_{j \in J}$ is a subalgebra structure for all $m \in M$, then $\prod\{\cup\{P_{j,m}\}_{j \in J}\}_{m \in M}$ is a subalgebra structure.

We assume, now, that J is a ring. A left J-module W might be regarded as an algebra $(W, \{F_j\}_{j \in J})$ of similarity type μ with μ being constantly 1 on J. One defines $F_j(x)$ to be jx for all $x \in W$ and all $j \in J$.

We assume, further, that J has a multiplicative identity 1. We consider the class of all left J-modules which are non-trivial, faithful and unitary with respect to 1. If V is such a module, we let P_V be the the structure in V which sends ϕ onto ϕ and each non-empty subset X of V onto the set of all finite linear combinations of elements of X. We will prove the following proposition:

Proposition III-1K

J is a division ring if and only if there is a non-trivial, faithful and unitary J-module V such that P_V has the exchange property.

Proof: We assume that J is a division ring and let V be the left J-module of all (r,s) such that $r \in J$ and $s \in J$. Then V is a vector space; hence, P_V has the exchange property. The "only if" part follows. Now, we assume that there is a non-trivial, faithful and unitary left J-module V such that P_V has the exchange property while r is a non-zero element of J. Since V is non-trivial, we choose a non-zero element x of V. It follows that $rx \in P_V(\phi \cup \{x\})$ while $rx \notin P_V(\phi)$; hence, since P_V has the exchange property, it follows that $x \in P_V(\phi \cup \{rx\}) = P_V(\{rx\})$. Therefore, $x = s(rx)$ for some $s \in J$; hence, $(1-sr)x$ is the zero element of V while V is faithful. It follows that $1 = rs$. We repeat the argument with s instead of r and obtain an element t of J such that $1 = ts$. Then $ts = sr$ and, hence, it follows that $t = t1 = t(sr) = (ts)r = (sr)r = 1r = r$. Therefore, $sr = 1$ and $rs = 1$. It follows that s is the multiplicative inverse of r. Therefore, J is a division ring. This completes a proof of the proposition.

The reader might compart Proposition III-1H with the following result appearing in [7]:

A ring J is primitive if and only if there is a faithful irreducible J-module.

2. Additive Structures

Throughout this section it is assumed that c is a cardinal number exceeding 1. A structure P in a set V is <u>c-additive</u> if and only if the following condition is satisfied:

If H is a family of subsets of V such that
$|H| \leq c$, then $P(\cup H) = \cup\{P(Z)\}_{Z\in H}$.

We have the following propositions: (Proofs of Propositions III-2B III-2C and III-2D appear in [5].)

<u>Proposition III-2A</u>

Every c-additive structure is monotone.

<u>Proposition III-2B</u>

Every family of c-additive structures in a set has a c-additive union.

<u>Proposition III-2C</u>

If P and Q are c-additive structures in a set, then PQ is c-additive.

<u>Proposition III-2D</u>

If P is a c-additive structure, then M_P, E_P, P_α, P^n $(n\in N)$ and P^∞ are c-additive.

<u>Proposition III-2E</u>

If P is a c-additive structure, then $E_{P^\infty} = (E_P)^\infty$.

<u>Proposition III-2F</u>

If Q is a structure in a set W and $g: V \longrightarrow W$ is a map from a set V, then Q_g is c-additive if and only if Q is c-additive.

Proposition III-2G

A monotone structure P in a set V is c-additive if and only if the following condition is satisfied:

If H is a family of subsets of V such that $|H| \leq c$, then

$\cup\{P(Z)\}_{Z \subseteq \cup H, Z \notin H} \subseteq \cup\{P(Z)\}_{Z \in H}$.

Proposition III-2H

A monotone structure having α-character in a set V is c-additive if and only if the following condition is satisfied:

If H is a family of subsets of V such that $|H| \leq c$, then

$\cup\{P(Z)\}_{Z \subseteq \cup H, |Z| < \alpha, Z \notin H} \subseteq \cup\{P(Z)\}_{Z \in H}$.

Theorem III-2I

Every structure P in a set V contains a maximal c-additive structure in V, namely, the union of all c-additive structures Q such that $Q \subseteq P$.

Theorem III-2J

The Cartesian product of any family of c-additive structures is c-additive.

Proofs: Assuming that P is a c-additive structure in a set V and $X \subseteq Y \subseteq V$, it follows that $P(X) \subseteq P(X) \cup P(Y) = P(X \cup Y) = P(Y)$ [since $c \geq 2$]. Proposition III-2A follows.

Assuming that G is a family of c-additive structures in a set V and that H is a family of subsets of V such that $|H| \leq c$, it follows that

$$\begin{aligned}(\cup G)(\cup H) &= \cup\{P(\cup H)\}_{P \in G}\\ &= \cup\{\cup\{P(Z)\}_{Z \in H}\}_{P \in G}\\ &= \cup\{\cup\{P(Z)\}_{P \in G}\}_{Z \in H}\\ &= \cup\{(\cup G)(Z)\}_{Z \in H}.\end{aligned}$$

Proposition III-2B follows.

Assuming that P and Q are c-additive structures in a set V and that H is a family of subsets of V such that $|H| \leq c$, it follows that $(PQ)(\cup H) = P(\cup\{Q(Z)\}_{Z\in H}) = \cup\{P(Q(Z))\}_{Z\in H} = \cup\{(PQ)(Z)\}_{Z\in H}$. Proposition III-2C follows.

Assuming that P is a c-additive structure in a set V, then P is monotone [Proposition III-2A] and, hence, $P = M_P$. The c-additivity of P^n (with $n\in N$) follows from Proposition III-2C by induction. The c-additivity of P^ω follows from the c-additivity of the members of $\{P^n\}_{n\in N}$ and Proposition III-2B. We suppose that H is a family of subsets of V such that $|H| \leq c$. It follows that $E_P(\cup H) = (\cup H) \cup P(\cup H) = \cup\{Z \cup P(Z)\}_{Z\in H}$. The c-additivity of E_P follows. Furthermore,

$$\begin{aligned} P_\alpha(\cup H) &= \cup\{P(Z)\}_{Z\subseteq \cup H, |Z| < \alpha} \\ &= \cup\{P(\cup\{Z\cap X\}_{X\in H})\}_{Z\subseteq \cup H, |Z|< \alpha} \\ &= \cup\{\cup\{P(Z\cap X)\}_{X\in H}\}_{Z\subseteq \cup H, |Z| < \alpha} \\ &= \cup\{\cup\{P(Z\cap X)\}_{Z\subseteq \cup H, |Z| < \alpha}\}_{X\in H} \\ &= \cup\{\cup\{P(Z)\}_{Z\subseteq X, |Z|< \alpha}\}_{X\in H} \\ &= \cup\{P_\alpha(X)\}_{X\in H}. \end{aligned}$$

The c-additivity of P_α follows. This completes a proof of Proposition III-2D.

Since $c \geq 2$, it follows that $(E_P)^2(X) = X \cup P(X) \cup P(X \cup P(X))$ for all $X \subseteq V$ while $P(X \cup P(X)) = P(X) \cup P^2(X)$ if P is a c-additive structure in a set V. It follows by induction that if P is a c-additive structure in a set V and $X \subseteq V$, then $(E_P)^n(X) = \cup\{P^i(X)\}_{i\in(N\cup\{0\}, i\leq n}$; hence, it follows that $(E_P)^{\infty}(X) = \cup\{\cup\{P^i(X)\}_{0\leq i\leq n}\}_{n\in N} = \cup\{P^n(X)\}_{n\in(N\cup\{0\})} = E_{P^{\infty}}(X)$. Proposition III-2E follows.

Assuming that Q is a structure in a set W and $g: V \longrightarrow W$ is a map from a set V, if Q is c-additive and H is a family of subsets of V_g such that $|H| \leq c$, then $Q(g(\cup(\cup H))) = Q(\cup\{g(\cup\beta)\}_{\beta\in H}) = \cup\{Q(g(\cup\beta))\}_{\beta\in H}$ and, hence, $Q_g(\cup H) = \cup\{Q_g(\beta)\}_{\beta\in H}$. If Q_g is c-additive while H is a family of subsets of V such that $|H| \leq c$, then

$$Q_g(\{g^{-1}(y)\}_{y\in\cup H}) = Q_g(\cup\{g^{-1}(y)\}_{y\in X}\}_{X\in H}) = \cup\{Q_g(\{g^{-1}(y)\}_{y\in X}\}_{X\in H};$$

hence, it follows that

$$\{g^{-1}(y)\}_{y\in Q(g(\cup H))} = \cup\{\{g^{-1}(y)\}_{y\in Q(g(X))}\}_{X\in H} = \cup\{g^{-1}(y)\}_{y\in\cup\{Q(g(X))\}_{X\in H}}.$$

Therefore, the index sets, $Q(g(\cup H))$ and $\cup\{Q(g(X))\}_{X\in H}$ coincide. Proposition III-2F follows.

If P is a monotone structure in a set V while H is a family of subsets of V such that $|H| \leq c$, then $P = M_P$ and, hence, it follows that $P(\cup H) = \cup\{P(Z)\}_{Z\subseteq\cup H} = [\cup\{P(Z)\}_{Z\in H}]\cup[\cup\{P(Z)\}_{Z\subseteq\cup H, Z\notin H}]$. Proposition III-2G follows.

An argument similar to the one proving Proposition III-2G suffices as a proof of Proposition III-2H. Theorem III-2I follows from Proposition III-2B.

We assume that $\{(P_m, V_m)\}_{m\in M}$ is a non-empty family such that V_m is a set and P_m is a c-additive structure in V_m for all $m\in M$. We suppose that H is a family of subsets of $\prod\{V_m\}_{m\in M}$ such that $|H| \leq c$. We let $X_{m,\beta}$ denote $\cup\{h(m)\}_{h\in\beta}$ for all $\beta \subseteq \prod\{V_m\}_{m\in M}$. It follows that $X_{m,\cup H}$ is $\cup\{X_{m,\beta}\}_{\beta\in H}$; therefore, it follows from Propositions II-5A and II-5B that

$$\begin{aligned}Q(\cup H) &= \prod\{P_m(X_{m,\cup H})\}_{m\in M}\\ &= \prod\{P_m(\cup\{X_{m,\beta}\}_{\beta\in H})\}_{m\in M}\\ &= \prod\{\cup\{P_m(X_{m,\beta})\}_{\beta\in H}\}_{m\in M}\\ &= \cup\{\prod\{P_m(X_{m,\beta})\}_{m\in M}\}_{\beta\in H}\\ &= \cup\{Q(\beta)\}_{\beta\in H}, \text{ where } Q = \prod\{P_m\}_{m\in M}.\end{aligned}$$

Theorem III-3J follows.

A structure P in a set V is finitely additive if and only if the following condition is satisfied:

If H is a finite family of subsets of V, then

$P(\cup H) = \cup\{P(X)\}_{X\in H}$.

Proposition III-2K

A structure in a set is finitely additive if and only if it is 2-additive.

Proof: It is clear that finitely additive structures are 2-additive. We assume that P is a 2-additive structure in a set V while $n\in N$ and H is a family of subsets of V such that $|H| = n+1$. We let $X\in H$. Since P is 2-additive, it follows that $P([\cup(H-\{X\})]\cup\{X\}) = P(\cup[H-\{X\}])\cup P(X)$. It follows by induction that 2-additive structures are finitely additive.

A structure P in a set V is universally additive if and only if the following condition is satisfied:

If H is a family of subsets of V, then $P(\cup H) = \cup\{P(Z)\}_{Z\in H}$.

Proposition III-2L

Every universally additive structure is c-additive and satisfies SO.

Upon considering universal additivity instead of c-additivity, Proposition III-2B through III-2J remain valid. The modified propositions in the cases of III-2B, III-2C and III-2D (for α finite) were proved in [5] by Hammer.

If P is a structure in a set V, then A_P denotes the structure in V which sends each subset X of V onto $\cup\{P(\{x\})\}_{x\in X}$.

Proposition III-2M

Suppose that P is a structure in a set V.

a) $A_P(\phi) = \phi$.

b) $P_2(X) = A_P(X) \cup P(\phi)$ for all $X \subseteq V$.

c) A_P is universally additive and has α-character for all $\alpha \geq 2$.

d) A_P has the exchange property if and only if the following condition is satisfied:

If $y\in V$ and $x\in P(\{y\})$, then $y\in P(\{x\})$.

Proof: We assume that P is a structure in a set V. Since ϕ contains no elements, part a) follows. If $X \subseteq V$, then

$$P_2(X) = \cup\{P(Y)\}_{Y\subseteq X, |Y|<2} = P(\phi) \cup [\cup\{P(\{x\})\}_{x\in X}] = P(\phi) \cup A_P(X).$$

Part b) follows. If H is a family of subsets of V, then

$$A_P(\cup H) = \cup\{P(\{x\})\}_{x\in \cup H} = \cup\{\cup\{P(\{x\})\}_{x\in X}\}_{X\in H} = \cup\{A_P(X)\}_{X\in H}.$$

It follows that A_P is universally additive. It is clear that A_P has α-character for all $\alpha \geq 2$. Part c) follows. We assume that A_P has the exchange property while $y\in V$ and $x\in P(\{y\})$. It follows from part a) of this proposition and the definition of A_P that $x\in[A_P(\phi\cup\{y\}) - A_P(\phi)]$; hence, $y\in A_P(\phi\cup\{x\}) = P(\{x\})$. Therefore, if A_P has the exchange property, then the condition in d) is satisfied. Conversely, we assume that the condition in d) is satisfied while $X \subseteq V$, $y \in V$ and $x \in [A_P(X \cup \{y\}) - A_P(X)]$. We choose $z \in (X \cup \{y\})$ such that $x \in P(\{z\})$. Since $x \notin A_P(X)$, then $z \notin X$. It follows that $z = y$ and, hence, that $x \in P(\{y\})$. We deduce from the condition that $y \in P(\{x\}) \subseteq A_P(X \cup \{x\})$. It follows that A_P has the exchange property. Part d) follows. Now, we assume that $n \in N$ such that $(A_P)^n \subseteq A_{P^n}$. We suppose that $X \subseteq V$ and $x \in (A_P)^{n+1}(X)$. We choose $y \in (A_P)^n(X)$ such that $x \in P(\{y\})$. Then $y \in A_{P^n}(X)$. We choose $z \in X$ such that $y \in P^n(\{z\})$. It follows that $P(\{y\}) \subseteq P(P^n(\{z\})) = P^{n+1}(\{z\}) \subseteq A_{P^{n+1}}(X)$; hence, $x \in A_{P^{n+1}}(X)$. Part e) follows by induction. This completes a proof of the proposition.

Corollary 1

If P is a structure, then the following are equivalent:

i) P is universally additive;

ii) $P = A_P$;

iii) $P = P_2$

Proof: Assuming that P is a structure in a set V, it suffices to show that i) implies ii) and apply parts b) and c) of the proposition. But if P is universally additive and $X \subseteq V$, then it follows that $P(X) = P(\cup\{x\}_{x \in X}) = \cup\{P(\{x\})\}_{x \in X} = A_P(X)$. The corollary follows.

Corollary 2

If P is a structure, then A_P inherits extensiveness and idempotence from P.

Corollary 3

If P is a universally additive closure structure in a set V, then, in order that P satisfy S5 the following condition is sufficient:

If $y \in V$ and $x \in P(\{y\})$, then $y \in P(\{x\})$.

Proof: We assume that the condition is satisfied. It follows from Corollary 1 above and part c) of this proposition that P has the exchange property and finite character. Therefore, it follows from Corollary 2 to Theorem II-3L that any two P-independent P-equivalent subsets of V have the same cardinal number. It follows from Proposition III-2L that P satisfies S0. Therefore, it follows from Theorem II-3G that every subset of V includes a P-independent subset to which it is P-equivalent; hence, it follows from Proposition II-3K that P satisfies S5. The corollary follows.

Proposition III-2N

If P is a universally additive structure in a set V, then the following condition is satisfied:

If $y \in V$ and $x \in P(\{y\})$, then there is a maximal subset X of V such that $x \in X$ and $y \notin P(X)$.

Proof: We assume that P is a universally additive structure in a set V while $y \in V$ and $x \in P(\{y\})$. If $y \in P(\{x\})$, then we let $X = \{x\}$. If $y \notin P(\{x\})$, then since P satisfies S0 [Proposition III-3L] it follows that every increasing chain G of subsets Y of V such that $x \in Y$ and $y \notin P(Y)$ is such that $P(\cup G) \subseteq \cup\{P(Z)\}_{Z \in G}$. Therefore, if $y \notin P(\{x\})$, then there is a maximal subset X of V such that $y \notin P(X)$. The proposition follows.

Proposition III-2O

If P is a universally additive closure structure in a set V, then the following are equivalent:

i) P has the exchange property;

ii) If $X \subseteq V$, then $\{P(\{x\})\}_{x \in X}$ is a pairwise disjoint family whose union is P(X).

Proof: We assume that P is a universally additive closure structure in a set V. We suppose that i) is true while $X \subseteq V$, $x \in X$ and $y \in X$ such that $P(\{x\}) \cap P(\{y\}) \neq \phi$. Let $z \in [P(\{x\}) \cap P(\{y\})]$. Since P is monotone and idempotent, it follows that $P(\{z\}) \subseteq P(\{x\}) \cap P(\{y\})$. Since P has the exchange property, it follows from part d) of Proposition III-2M that $x \in P(\{z\})$ and $y \in P(\{z\})$; hence, since P is monotone and idempotent, it follows that $P(\{x\}) \cup P(\{y\}) \subseteq P(\{z\})$. Therefore, $P(\{x\}) = P(\{z\}) = P(\{y\})$. It follows that $\{P(\{x\})\}_{x \in X}$ is a pairwise disjoint family whose union is $P(X)$. (See Corollary 1 to Theorem III-2M.) This shows that i) implies ii). Conversely, we suppose that ii) is satisfied while $y \in V$ and $x \in P(\{y\})$. Then $\{P(\{z\})\}_{z \in P(\{y\})}$ is a pairwise disjoint family whose union is $P(\{y\})$. Also, $P(\{x\})$ and $P(\{y\})$ are members of the family $\{P(\{z\})\}_{z \in P(\{y\})}$ and $x \in [P(\{x\}) \cap P(\{y\})]$. It follows that $P(\{x\}) = P(\{y\})$ while $y \in P(\{y\})$. This shows that if $y \in V$ and $x \in P(\{y\})$, then $y \in P(\{x\})$. Therefore, it follows from part d) of Proposition III-2M that P has the exchange property; hence, ii) implies i). The proposition follows.

3. Generalized Compactness

We assume that α is a cardinal number at least 2 and that P is a structure in a set V. A non-empty family G of P-closed subsets of V has the α-intersection property if and only if every subfamily H of G such that $0 < |H| < \alpha$ has a non-empty intersection. If $X \subseteq V$, then (X,P) is α-compact if and only if the following condition is satisfied:

> If G is a non-empty family of P-closed subsets of V such that $X \subseteq \cup\{V-Y\}_{Y \in G}$, then there is a subfamily H of G such that $0 < |H| < \alpha$ and $X \subseteq \cup\{V-Y\}_{Y \in H}$.

Proposition III-3A

(V,P) is α-compact if and only if every non-empty family of P-closed subsets of V having the α-intersection property has a non-empty intersection.

Proof: We assume that (V,P) is α-compact. We suppose that G is a non-empty family of P-closed subsets of V such that $\cap G = \emptyset$. It follows that $V = V - \cap G = \cup\{V-Y\}_{Y\in G}$. We use the α-compactness of (V,P) to obtain a subfamily H of G such that $0 < |H| < \alpha$ and $V = \cup\{V-Y\}_{Y\in H} = V - \cap H$. It follows that $\cap H = \emptyset$. Therefore, G does not have the α-intersection property. It follows that if G is a non-empty family of P-closed subsets of V having the α-intersection property, then G has a non-empty intersection. The "only if" part of the proposition follows. Conversely, we assume that every non-empty family of P-closed subsets of V having the α-intersection property has a non-empty intersection. We suppose that G is a non-empty family of P-closed subsets of V such that $V = \cup\{V-Y\}_{Y\in G}$. Then $\cap G = \emptyset$, and it follows from the latter assumption that G does not have the α-intersection property. Therefore, we choose a subfamily H of G such that $0 < |H| < \alpha$ and $\cap H = \emptyset$. It follows that $V = V - \cap H = \cup\{V-Y\}_{Y\in H}$. Therefore, (V,P) is α-compact. The "if" part of the proposition follows. The proposition is proved.

Proposition III-3B

If $\alpha > 2$ and (V,P) is α-compact, then (X,P) is α-compact for every non-empty P-closed subset X of V.

Proof: We assume that $\alpha > 2$, that (V,P) is α-compact and that X is a non-empty subset of V which is P-closed. We suppose that G is a non-empty family of P-closed subsets of V such that $X \subseteq \cup\{V-Y\}_{Y\epsilon G}$. Then $G\cup\{X\}$ is a non-empty family of P-closed subsets of V and $V = \cup\{V-Y\}_{Y\epsilon(G\cup\{X\})}$. Since (V,P) is α-compact, then $V = \cup\{V-Y\}_{Y\epsilon H}$ for some subfamily H of $G\cup\{X\}$ such that $0 < |H| < \alpha$. If $X\epsilon H$, then $X \subseteq \cup\{V-Y\}_{Y\epsilon(H-\{X\})}$ while $H-\{X\}$ is a subfamily of G and $0 < |H| < \alpha$ [since X is non-empty]. If $X\notin H$, then H is a subfamily of G. It follows that (X,P) is α-compact. The proposition is proved.

Proposition III-3C

If $\{(V_m,P_m)\}_{m\epsilon M}$ is a non-empty family such that V_m is a set, P_m is a structure in V_m and (V_m,P_m) is α-compact for all $m\epsilon M$, then (W,Q) is α-compace, where $W = \prod\{V_m\}_{m\epsilon M}$ and $Q = \prod\{P_m\}_{m\epsilon M}$.

Proof: We assume the hypothesis of the proposition. We suppose that G is a non-empty family of subsets of W such that $\{Q(\beta)\}_{\beta\epsilon G}$ has the α-intersection property. If $m\epsilon M$ and $\beta\epsilon G$, we let $X_{m,\beta} = \cup\{h(m)\}_{h\epsilon\beta}$. We will show that $\{P_m(X_{m,\beta})\}_{\beta\epsilon G}$ has the α-intersection property for all $m\epsilon M$. We suppose, further, that $m\epsilon M$ and that H is a subfamily of G such that $0 < |H| < \alpha$. Since $\{Q(\beta)\}_{\beta\epsilon G}$ has the α-intersection property, then $\cap\{Q(\beta)\}_{\beta\epsilon H} \neq \phi$. It follows from Proposition II-5A that if $\beta\epsilon G$, then $Q(\beta) = \prod\{P_n(X_{n,\beta})\}_{n\epsilon N}$. Since $\cap\{Q(\beta)\}_{\beta\epsilon H} \neq \phi$, we choose $f\epsilon \cap\{Q(\beta)\}_{\beta\epsilon H}$ and deduce that $f(n)\epsilon P_n(X_{n,\beta})$ for all $n\epsilon N$ and all $\beta\epsilon H$. It follows that $f(m)\epsilon P_m(X_{m,\beta})$ for all $\beta\epsilon H$. Consequently, $\{P_m(X_{m,\beta})\}_{\beta\epsilon G}$ has the α-intersection property. Since m was taken as an arbitrary element of M, then $\{P_m(X_{m,\beta})\}_{\beta\epsilon G}$ has the α-intersection property for all $m\epsilon M$. Since (V_m,P_m) is α-compact for all $m\epsilon M$, it follows from Proposition III-3A that $\{P_m(X_{m,\beta})\}_{\beta\epsilon G}$ has a non-void intersection for all $m\epsilon M$. Therefore, we construct $g\epsilon W$ by choosing $g(m)\epsilon \cap\{P_m(X_{m,\beta})\}_{\beta\epsilon G}$ for all $m\epsilon M$. It follows that $g(m)\epsilon P_m(X_{m,\beta})$ for all $m\epsilon M$ and all $\beta\epsilon G$. Therefore, if $\beta\epsilon G$, then $g(m)\epsilon P_m(X_{m,\beta})$ for all $m\epsilon M$; hence, it follows from Proposition II-5A that $g\epsilon \prod\{P_m(X_{m,\beta})\}_{m\epsilon M} = Q(\beta)$ for all $\beta\epsilon G$. Consequently, $g\epsilon \cap\{Q(\beta)\}_{\beta\epsilon G}$. It follows that $\cap\{Q(\beta)\}_{\beta\epsilon G} \neq \phi$. Therefore, every non-empty family of Q-closed subsets of W having the α-intersection property has a non-empty intersection. It follows from Proposition III-3A that (W,Q) is α-compact. The proposition follows.

Proposition III-3D

If Q is a structure in a set W and $g: V \longrightarrow W$ is a (P,Q)-continuous map while (V,P) is α-compact, then $(g(V),Q)$ is α-compact.

Proof: We assume that Q is a structure in a set W and $g: V \longrightarrow W$ is a (P,Q)-continuous map while (V,P) is α-compact. We suppose that G is a non-empty family of Q-closed subsets of W such that $g(V) \subseteq \{W-Y\}_{Y\epsilon G}$. Then $\{g^{-1}(Y)\}_{Y\epsilon G}$ is a non-empty family of P-closed subsets of V such that $V = \cup\{V-g^{-1}(Y)\}_{Y\epsilon G}$. Since (V,P) is α-compact, then $V = \cup\{V-g^{-1}(Y)\}_{Y\epsilon H}$ for some subfamily H of G such that $0 < |H| < \alpha$. Therefore, $\{g^{-1}(Y)\}_{Y\epsilon H}$ has an empty intersection; hence, $g(V) = g(V) - g(\cup\{g^{-1}(Y)\}_{Y\epsilon H}) \subseteq \cup\{W-Y\}_{Y\epsilon H}$. It follows that $(g(V),Q)$ is α-compact. The proposition follows.

The reader might compare Proposition III-3E below with the following result characterizing compactness of the unit sphere in a normed linear space:

> The unit sphere in a normed linear space is compact if and only if the space is finite dimensional.

Proposition III-3E

If P is extensive and X is a non-empty P-independent subset of V, then (X,P) is α-compact if and only if $|X| < \alpha$.

Proof: We assume that P is extensive and that X is a non-empty P-independent subset of V. We suppose that (X,P) is α-compact. It follows from the P-independence of X that $X \subseteq \cup\{V-P(X-\{x\})\}_{x\epsilon X}$. Therefore, we choose $Y \subseteq X$ such that $0 < |Y| < \alpha$ and $X \subseteq \cup\{V-P(X-\{x\})\}_{x\epsilon Y}$. Since P is extensive, it follows that $V-P(X-\{x\}) \subseteq (V-X) \quad \{x\}$ for all $x\epsilon Y$; hence, it follows that $X \subseteq (V-X)\cup Y$. Therefore, $X \subseteq Y \subseteq X$. It follows that if (X,P) is α-compact, then $|X| < \alpha$. Conversely, we assume that $|X| < \alpha$. We suppose that G is a non-empty family of P-closed subsets of V such that $X \subseteq \cup\{V-Y\}_{Y\epsilon G}$. We choose a family $\{Y_x\}_{x\epsilon X}$ of elements of G such that $x\epsilon(V-Y_x)$ for all $x\epsilon X$. It follows that $0 < |X| = |\{Y_x\}_{x\epsilon X}| < \alpha$ while $X \subseteq \cup\{V-Y_x\}_{x\epsilon X}$ and $\{Y_x\}_{x\epsilon X}$ is a subfamily of G. Therefore, (X,P) is α-compact. It follows that if $|X| < \alpha$, then (X,P) is α-compact. The proposition follows.

IV. APPENDIX

1. Examples of Closure Structures

1A Let V be a non-empty set, $A \subseteq V$ and F be a non-empty family of subsets of V. It was shown in the proof of Theorem II-2D that the following defines a closure structure sending ϕ onto A:

$$\text{If } X \subseteq V, \text{ then } P(X) = \begin{cases} V \text{ if } Y \subseteq X-A \text{ for some } Y \epsilon F; \\ X \cup A \text{ otherwise} \end{cases}$$

Let μ be a cardinal number such that $\mu < |V|$.

a) If $F = \{X\}_{X \subseteq V, |X| = \mu}$, then P has the exchange property.

b) If $F \subseteq \{X\}_{X \subseteq V, |X| = \mu}$, then P satisfies S5.

c) If $\mu < \alpha$ and $F \subseteq \{X\}_{X \subseteq V, |X| = \mu}$, then P has α-character.

1B Let V be an infinite set and F be a non-empty family of infinite subsets of V. As was shown in the proof of Theorem II-2D, the following defines a closure structure having the exchange property:

$$\text{If } X \subseteq V, \text{ then } P(X) = \begin{cases} V \text{ if } |Y-X| < \infty \text{ and } |X-Y| \geq |Y-X| \text{ for some } Y \epsilon F \\ X \text{ otherwise} \end{cases}$$

1C Let $(V, \{F_j\}_{j \epsilon J})$ be an algebra of similarity type μ as defined in Section 1 of Chapter III. The structure sending each subset X of V onto the intersection of all subalgebras Y of V such that $X \subseteq Y$ is a closure structure.

1D Let V be a topological space. The topological closure operation on the subsets of V is a closure structure.

1E Let F be a ring with multiplicative identity 1 and P be a structure in an F-module V such that $P(\phi) \varepsilon \{\phi, \{0\}\}$, with $0 + x = x$ for all $x \varepsilon V$.

a) If $P(X)$ is the set of all finite linear combinations of elements of X (for all non-empty subsets X of V), then P is a closure structure having finite character and satisfying S0.

b) If $P(X)$ is the set of all finite linear combinations of elements of X with coefficients summing to 1 (for all non-empty subsets X of V), then P is a closure structure having finite character and satisfying S0.

c) If F is an ordered ring and $P(X)$ is the set of all finite linear combinations of elements of X with coefficients summing to 1 and all coefficients being at least the zero in F (for all non-empty subsets X of V), then P is a closure structure having finite character and satisfying S0; the structure defined in the same manner with respect to positive coefficients has the same properties.

d) In the event that F is a division ring, all structures defined in a), b) and c) above have the exchange property and satisfy S5.

1F Let V be either a real or complex inner product space. The structure sending each subset X of V onto the closed linear manifold generated by X is a closure structure having the exchange property and satisfying S5. However, such a structure does not have finite character if V is infinite dimensional.

1G Let V be the set of all non-negative real numbers. Let $P(\phi)$ be either ϕ or $\{0\}$. If X is a non-empty subset of V, we let $P(X)$ be the set of all $x \in V$ such that $0 \leq X \leq \text{lub}(X)$ if X is bounded; otherwise, we let $P(X)$ be V. Then P is a closure structure having the exchange property while P neither has finite character nor satisfies SO. Indeed, if $X_n = \{x \in V: 0 \leq x < 1-n^{-1}\}$ for all positive integers n exceeding 1, then $\{X_n\}$ is increasing, $2^{-1} \notin P(X)$ for all finite subsets X of X_2, $P(\cup\{X_n\})$ is $\{x \in V: 0 \leq x \leq 1\}$ and $\cup\{P(X_n)\} = \{x \in V: 0 \leq x < 1\}$.

1H Let V be a set. We consider a collection of subsets of V (to be referred to as "lines") subject to the following:

i) Each two elements of V belong to just one line.

ii) Each line contains at least two elements of V.

If $x \in V$ and $y \in V$, then we let $L(x,y) = \{x\}$ if $x = y$, and we let $L(x,y)$ be the line to which both x and y belong if $x \neq y$. We let $P(X)$ be the union of the class of all $L(x,y)$ such that $x \in X$ and $y \in X$ (for all subsets X of V). It follows that $P(\phi) = \phi$ while P is monotone and extensive and P has 3-character. Therefore, it follows from Proposition II-1D that P^n has 3^n-character for all $n \in N$; hence, P^∞ has finite character. Then it follows from Corollary 4 to Theorem II-1F and Corollary 2 to Theorem II-2C that P^∞ is a closure structure satisfying SO.

2 Miscellaneous Examples

2A Let P be a structure in a set V such that $|V| \geq 3$ and let a, b and c be distinct elements of V. Let Q be defined as follows:

$$\text{If } X \subseteq V, \text{ then } Q(X) = \begin{cases} \{a,b\} \text{ if } \phi \neq X \subseteq \{a,b\}; \\ P(X) \text{ otherwise} \end{cases}$$

Then Q is not monotone while Q satisfies any axiom (or S0) that P satisfies. (See the proof of Theorem II-2D.)

2B Let V be a vector space of dimension at least 2 over a division ring. Let $P(\phi)$ be either ϕ or $\{0\}$, with $0+x = x$ for all $x \in V$. If X is a non-empty subset of V, let P(X) be the set of all finite linear combinations of elements of X. Then P_3 is not idempotent while P_3 is monotone, is extensive, has the exchange property, has α-character for all $\alpha \geq 3$, satisfies S0 and satisfies S5. (See the proof of Theorem II-2D.)

2C Let V be a non-empty set. The structure sending each subset of V onto a fixed subset X of V is monotone, is idempotent, has the exchange property, has α-character for all $\alpha \geq 1$, satisfies S0 and satisfies S5. Such a structure is extensive if and only if the set X is V.

2D Let F be a ring containing an element p other than the zero of F, and let V be a left F-module. Let x_0 be the identity in the Abelian group V and $P(\phi)$ be either ϕ or $\{x_0\}$. If X is a non-empty subset of V, let P(X) be the set of all finite linear combinations of elements of X with coefficients summing to p. Then P is monotone, has finite character and satisfies S0. If X is a non-empty subset of V, then $P^n(X)$ is the set of all finite linear combinations of elements of X with coefficients summing to p^n for all $n \in N$. Therefore, if F has a multiplicative identity 1 and $p^n = 1$ for some $n \in N$, then P^n is a closure structure. According to Corollary 4 to Theorem II-1F, $(E_p)^\infty$ is idempotent.

If p is neither 1 (the multiplicative identity in F) nor -1, then P^n is not idempotent for all $n \in N$.

2E Let F be an ordered ring containing an element p other than the zero of F. Let P be defined as in Example 2D above with the additional requirement that all coefficients in linear combinations be positive. P has the same properties as the structure defined in Example 2D. The same is true if coefficients are taken as non-negative elements of F.

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