WORKFORCE SUPPLY AND FACILITY LOCATION

by

NATHAN CURTIS PALMER

A DISSERTATION

Submitted in partial fulfillment of the requirements for
the degree of Doctor of Philosophy in the
Department of Information Systems,
Statistics, and Management Science
in the Graduate School of
The University of Alabama

TUSCALOOSA, ALABAMA

2009

ABSTRACT

It is important for every company to minimize its costs. This includes labor costs. We develop mathematical models that allow a company to minimize its labor costs by deciding from where to hire workers and the amount that will be paid to those workers within a similar region. These decisions are particularly important when the company has multiple facilities that compete amongst themselves for labor resources. In areas that are experiencing economic growth or in developing countries labor resources are limited and labor decisions are critical.

With this motivation, this work investigates the labor and facility location decisions of a company that has decided to build many new facilities in close proximity to each other. One example is a large manufacturing firm that seeks simultaneously to locate a new assembly plant and supplier facilities. Concentrating all, or much, of the supply chain together will cause already limited labor resources to be depleted even further. Higher wages are paid and higher labor costs are incurred by the company, as a result. On the other hand, greater transportation costs are incurred as the distances between the plant and its suppliers increase. For each of the supply chain facilities, the location of the facility, the labor markets from which to hire workers, and the wages offered must be determined. While considering these decisions, another potential factor in choosing the location for each facility is the cost of the site. This dissertation introduces this real-world problem, formulates it mathematically, and provides managerial insights for companies faced with these decisions.

ACKNOWLEDGEMENTS

I take this opportunity to thank all those who have played such a big part in my completing this dissertation. First, I thank my family. I am especially grateful to my patient and wonderful wife, Rachel. She has put up with many late nights and long days of research. I also thank my children for their patience, prayers and encouragement throughout this process. I must also include my parents, Timothy and Karen Palmer, for the dedication and support to help me in all of my educational and other pursuits. Without each of these persons as well as my brothers and sisters I would not have accomplished this goal.

I must also thank my advisors and dissertation committee co-chairs, Dr. Charles Sox and Dr. John Mittenthal. I thank them for pushing me to improve in my ability to research and write. I thank them for the research ideas that have made my reaching this point possible. Thank you to each of the other committee members – Dr. Samuel N. Addy, Dr. Burcu B. Keskin, and Dr. Denise McManus. I especially appreciate Dr. Addy for providing me with an economics perspective and better understanding of the real applications of the problems we address in the dissertation. I thank Dr. Keskin for all her help on learning and using LaTeX. I also thank her for pushing me to be a better researcher. I am indebted to Dr. McManus for the time she spent with me and her words of encouragement.

I acknowledge the help of friends and colleagues throughout this process. I thank both the University of Alabama and Southern Utah University. I thank the Saccomanno Higher Education Foundation for their financial support.

Lastly, I thank my Father in Heaven for all of the blessings He has given me including His help in conducting this research.

CONTENTS

LIST OF TABLES

LIST OF FIGURES

1 INTRODUCTION

Many recent political and economic issues have involved, either directly or indirectly, the issue of labor costs incurred by companies. For many years there has been much debate about whether U.S. companies should outsource manufacturing and other business functions to other countries. One of the major reasons that is given for U.S. companies outsourcing is the cheapness of labor in countries such as China, India, and Mexico. The recent economic downturn has resulted in a government bailout of General Motors and other companies. Many people blamed the labor unions involved with these companies for at least a part of their economic struggles. Their reasoning was that the labor unions forced the companies to have labor costs that were unsustainable. Most recently, some Congressmen and U.S. citizens are arguing that a new health insurance system is needed for the country. One reason that is cited by proponents is the rising costs of health care. These rising costs have caused some employers to cutback or eliminate health care benefits for their employees in order to cut labor costs.

These issues highlight the importance of labor costs to companies and countries, as well as the importance of wages to employees. It is well-studied that manufacturing companies consider labor costs to be an important factor in choosing facility locations; see, for example, Tong and Walter (1980), Bartik (1985), Swamidass (1990), and Ulgado (1996). In the literature review that follows this introduction, we discuss some studies of important factors in facility location. Most of this research is done on a macro scale where companies consider the labor costs that would be incurred in different countries, states, or regions. However, the problem of a company minimizing its labor costs on a micro level is not as well studied.

One example of a micro level problem, that is also one of the problems considered in this dissertation, is that of a newly located facility deciding how many workers it will hire from the different labor markets in the area and how much it will pay those workers. Another example considers a facility that is choosing a facility location from a set of possible locations in the same state or region. Such a facility can consider the precise costs and benefits of each possible facility location, including labor costs, and choose the best one.

The decisions made by companies on the micro level are especially important when the availability of workers is limited. This is the case when the unemployment rate is low in the region. Another instance of limited worker availability occurs in developing countries where the amount of skilled labor is small. Sometimes the decision of a company to locate in a region causes the availability of workers to be limited.

In this dissertation, we examine two problems where a manufacturing company having multiple facilities makes labor decisions in order to minimize its total costs. Both of these problems are new to the literature. However, we do use concepts common to the competitive location literature in order to formulate the problems. In both problems, facilities belonging to the company compete against each other and with other facilities for workers.

A central element to both of the problems is measuring the attractiveness of an employer to potential workers. There are many things a facility can offer to attract workers. These include wages, benefits, days off, clean and positive work environments, and the possibility for advancement. Not all of these are easily measured and implemented into a mathematical model. In both problems we assume that the attractiveness of an employer to a potential worker is determined by the distance between the facility and the worker's labor market and the total amount of wages and benefits paid by the facility. The more attractive a facility is to workers from a market the more workers it hires from that market. In the first problem the location of each of the company's facilities is fixed. Thus, the only way a facility is able to attract more workers from a market is to increase the amount of wage and benefits offered to workers from that labor market. In the second problem, the company makes facility location and labor decisions.

In the first problem, we consider a company with multiple established facilities in the same region. The goal for the company is to minimize its labor costs, while determining how much each facility will pay workers and how many workers from each of the labor markets are hired to work at each facility. We refer to this problem as the "Workforce Supply Problem" (WSP).

A mathematical model that allows a company to optimize these decisions is presented. If all workers are hired from a single labor market, the model has a closed-form solution. If there are multiple labor markets from which workers are hired by the company, then it is shown that the Karush-Kuhn-Tucker conditions are sufficient and necessary for a solution to the model to be optimal. A reduced gradient algorithm is used to find the optimal solution to the model.

We analyze the sensitivity of the solution to the workforce supply problem to different parameters to obtain some managerial insights into the labor decisions. We also discuss the benefit that is gained from the facilities making coordinated labor decisions as opposed to each facility making its own labor decisions independently. Lastly, we discuss some real-world aspects of the labor decisions that are not addressed by the workforce supply problem and the direction that our future research might take to address these aspects.

For the second problem we address in this dissertation, we consider a company which must establish multiple facilities in the same region. It is assumed that some of the company's facilities are already located in the region. The rest of the company's facilities must be located at a site chosen from a set of potential locations.

Once again, the goal of the company is to lower its total costs. The total costs are composed of labor costs and fixed site costs. We refer to this problem as the "Workforce Supply and Facility Location Problem" (WSFLP). The labor costs incurred by the facilities in the WSFLP are the same as those incurred by facilities in the WSP. However, in this case, a facility is able to attract more workers from a market by increasing the amount of wage and benefits offered to workers from the labor market or by choosing to locate at a site that is closer to the market.

The fixed site costs incurred by a facility are composed of the cost of purchasing and preparing the site and the cost of transporting goods between the facility and the other company facilities already located in the region. The site purchase and preparation costs are not affected by the proximity to other company facilities. On the other hand, the transportation cost component of a facility's fixed site cost depends on the proximity of the site chosen to the company's facilities that are already located.

There is a tradeoff between the labor costs and the fixed site costs of a facility. Depending on the site purchase and preparation costs, the fixed site costs are likely to be lowest near other company facilities that are already located. However, choosing a site in close proximity to other facilities means that higher wages must be paid by the facilities in order to attract from the labor markets. This is because the competition for workers at nearby labor markets is high when many facilities are concentrated in one area. In order to decrease its labor costs, a facility it may choose a rural location. Bastian (2002) makes the argument that manufacturing firms are, and should be, considering rural areas in making location decisions. She points out that the workforce is one of the greatest benefits to locating in a rural areas. The workers in these areas are typically loyal, educated, and hard working. Other benefits include good road access, high-quality telecommunications, cheap land, lower crime rates, and high quality of life.

We develop a mathematical model to solve the WSFLP. For a given set of facility locations, the WSFLP reduces to the WSP. We propose two heuristics for solving the problem. In the first heuristic, we estimate the labor costs of each facility at each potential site. This reduces the problem to an assignment problem which is easily solved. This algorithm provides us with both a lower and upper bound on the optimal solution to the problem. The second algorithm is an application of tabu search. The third procedure we propose is a branch and bound technique for solving the problem to optimality.

None of these solution procedures are developed fully. We simply propose the methods and a plan for developing them more fully in the future. Some of the procedures will require experimentation to determine appropriate values for their parameters.

The major benefit of this study will be the development of solvable models that are useful to companies making decisions about hiring, wages, and possibly locating one or more facilities when faced with limited labor resources. These models are new to the operations management literature. One of the primary focuses of this research is to introduce these problems to the literature, propose mathematical models to solve them, and offer solution procedures and managerial insights.

This work is related to two widely studied problems. These are the competitive location problem and the facility location problem. The competitive location problem, sometimes referred to as "the maxcap problem," seeks to maximize the market share captured by a company. The model assumes that there are competitors outside of the decision making firm that are also competing for market share. Some models measure the market share in a dollar amount and allow for some costs to contribute to the amount of the market that is captured. The location decisions made in these problems become more complicated as the number of facilities increases. This is due to the competition that occurs between facilities that belong to the same company. The assumptions made about the attraction of customers to firms are key to these models. Many different attraction models have been developed. Some of these have been verified by gathering real data about consumer behavior. Many of the basic models presented in this area have been solved optimally. Other more complicated models require heuristics to generate feasible solutions.

The objective of the facility location problem is to minimize costs. Brandeau and Chiu (1989) describe these costs as a fixed investment cost and a distribution cost. There are other considerations that are not easily assigned a dollar value cost, but are important to the decision of facility locations. Such considerations are not easily included in mathematical models. Some examples include: quality of life for potential employees, "goodness-of-fit" with company strategy, and potential for growth (Chopra and Meindl, 2007). Flaherty (1998) also points out that firms sometimes choose a site based on its communications and technology infrastructure. Newman and Sullivan (1988) discuss the effect of business tax levels on facility location decisions.

In Chapter 2, the literature from these two areas is discussed. It is shown that there is a need for the models presented in this work. The Workforce Supply Problem is presented in Chapter 3. This chapter includes the presentation of the problem and its mathematical model as well as the solution procedures and managerial insights. The Workforce Supply and Facility Location Problem along with its mathematical model and proposed solution procedures are discussed in Chapter 4. Finally, in Chapter 5, we highlight and summarize the major contributions to the operations management literature that are made in this dissertation.

2 LITERATURE REVIEW

To our knowledge both the Workforce Supply Problem (WSP) and the Workforce Supply and Facility Location Problem (WSFLP), along with their mathematical models, solution procedures, and managerial insight, are new to the research literature. This does not mean that our research is not reliant on the work of others. Labor decision science and facility location analysis are areas of study covered in the fields of operations management, economics, and geography. Typically, operations managers develop models to optimize the location of new facilities. Economists differ in that they use models to find the equilibrium in location decisions made by competing firms.

For the most part, the foundation for this dissertation is found in the operations management literature. Two problems that have been studied extensively are the facility location problem and the competitive location problem. The facility location problem is one where the location of a facility is sought in order to minimize total costs. The total cost objectives are usually dominated by transportation costs. Nowhere, that we have found, do the models developed directly deal with the wages paid at the new facility. We use concepts from this area of research to model the WSFLP and develop its solution procedures.

In the competitive location problem, a set of retail facilities belonging to the same firm compete amongst themselves and other retailers to capture the greatest share of the market. The greater the market share the firm captures, the greater their revenue. The facilities belonging to the same firm are at the same time allies and competitors. Most often the decision that must be made by the firm is the choice of locations for the retail facilities. Sometimes it is assumed that the firm must also make some other decision that affects the way consumers decide between facilities for shopping. Even though in the WSP and the WFLP deal with manufacturing facilities and not retailers, the competitive location problem provides insight into assumptions and models for determining how "shoppers" in a market choose between facilities. In our problems, the “shoppers” are not a set of consumers with some amount of buying power. They are, however, a set of workers that decide which facility receives their labor and efforts in exchange for a wage and benefits. The facilities belonging to the same firm in our problems can also be viewed as being both competitors and allies.

There are also empirical studies done to determine the factors that are most important to firms making facility location decisions. We discuss some of these at the end of this literature review.

There is abundant literature on both the facility location and competitive location problems. We present some of this literature that is pertinent to our problems in this literature review. First, we present the facility location problem because the competitive location problem is really a special case of the general facility location problem.

2.1 The Facility Location Problem

Alfred Weber (1909) is credited as the first to consider the problem of optimizing the location of a facility. Over the many years since, the knowledge of the problem has been expanded by many researchers. Drezner (1995) is the editor of an excellent survey on the contributions to location analysis. The survey by Owen and Daskin (1998) gives a helpful background on facility location with a focus on the recent contributions in stochastic and dynamic location analysis. In their textbook, Chopra and Meindl (2007) provide discussion and models for many of the basic facility location problems. Aikens (1985) also provides a useful survey of the major facility location models. A brief taxonomy of location models is given by Daskin (2008) along with a detailed taxonomy of discrete models.

Francis et al. (1992) develop a classification scheme for facility location problems. They classify location problems on six factors: new facility characteristics, existing facility locations, new and existing facility interactions, solution space characteristics, distance measure, and the objective. The following paragraphs characterize the Workforce Supply and Facility Location Problem (WSFLP) presented in this research according to these factors.

However, these factors are not enough to classify the problem completely. Other factors found in the competitive location literature will also be used.

New Facility Characteristics: In the Workforce Supply and Facility Location Problem, we consider the location of multiple new facilities. It will be assumed that these facilities choose from a discrete set of potential sites. The location decisions are interdependent due to the competition that occurs between firms for limited labor resources. It is assumed that the computation time used to solve the problem to optimality increases as the number of new facilities to be located increases.

Existing Facility Locations: The models proposed in this research will consider the existing facility locations to be static, deterministic and located at a single point. New facilities being located interact with other new facilities and existing facilities.

New and Existing Facility Interactions: This work assumes that this interaction, while static and deterministic, is a function of the facilities' locations, and involves decision variables – specifically, the number of workers that each facility hires from each labor market. The competitive location problem is a specific area of facility location theory. It includes models where the interactions among the new facilities and between existing facilities are centered around consumers' attraction toward each facility. The models in this research explore similar pulls on labor markets. However, instead of modeling the consumers' attraction, this work explores potential workers' attraction toward places of employment. Competitive location is explored in greater detail in later paragraphs.

Solution Space Characteristics: The solution space of the Workforce Supply and Facility Location Problem is discrete. The models includes the use of binary variables to indicate whether or not a facility is assigned to a certain site. The result is a mixed integer nonlinear program. Bussieck and Pruessner (2003) explain that this type of model is difficult to solve because it has characteristics of both nonlinear programs and mixed integer programs. We propose a branch and bound algorithm for optimizing the problem as well as two heuristic solution approaches.

Distance Measure: Because of the discretized solution space employed in the Workforce Supply and Facility Location the distance measures used between facilities themselves and between facilities and labor markets is the actual distance on the road network. This same distance metric is used in the Workforce Supply Problem where all the locations of facilities and markets is known and fixed.

The Objective: The objective of traditional facility location models is to minimize a total cost function. Transportation costs incurred by either shipping out to customers from locations or shipping into distribution centers from suppliers are usually a large part of the total cost function. Some models seek to minimize the maximum cost between facilities. In contrast, the objective of competitive location problems is to maximize the market share captured by the retail firm across all of its facilities. The assumption is that maximizing market share is equivalent to maximizing profit. The function that measures the attraction felt by customers towards a facility is often included in these objective functions.

For the WSP the objective is simply to minimize total labor costs across all of a company's facilities. In the WSFLP the objective function is to minimize both the total labor costs and fixed site costs across each of the facilities belonging to the same firm. As noted in the introduction, these costs conflict with each other. Usually lowering the labor costs will increase the transportation costs and vice versa.

A more in-depth review of competitive location models is needed because they form the major foundation for the development of the mathematical models in this work. Plastria (2001) provides a review of competitive location models and optimization. He categorizes the competitive location models using many of the same factors as Francis et al. (1992), but includes two other factors more specific to competitive location analysis.

2.2 The Competitive Location Problem

Plastria (2001) lists the “ingredients” of competitive location problems as follows: features of competition, features of the market, and features of decision space. Each of these categories has a set of subcategories. In the following paragraphs the subcategories of features of competition and features of the market are described. Furthermore, the WSP and the WSFLP are categorized according to these subcategories. The subcategories under features of decision space have already been discussed above.

2.2.1 Features of Competition

The first factor Plastria (2001) considers is the state of the competition. Competitive location problems fit into one of the three subcategories: static competition, competition with foresight, and dynamic and competitive equilibrium.

Static Competition: The simplest assumption that can be made about the competition is that all competitors are already present. All characteristics of the competition, including location, are known and static. This is the assumptions made for both problems discussed in this dissertation. Huff, in two seminal papers in 1964 and 1966, defines trading areas for retailers under the assumption of static competition.

Although Plastria (2001) does not explicitly consider it, some of the competitive facility location literature considers a situation where at least some of the already existing “competitors” belong to the same firm as the new facilities to be located. In such a case the problem is not to maximize the market share captured by the newly located facilities, but to maximize the market share won by all of the firm’s locations collectively. One example of such a situation is the work of McGarvey and Cavalier (2005). Their approach is valid to this research because it seeks to locate, simultaneously, multiple coordinated facilities competing for market share. In both problems labor decisions are made simultaneously by all coordinated facilities. Also, in the WSFLP multiple facilities are located simultaneously.

Competition with Foresight: Another competition assumption is that the market is virgin. This means that no competitors are present. Decisions are made knowing that soon after the new facility is located other competitors will enter the market. Shiode and Drezner (2003) consider such a situation. They also provide a background on the introduction and solution of this type of problem.

Dynamic Models and Competitive Equilibrium: The other possible assumption is that the competition is dynamic. This assumption often leads to discovering the existence or nonexistence of equilibrium solutions. Hotelling (1929), author of the first work on competitive facility location, considered two competing retailers located on a line segment. He discovered the equilibrium that exists for the location of these competitors when buying power is uniform on the segment and customers choose the closest retailer.

2.2.2 Features of the Market

Most relevant to our work is the next factor listed by Plastria (2001) for categorizing competitive facility location models – the features of the market. He defines a market to be "the collection of all customers and their demand." His list of market features includes point versus regional demand, the patronizing behavior, and the attraction function. These features are relevant to our work because the attraction felt by customers and their patronizing behavior is comparable to that of potential employees deciding for which firm they will work. This is especially true when labor resources are limited due to a low unemployment rate or economic growth in an underdeveloped country.

Point vs. Regional Demand: Most of the models presented in the competitive facility location literature assume that demand is discrete or point. The model proposed by Huff (1964) and (1966) assumes point demand. Drezner and Drezner (1997) extend the work of Huff by assuming that demand is continuous across the plane. Double integrals are used to evaluate the demand of an area. They argue that assuming point demand introduces inaccuracies into the solutions obtained by the model. Finding an optimal facility location, though, is more difficult. They also introduce the distance correction approach to approximating the continuous distribution of the demand. This approach allows for the use of points that represents subareas. It uses the average distance between a facility and all the points in the subarea as an approximation of the distance between the facility and the demand points. This is a useful development because all potential employees are not found at one point in a labor market. Finding optimal solutions is more difficult if the demand in a market is assumed to be continuous.

Since the different facilities in the WSP and the WSFLP compete for workers and not consumer dollars, we say that there exists only workers at a market and not demand. We assume that all workers belonging to the same market are at a single point in both problems. Future work may incorporate the continuous demand assumption or the distance correction approach.

Patronizing Behavior: Competitive facility location assumes, in general, that each customer feels a variable attraction to each of the competing facilities. Given this attraction, the patronizing behavior of the market describes the way customers determine which facility to patronize. Plastria (2001) describes a deterministic and a probabilistic patronizing behavior. The deterministic assumption is the simplest conceptually and the most common. It is that the full demand of each customer is met by the facility to which he feels the greatest attraction. One example is the rule that each customer patronizes the closest facility. Important to the assumption of a deterministic patronizing behavior is the rule for cases where a new facility and one or more competitors tie as the most attractive facility to a market. Some rules split the demand of the market. Other rules allow for the new facility to capture either all or none of the demand from that market. Hotelling (1929), Hakimi (1986), Shiode and Drezner (2003), and Suárez-Vega et al. (2004a) and (2004b) consider models with the assumption that customers patronize facilities in a deterministic way.

The probabilistic patronizing behavior rule is that the demand of a market is divided between the different facilities. The attraction customers feel toward a facility is used to determine the probability that a customer will patronize that facility. Huff (1964) and (1966) was the first to use such a rule to determine what he termed "trading areas" for retailers. This rule is used under the assumption that the product being sold at the facilities is nonessential. Usually the attraction felt by customers toward a facility is not just a function of the distance between it and the customer.

When employment-seeking workers are considered, neither a deterministic nor probabilistic model is perfect. It is logical to assume that every worker desires to work for the most attractive employing facility. This would suggest that a deterministic rule should be used to model their behavior in choosing an employer. The problem with this assumption is that the most attractive of the facilities competing for workers in a labor market may not need as many workers as are available in that market. Thus, to say that all workers will be employed at the most attractive facility is not a logical assumption. We hope to address this point in the future using the assumption that the facility most attractive to a labor market will be able to employ all the workers that facility wants from that market. The facility with the second highest attractiveness in the market would be able to hire all that it wants from the remaining number of workers in the market. The other competing facilities would attract workers in a similar way as long as the labor force in the market is not depleted.

The alternative is to assume that workers select an employing facility probabilistically, where the probability that a worker chooses to work at a facility is determined by the attraction that is felt toward that facility in comparison to the other competing facilities. Underlying this assumption is that the preferences of different workers in the same market vary. One facility may be the most attractive to one worker but not to another. This assumption is especially reasonable when unemployment rates are low in a region and workers have many choices in choosing an employer. This assumption eliminates the problem of a facility attracting more workers than it needs. It is assumed that it is costly, because it requires a higher wage, for a facility to make itself more attractive to workers. Thus, a location will not incur more cost than is necessary to attract the minimum number of workers it needs. The patronizing behavior assumption made for both problems we discuss is that workers choose to work at an employing facility probabilistically.

Attraction Function: Much has been said already of the attraction that is felt by worker or customers toward the competing facilities. The function that describes this attraction is an important distinction of competitive facility location and also to this work. Plastria (2001) considers the attraction functions found in the competitive location literature to be either uniform or multiform. A uniform attraction function assumes that a facility's distance from the customer is the only thing that distinguishes it from its competitors. Such an attraction function is usually used in conjunction with deterministic patronizing behavior. When multiform attraction functions are used, customers still consider the locations of the competing facilities when choosing which facility to patronize. However, other differences that exist between the competing facilities influence the attraction felt by the customers. These differences are included in the attraction function.

Plastria (2001) further divides multiform attraction functions into two groups. It is common in economics to use additive attraction functions. The observed price of a facility's product is considered a major factor in customers' choices. The actual price paid by a customer includes the cost to travel to the facility to purchase the product. The total cost of the product is the price charged and the travel cost, determined in an additive way. The other group of attraction functions is multiplicative. These functions incorporate many factors that would be attractive to customers such as "floor area, number of cashing counters, product mix, publicity, *etc.*" These factors can be considered all together in a single measure referred to as "quality." Drezner et al.

(1996) criticize this type of attraction function because, when modeled mathematically, it allows a customers' choice of facility to change during their trip to a facility.

Drezner et al. (1998) review the major attraction function assumptions. Two of these functions are uniform. The other three are multiform. The attraction functions can also be categorized according to the two patronizing behavior assumptions. The two major uniform functions and one of the multiform functions are used with a deterministic patronizing behavior assumption. All of the other multiform functions are used under the probabilistic assumption.

The models that assume a deterministic patronizing behavior are the allocation-by-proximity, location-allocation, and deterministic utility models. When Hotelling (1929) introduced competitive facility location analysis, he assumed that each customer patronizes the closest facility. This rule is called binary by some authors. Location-allocation models still assume that customers patronize the closest facility. They differ because the markets are divided using Voronoi diagrams. A Voronoi diagram is a tessellation of the plane into a set of polygons. Each polygon is associated with a point from a given set. A set of demand points is one example of such a set. For a more detailed example, see Suzuki and Okabe's chapter in Drezner's survey (1995). When each customer chooses the facility with the highest utility, it is a deterministic utility model. The utility is determined by the facility attractiveness and the distance from the market. All customers in a market have the same utility function. Thus, they all select the same facility.

The random utility model is the extension by Drezner and Drezner (1996) of the deterministic utility model. In the random utility model each customer uses a different utility function for any given facility. The normal distribution is used to assign the values of the utility function components.

Huff (1964) and (1966) was the first to present a model with a probabilistic patronizing behavior assumption. His rule has come to be called the gravity model. He explained that "spatial interaction between points of origin and places of destination has been found to vary directly with some function of the 'attraction' of the places of destination and inversely with some function of the distance separating the points of origination and destination." Huff uses the size of the facility as the attractiveness factor. In order to capture these relationships between facilities and customers, Huff presents the following attraction function:

$$P_{ij} = \frac{\frac{S_j}{T_{ij}^{\lambda}}}{\sum_{j=1}^{n} \frac{S_j}{T_{ij}^{\lambda}}}.$$

P_{ij} is defined as the probability of a consumer at market i traveling to a particular shopping center j. S_j indicates the size of a shopping center j in terms of the square footage of the selling area dedicated to a class of goods. T_{ij} equals the travel time involved in getting from a consumer's travel base i to given shopping center j. λ reflects the effect of travel time on various kinds of shopping trips. This parameter is estimated empirically. Huff explains that λ will differ depending on the product being considered. A smaller λ indicates that consumers are more willing travel the same distance to shop for a product than one where a larger λ is used. Huff estimates λ to be 2.723 if the product is furniture and 3.191 if it is clothing. E_{ij}, the expected number of consumers at market i that are likely to travel to shopping center j is found using the following equation:

$$E_{ij} = P_{ij} \cdot C_i.$$

C_i is the number of consumers at market i.

The mathematical models developed for both problems in this dissertation use Huff's gravity model of attraction. We assume that the different facilities compete for workers in a market based on the distance between the facilities and the market and the wages and benefits offered.

2.2.3 Other Contributions

Although both problems we present in this dissertation are new, we have established its similarities and differences with existing literature. We have also discussed the modeling techniques we use that are common to the facility location and competitive location literature. Fortunately, the literature also provides ideas and direction for formulating solution procedures for the Workforce Supply Problem and the Workforce Supply and Facility Location Problem. Future work, beyond the scope of this dissertation, is to validate the use of the gravity attraction model for solving these two problems. The rest of this literature review highlights validation techniques, solution approaches, and some special modeling novelties used in the competitive location literature.

In 1974, Nakanishi and Cooper use least squares techniques to estimate the parameters necessary to use Huff's gravity model. Drezner and Drezner (2002) validate the use of the gravity model. Sales volumes of competing facilities are used to measure the amount of market share that is captured. This may provide insight into using data to estimate the parameters needed for our models.

Peeters and Plastria (1998) and Suárez-Vega et al. (2004a, 2007) employ gravity models of attraction under the assumption that the decision space is discrete. Peeters and Plastria show that the optimal location for a single facility will be on one of the vertices of the network under the assumption that the distance measure is a concave function. Suárez-Vega et al. (2004a) treat the quality of a facility as a decision variable. They develop discretization results and solve for the optimal location using a branch and bound procedure. They consider both a single facility and multiple facilities. In their 2007 paper, they discretize the problem of locating multiple new facilities where customers demand that a minimum threshold of attraction be met before a facility is patronized at all.

Drezner (1995) demonstrates that the objective of maximizing the market share captured by a single new facility has a nonlinear function that is neither convex nor concave. This is established under the following assumptions: a gravity attraction function, a continuous decision space, and continuous demand. He also points out that employing the distance correction approach for estimating continuous demand in place of point demand results in an objective function with fewer local optima.

Given the nonlinear nature of the objective functions for these problems, solving them is difficult. Drezner (1994) solves the single competitive facility location model based on the gravity model and a planar decision space. Drezner (1998) also solves the problem for multiple facilities. Two models are solved. One model considers the attractiveness of the facilities as a decision variable constrained by a limited budget for all facilities. The attractiveness of a facility is a known parameter in the other model. Both of the solutions are found using a heuristic ascent algorithm. Drezner et al. (2002) propose four heuristic approaches in addition to the ascent algorithm suggest by Drezner (1998). The conclusion is that a procedure combining simulated annealing and an ascent algorithm yields the best solutions.

An optimal solution to the single competitive facility location problem using the gravity attraction model over a planar decision space is found by Drezner and Drezner (2004). They use a branch and bound procedure by dividing the convex hull of the demand points into triangles that do not contain any of the demand points.

McGarvey and Cavalier (2005) introduce many new aspects to the problem of locating multiple new facilities on the plane. The gravity model is the basis for the attraction function they employ. They use the capacity of a facility to fill customer orders as the attractiveness of a facility. They introduce a new problem formulation with elastic gravity-based demand. Three novel constraints are presented: facility capacity, forbidden regions, and a budget function. The facility capacity constraint is that no facility can capture more demand than it has capacity to meet. Forbidden regions restrict the possible locations for the new facilities. This is useful to capture the reality that bodies of water, mountains, zoning restrictions, and other factors may limit the choice of location. The budget constraint limits the amount of capacity that can be assigned to a new facility. Two heuristics are suggested for solving the problem.

Another interesting extension to Huff's work has been recently explored by Fernández et al. (2007). They extend the work of Suárez-Vega et al. (2004a) by assuming that the same level of quality at a facility may be perceived differently by each demand point. This reflects the socio-economic population characteristics including average age and income class. Fernández et al. use an objective function that seeks to maximize profit. The market share captured by the new facility is transformed into expected sales. The cost of attaining the chosen level of quality at the new plant is subtracted from the expected sales. An example of this objective function is shown to be neither concave nor convex and to have multiple local optima.

Lastly, we discuss some empirical studies that have been conducted to determine the important factors considered by manufacturing firms choosing a new location in the United States. Tong and Walter (1980) surveyed firms with foreign owners asking them to rate the importance of 32 different facility location factors. The top fifteen results include availability of transportation services, labor attitudes, availability of skilled labors, salary and wage rates, and cost of transportation services. Bartik (1985) uses a conditional logit model to show that low levels unionization, low wages and high amounts of existing manufacturers were all factors that firms seeking a location for a new facility. On the other hand, Swamidass (1990) compare foreign and domestic firms and finds that labor costs and unemployment levels are not major factors for either type of firm. His study shows that the size of the consumer market is the most important factor. Finally, Ulgado (1996) that the top ten facility location attributes for domestic firms includes local salary and wage levels, and local labor attitudes. For foreign firms the important factors did not change very much. These findings emphasize the importance of our work to quantify and minimize the labor and shipping costs of new manufacturing facilities.

The Workforce Supply Problem and the Workforce Supply and Facility Location Problem are unique problems in the operations management literature. However, we are able to categorize them using factors of facility location and competitive facility location problems. By doing so, we can draw on similar problems to gain insight into developing models and finding solutions for our problems. Of particular importance to our problems is Huff's 1964 and 1966 attraction function and the extensions that have been made to his model. In the next chapter, we explain how Huff's attraction function is applied to the WSP to model the way workers choose an employer.

3 WORKFORCE SUPPLY PROBLEM

The objective of the Workforce Supply Problem is to minimize the total labor cost of a collection of facilities that compete for workers from a common set of labor markets. These facilities decide from where to hire workers and what level of wage to pay. We will refer to these facilities as "internal" facilities since these facilities constitute elements of a common supply chain. There are also other competing facilities that seek to hire from the same group of qualified workers in these labor markets. We will call these facilities "external" facilities. We say that the internal and external facilities form an "industry." Throughout this research, "workers" are defined to be persons with the ability to work in the industry. There are a limited number of workers available in a labor market. The locations of these markets and their associated populations of potential workers are assumed to be known. We approximate the distance between any worker in a labor market and a facility by the distance between the center of the labor market and the facility. The locations and wages of the external facilities are also known and fixed while only the locations of the internal facilities are assumed to be known and fixed. The use of the term "wage" in this work represents the company's cost of an employee including salary, benefits, retirement plans, etc.

The following notation and assumptions are used to formulate a model for the WSP.

Parameters

K	=	number of labor markets
M	=	number of external facilities
N	=	number of internal facilities
i	=	facility index where $i = 1, ..., N$ refers to internal facilities and $i = N + 1, ..., N + M$ refers to external facilities
B_k	=	number of workers at labor market k, $k = 1, ..., K$
R_i	=	number of workers required by internal facility i, $i = 1, ..., N$
D_{ik}	=	distance between facility i and market k, for $i = 1, ..., N + M$ and $k = 1, ..., K$
λ_k	=	attraction parameter for market k, $k = 1, ..., K$
w_{ik}	=	annual wage paid by external facility i to workers from market k, for $i = N + 1, ..., N + M$ and $k = 1, ..., K$

Decision Variables

w_{ik}	=	annual wage paid by internal facility i to workers from market k, for $i = 1, ..., N$ and $k = 1, ..., K$

Vectors and Matrices

$\mathbf{W}$	=	matrix of wages paid by each internal facility to workers from each market;
	=	(w_{ik})
$\bar{w}_k$	=	kth column of matrix $\mathbf{W}$;
	=	$(w_{1k}, ..., w_{Nk})^T$

Assumptions

1. In every market k there exists at least one external facility, $i \in \{N + 1, ..., N + M\}$, for which $w_{ik} > 0$.

2. The distance between any facility and any labor market is strictly positive, $D_{ik} > 0,\ i = 1, ..., N + M,\ k = 1, ..., K$.

3. The attraction parameter is always positive, $\lambda_k > 0,\ k = 1, ..., K$.

4. Each internal facility requires at least one worker, $R_i \geq 1,\ i = 1, ..., N$.

5. The total number of workers required by all company facilities is strictly less than the total number of available workers, $\sum_{i=1}^{N} R_i < \sum_{k=1}^{K} B_k$.

6. Labor attraction of market k toward facility i is defined as $\frac{w_{ik}}{D_{ik}^{\lambda_k}}$.

7. Labor markets split their workforce among the various facilities proportionally to their attraction.

The last three assumptions merit further discussion.

Rynes et al. (2004) provide a list of empirical studies that have investigated the different factors workers consider when choosing an employer. Pay ranks from second to seventh as the most important factor in the different studies. However, they also argue that pay has a greater effect on the behavior of workers than they report in studies. Manning (2003) argues that there is a positive correlation between wages and commuting distance. He explains that workers who commute a farther distance receive a higher wage, but the increase in pay may not be proportional to the increase is commuting distance. Also, van den Berg and Gorter (1997) present a job search model allowing jobs to have different combinations of wage and commuting time. The model shows that workers have a decreasing utility for longer commuting times. These studies emphasize the importance of wage in the way workers choose between employers. They also show that the attractiveness of the wage paid to workers depends on the distance or time that the worker must commute to work.

Thus, we make the assumption that labor attraction is a factor of wage and commute distance. We let the attraction be proportional to the wage paid. Also, we let the attraction be inversely proportional to the distance between the facility i and the market k. The parameter λ_k determines the effect of distance on the attractiveness of a facility to market k. Figure 1 shows the effect of λ_k on the attraction of a facility. As λ_k increases, distance has a greater effect on the facility's attraction of labor. For the same distance between markets and facility i and the same wage paid to workers from both markets by facility i, $\lambda_k > \lambda_l$ means that workers from market l will have a greater attraction for a facility. The total attraction of labor market k toward all external facilities is given by

$$A_k = \sum_{i=N+1}^{N+M} \frac{w_{ik}}{D_{ik}^{\lambda_k}}. \tag{1}$$

Because of assumptions 1 and 2, $A_k > 0,\ \forall k$. From this point forward, when we use the subscript, i, we refer only to the facilities $1, ..., N$.

Assumption 7 is similar to an assumption made by Huff (1964) concerning retail markets. He assumes that the attraction of shoppers to retail locations is comprised of the size of

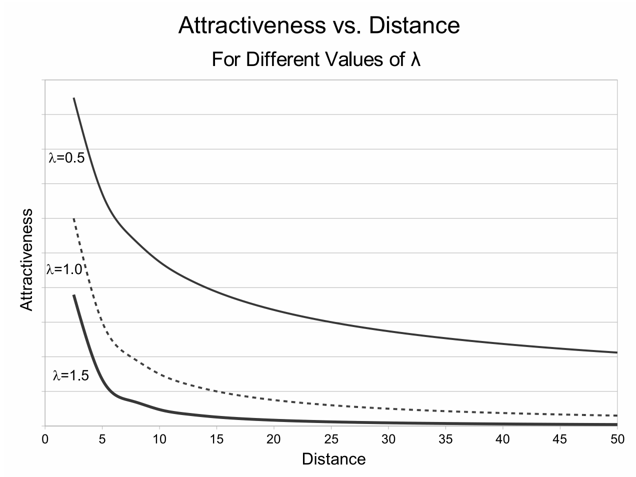

Figure 1: Effect of λ on Attraction of Labor

the retailers store and the amount of travel time between the shoppers and the store. The proportion of a market attracted to a store is equal to the attraction of the retailer compared to the attraction of all retailers. In the WSP problem, the wage paid to workers by an internal facility corresponds with the size of a retailer. Also, the distance between a labor market and an internal facility of the WSP model corresponds to the length of time a shopper travels to a retailer in Huff's model. Thus, in the WSP problem the proportion of workers from market k that facility i is able to recruit is defined as $\frac{\frac{w_{ik}}{D_{ik}^{\lambda_k}}}{\sum_{l=1}^{N}\frac{w_{lk}}{D_{lk}^{\lambda_k}}+A_k}$. Thus, the number of the workers at labor market k attracted by facility i is given by

$$g_{ik}(\bar{w}_k) = B_k \left[\frac{\frac{w_{ik}}{D_{ik}^{\lambda_k}}}{\sum_{l=1}^{N} \frac{w_{lk}}{D_{lk}^{\lambda_k}} + A_k} \right].$$

As one would expect, as the wage paid to workers in market k by facility i, w_{ik}, increases the number of workers increases as well. On the other hand, the number of workers is decreased as the distance, D_{ik}, increases. Also, if an internal facility $j \neq i$ increases the wage paid to workers at market k, w_{jk}, the number of workers attracted to facility i is decreased. This shows the interesting relationship between the different internal facilities. They compete with each other for the limited labor resources at the same time they work together to minimize the total labor costs of the internal facilities. The total labor cost of hiring workers from labor market k to work at internal facility i is given by

$$f_{ik}(\bar{w}_k) = w_{ik}g_{ik}(\bar{w}_k) = B_k \left[\frac{\frac{w_{ik}^2}{D_{ik}^{\lambda_k}}}{\sum_{l=1}^{N} \frac{w_{lk}}{D_{lk}^{\lambda_k}} + A_k} \right].$$

Using these function to model the WSP mathematically requires that $\sum_{i=1}^{N} R_i < \sum_{k=1}^{K} B_k$. Otherwise, the wage required to hire every worker in the markets is infinite. This is the reason for assumption 5.

Given this notation and these assumptions, we present the general model of the workforce supply problem.

$$\begin{aligned} \min \sum_{i=1}^{N}\sum_{k=1}^{K} & f_{ik}(\bar{w}_k) \\ \text{s.t.} \sum_{k=1}^{K} g_{ik}(\bar{w}_k) & \geq R_i, \ \forall i \in \{1, ..., N\} \\ w_{ik} & \geq 0, \ \forall i \in \{1, ..., N\}, \ \forall k \in \{1, ..., K\}. \end{aligned}$$

This model has $N \cdot K$ decision variables, N nonlinear constraints and $N \cdot K$ nonnegativity constraints. The objective function is also nonlinear. The number of workers hired from each market is chosen implicitly in the model. Note that setting $w_{ik} = 0$ is equivalent to facility i not hiring workers from labor market k. We investigate the structure of this model in more detail in the next section.

3.1 Basic Structural Results

First, we show that $f_{ik}(\bar{w}_k)$ is a convex function and that $g_{ik}(\bar{w}_k)$ is a quasimonotone function. In order to show the result for $f_{ik}(\bar{w}_k)$, some lemmas are helpful. It is also helpful to rewrite the function $f_{ik}(\bar{w}_k)$ as $f_{ik}(\bar{w}_k) = \frac{B_k w_{ik}^2}{w_{ik} + \sum_{l \neq i}^{N} \frac{D_{ik}^{\lambda_k}}{D_{lk}^{\lambda_k}} w_{lk} + A_k D_{ik}^{\lambda_k}}$. Note that we will use the notation $\sum_{l \neq i} x_l$ to denote $x_1 + x_2 + ... + x_{i-1} + x_{i+1} + ... + x_{N-1} + x_N$ throughout this work. For ease of notation in the subsequent lemmas and theorem, we define

$$Z_{ik}(\bar{w}_k) = w_{ik} + \sum_{l \neq i} \frac{D_{ik}^{\lambda_k}}{D_{lk}^{\lambda_k}} w_{lk} + A_k D_{ik}^{\lambda_k}$$

and

$$\begin{aligned} Y_{ik}(\bar{w}_k) &= \sum_{l \neq i} \frac{D_{ik}^{\lambda_k}}{D_{lk}^{\lambda_k}} w_{lk} + A_k D_{ik}^{\lambda_k} \\ &= Z_{ik}(\bar{w}_k) - w_{ik}. \end{aligned}$$

Both $Z_{ik}(\bar{w}_k)$ and $Y_{ik}(\bar{w}_k)$ are strictly positive for $i = 1, ..., N,\ k = 1, ..., K$. This is because $A_k > 0,\ k = 1, ..., K,\ D_{ik}^{\lambda_k} > 0,\ i = 1, ..., N,\ k = 1, ..., K$, and $w_{ik} \geq 0,\ i = 1, ..., N,\ k = 1, ..., K$. Thus, for any facility i and market k we know that $A_k D_{ik}^{\lambda_k} > 0$ and $\frac{D_{ik}^{\lambda_k}}{D_{lk}^{\lambda_k}} w_{lk} \geq 0,\ l \neq i$. This implies that $Z_{ik}(\bar{w}_k) > 0$ and $Y_{ik}(\bar{w}_k) > 0$ even if $w_{jk} = 0,\ j = 1, ..., N$.

Since the labeling of facilities with subscripts is arbitrary, we can consider facility $i = 1$ and thus $f_{1k}(\bar{w}_k)$ in the lemmas and theorem that follow without any loss of generality.

Lemma 1 *The gradient of $f_{1k}(\bar{w}_k)$ is completely defined by the following:*

$$\begin{aligned} \frac{\partial f_{1k}(\bar{w}_k)}{\partial w_{1k}} &= \frac{2B_k w_{1k} Y_{1k}(\bar{w}_k) + B_k w_{1k}^2}{Z_{1k}(\bar{w}_k)^2} \\ \frac{\partial f_{1k}(\bar{w}_k)}{\partial w_{jk}} &= -\frac{B_k w_{1k}^2 \frac{D_{1k}^{\lambda_k}}{D_{jk}^{\lambda_k}}}{Z_{1k}(\bar{w}_k)^2} \qquad \forall j \neq 1. \end{aligned}$$

Proof In this proof, we employ differentiation and algebra.

$$\begin{aligned}
\frac{\partial f_{1k}(\bar{w}_k)}{\partial w_{1k}} =& \frac{\partial}{\partial w_{1k}}\left(\frac{\frac{B_k w_{ik}^2}{D_{ik}^{\lambda_k}}}{\sum_{l=1}^{N}\frac{w_{lk}}{D_{lk}^{\lambda_k}} + A_k}\right) \\
=& \frac{2B_k w_{1k}\left(w_{1k} + \sum_{l\neq 1}\frac{D_{1k}^{\lambda_k}}{D_{lk}^{\lambda_k}} w_{lk} + A_k D_{1k}^{\lambda_k}\right) - B_k w_{1k}^2}{\left(w_{1k} + \sum_{l\neq i}\frac{D_{1k}^{\lambda_k}}{D_{lk}^{\lambda_k}} w_{lk} + A_k D_{1k}^{\lambda_k}\right)^2} \\
=& \frac{2B_k w_{1k}\left(\sum_{l\neq 1}\frac{D_{1k}^{\lambda_k}}{D_{lk}^{\lambda_k}} w_{lk} + A_k D_{1k}^{\lambda_k}\right) + 2B_k w_{1k}^2 - B_k w_{1k}^2}{Z_{1k}(\bar{w}_k)^2} \\
=& \frac{2B_k w_{1k} Y_{1k}(\bar{w}_k) + B_k w_{1k}^2}{Z_{1k}(\bar{w}_k)^2} \\
\frac{\partial f_{1k}(\bar{w}_k)}{\partial w_{jk}} =& \frac{0 \cdot \left(w_{1k} + \sum_{l\neq 1}\frac{D_{1k}^{\lambda_k}}{D_{lk}^{\lambda_k}} w_{lk} + A_k D_{1k}^{\lambda_k}\right) - B_k w_{1k}^2 \frac{D_{1k}^{\lambda_k}}{D_{lk}^{\lambda_k}}}{\left(w_{1k} + \sum_{l\neq 1}\frac{D_{1k}^{\lambda_k}}{D_{lk}^{\lambda_k}} w_{lk} + A_k D_{1k}^{\lambda_k}\right)^2} \\
=& -\frac{B_k w_{1k}^2 \frac{D_{1k}^{\lambda_k}}{D_{jk}^{\lambda_k}}}{Z_{1k}(\bar{w}_k)^2} \qquad \forall j \neq 1 \blacksquare
\end{aligned}$$

Now, we use the results for the gradient of $f_{1k}(\bar{w}_k)$ to develop its Hessian matrix.

Lemma 2 *The Hessian matrix, H, of $f_{1k}(\bar{w}_k)$ is defined by the following:*

$$\begin{aligned}
H_{11} &= \frac{2B_k}{Z_{1k}(\bar{w}_k)^3} Y_{1k}(\bar{w}_k)^2 && > 0 \\
H_{u1} = H_{1u} &= -\frac{2B_k}{Z_{1k}(\bar{w}_k)^3}\left(w_{1k}\frac{D_{1k}^{\lambda_k}}{D_{uk}^{\lambda_k}} Y_{1k}(\bar{w}_k)\right) && \leq 0, \quad u \neq 1 \\
H_{uv} &= \frac{2B_k}{Z_{1k}(\bar{w}_k)^3}\left(w_{1k}^2 \frac{D_{1k}^{\lambda_k}}{D_{uk}^{\lambda_k}} \frac{D_{1k}^{\lambda_k}}{D_{vk}^{\lambda_k}}\right) && \geq 0, \quad u, v \neq 1,
\end{aligned}$$

where $H_{uv} = \frac{\partial^2 f_{ik}(\bar{w}_k)}{\partial w_{uk}\partial w_{vk}}$.

Proof In this proof we employ differentiation and algebra.

$$
\begin{aligned}
H_{11} &= \frac{Z_{1k}(\bar{w}_k)^2\left(2B_kY_{1k}(\bar{w}_k)+2B_kw_{1k}\right)-2Z_{1k}(\bar{w}_k)\left(2B_kw_{1k}Y_{1k}(\bar{w}_k)+B_kw_{1k}^2\right)}{Z_{1k}(\bar{w}_k)^4} \\
&= \frac{2B_kZ_{1k}(\bar{w}_k)\left(w_{1k}+Y_{1k}(\bar{w}_k)\right)^2-2Z_{1k}(\bar{w}_k)\left(2B_kw_{1k}Y_{1k}(\bar{w}_k)+B_kw_{1k}^2\right)}{Z_{1k}(\bar{w}_k)^4} \\
&= \frac{2B_k}{Z_{1k}(\bar{w}_k)^3}\left(w_{1k}^2+2w_{1k}Y_{1k}(\bar{w}_k)+Y_{1k}(\bar{w}_k)^2-2w_{1k}Y_{1k}(\bar{w}_k)-w_{1k}^2\right) \\
&= \frac{2B_k}{Z_{1k}(\bar{w}_k)^3}Y_{1k}(\bar{w}_k)^2
\end{aligned}
$$

$H_{11}>0$ because, for any market k, B_k, $Z_{1k}(\bar{w}_k)$, and $Y_{1k}(\bar{w}_k)$ are all strictly positive.

$$
\begin{aligned}
H_{1u} &= \frac{2B_kw_{1k}\frac{D_{1k}^{\lambda_k}}{D_{uk}^{\lambda_k}}Z_{1k}(\bar{w}_k)^2-2Z_{1k}(\bar{w}_k)\frac{D_{1k}^{\lambda_k}}{D_{uk}^{\lambda_k}}\left(2B_kw_{1k}Y_{1k}(\bar{w}_k)+B_kw_{1k}^2\right)}{Z_{1k}(\bar{w}_k)^4} \\
&= \frac{2B_k}{Z_{1k}(\bar{w}_k)^3}\left(w_{1k}\frac{D_{1k}^{\lambda_k}}{D_{uk}^{\lambda_k}}Z_{1k}(\bar{w}_k)-\frac{D_{1k}^{\lambda_k}}{D_{uk}^{\lambda_k}}w_{1k}\left(2Y_{1k}(\bar{w}_k)+w_{1k}\right)\right) \\
&= \frac{2B_k}{Z_{1k}(\bar{w}_k)^3}\left(w_{1k}\frac{D_{1k}^{\lambda_k}}{D_{uk}^{\lambda_k}}Z_{1k}(\bar{w}_k)-\frac{D_{1k}^{\lambda_k}}{D_{uk}^{\lambda_k}}w_{1k}\left(Y_{1k}(\bar{w}_k)+Z_{1k}(\bar{w}_k)\right)\right) \\
&= -\frac{2B_k}{Z_{1k}(\bar{w}_k)^3}\left(w_{1k}\frac{D_{1k}^{\lambda_k}}{D_{uk}^{\lambda_k}}Y_{1k}(\bar{w}_k)\right)
\end{aligned}
$$

$$
\begin{aligned}
H_{u1} &= -\left(\frac{2B_kw_{1k}\frac{D_{1k}^{\lambda_k}}{D_{uk}^{\lambda_k}}Z_{1k}(\bar{w}_k)-2B_kw_{1k}^2\frac{D_{1k}^{\lambda_k}}{D_{uk}^{\lambda_k}}}{Z_{1k}(\bar{w}_k)^4}\right) \\
&= -\frac{2B_k}{Z_{1k}(\bar{w}_k)^3}\left(w_{1k}\frac{D_{1k}^{\lambda_k}}{D_{uk}^{\lambda_k}}\left(w_{1k}+Y_{1k}(\bar{w}_k)\right)-w_{1k}^2\frac{D_{1k}^{\lambda_k}}{D_{uk}^{\lambda_k}}\right) \\
&= -\frac{2B_k}{Z_{1k}(\bar{w}_k)^3}\left(w_{1k}\frac{D_{1k}^{\lambda_k}}{D_{uk}^{\lambda_k}}Y_{1k}(\bar{w}_k)\right)
\end{aligned}
$$

We know that $B_k>0$, $Z_{1k}(\bar{w}_k)>0$, $Y_{1k}(\bar{w}_k)>0$, and $w_{1k}\frac{D_{1k}^{\lambda_k}}{D_{uk}^{\lambda_k}}\leq 0$ for any market k. Thus, $H_{1u}=0$ and $H_{u1}=0$ if $w_{1k}=0$. Also, $H_{1u}<0$ and $H_{u1}<0$ if $w_{1k}>0$.

$$
\begin{aligned}
H_{uv} &= -\left(\frac{Z_{1k}(\bar{w}_k)^2 \cdot 0 - 2B_k w_{1k}^2 Z_{1k}(\bar{w}_k)\frac{D_{1k}^{\lambda_k}}{D_{uk}^{\lambda_k}}\frac{D_{1k}^{\lambda_k}}{D_{vk}^{\lambda_k}}}{Z_{1k}(\bar{w}_k)^4}\right) \\
&= \frac{2B_k}{Z_{1k}(\bar{w}_k)^3}\left(w_{1k}^2 \frac{D_{1k}^{\lambda_k}}{D_{uk}^{\lambda_k}}\frac{D_{1k}^{\lambda_k}}{D_{vk}^{\lambda_k}}\right)
\end{aligned}
$$

For any market k, we know that $B_k > 0$, $Z_{1k}(\bar{w}_k) > 0$, $D_{jk} > 0$, $j = 1, ..., N$, and $w_{1k}^2 \geq 0$. This implies that $H_{uv} \geq 0$. ■

Following Bazaraa et al. (2006), page 771, we now summarize concepts of submatrices, principal submatrices, minors and principal minors of a matrix used in the subsequent analysis. Let Q be an N x N matrix . Let $p \in \{1, ...N\}$. Also, let $\beta = \{j_1, ..., j_p\}$ be composed of p distinct indices where $1 \leq j_1 < j_2 < ... < j_p \leq N$. Finally, let $\gamma = \{l_1, ..., l_p\}$ be composed of p distinct indices where $1 \leq l_1 < l_2 < ... < l_p \leq N$. Note that the set of rows, β, may be, but is not necessarily, equivalent to the set of columns, l_p. We define $Q_{\beta,\gamma,p}$ to be a p x p submatrix of Q formed by selecting the elements of Q that are located at the intersection of its rows $j_1, ..., j_p$ with its columns $l_1, ..., l_p$. A principal submatrix, $Q_{\gamma,p}$, of Q is formed by selecting the elements of Q that are located at the intersection of its rows $j_1, ..., j_p$ with its columns $j_1, ..., j_p$ of Q. Thus, we define a minor, $M_{\beta,\gamma,p}(Q)$, of Q to be $\det Q_{\beta,\gamma,p}$. A principal minor, $M_{\gamma,p}(Q)$, of Q is $\det Q_{\gamma,p}$.

Lemma 3 *For any pair of distinct rows β and any pair of distinct columns γ, $M_{\beta,\gamma,2}(H) = 0$.*

Proof Following Lemma 2 the Hessian matrix of $f_{1k}(\bar{w}_k)$ is

$$
H = \frac{2B_k}{Z_{1k}(\bar{w}_k^3)}\begin{bmatrix}
Y^2 & -w_{1k}\frac{D_{1k}^{\lambda_k}}{D_{2k}^{\lambda_k}}Y & \dots & -w_{1k}\frac{D_{1k}^{\lambda_k}}{D_{Nk}^{\lambda_k}}Y \\
-w_{1k}\frac{D_{1k}^{\lambda_k}}{D_{2k}^{\lambda_k}}Y & w_{1k}^2\left(\frac{D_{1k}^{\lambda_k}}{D_{2k}^{\lambda_k}}\right)^2 & \dots & w_{1k}^2\frac{D_{1k}^{\lambda_k}}{D_{2k}^{\lambda_k}}\frac{D_{1k}^{\lambda_k}}{D_{Nk}^{\lambda_k}} \\
\vdots & \vdots & \ddots & \vdots \\
-w_{1k}\frac{D_{1k}^{\lambda_k}}{D_{Nk}^{\lambda_k}}Y & w_{1k}^2\frac{D_{1k}^{\lambda_k}}{D_{2k}^{\lambda_k}}\frac{D_{1k}^{\lambda_k}}{D_{Nk}^{\lambda_k}} & \dots & w_{1k}^2\left(\frac{D_{1k}^{\lambda_k}}{D_{Nk}^{\lambda_k}}\right)^2
\end{bmatrix}
$$

where $Z_{1k}(\bar{w}_k) = w_{1k} + \sum_{l\neq 1}^{N} \frac{D_{1k}^{\lambda_k}}{D_{lk}^{\lambda_k}} w_{lk} + A_k D_{1k}^{\lambda_k}$ and $Y = \sum_{l\neq 1}^{N} \frac{D_{1k}^{\lambda_k}}{D_{lk}^{\lambda_k}} w_{lk} + A_k D_{1k}^{\lambda_k}$. We define a matrix J such that $J\frac{2B_k}{Z_{1k}(\bar{w}_k)^3} = H$. Since $\frac{2B_k}{Z_{1k}(\bar{w}_k)^3} > 0$, showing that $M_{\beta,\gamma,p}(H) = 0$ is equivalent to showing that $M_{\beta,\gamma,p}(J) = 0$.

Now, we consider four cases. The first case is $\beta_1 = \{1, s\}$, where s is any row except 1, and $\gamma_1 = \{1, t\}$, where t is any column except 1. It is possible, but not necessary, that $s = t$. The submatrix formed is

$$J_{\beta_1,\gamma_1,2} = \begin{bmatrix} Y^2 & -w_{1k}\frac{D_{1k}^{\lambda_k}}{D_{tk}^{\lambda_k}}Y \\ -w_{1k}\frac{D_{1k}^{\lambda_k}}{D_{sk}^{\lambda_k}}Y & w_{1k}^2\frac{D_{1k}^{\lambda_k}}{D_{sk}^{\lambda_k}}\frac{D_{1k}^{\lambda_k}}{D_{tk}^{\lambda_k}} \end{bmatrix}.$$

So, $M_{\beta_1,\gamma_1,2}(J) = Y^2 w_{1k}^2 \frac{D_{1k}^{\lambda_k}}{D_{sk}^{\lambda_k}}\frac{D_{1k}^{\lambda_k}}{D_{tk}^{\lambda_k}} - Y^2 w_{1k}^2 \frac{D_{1k}^{\lambda_k}}{D_{sk}^{\lambda_k}}\frac{D_{1k}^{\lambda_k}}{D_{tk}^{\lambda_k}} = 0$.

The second case is $\beta_2 = \{1, s\}$, where s is any row except 1, and $\gamma_2 = \{t, u\}$, where t and u are any two distinct columns different from 1. It is possible, but not necessary, that $s = t$ or $s = u$. The submatrix formed is

$$J_{\beta_2,\gamma_2,2} = \begin{bmatrix} -w_{1k}\frac{D_{1k}^{\lambda_k}}{D_{tk}^{\lambda_k}}Y & -w_{1k}\frac{D_{1k}^{\lambda_k}}{D_{uk}^{\lambda_k}}Y \\ w_{1k}^2\frac{D_{1k}^{\lambda_k}}{D_{sk}^{\lambda_k}}\frac{D_{1k}^{\lambda_k}}{D_{tk}^{\lambda_k}} & w_{1k}^2\frac{D_{1k}^{\lambda_k}}{D_{sk}^{\lambda_k}}\frac{D_{1k}^{\lambda_k}}{D_{uk}^{\lambda_k}} \end{bmatrix}.$$

So, $M_{\beta_2,\gamma_2,2}(J) = -Yw_{1k}^3 \frac{D_{1k}^{\lambda_k}}{D_{tk}^{\lambda_k}}\frac{D_{1k}^{\lambda_k}}{D_{sk}^{\lambda_k}}\frac{D_{1k}^{\lambda_k}}{D_{uk}^{\lambda_k}} + Yw_{1k}^3 \frac{D_{1k}^{\lambda_k}}{D_{tk}^{\lambda_k}}\frac{D_{1k}^{\lambda_k}}{D_{sk}^{\lambda_k}}\frac{D_{1k}^{\lambda_k}}{D_{uk}^{\lambda_k}} = 0$.

The third case is $\beta_3 = \{s, t\}$, where s and t are any two distinct rows different from 1, and $\gamma_3 = \{1, u\}$, where u is any column different from 1. It is possible, but not necessary, that $s = u$ or $t = u$. The submatrix formed is

$$J_{\beta_3,\gamma_3,2} = \begin{bmatrix} -w_{1k}\frac{D_{1k}^{\lambda_k}}{D_{sk}^{\lambda_k}}Y & w_{1k}^2\frac{D_{1k}^{\lambda_k}}{D_{sk}^{\lambda_k}}\frac{D_{1k}^{\lambda_k}}{D_{uk}^{\lambda_k}} \\ -w_{1k}\frac{D_{1k}^{\lambda_k}}{D_{tk}^{\lambda_k}}Y & w_{1k}^2\frac{D_{1k}^{\lambda_k}}{D_{tk}^{\lambda_k}}\frac{D_{1k}^{\lambda_k}}{D_{uk}^{\lambda_k}} \end{bmatrix}.$$

So, $M_{\beta_3,\gamma_3,2}(J) = -Yw_{1k}^3 \frac{D_{1k}^{\lambda_k}}{D_{sk}^{\lambda_k}}\frac{D_{1k}^{\lambda_k}}{D_{tk}^{\lambda_k}}\frac{D_{1k}^{\lambda_k}}{D_{uk}^{\lambda_k}} + Yw_{1k}^3 \frac{D_{1k}^{\lambda_k}}{D_{sk}^{\lambda_k}}\frac{D_{1k}^{\lambda_k}}{D_{tk}^{\lambda_k}}\frac{D_{1k}^{\lambda_k}}{D_{uk}^{\lambda_k}} = 0$.

The fourth case is $\beta_4 = \{s, t\}$, where s and t are any two distinct rows different from 1, and $\gamma_4 = \{u, v\}$, where u and v are any two distinct columns different from 1. It is possible, but not necessary, that $s = u$, $s = v$, $t = u$, or $t = v$. The submatrix formed is

$$J_{\beta_4,\gamma_4,2} = \begin{bmatrix} w_{1k}^2 \frac{D_{1k}^{\lambda_k}}{D_{sk}^{\lambda_k}} \frac{D_{1k}^{\lambda_k}}{D_{uk}^{\lambda_k}} & w_{1k}^2 \frac{D_{1k}^{\lambda_k}}{D_{sk}^{\lambda_k}} \frac{D_{1k}^{\lambda_k}}{D_{vk}^{\lambda_k}} \\ w_{1k}^2 \frac{D_{1k}^{\lambda_k}}{D_{tk}^{\lambda_k}} \frac{D_{1k}^{\lambda_k}}{D_{uk}^{\lambda_k}} & w_{1k}^2 \frac{D_{1k}^{\lambda_k}}{D_{tk}^{\lambda_k}} \frac{D_{1k}^{\lambda_k}}{D_{vk}^{\lambda_k}} \end{bmatrix}.$$

So, $M_{\beta_4,\gamma_4,2}(J) = w_{1k}^4 \frac{D_{1k}^{\lambda_k}}{D_{sk}^{\lambda_k}} \frac{D_{1k}^{\lambda_k}}{D_{uk}^{\lambda_k}} \frac{D_{1k}^{\lambda_k}}{D_{tk}^{\lambda_k}} \frac{D_{1k}^{\lambda_k}}{D_{vk}^{\lambda_k}} - w_{1k}^4 \frac{D_{1k}^{\lambda_k}}{D_{sk}^{\lambda_k}} \frac{D_{1k}^{\lambda_k}}{D_{uk}^{\lambda_k}} \frac{D_{1k}^{\lambda_k}}{D_{tk}^{\lambda_k}} \frac{D_{1k}^{\lambda_k}}{D_{vk}^{\lambda_k}} = 0$. ■

Lemma 3 is a step towards showing that the Hessian matrix, H, of $f_{1k}(\bar{w}_k)$ is a positive semidefinite matrix for the set $\{\bar{w}_k | \bar{w}_k \geq 0\}$. We now show that any p x p, $p \geq 2$, matrix has a determinant of 0.

Lemma 4 *For $p > 2$, any β, a set of p distinct rows, and any γ, a set of p distinct columns, $M_{\beta,\gamma,p}(H) = 0$.*

Proof Again, we consider matrix J such that $J\frac{2B_k}{Z_{1k}(\bar{w}_k)^3} = H$ because showing that $M_{\beta,\gamma,p}(J) = 0$ is equivalent to showing that $M_{\beta,\gamma,p}(H) = 0$. We use induction to show that for any β, any γ, and $p > 2$, $M_{\beta,\gamma,p}(J) = 0$. Lemma 3 states that for any β and any γ, $M_{\beta,\gamma,2}(J) = 0$. Now, we assume that for any β and any γ, $M_{\beta,\gamma,n-1}(J) = 0$. Now, consider the minors $M_{\beta,\gamma,n}(J)$. The determinant of a n x n matrix $Q = \begin{bmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & \ddots & \cdots \\ q_{n1} & \cdots & q_{nn} \end{bmatrix}$ is defined as $\det Q = \sum_{u=1}^{n} q_{uv}(-1)^{u+v} M_{\Gamma\backslash u,\Gamma\backslash v,n-1}(Q)$, where $\Gamma = \{1, ..., n\}$ and where v is some fixed column in Q. Since for any β and any γ, $M_{\beta,\gamma,n-1}(J) = 0$, $M_{\beta,\gamma,n}(J) = \det J_{\beta,\gamma,n} = \sum_{u=1}^{n}(j_{uv}(-1)^{u+v} \cdot 0) = 0$ for any β and any γ. Since $M_{\beta,\gamma,p}(J) = 0$ for every β and γ and $\frac{2B_k}{Z_{1k}(\bar{w}_k)^3} > 0$ we know that $M_{\beta,\gamma,p}(H) = 0$ for every β and γ. ■

Bazaraa et al. (2006), Theorem 3.3.7, state that a twice differentiable function $f : S \to E_1$, where S is an open convex set, is convex if and only the Hessian matrix is positive semidefinite at each point in S. So, in order to show that $f_{ik}(\bar{w}_k)$ is a convex function of $\bar{w}_k \geq 0$, we show that H is positive semidefinite. Chong and Żak (2001), page 30, explain that a symmetric matrix Q is positive semidefinite if and only if every principal minor of Q is nonnegative.

Theorem 1 *For $i = 1, ..., N$ and $k = 1, ..., K$, the function $f_{ik}(\bar{w}_k)$ is a convex function over $T = \{\bar{w}_k | \bar{w}_k \geq 0\}$.*

Proof We define an open set $S = \{\bar{w}_k | Z_{1k}(\bar{w}_k) > 0\}$. Recall that $Z_{1k}(\bar{w}_k) < 0$. Thus, $T = \{\bar{w}_k | \bar{w}_k \geq 0\}$ is a closed subset of S.

The first principal minors of H are the elements of the diagonal of H: $H_{ii} = \frac{2B_k\left(\sum_{l \neq i}^{N} \frac{D_{ik}^{\lambda_k}}{D_{lk}^{\lambda_k}} w_{lk} + A_k D_{ik}^{\lambda_k}\right)^2}{Z_{1k}(\bar{w}_k)^3}$ and $H_{uu} = \frac{2B_k w_{ik}^2 \left(\frac{D_{ik}^{\lambda_k}}{D_{uk}^{\lambda_k}}\right)^2}{Z_{1k}(\bar{w}_k)^3}$, $u \neq i$. Both H_{ii} and H_{uu} are nonnegative since $Z_{1k}(\bar{w}_k)$ is strictly positive. Thus they are positive on S and on T. We have shown, in Lemmas 3 and 4, that for any β, any γ, and $p \geq 2$, $M_{\beta,\gamma,p}(H) = 0$ over the set of all real values of $\bar{w}_k$. Thus, every principal minor is nonnegative on the sets S and T. Thus, H is positive semidefinite for values on T which implies that $f_{ik}(\bar{w}_k)$ is convex over T. ■

Since we have shown that $f_{ik}(\bar{w}_k)$ is a convex function over the set $\{\bar{w}_k | \bar{w}_k \geq 0\}$, we are now able to show that the objective function of WSP is convex over the set $\{\mathbf{W} | \mathbf{W} \geq 0\}$.

Corollary 1 *The objective function of model WSP, $\sum_{i=1}^{N} \sum_{k=1}^{K} f_{ik}(\bar{w}_k)$, is a convex function of $\mathbf{W} \geq 0$.*

Proof Hillier and Lieberman (2001), page 1163, explain that the sum of convex functions is a convex function. According to Theorem 1, for $i = 1, ..., N$ and $k = 1, ..., K$, the function $f_{ik}(\bar{w}_k)$ is a convex function of $\bar{w}_k \geq 0$. Thus, the objective function is convex on $\mathbf{W} \geq 0$. ■

Now we will analyze the function $g_{ik}(\bar{w}_k)$. Avriel et al. (1988), page 68, explain that a quasimonotone function is both quasiconcave and quasiconvex. Following Floudas (1995), Theorem 2.3.1, for h defined on the convex set $T \subset \mathbb{R}^n$, $h(\bar{x})$ is quasiconvex on T if $T_\beta = \{x | x \in T, h(\bar{x}) \leq \beta\}$ is convex for all $\beta \in \mathbb{R}$. Similarly, $h(\bar{x})$ is quasiconcave on T if $T_\beta = \{x | x \in T, h(\bar{x}) \geq \beta\}$ is convex for all $\beta \in \mathbb{R}$. So, $h(\bar{x})$ is quasimonotonic on T if $T_\beta = \{x | x \in T, h(\bar{x}) = \beta\}$ is convex for all $\beta \in \mathbb{R}$.

Theorem 2 *The function $g_{ik}(\bar{w}_k)$ is a quasimonotone function on $T = \{\bar{w}_k | \bar{w}_k \geq 0\}$.*

Proof We define $T_\beta = \{\bar{w}_k | \bar{w}_k \in T,\ g_{ik}(\bar{w}_k) = \beta\}$.

First we consider the cases where T_β is the empty set or where T_β contains only one vector, $\bar{w}_k$. In either case the set is convex.

Now, we consider the case where T_β has at least two component vectors. Then we show that for any two distinct vectors, $\bar{w}_k^1$ and $\bar{w}_k^2$, the set T_β is convex. Since $\bar{w}_k^1$ and $\bar{w}_k^2$ are elements of T_β, $g_{ik}(\bar{w}_k^1) = \beta$ and $g_{ik}(\bar{w}_k^2) = \beta$. So,

$$B_k \frac{\frac{w_{ik}^1}{D_{ik}^{\lambda_k}}}{\sum_{l=1}^N \frac{w_{lk}^1}{D_{lk}^{\lambda_k}} + A_k} = \beta \text{ and } B_k \frac{\frac{w_{ik}^2}{D_{ik}^{\lambda_k}}}{\sum_{l=1}^N \frac{w_{lk}^2}{D_{lk}^{\lambda_k}} + A_k} = \beta. \tag{2}$$

This implies that

$$B_k \frac{w_{ik}^1}{D_{ik}^{\lambda_k}} = \beta(\sum_{l=1}^N \frac{w_{lk}^1}{D_{lk}^{\lambda_k}} + A_k) \text{ and } B_k \frac{w_{ik}^2}{D_{ik}^{\lambda_k}} = \beta(\sum_{l=1}^N \frac{w_{lk}^2}{D_{lk}^{\lambda_k}} + A_k).$$

So,

$$g_{ik}(\pi\bar{w}_k^1 + (1-\pi)\bar{w}_k^2) = \frac{\frac{\pi B_k w_{ik}^1 + (1-\pi) B_k w_{ik}^2}{Dik^{\lambda_k}}}{\sum_{l=1}^N \frac{\pi w_{lk}^1 + (1-\pi) w_{lk}^2}{D_{lk}^{\lambda_k}} + A_k},$$

where $0 \leq \pi \leq 1$. From Equations 2,

$$\begin{aligned} g_{ik}(\pi\bar{w}_k^1 + (1-\pi)\bar{w}_k^2) &= \frac{\pi\beta(\sum_{l=1}^N \frac{w_{lk}^1}{D_{lk}^{\lambda_k}} + A_k) + (1-\pi)\beta(\sum_{l=1}^N \frac{w_{lk}^2}{D_{lk}^{\lambda_k}} + A_k)}{\sum_{l=1}^N \frac{\pi w_{lk}^1 + (1-\pi) w_{lk}^2}{D_{lk}^{\lambda_k}} + A_k} \\ &= \frac{\beta(\sum_{l=1}^N \frac{\pi w_{lk}^1 + (1-\pi) w_{lk}^2}{D_{lk}^{\lambda_k}} + A_k)}{\sum_{l=1}^N \frac{\pi w_{lk}^1 + (1-\pi) w_{lk}^2}{D_{lk}^{\lambda_k}} + A_k} \\ &= \beta. \end{aligned}$$

So, T_β is convex for all $\beta \in \mathbb{R}$. Thus, $g_{ik}(\bar{w}_k)$ is quasimonotone on T. ■

We have shown in Theorem 2 that $g_{ik}(\bar{w}_k)$ is quasimonotone. However, the sum of quasimonotone functions is not necessarily quasimonotone. Furthermore, we are unable to show that the constraints form a convex set of feasible solutions. This means that a solution to WSP satisfying the KKT conditions is not necessarily an optimal solution. Because of that we will examine the multiple facility, single market workforce supply problem after discussing some other basic results.

The function g_{ik} has other nice properties. Theorems 3, 4, and 5 establish these properties.

Theorem 3 *The function $g_{ik}(\bar{w}_k)$ is strictly increasing in w_{ik}.*

Proof Anton (1998), Definition 5.1.1, states that a function, h, is strictly increasing in x if $x^1 < x^2$ implies that $h(x^1) < h(x^2)$.

Let $w_{ik}^1 < w_{ik}^2$ and $w_{lk}^1 = w_{lk}^2,\ \forall l \neq i$. Then

$$\begin{aligned}
g_{ik}(\bar{w}_k^1) - g_{ik}(\bar{w}_k^2) &= \frac{\frac{B_k w_{ik}^1}{D_{ik}^{\lambda_k}}}{\frac{w_{ik}^1}{D_{ik}^{\lambda_k}} + \sum_{l\neq i} \frac{w_{lk}^1}{D_{lk}^{\lambda_k}} + A_k} - \frac{\frac{B_k w_{ik}^2}{D_{ik}^{\lambda_k}}}{\frac{w_{ik}^2}{D_{ik}^{\lambda_k}} + \sum_{l\neq i} \frac{w_{lk}^2}{D_{lk}^{\lambda_k}} + A_k} \\
&= \frac{\frac{B_k w_{ik}^1}{D_{ik}^{\lambda_k}}\left(\frac{w_{ik}^2}{D_{ik}^{\lambda_k}} + \sum_{l\neq i} \frac{w_{lk}^2}{D_{lk}^{\lambda_k}} + A_k\right) - \frac{B_k w_{ik}^2}{D_{ik}^{\lambda_k}}\left(\frac{w_{ik}^1}{D_{ik}^{\lambda_k}} + \sum_{l\neq i} \frac{w_{lk}^1}{D_{lk}^{\lambda_k}} + A_k\right)}{\left(\sum_{l=1}^N \frac{w_{lk}^1}{D_{lk}^{\lambda_k}} + A_k\right)\left(\sum_{l=1}^N \frac{w_{lk}^2}{D_{lk}^{\lambda_k}} + A_k\right)} \\
&= \left(\frac{w_{ik}^1 - w_{ik}^2}{D_{ik}^{\lambda_k}}\right) \frac{B_k\left(\sum_{l\neq i} \frac{w_{lk}^1}{D_{lk}^{\lambda_k}} + A_k\right)}{\left(\sum_{l=1}^N \frac{w_{lk}^1}{D_{lk}^{\lambda_k}} + A_k\right)\left(\sum_{l=1}^N \frac{w_{lk}^2}{D_{lk}^{\lambda_k}} + A_k\right)}
\end{aligned}$$

All the components of this function are positive with the exception of $w_{ik}^1 - w_{ik}^2$ because $w_{ik}^1 < w_{ik}^2$. This means the difference is negative. So, $g_{ik}(\bar{w}_k^1) < g_{ik}(\bar{w}_k^2)$. ■

Theorem 4 establishes the relationship that exists between the different internal facilities.

Theorem 4 *The function $g_{ik}(w_k)$ is strictly decreasing in $w_{qk},\ q \neq i$.*

Proof Anton (1998), Definition 5.1.1, also states that a function, h, is strictly decreasing in x if $x^1 < x^2$ implies that $h(x^1) > h(x^2)$.

Let $w_{qk}^1 < w_{qk}^2$, $w_{lk}^1 = w_{lk}^2$, $\forall l \neq q, i$, and $w_{ik}^1 = w_{ik}^2$. Then

$$\begin{aligned}
g_{ik}(\bar{w}_k^1) - g_{ik}(\bar{w}_k^2) &= \frac{\frac{B_k w_{ik}^1}{D_{ik}^{\lambda_k}}}{\frac{w_{qk}^1}{D_{qk}^{\lambda_k}} + \sum_{l\neq q} \frac{w_{lk}^1}{D_{lk}^{\lambda_k}} + A_k} - \frac{\frac{B_k w_{ik}^2}{D_{ik}^{\lambda_k}}}{\frac{w_{qk}^2}{D_{qk}^{\lambda_k}} + \sum_{l\neq q} \frac{w_{lk}^2}{D_{lk}^{\lambda_k}} + A_k} \\
&= \frac{\frac{B_k w_{ik}^1}{D_{ik}^{\lambda_k}} \left(\frac{w_{qk}^2}{D_{qk}^{\lambda_k}} + \sum_{l\neq q} \frac{w_{lk}^2}{D_{lk}^{\lambda_k}} + A_k \right) - \frac{B_k w_{ik}^2}{D_{ik}^{\lambda_k}} \left(\frac{w_{qk}^1}{D_{qk}^{\lambda_k}} + \sum_{l\neq q} \frac{w_{lk}^1}{D_{lk}^{\lambda_k}} + A_k \right)}{\left(\sum_{l=1}^N \frac{w_{lk}^1}{D_{lk}^{\lambda_k}} + A_k \right) \left(\sum_{l=1}^N \frac{w_{lk}^2}{D_{lk}^{\lambda_k}} + A_k \right)} \\
&= \frac{B_k \frac{w_{ik}^1}{D_{ik}^{\lambda_k}} \left(\frac{w_{qk}^2 - w_{qk}^1}{D_{qk}^{\lambda_k}} \right)}{\left(\sum_{l=1}^N \frac{w_{lk}^1}{D_{lk}^{\lambda_k}} + A_k \right) \left(\sum_{l=1}^N \frac{w_{lk}^2}{D_{lk}^{\lambda_k}} + A_k \right)}
\end{aligned}$$

All the components of this function are positive including $w_{qk}^2 - w_{qk}^1$ because $w_{qk}^1 < w_{qk}^2$. This means the difference is positive. So, $g_{ik}(\bar{w}_k^1) > g_{ik}(\bar{w}_k^2)$. ■

Theorems 3 and 4 show that there are two ways for a company to increase the number of workers attracted from a labor market, k, to a facility, i. The first way is to increase the wage paid by facility i, w_{ik}. However, this also results in a decrease in the number of workers attracted to every other internal facility. The other way is to reduce the wage paid by a different internal facility, w_{qk} for $q \neq i$. However, this also decreases the number of workers attracted to work at facility q.

Theorem 5 $\lim_{w_{ik}\to\infty} g_{ik}(\bar{w}_k) = B_k$

Proof We define the numerator and denominator of $g_{ik}(\bar{w}_k)$ to be $\bar{g}_{ik}(w_{ik})$ and $\underline{g}_{ik}(\bar{w}_k)$, respectively. According to Anton (1998), Theorem 4.7.1, l'Hospital's rule states that if $\lim \bar{g}_{ik}(w_{ik})$ and $\lim \underline{g}_{ik}(\bar{w}_k)$ are both infinity and if $\lim \frac{\bar{g}'_{ik}(w_{ik})}{\underline{g}'_{ik}(\bar{w}_k)}$ has a finite value then $\lim \frac{\bar{g}_{ik}(w_{ik})}{\underline{g}_{ik}(\bar{w}_k)} = \lim \frac{\bar{g}'_{ik}(w_{ik})}{\underline{g}'_{ik}(\bar{w}_k)}$.

Thus for $g_{ik}(\bar{w}_k) = \frac{\frac{B_k w_{ik}}{D_{ik}^{\lambda_k}}}{\sum_{l=1}^N \frac{w_{lk}}{D_{lk}^{\lambda_k}} + A_k}$,

$$\bar{g}_{ik}(w_{ik}) = B_k \frac{w_{ik}}{D_{ik}^{\lambda_k}} \quad \text{and} \quad \underline{g}_{ik}(\bar{w}_k) = \sum_{l=1}^N \frac{w_{lk}}{D_{lk}^{\lambda_k}} + A_k.$$

This implies that

$$\lim_{w_{ik}\to\infty} \bar{g}_{ik}(w_{ik}) \quad = \quad \infty \quad \text{and} \quad \lim_{w_{ik}\to\infty} \underline{g}_{ik}(\bar{w}_k) \quad = \quad \infty.$$

The derivatives of the functions are

$$\tfrac{\partial \bar{g}_{ik}(w_{ik})}{\partial w_{ik}} \quad = \quad \tfrac{B_k}{D_{ik}^{\lambda_k}} \quad \text{and} \quad \tfrac{\partial \underline{g}_{ik}(\bar{w}_k)}{\partial w_{ik}} \quad = \quad \tfrac{1}{D_{ik}^{\lambda_k}}.$$

So $\lim_{w_{ik}\to\infty} g_{ik}(\bar{w}_k) = \frac{\frac{B_k}{D_{ik}^{\lambda_k}}}{\frac{1}{D_{ik}^{\lambda_k}}} = B_k$. ■

As the wage paid by an internal facility to workers from a labor market is increased the number of workers attracted approaches the entire population of the labor market. However, since $\displaystyle\sum_{l=1}^{N} \frac{w_{lk}}{D_{lk}^{\lambda_k}} + A_k > \frac{w_{ik}}{D_{ik}^{\lambda_k}}$, $g_{ik}(\bar{w}_k)$ can never be equal to B_k.

The structure of model WSP shown in the previous lemmas and theorems allows us to understand the relationships between the different internal facilities and between internal and external facilities. It also provides insight into the effect that wage decisions have on all internal facilities. At the same time, the structure of the nonlinear model, WSP, does not fit the Karush-Kuhn-Tucker sufficient conditions for a solution to be optimal. Thus, we present a simplified model, the Multiple Facility, Single Market WSP, in preparation for reformulating the Multiple Facility, Multiple Market WSP. The reformulation captures all the same relationships but can be shown to satisfy the KKT sufficiency conditions for an optimal solution.

3.2 Multiple Facility, Single Market Workforce Supply Model

Now, we present the multiple facility, single market model of the workforce supply problem. It is assumed that there are N company facilities in the region that are coordinating their labor decisions. The wage paid to workers at the single market must be determined for each of the internal facilities. The model is

WSP-SM

$$\min \sum_{i=1}^{N} f_{i1}(\bar{w}_1)$$

$$\text{s.t.} \quad g_{i1}(\bar{w}_1) \geq R_i, \ \forall i \in \{1, ..., N\} \tag{3}$$

$$w_{i1} \geq \ 0, \ \forall i \in \{1, ..., N\}. \tag{4}$$

Despite the nice properties of the functions $g_{ik}(\bar{w}_k)$ and $f_{ik}(\bar{w}_k)$, it is useful to simplify the multiple facility, single market WSP model even further. The constraints can be linearized.

Theorem 6 *The constraint set for WSP-SM can be linearized.*

Proof The constraint set 4 is already linear for all i. The constraint set 3 can be written as $B_1 \frac{\frac{w_{i1}}{D_{i1}^{\lambda_1}}}{\sum_{l=1}^{N} \frac{w_{l1}}{D_{l1}^{\lambda_1}} + A_1} \geq R_1$. Using algebra, this implies that

$$\frac{w_{i1}}{D_{i1}^{\lambda_1}} \geq \frac{R_i}{B_1} \left(\sum_{l=1}^{N} \frac{w_{l1}}{D_{l1}^{\lambda_1}} + A_1 \right)$$

$$\Leftrightarrow \quad w_{i1} \geq \frac{R_i D_{i1}^{\lambda_1}}{B_1} \left(\sum_{l=1}^{N} \frac{w_{l1}}{D_{l1}^{\lambda_1}} + A_1 \right)$$

$$\Leftrightarrow w_{i1} - \frac{R_i D_{i1}^{\lambda_1}}{B_1} \sum_{l=1}^{N} \frac{w_{l1}}{D_{l1}^{\lambda_1}} \geq \frac{R_i A_1 D_{i1}^{\lambda_1}}{B_1}.$$

This is a linear function in the variables $w_{l1}(l = 1..N)$. Constraint set 3 can be linearized for each facility i, $i = 1, ..., N$. ■

This is useful because we are able to show that there exists a solution to the system of linear equations. Then, we are able to show that the optimal solution to WSP-SM must be on the boundary of the feasible region. However, first, we present an optimal solution to WSP-SM and then show uniqueness of that solution.

Theorem 7 $w_{i1}(\bar{r}) = \frac{A_1 D_{i1}^{\lambda_1} R_i}{B_1 - \sum_{l=1}^{N} R_l}$ *is a feasible solution to WSP-SM.*

Proof If $w_{i1}(\bar{r}) = \frac{A_1 D_{i1}^{\lambda_1} R_i}{B_1 - \sum_{l=1}^{N} R_l}$, then the linearized required workforce constraint becomes

$$\begin{aligned}
& \frac{A_1 D_{i1}^{\lambda_1} R_i}{B_1 - \sum_{l=1}^{N} R_l} - \frac{R_i D_{i1}^{\lambda_1}}{B_1} \left(\sum_{l=1}^{N} \frac{\frac{A_1 D_{l1}^{\lambda_1} R_l}{B_1 - \sum_{m=1}^{N} R_m}}{D_{l1}^{\lambda_1}} \right) \geq \frac{R_i A_1 D_{i1}^{\lambda_1}}{B_1} \\
\Rightarrow \quad & \frac{B_1 A_1}{B_1 - \sum_{l=1}^{N} R_l} - \sum_{l=1}^{N} \frac{A_1 R_l}{B_1 - \sum_{m=1}^{N} R_m} \geq A_1 \\
\Rightarrow \quad & A_1 \left(B_1 - \sum_{l=1}^{N} R_l \right) \geq A_1 \left(B_1 - \sum_{m=1}^{N} R_m \right).
\end{aligned}$$

So, the required workforce constraint for internal facilities i, $i = 1, ..., N$, are satisfied as equalities. Note that $w_{ik}(\bar{r}) = \frac{A_k D_{ik}^{\lambda_k} R_i}{B_k - \sum_{l=1}^{N} R_l} > 0$ for all $i = 1, ..., N$. Thus, this solution is a feasible solution to WSP-SM. We have shown that $w_{i1}(\bar{r}) = \frac{A_1 D_{i1}^{\lambda_1} R_i}{B_1 - \sum_{l=1}^{N} R_l}$ solves WSP-SM. At the same time, we have shown that this solution satisfies each of the constraints in set 3 as equalities. ■

In the proof of Theorem 7, we show that $w_{ik}(\bar{r}) = \frac{A_k D_{ik}^{\lambda_k} R_i}{B_k - \sum_{l=1}^{N} R_l}$ is not only a feasible solution to WSP-SM, but it solves the system of equations formed from constraint set 3. Now, we argue that the solution to the system of equations, $w_{ik}(\bar{r})$, is unique. We will examine the objective value of this solution in Theorem 9 below.

Now let $\mathbf{A}$ be the matrix consisting of the coefficients of the system of equations:

$$w_{i1} - \frac{R_i D_{i1}^{\lambda_1}}{B_1} \sum_{l=1}^{N} \frac{w_{l1}}{D_{l1}^{\lambda_1}} = \frac{R_i A_1 D_{i1}^{\lambda_1}}{B_1}, \; i = 1, ..., N. \tag{5}$$

Thus, the N x N matrix is

$$\mathbf{A} = \begin{bmatrix} 1 - \frac{R_1}{B_1} & -\frac{R_1 D_{11}^{\lambda_1}}{B_1 D_{21}^{\lambda_1}} & \cdots & -\frac{R_1 D_{11}^{\lambda_1}}{B_1 D_{N1}^{\lambda_1}} \\ -\frac{R_2 D_{21}^{\lambda_1}}{B_1 D_{11}^{\lambda_1}} & 1 - \frac{R_2}{B_1} & \cdots & -\frac{R_2 D_{21}^{\lambda_1}}{B_1 D_{N1}^{\lambda_1}} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{R_N D_{N1}^{\lambda_1}}{B_1 D_{11}^{\lambda_1}} & -\frac{R_N D_{N1}^{\lambda_1}}{B_1 D_{21}^{\lambda_1}} & \cdots & 1 - \frac{R_N}{B_1} \end{bmatrix}.$$

In order to show that the solution shown in Theorem 7 is unique we present the inverse of $\mathbf{A}$ and show that it always exists.

Lemma 5 *The inverse of* $\mathbf{A}$ *is*

$$\mathbf{A}^{-1} = \frac{1}{B_1 - \sum_{i=1}^{N} R_i} \begin{bmatrix} B_1 - \sum_{i\neq 1} R_i & R_1 \frac{D_{11}^{\lambda_1}}{D_{21}^{\lambda_1}} & \dots & R_1 \frac{D_{11}^{\lambda_1}}{D_{N1}^{\lambda_1}} \\ R_2 \frac{D_{21}^{\lambda_1}}{D_{11}^{\lambda_1}} & B_1 - \sum_{i\neq 2} R_i & \dots & R_2 \frac{D_{21}^{\lambda_1}}{D_{N1}^{\lambda_1}} \\ \vdots & \vdots & \ddots & \vdots \\ R_N \frac{D_{N1}^{\lambda_1}}{D_{11}^{\lambda_1}} & R_N \frac{D_{N1}^{\lambda_1}}{D_{21}^{\lambda_1}} & \dots & B_1 - \sum_{i\neq N} R_i \end{bmatrix}.$$

Proof We show that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ where $\mathbf{I}$ is the identity matrix. We show this in two steps. The first step is to show that the diagonal components of $\mathbf{A}^{-1}\mathbf{A}$ are 1. The ith diagonal component is the product of the ith row of $\mathbf{A}^{-1}$ and the ith column of $\mathbf{A}$. That is

$$\frac{1}{B_1 - \sum_{i=1}^{N} R_i} \left[R_i \frac{D_{i1}^{\lambda_1}}{D_{11}^{\lambda_1}}, ..., B_1 - \sum_{j\neq i} R_j, ..., R_i \frac{D_{i1}^{\lambda_1}}{D_{N1}^{\lambda_1}} \right] \text{ x } \left[-\frac{R_1 D_{11}^{\lambda_1}}{B_1 D_{i1}^{\lambda_1}}, ..., 1 - \frac{R_i}{B_1}, ..., -\frac{R_N D_{N1}^{\lambda_1}}{B_1 D_{i1}^{\lambda_1}} \right]^T$$

$$\begin{aligned}
=&\frac{1}{B_1 - \sum_{j=1}^{N} R_j} \left(\frac{(B_1 - R_i)(B_1 - \sum_{j\neq i} R_j)}{B_1} - \sum_{j\neq i} \frac{R_i D_{i1}^{\lambda_1}}{D_{j1}^{\lambda_1}} \frac{R_j D_{j1}^{\lambda_1}}{B_1 D_{i1}^{\lambda_1}} \right) \\
=&\frac{1}{B_1 - \sum_{j=1}^{N} R_j} \left(\frac{B_1^2 - B_1 R_i - B_1 \sum_{j\neq i} R_j + R_i \sum_{j\neq i} R_j}{B_1} - \frac{R_i}{B_1} \sum_{j\neq i} R_j \right) \\
=&\frac{1}{B_1 - \sum_{j=1}^{N} R_j} \left(\frac{B_1^2 - B_1 R_i - B_1 \sum_{j\neq i} R_j}{B_1} \right) \\
=&\frac{1}{B_1 - \sum_{j=1}^{N} R_j} \left(B_1 - \sum_{j=1}^{N} R_j \right) \\
=&1.
\end{aligned}$$

The second step is to show that the off-diagonal components of $\mathbf{A}^{-1}\mathbf{A}$ are 0. We consider the ith row of $\mathbf{A}^{-1}$ and the jth column of $\mathbf{A}$. That is

$$\frac{1}{B_1 - \sum_{i=1}^{N} R_i} \left[R_i \frac{D_{i1}^{\lambda_1}}{D_{11}^{\lambda_1}}, ..., B_1 - \sum_{j\neq i} R_j, ..., R_i \frac{D_{i1}^{\lambda_1}}{D_{N1}^{\lambda_1}} \right] \text{ x } \left[-\frac{R_1 D_{11}^{\lambda_1}}{B_1 D_{j1}^{\lambda_1}}, ..., 1 - \frac{R_j}{B_1}, ..., -\frac{R_N D_{N1}^{\lambda_1}}{B_1 D_{j1}^{\lambda_1}} \right]^T$$

$$
\begin{aligned}
=&\frac{1}{B_1-\sum_{l=1}^{N}R_l}\left(\frac{R_iD_{i1}^{\lambda_1}}{B_1D_{j1}^{\lambda_1}}(B_1-R_j)-\frac{R_iD_{i1}^{\lambda_1}}{B_1D_{j1}^{\lambda_1}}\left(B_1-\sum_{l\neq i}R_l\right)-\sum_{l\neq i,j}\frac{R_lD_{l1}^{\lambda_1}}{B_1D_{j1}^{\lambda_1}}\frac{R_iD_{i1}^{\lambda_1}}{D_{l1}^{\lambda_1}}\right)\\
=&\frac{1}{B_1-\sum_{l=1}^{N}R_l}\left(\frac{R_iD_{i1}^{\lambda_1}}{B_1D_{j1}^{\lambda_1}}\left(\sum_{l\neq i}R_l-R_j\right)-\frac{R_iD_{i1}^{\lambda_1}}{B_1D_{j1}^{\lambda_1}}\sum_{l\neq i,j}R_l\right)\\
=&\frac{1}{B_1-\sum_{l=1}^{N}R_l}\left(\frac{R_iD_{i1}^{\lambda_1}}{B_1D_{j1}^{\lambda_1}}\sum_{l\neq i}R_l-\frac{R_iD_{i1}^{\lambda_1}}{B_1D_{j1}^{\lambda_1}}\sum_{l\neq i}R_l\right)\\
=&0.
\end{aligned}
$$

Since the product of $\mathbf{A}^{-1}$ and $\mathbf{A}$ is a matrix where every diagonal component is 1 and every off-diagonal component is 0, the product is the N x N identity matrix. ■

Note that the only way $\mathbf{A}^{-1}$ does not exist is if $B_1-\sum_{i=1}^{N}R_i=0$. However, by assumption 5, $\sum_{i=1}^{N}R_i<B_1$. Thus, $B_1-\sum_{i=1}^{N}R_i$ is always positive.

Theorem 8 $w_{ik}(\bar{r})=\frac{A_kD_{ik}^{\lambda_k}R_i}{B_k-\sum_{l=1}^{N}R_l}$ *is the unique solution to the system of equations,*

$$
w_{i1}-\frac{R_iD_{i1}^{\lambda_1}}{B_1}\sum_{l=1}^{N}\frac{w_{l1}}{D_{l1}^{\lambda_1}}=\frac{R_iA_1D_{i1}^{\lambda_1}}{B_1},\ i=1,...,N,
$$

derived from the constraint set 3 of WSP-SM.

Proof Friedberg et al. (1997), in Theorem 3.10, show that a system of equations, $\mathbf{A}\bar{w}_1=b$ has exactly one solution if and only if $\mathbf{A}$ is invertible. Since we have shown, in Lemma 5, that $\mathbf{A}^{-1}$ always exists we know that the system of equations has a unique solution. ■

We have shown that there exists a unique solution to WSP-SM that lies on the boundary of the feasible region. However, there are many feasible solutions that are not on the boundary. Now we show that the unique solution to the system of equations 5 is the optimal solution to WSP-SM.

Theorem 9 *The optimal solution to the multiple facility, single market WSP model is on the boundary of the feasible region where every required workforce constraint, Equation 3, is satisfied as an equality. Specifically, the optimal solution is* $w_{i1}^*(\bar{R})=\frac{A_1D_{i1}^{\lambda_1}R_i}{B_1-\sum_{l=1}^{N}R_l}$.

Proof We have shown that there exists a unique feasible solution where every required workforce constraint is satisfied as an equality. This solution is $\bar{w}_1^1 = \{w_{i1}^1(\bar{R}) = \frac{A_1 D_{i1}^{\lambda_1} R_i}{B_1 - \sum_{l=1}^N R_l},\ i = 1...N\}$. Now, we compare this solution to a different feasible solution that is not on the boundary. In particular, we consider the solution where the number of workers attracted to work at some facility, q, is $\hat{R} > R_q$ while all other facilities attract $R_i, i = 1, ..., q-1, q+1, ..., N$. Since we have shown that Equation 3 can be linearized and that there is a unique solution to the resulting system of linear equations. The only change to the solution is $\hat{R}$ replacing R_q in that solution. Thus,

$$\bar{w}_1^2 = \{w_{i1}^2(\bar{R}) = \frac{A_1 D_{i1}^{\lambda_1} R_i}{B_1 - \sum_{l \neq q} R_l - \hat{R}},\ i \neq q\} \bigcup \{w_{q1}^2 = \frac{A_1 D_{i1}^{\lambda_1} \hat{R}}{B_1 - \sum_{l \neq q} R_l - \hat{R}}\}$$

is the solution not on the boundary.

In order to compare the objective function values of each solution to the problem, we use the fact that $g_{i1}(\bar{w}_1^1) = R_i, i = 1...N$ and $g_{i1}(\bar{w}_1^2) = R_i, i \neq q$ and $g_{q1}(\bar{w}_1^2) = \hat{R}$. So the objective function value for the solution on the boundary is

$$\sum_{i=1}^N w_{i1}^1 g_{i1}(\bar{w}_1^1) = \sum_{i=1}^N w_{i1}^1 R_i = \sum_{i=1}^N \frac{A_1 D_{i1}^{\lambda_1} R_i^2}{B_1 - \sum_{l=1}^N R_l}.$$

The objective function value for the solution not on the boundary is

$$\sum_{i=1}^N w_{i1}^2 g_{i1}(\bar{w}_1^2) = \sum_{i \neq q} w_{i1}^2 R_i + w_{q1}^2 \hat{R} = \sum_{i \neq q} \frac{A_1 D_{i1}^{\lambda_1} R_i^2}{B_1 - \sum_{l \neq q} R_l - \hat{R}} + \frac{A_1 D_{q1}^{\lambda_1} \hat{R}^2}{B_1 - \sum_{l \neq q} R_l - \hat{R}}.$$

We first compare the labor costs of every facility except q. The only difference in the costs of the two solutions is that they have a different denominator. Since $\hat{R} > R_q$,

$$B_1 - \sum_{l \neq q} R_l - \hat{R} < B_1 - \sum_{l=1}^N R_l.$$

Thus, the numerator for the solution on the boundary, $\bar{w}_1^1$, is larger. This implies that

$$\frac{A_1 D_{i1}^{\lambda_1} R_i^2}{B_1 - \sum_{l \neq q} R_l - \hat{R}} > \frac{A_1 D_{i1}^{\lambda_1} R_i^2}{B_1 - \sum_{l=1}^N R_l}, \forall i \neq q.$$

Now we compare the labor costs of facility q. Again, the denominator of the solution on the boundary, $\bar{w}_1^1$, is larger because $B_1 - \sum_{l \neq q} R_l - \hat{R} < B_1 - \sum_{l=1}^N R_l$. We compare the numerators of the two terms for facility q. Because $\hat{R} > R_q$ we know that $\hat{R}^2 > R_q^2$ since $\hat{R} > R_q \geq 1$. Thus, $A_1 D_{i1}^{\lambda_1} \hat{R}^2 > A_1 D_{i1}^{\lambda_1} R_q^2$. For $\bar{w}_1^1$ the qth term has a smaller numerator and a larger denominator. This implies that

$$\frac{A_1 D_{i1}^{\lambda_1} \hat{R}^2}{B_1 - \sum_{l \neq q} R_l - \hat{R}} > \frac{A_1 D_{i1}^{\lambda_1} R_q^2}{B_1 - \sum_{l=1}^{N} R_l}.$$

For every facility the labor costs of the solution on the boundary are lower than the labor costs of the solution where some facility attracts more than its required workforce. So, a solution on the boundary is always better than one that is not. Thus, the optimal solution is $\bar{w}_1^1$ and the objective function value is $\displaystyle\sum_{i=1}^{N} \frac{A_1 D_{i1}^{\lambda_1} R_i^2}{B_1 - \sum_{l=1}^{N} R_l}$. ■

There are two immediate corollaries to the result above. In the first one we show the closed-form solution to WSP-SM for the special case where there is a single facility that is making the wage and labor decisions. We do not provide a proof because the result is a simple modification of the closed-form solution.

Corollary 2 *If there is only a single facility the optimal solution is* $w_{11}^*(\bar{R}) = \frac{A_1 D_{11}^{\lambda_1} R_1}{B_1 - R_1}$ *and the total labor costs of the facility are* $\frac{A_1 D_{11}^{\lambda_1} R_1^2}{B_1 - R_1}$.

In the second corollary to Theorem 9 we show, using the closed form solution to WSP-SM, that every facility that requires workers from the labor market will pay a strictly positive wage to those workers.

Corollary 3 *Let* $\bar{w}_k^*$ *be the optimal solution to the WSP-SM. Then* $w_{i1}^* > 0$

Proof The key to this result is that $w_{i1}^*(\bar{r}) = \frac{A_1 D_{i1}^{\lambda_1} R_i}{B_1 - \sum_{l=1}^{N} R_l}$, $i = 1...N$. We have assumed that $A_1 > 0$, $B_1 - \sum_{l=1}^{N} R_l > 0$, $D_{i1}^{\lambda_1} > 0$. $i = 1, ..., N$, and $R_i > 0$, $i = 1, ..., N$. Thus, $w_{i1}^*(\bar{r}) > 0$ because it is a quotient of positive numbers. ■

In summary, we have shown that the multiple facility, single market workforce supply model can be linearized, has a solution on the boundary of the feasible region where the required workforce constraints are satisfied as equalities, and that the optimal solution is $w_{i1}^* = \frac{A_1 D_{i1}^{\lambda_1} R_i}{B_1 - \sum_{l=1}^{N} R_l}$, $i = 1...N$. The wage paid by any internal facility i to workers from a market varies linearly with A_1 and $D_{i1}^{\lambda_1}$. It is easy to see that $\lim_{B_1 \to \infty} w_{i1}^* = 0$. The wage paid by any of

Figure 2: The relationship between the wage paid by some facility and the population of the labor market

the internal facilities decreases asymptotically toward 0 as the available workforce increases. This relationship is shown in Figure 2. Next, we examine the effect of R_1 on w_{i1}^*. We have

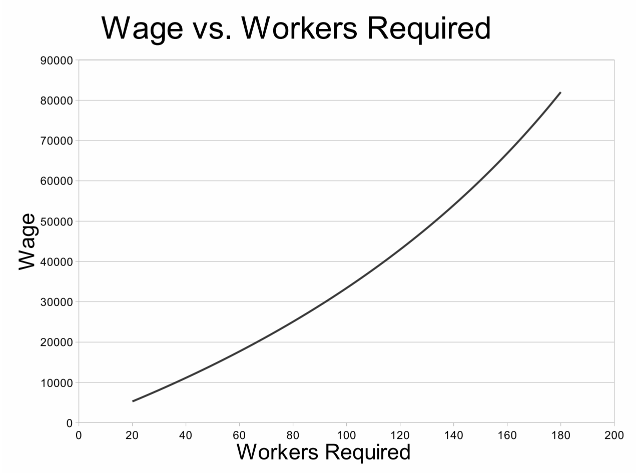

Figure 3: The relationship between the wage paid by some facility and the number of workers that facility requires

assumed that $\sum_{i=1}^{N} R_i < B_1$. Thus, $R_1 < B_1 - \sum_{j \neq 1} R_j$. It is clear that $\lim_{R_1 \to B_1 - \sum_{j \neq 1} R_j} w_{i1}^* = \infty$

because the denominator approaches 0 while the numerator approaches $A_1 D_{i1}^{\lambda_1}(B_1 - \sum_{j \neq 1} R_j)$ and $A_1 D_{i1}^{\lambda_1} R_{i1}$ for w_{11} and w_{i1}, respectively. Figure 3 demonstrates this relationship.

We now investigate how sensitive the total labor costs are to the different parameters in the model. To do so, we compare two problems where a single parameter is different in each problem. Sometimes we assume that the difference in the two parameters is additive and other times we assume it is multiplicative. We use the type of difference that is most convenient and informative for each parameter.

First, we determine the sensitivity of the total labor costs to the level of competition from external facilities competing in the market. We compare two separate WSP-SM problems where every parameter is the same except the level of external competition. Let A_1 be the level of external competition in Problem 1 and A'_1 that of Problem 1′. Let $A'_1 = \alpha_1 A_1$, where $\alpha_1 > 1$. This means that $A_1 < A'_1$. The relationship between the wages paid by each facility i is

$$w'_{i1}(\bar{R}) = \frac{A'_1 D_{i1}^{\lambda_1} R_i}{B_1 - \sum_{l=1}^{N} R_l} = \alpha_1 \frac{A_1 D_{i1}^{\lambda_1} R_i}{B_1 - \sum_{l=1}^{N} R_l} = \alpha_1 w_{i1}(\bar{R}), \ i = 1, ..., N.$$

The relationship between the optimal objective function values of the two problems is

$$TC' = \sum_{i=1}^{N} \frac{A'_1 D_{i1}^{\lambda_1} R_i^2}{B_1 - \sum_{l=1}^{N} R_l} = \sum_{i=1}^{N} \alpha_1 \frac{A_1 D_{i1}^{\lambda_1} R_i^2}{B_1 - \sum_{l=1}^{N} R_l} = \alpha_1 \sum_{i=1}^{N} \frac{A_1 D_{i1}^{\lambda_1} R_i^2}{B_1 - \sum_{l=1}^{N} R_l} = \alpha_1 TC.$$

Thus, the objective function value differs by the same factor, α_1, as the difference between the external competition amounts of the two problems. Wages and total cost are proportional to $\alpha_1 = \frac{A'_1}{A_1}$. Also, $\alpha_1 = \frac{w'_{i1}(\bar{R})}{w_{i1}(\bar{R})} = \frac{TC'}{TC}$.

Next, consider two separate WSP-SM problems where every parameter is equivalent except the attraction parameter. Let the parameter λ_1 be for Problem 2 and λ'_1 be the parameter for Problem 2′. Let $\lambda'_1 = \lambda_1 + \delta_2$, where $\delta_2 > 0$. Assume all other parameters of the two problems are equivalent. We have shown that the optimal wage paid in Problem 2 is

$$w_{i1}(\bar{R}) = \frac{A_1 D_{i1}^{\lambda_1} R_i}{B_1 - \sum_{l=1}^{N} R_l}, \ i = 1, ..., N.$$

The optimal value of the objective function of Problem 2 is $\sum_{i=1}^{N} \frac{A_1 D_{i1}^{\lambda_1} R_i^2}{B_1 - \sum_{l=1}^{N} R_l}$. The optimal wage paid by facilities in Problem 2′ is

$$\begin{aligned} w'_{i1}(\bar{R}) = & \frac{A_1 D_{i1}^{\lambda'_1} R_i}{B_1 - \sum_{l=1}^{N} R_l} \\ = & \frac{A_1 D_{i1}^{\lambda_1+\delta_2} R_i}{B_1 - \sum_{l=1}^{N} R_l} \\ = & D_{i1}^{\delta_2} \frac{A_1 D_{i1}^{\lambda_1} R_i}{B_1 - \sum_{l=1}^{N} R_l}, \quad i = 1, ..., N. \end{aligned}$$

The optimal objective function value of Problem 2′ is

$$\sum_{i=1}^{N} D_{i1}^{\delta_2} \frac{A_1 D_{i1}^{\lambda_1} R_i^2}{B_1 - \sum_{l=1}^{N} R_l}.$$

By comparing the optimal wage values in each problem, $\frac{w'_{i1}(\bar{R})}{w_{i1}(\bar{R})} = D_{i1}^{\delta_2}, \ i = 1, ..., N$, we see that they increase exponentially in Problem 2 as δ_2 increases. The difference between the two objective function values is

$$TC' - TC = \sum_{i=1}^{N} (D_{i1}^{\delta_2} - 1) \frac{A_1 D_{i1}^{\lambda_1} R_i^2}{B_1 - \sum_{l=1}^{N} R_l}.$$

Since $D_{i1}^{\delta_2} > 1$ the difference is positive, indicating that the objective function value of Problem 2 is smaller than that of Problem 2′.

The only characteristic of the market that we have not investigated yet is the population size. Now we investigate the sensitivity of the objective function value of WSP-SM to different sizes of market population, B_1. Again we consider two problems Problem 3 and Problem 3′ where every parameter is equivalent except $B_1 + \delta_3 = B'_1$, where $\delta_3 > \sum_{i=1}^{N} R_i - B_1$. We have shown the optimal wage and objective function values for Problem 3. The optimal wage for Problem 3′ is $w'_{i1} = \frac{A_1 D_{i1}^{\lambda_1} R_i}{B_1 + \delta_3 - \sum_{l=1}^{N} R_l}, \ i = 1, ..., N$. The optimal objective function value is $\displaystyle\sum_{i=1}^{N} \frac{A_1 D_{i1}^{\lambda_1} R_i^2}{B_1 + \delta_3 - \sum_{l=1}^{N} R_l}$. The difference between the two objective function values is

$$\begin{aligned}
TC - TC' =& \sum_{i=1}^{N} \frac{A_1 D_{i1}^{\lambda_1} R_i^2}{B_1 - \sum_{l=1}^{N} R_l} - \sum_{i=1}^{N} \frac{A_1 D_{i1}^{\lambda_1} R_i^2}{B_1 + \delta_3 - \sum_{l=1}^{N} R_l} \\
=& \frac{(B_1 + \delta_3 - \sum_{l=1}^{N} R_l) \sum_{i=1}^{N} A_1 D_{i1}^{\lambda_1} R_i^2 - (B_1 - \sum_{l=1}^{N} R_l) \sum_{i=1}^{N} A_1 D_{i1}^{\lambda_1} R_i^2}{(B_1 - \sum_{l=1}^{N} R_l)(B_1 + \delta_3 - \sum_{l=1}^{N} R_l)} \\
=& \frac{\delta_3 \sum_{i=1}^{N} A_1 D_{i1}^{\lambda_1} R_i^2}{(B_1 - \sum_{l=1}^{N} R_l)(B_1 + \delta_3 - \sum_{l=1}^{N} R_l)} \\
=& \frac{\delta_3}{B_1 + \delta_3 - \sum_{l=1}^{N} R_l} \sum_{i=1}^{N} \frac{A_1 D_{i1}^{\lambda_1} R_i^2}{B_1 - \sum_{l=1}^{N} R_l} \\
=& \frac{\delta_3}{B_1 + \delta_3 - \sum_{l=1}^{N} R_l} TC
\end{aligned}$$

Thus, we can rewrite $TC' = (1 - \frac{\delta_3}{B_1+\delta_3-\sum_{l=1}^{N} R_l})TC$. By assumption, $B_1 - \sum_{l=1}^{N} R_l > 0$. So, $B_1 + \delta_3 - \sum_{l=1}^{N} R_l > \delta_3$ and $0 < \frac{\delta_3}{B_1+\delta_3-\sum_{l=1}^{N} R_l} < 1$. This means that $TC' > TC$.

Now, we show how sensitive the objective function is to changes in facility characteristics. Since the indexing of facilities is arbitrary, we always consider that the changes in facility characteristics are in facility 1. First, we consider two WSP-SM problems where every parameter is the same except the distance that a single facility is from the market. Let D_{11} be the distance between the market and facility 1 in Problem 4. Let D'_{11} be the distance between the market and facility 1 in Problem 4′. Suppose that $D'_{11} = \alpha_4 D_{11}$, where $\alpha_4 > 1$. It is known that the optimal wage and objective function values of Problem 4 are $w_{i1}(\bar{R}) = \frac{A_1 D_{i1}^{\lambda_1} R_i}{B_1-\sum_{l=1}^{N} R_l}$, $i = 1, ..., N$ and $\displaystyle\sum_{i=1}^{N} \frac{A_1 D_{i1}^{\lambda_1} R_i^2}{B_1 - \sum_{l=1}^{N} R_l}$, respectively. The optimal wage value for facility 1 in Problem 4′ is $w'_{11}(\bar{R}) = \frac{A_1(\alpha_4 D_{11})^{\lambda_1} R_1}{B_1-\sum_{l=1}^{N} R_l} = \alpha_4^{\lambda_1} \frac{A_1 D_{11}^{\lambda_1} R_1}{B_1-\sum_{l=1}^{N} R_l} = \alpha_4^{\lambda_1} w_{11}(\bar{R})$. The optimal wage values for all other facilities in Problem 4′ are the same as those in Problem 4. The optimal objective function value of Problem 4′ is $\alpha_4^{\lambda_1} \frac{A_1 D_{11}^{\lambda_1} R_1^2}{B_1-\sum_{l=1}^{N} R_l} + \sum_{i=2}^{N} \frac{A_1 D_{i1}^{\lambda_1} R_i^2}{B_1-\sum_{l=1}^{N} R_l}$. The wage paid to workers by facility 1 changes by a factor of $\alpha_4^{\lambda_1}$. Thus, we know that $\alpha_4 = \frac{D'_{11}}{D_{11}}$ and

$\alpha_4^{\lambda_1} = \frac{w'_{11}(\bar{R})}{w_{11}(\bar{R})}$. The difference in the objective function values of Problem 4 and Problem 4′ is

$$TC - TC' = (1 - \alpha_4^{\lambda_1})\frac{A_1 D_{11}^{\lambda_1} R_1^2}{B_1 - \sum_{l=1}^{N} R_l}.$$

Since $\alpha_4 > 1$ and $\lambda_1 > 0$ by assumption, the optimal wage and objective function values of Problem 4′ are both larger then those of Problem 4. The change in both values is polynomial in α_4. However, the rate of change in the values also depends on the value of λ_1. We examine the function $c(\alpha_4) = \alpha_4^{\lambda_1}$. If $\lambda_1 < 1$ then $c(\alpha_4)$ is concave. If $\lambda_1 = 1$ then $c(\alpha_4)$ is linear. Lastly, if $\lambda_1 > 1$ then $c(\alpha_4)$ is convex.

Next, we investigate the sensitivity of the the objective function value of WSP-SM to differences in the required number of workers at some facility. In particular, we consider that R_1 is the required number of workers at facility 1 in Problem 5 and R'_1 is the required number of workers at facility 1 in Problem 5′. Let $R'_1 = R_1 + \delta_5$, where $0 < \delta_5 < B_1 - \sum_{i=1}^{N} R_i$. Using the results shown above, we know the optimal wage and objective function values for Problem 5. The optimal wage values for problem 5′ is $w'_{11} = \frac{A_1 D_{11}^{\lambda_1}(R_1+\delta_5)}{B_1 - \sum_{l=1}^{N} R_l - \delta_5}$ and $w'_{j1} = \frac{A_1 D_{j1}^{\lambda_1} R_j}{B_1 - \sum_{l=1}^{N} R_l - \delta_5}$, $j = 2, ..., N$. The optimal objective function value of Problem 5′ is

$$\frac{A_1 D_{11}^{\lambda_1}(R_1 + \delta_5)^2}{B_1 - \sum_{l=1}^{N} R_l - \delta_5} + \sum_{j=2}^{N} \frac{A_1 D_{j1}^{\lambda_1} R_j^2}{B_1 - \sum_{l=1}^{N} R_l - \delta_5}.$$

The difference between the objective function values is

$$
\begin{aligned}
&TC - TC' \\
=&\frac{A_1 D_{11}^{\lambda_1} R_1^2}{B_1 - \sum_{l=1}^N R_l} - \frac{A_1 D_{11}^{\lambda_1}(R_1+\delta_5)^2}{B_1 - \sum_{l=1}^N R_l - \delta_5} + \sum_{j=2}^N \frac{A_1 D_{j1}^{\lambda_1} R_j^2}{B_1 - \sum_{l=1}^N R_l} - \sum_{j=2}^N \frac{A_1 D_{j1}^{\lambda_1} R_j^2}{B_1 - \sum_{l=1}^N R_l - \delta_5} \\
=&\frac{(B_1 - \sum_{l=1}^N R_l - \delta_5) A_1 D_{11}^{\lambda_1} R_1^2 - (B_1 - \sum_{l=1}^N R_l) A_1 D_{11}^{\lambda_1} (R_1+\delta_5)^2}{(B_1 - \sum_{l=1}^N R_l)(B_1 - \sum_{l=1}^N R_l - \delta_5)} \\
&+ \frac{(B_1 - \sum_{l=1}^N R_l - \delta_5)\sum_{j=2}^N A_1 D_{j1}^{\lambda_1} R_j^2 - (B_1 - \sum_{l=1}^N R_l)\sum_{j=2}^N A_1 D_{j1}^{\lambda_1} R_j^2}{(B_1 - \sum_{l=1}^N R_l)(B_1 - \sum_{l=1}^N R_l - \delta_5)} \\
=&\frac{(B_1 - \sum_{l=1}^N R_l) A_1 D_{11}^{\lambda_1} R_1^2 - \delta_5 A_1 D_{11}^{\lambda_1} R_1^2}{(B_1 - \sum_{l=1}^N R_l)(B_1 - \sum_{l=1}^N R_l - \delta_5)} \\
&+ \frac{-(B_1 - \sum_{l=1}^N R_l) A_1 D_{11}^{\lambda_1} R_1^2 - 2(B_1 - \sum_{l=1}^N R_l) A_1 D_{11}^{\lambda_1} R_1 \delta_5}{(B_1 - \sum_{l=1}^N R_l)(B_1 - \sum_{l=1}^N R_l - \delta_5)} \\
&+ \frac{-(B_1 - \sum_{l=1}^N R_l) A_1 D_{11}^{\lambda_1} \delta_5^2 - \delta_5 \sum_{j=2}^N A_1 D_{j1}^{\lambda_1} R_j^2}{(B_1 - \sum_{l=1}^N R_l)(B_1 - \sum_{l=1}^N R_l - \delta_5)} \\
=&\frac{-\delta_5 A_1 \sum_{i=1}^N D_{i1}^{\lambda_1} R_i^2}{(B_1 - \sum_{l=1}^N R_l)(B_1 - \sum_{l=1}^N R_l - \delta_5)} - \frac{A_1 D_{11}^{\lambda_1} \delta_5 (2R_1 + \delta_5)}{B_1 - \sum_{l=1}^N R_l - \delta_5} \\
=&-\frac{\delta_5}{B_1 - \sum_{l=1}^N R_l - \delta_5}\left(\sum_{i=1}^N \frac{A_1 D_{i1}^{\lambda_1} R_i^2}{B_1 - \sum_{l=1}^N R_l} + A_1 D_{11}^{\lambda_1}(2R_1 + \delta_5)\right).
\end{aligned}
$$

It is difficult to characterize the difference in the optimal objective function values. However, because $\delta_5 > 0$ we know that the difference is negative. So, the objective function value of Problem $5'$ is greater than that of Problem 5.

The relationships and sensitivity analysis that we have shown above hold with the knowledge that companies compete for workers in areas with little competition or large labor pools. They also suggest that companies already close to a market will not gain much benefit from relocating closer to the market. Whether the distance to the market is near or far, the benefit of reducing the number of workers required helps to greatly reduce labor costs.

As we will show in the next section, the closed form solution of the multiple facility, single market workforce supply model, WSP-SM, is useful in solving the multiple facility, multiple market workforce supply model, WSP.

3.3 Reformulated Multiple Facility, Multiple Market Workforce Supply Model

We discussed some of the properties of the multiple facility, multiple market workforce supply model, WSP, in Section 3.1. However, we have not yet discussed in detail its solution. This is because it is a nonlinear model and we are unable to prove that WSP has a convex feasible region. Theorem 2, states that $g_{ik}(w_k)$ is a quasimonotone function. This does not help show that $\{w_{ik}| \sum_{k=1}^{K} g_{ik}(\bar{w}_k) \geq R_i,\ \forall i \in \{1, ..., N\}\}$ forms a convex set because the sum of quasimonotone functions is not necessarily quasimonotone. This means that a solution to WSP satisfying the Karush-Kuhn-Tucker (KKT) conditions is not guaranteed to be a global optimum.

All of the above is true with the exception of the single facility version of the problem. In this case, the constraint functions are a single quasimonotone function. This means that the feasible region of a single facility, multiple market workforce supply problem is a convex region. Any solution to this special case problem satisfying the KKT conditions is optimal.

If there is more than one facility then we must reformulate the model such that the KKT conditions are also sufficient. This reformulation also provides us with managerial insights into the labor decision that companies should make. We will discuss the details of the KKT conditions later.

We call the reformulation of the model a convex allocation problem (CAP). The reformulation has a master problem that determines the number of workers allocated to each company facility from each market. It also has a subproblem for each market that determines the wage that will be paid by each facility in order to attract the allocated number of workers from the market. To present this reformulation, we first introduce some new notation. Let r_{ik} be the number of workers allocated from market k to company facility i. Thus, $\bar{r}_k = (r_{1k}, ..., r_{Nk})$ is the column vector of the number of workers allocated from market k to each of the company facilities.

The market k subproblem is

CAP-SM

$$
\begin{aligned}
h_k(\bar{r}_k) = \quad \min \quad & \sum_{i=1}^{N} f_{ik}(\bar{w}_k) \\
\text{s.t.} \quad & g_{ik}(\bar{w}_k) \geq r_{ik}, \ \forall i \in \{1, ...N\} \\
& w_{ik} \geq 0, \quad \forall i \in \{1, ..., N\}, k \in \{1, ..., K\}.
\end{aligned}
$$

The master problem is

CAP-MP

$$
\begin{aligned}
\min & \sum_{k=1}^{K} h_k(\bar{r}_k) \\
\text{s.t.} \quad & \sum_{k=1}^{K} r_{ik} \geq R_i, \ \forall i \in \{1, ...N\} \qquad (6) \\
& r_{ik} \geq 0, \quad \forall i \in \{1, ..., N\}, k \in \{1, ..., K\}.
\end{aligned}
$$

Each market subproblem is a multiple facility, single market model, WSP-SM. Thus, we have a closed-form solution to each of the subproblems, given $\bar{r}_k$. Before showing the usefulness of this formulation, we show that the convex allocation formulation of the problem is equivalent to the workforce supply formulation. We say Formulation A of a problem is "equivalent" to Formulation B if the following conditions are met:

1. Every feasible solution of Formulation A is a feasible solution of Formulation B
2. A solution to Formulation A has the same objective function value in Formulation B.

Theorem 10 *The convex allocation, CAP, formulation of the multiple facility, multiple market workforce supply problem is equivalent to the workforce supply, WSP, formulation.*

Proof First, we show that the feasible regions of the two formulations are equivalent. Let $\mathbf{w}^* = (w^*_{ik})$ be a feasible solution of the CAP formulation. This means that

$$w^*_{ik} \geq 0 \ \forall i \in \{1, ..., N\}, k \in \{1, ..., K\}$$

and

$$g_{ik}(\bar{w^*}_k) \geq r_{ik}, \ \forall i \in \{1, ...N\}$$

because the constraints of RCAP-SM are satisfied. The constraint set 6 ensures that

$$\sum_{k=1}^{K} r_{ik} \geq R_i, \ \forall i \in \{1, ...N\}.$$

Because the market subproblems are multiple facility, single market models, we appeal to Theorem 9 to establish that the required workforce constraints are satisfied as equalities. So, $g_{ik}(\bar{w}^*_k) = r_{ik}$. Thus, in the master problem, $\sum_{k=1}^{K} r_{ik} \geq R_i$ can be replaced with $\sum_{k=1}^{K} g_{ik}(\bar{w^*}_k) \geq R_i$. So, $\mathbf{w}^*$ is also a feasible solution to the WSP formulation.

Next, we show that a solution to the CAP formulation of the problem has the same objective function value as an equivalent solution to the WSP formulation. Let $\mathbf{w}^* = (w^*_{ik})$ be a solution to the CAP formulation. The market k subproblem has an objective function value of $h_k(\bar{r}_k) = \sum_{i=1}^{N} f_{ik}(\bar{w}^*_k)$. Thus, the objective function of CAP-MP is $\sum_{k=1}^{K} h_k(\bar{r}_k) = \sum_{k=1}^{K} \sum_{i=1}^{N} f_{ik}(\bar{w}^*_k)$. For the same solution, $\mathbf{w}^* = (w^*_{ik})$, the objective function value of this solution in the WSP formulation is also $\sum_{i=1}^{N} \sum_{k=1}^{K} f_{ik}(\bar{w}^*_k)$. ■

The CAP formulation of the multiple facility, multiple market WSP model is useful because the multiple facility, single market subproblems have known closed-form solutions and we can show that the master problem has a solution that satisfies the KKT sufficient conditions. We showed the closed form solution to WSP-SM in Section 3.2. We also discussed the structure of the problem and the interactions between internal facilities. All of the results apply to the CAP-SM subproblems. Thus, the optimal closed-form solution to the CAP-SM subproblem for market k is $w^*_{ik}(\bar{r}) = \frac{A_k D_{ik}^{\lambda_k} r_{ik}}{B_k - \sum_{l=1}^{N} r_{lk}}$ for a given allocation of workers $\bar{r}$. This minimizes each of the market subproblems so that $h_k(\bar{r}_k) = \sum_{i=1}^{N} \frac{A_k D_{ik}^{\lambda_k} r_{ik}^2}{B_k - \sum_{l=1}^{N} r_{lk}}$. Thus, we present the revised convex allocation (RCAP) formulation.

RCAP

$$
\begin{aligned}
\min \sum_{k=1}^{K} & h_k(\bar{r}_k) \\
\text{s.t.} \quad \sum_{k=1}^{K} r_{ik} & \geq R_i, \ \forall i \in \{1, ...N\} \\
r_{ik} & \geq 0, \quad \forall i \in \{1, ..., N\}, k \in \{1, ..., K\} \\
r_{ik} & \in \Omega, \quad \forall i \in \{1, ..., N\}, k \in \{1, ..., K\}
\end{aligned}
\tag{7}
$$

where $\Omega = \{r_{ik} | \sum_{i=1}^{N} r_{ik} < B_k\}$.

After solving RCAP the wages can be determined by the equation $w_{ik}(\bar{r}) = \frac{A_k D_{ik}^{\lambda_k} r_{ik}}{B_k - \sum_{l=1}^{N} r_{lk}}$. By defining Ω as we have, we ensure that $w_{ik}(\bar{r}) \geq 0$.

RCAP is a nonlinear program because the function $h_k(\bar{r}_k)$, and thus the objective function, is a nonlinear function. The constraints are all linear. They form a convex feasible region.

Next, we show that $\sum_{k=1}^{K} h_k(\bar{r}_k)$ is a convex function. It is useful to define $h_{ik}(\bar{r}_k) = \frac{A_k D_{ik}^{\lambda_k} r_{ik}^2}{B_k - \sum_{l=1}^{N} r_{lk}}$ to be the ith term of the summation $h_k(\bar{r}_k)$. Since the labeling of facilities and markets with subscripts is arbitrary, we can consider facility $i = 1$ and market $k = 1$ in the lemmas and theorem that follow without any loss of generality. We show some properties of $h_{11}(\bar{r}_1)$ which hold for $h_{ik}(\bar{r}_k), \ i = 1, ..., N, \ k = 1, ..., K$.

Lemma 6 *The gradient of $h_{11}(\bar{r}_1)$ is completely defined by the following:*

$$
\begin{aligned}
\frac{\partial h_{11}(\bar{r}_1)}{\partial r_{11}} &= \frac{A_1 D_{11}^{\lambda_1}}{\left(B_1 - \sum_{l=1}^{N} r_{l1}\right)^2} \left[2 r_{11} \left(B_1 - \sum_{l \neq 1} r_{l1} \right) - r_{11}^2 \right] \\
\frac{\partial h_{11}(\bar{r}_1)}{\partial r_{j1}} &= \frac{A_1 D_{11}^{\lambda_1}}{\left(B_1 - \sum_{l=1}^{N} r_{l1}\right)^2} r_{11}^2 \qquad \forall j \neq 1.
\end{aligned}
$$

Proof

$$\begin{aligned}
\frac{\partial h_{11}(\bar{r}_1)}{\partial r_{11}} =& \frac{2A_1 D_{11}^{\lambda_1}\left(B_1 - \sum_{l=1}^{N} r_{l1}\right) r_{11} + A_1 D_{11}^{\lambda_1} r_{11}^2}{\left(B_1 - \sum_{l=1}^{N} r_{l1}\right)^2} \\
=& \frac{2A_1 D_{11}^{\lambda_1}\left(B_1 - \sum_{l\neq 1} r_{l1}\right) r_{11} - A_1 D_{11}^{\lambda_1} r_{11}^2}{\left(B_1 - \sum_{l=1}^{N} r_{l1}\right)^2} \\
=& \frac{A_1 D_{11}^{\lambda_1}}{\left(B_1 - \sum_{l=1}^{N} r_{l1}\right)^2}\left[2r_{11}\left(B_1 - \sum_{l\neq 1} r_{l1}\right) - r_{11}^2\right] \\
\frac{\partial h_{11}(\bar{r}_1)}{\partial r_{j1}} =& \frac{0 \cdot \left(B_1 - \sum_{l=1}^{N} r_{l1}\right) + A_1 D_{11}^{\lambda_1} r_{11}^2}{\left(B_1 - \sum_{l=1}^{N} r_{l1}\right)^2} \\
=& \frac{A_1 D_{11}^{\lambda_1}}{\left(B_1 - \sum_{l=1}^{N} r_{l1}\right)^2} r_{11}^2 \qquad \forall j \neq 1. \blacksquare
\end{aligned}$$

We now use the results for the gradient to develop the Hessian matrix for $h_{11}(\bar{r}_1)$.

Lemma 7 *The Hessian matrix, H, of $h_{11}(\bar{r}_1)$ is completely defined by the following:*

$$\begin{aligned}
H_{11} &= 2A_1 D_{11}^{\lambda_1} \frac{\left(B_1 - \sum_{l\neq 1} r_{l1}\right)^2}{\left(B_1 - \sum_{l=1}^{N} r_{l1}\right)^3} && > 0 \\
H_{u1} = H_{1u} &= 2A_1 D_{11}^{\lambda_1} \frac{\left(B_1 - \sum_{l\neq 1} r_{l1}\right) r_{11}}{\left(B_1 - \sum_{l=1}^{N} r_{l1}\right)^3} && \geq 0, \quad u \neq 1 \\
H_{uv} &= 2A_1 D_{11}^{\lambda_1} \frac{r_{11}^2}{\left(B_1 - \sum_{l=1}^{N} r_{l1}\right)^3} && \geq 0, \quad u, v \neq 1,
\end{aligned}$$

where $H_{uv} = \frac{\partial^2 h_{i1}(\bar{r}_1)}{\partial r_{u1} \partial r_{v1}}$.

Proof

$$
\begin{aligned}
H_{11} &= A_1 D_{11}^{\lambda_1} \frac{2\left(B_1 - \displaystyle\sum_{l=1}^{N} r_{l1}\right)\left(B_1 - \displaystyle\sum_{l\neq 1} r_{l1} - r_{11}\right) + 2\left[2\left(B_1 - \displaystyle\sum_{l\neq 1} r_{l1}\right) r_{11} - r_{11}^2\right]}{\left(B_1 - \sum_{l=1}^{N} r_{l1}\right)^3} \\
&= 2A_1 D_{11}^{\lambda_1} \frac{\left(B_1 - \sum_{l=1}^{N} r_{l1}\right)^2 + 2\left(B_1 - \displaystyle\sum_{l\neq 1} r_{l1}\right) r_{11} - r_{11}^2}{\left(B_1 - \sum_{l=1}^{N} r_{l1}\right)^3} \\
&= 2A_1 D_{11}^{\lambda_1} \frac{\left(B_1 - \sum_{l=1}^{N} r_{l1}\right)^2 + 2\left(B_1 - \sum_{l=1}^{N} r_{l1}\right) r_{11} + r_{11}^2}{\left(B_1 - \sum_{l=1}^{N} r_{l1}\right)^3} \\
&= 2A_1 D_{11}^{\lambda_1} \frac{\left(B_1 - \sum_{l=1}^{N} r_{l1} + r_{11}\right)^2}{\left(B_1 - \sum_{l=1}^{N} r_{l1}\right)^3} \\
&= 2A_1 D_{11}^{\lambda_1} \frac{\left(B_1 - \sum_{l\neq 1} r_{l1}\right)^2}{\left(B_1 - \sum_{l=1}^{N} r_{l1}\right)^3}.
\end{aligned}
$$

We know that $H_{11} > 0$ because $A_1 > 0$, $D_{11}^{\lambda_1} > 0$, and $B_1 - \displaystyle\sum_{l=1}^{N} r_{l1} > 0$.

$$
\begin{aligned}
H_{u1} &= A_1 D_{11}^{\lambda_1} \frac{-2\left(B_1 - \sum_{l=1}^{N} r_{l1}\right) r_{11} + 2\left[2\left(B_1 - \sum_{l\neq 1} r_{l1}\right) r_{11} - r_{11}^2\right]}{\left(B_1 - \sum_{l=1}^{N} r_{l1}\right)^3} \\
&= 2A_1 D_{11}^{\lambda_1} \frac{2\left(B_1 - \sum_{l\neq 1} r_{l1}\right) r_{11} - r_{11}^2 - \left(B_1 - \sum_{l=1}^{N} r_{l1}\right) r_{11}}{\left(B_1 - \sum_{l=1}^{N} r_{l1}\right)^3} \\
&= 2A_1 D_{11}^{\lambda_1} \frac{2\left(B_1 - \sum_{l=1}^{N} r_{l1}\right) r_{11} + r_{11}^2 - \left(B_1 - \sum_{l=1}^{N} r_{l1}\right) r_{11}}{\left(B_1 - \sum_{l=1}^{N} r_{l1}\right)^3} \\
&= 2A_1 D_{11}^{\lambda_1} \frac{\left(B_1 - \sum_{l=1}^{N} r_{l1}\right) r_{11} + r_{11}^2}{\left(B_1 - \sum_{l=1}^{N} r_{l1}\right)^3} \\
&= 2A_1 D_{11}^{\lambda_1} \frac{\left(B_1 - \sum_{l\neq 1} r_{l1}\right) r_{11}}{\left(B_1 - \sum_{l=1}^{N} r_{l1}\right)^3}.
\end{aligned}
$$

$$
\begin{aligned}
H_{1u} &= A_1 D_{11}^{\lambda_1} \frac{2\left(B_1 - \sum_{l=1}^{N} r_{l1}\right) r_{11} + 2r_{11}^2}{\left(B_1 - \sum_{l=1}^{N} r_{l1}\right)^3} \\
&= 2A_1 D_{11}^{\lambda_1} \frac{\left(B_1 - \sum_{l=1}^{N} r_{l1}\right) r_{11} + r_{11}^2}{\left(B_1 - \sum_{l=1}^{N} r_{l1}\right)^3} \\
&= 2A_1 D_{11}^{\lambda_1} \frac{\left(B_1 - \sum_{l\neq 1} r_{l1}\right) r_{11}}{\left(B_1 - \sum_{l=1}^{N} r_{l1}\right)^3}.
\end{aligned}
$$

Because $A_1 > 0$, $D_{11}^{\lambda_1} > 0$, and $B_1 - \sum_{l=1}^{N} r_{l1} > 0$ we know that $H_{u1} > 0$ and $H_{1u} > 0$ if $r_{11} > 0$. However, if $r_{11} = 0$ then $H_{u1} = H_{1u} = 0$.

$$
\begin{aligned}
H_{uv} &= A_1 D_{11}^{\lambda_1} \frac{0 \cdot \left(B_1 - \sum_{l=1}^{N} r_{l1}\right)^2 + 2r_{11}^2}{\left(B_1 - \sum_{l=1}^{N} r_{l1}\right)^3} \\
&= 2A_1 D_{11}^{\lambda_1} \frac{r_{11}^2}{\left(B_1 - \sum_{l=1}^{N} r_{l1}\right)^3}.
\end{aligned}
$$

Again, we know that $A_1 > 0$, $D_{11}^{\lambda_1} > 0$, and $B_1 - \sum_{l=1}^{N} r_{l1} > 0$. Thus, if $r_{11} > 0$ then $H_{u1} > 0$ and $H_{1u} > 0$. If $r_{11} = 0$ then $H_{u1} = H_{1u} = 0$. ∎

As we did in Section 3.1, we define $Q_{\beta,\gamma,p}$ to be the submatrix of the N x N matrix Q formed by picking a set of p distinct rows, β, and a set of p distinct columns, γ, of Q. We also define $Q_{\gamma,p}$ to be the principal submatrix of Q formed by a set of p distinct rows and columns, γ. Lastly, we define minors $M_{\beta,\gamma,p}(Q)$ and $M_{\gamma,p}$ to be $\det Q_{\beta,\gamma,p}$ and $\det Q_{\gamma,p}$, respectively. We now show, in the same way we did in Section 3.1, that each of the principal minors is nonnegative.

Lemma 8 *For any pair of distinct rows β and any pair of distinct columns γ, $M_{\beta,\gamma,2}(H) = 0$.*

Proof Following Lemma 7 the Hessian matrix of $h_{11}(\bar{w}_1)$ is

$$H = \frac{2A_1 D_{11}^{\lambda_1}}{\left(B_1 - \sum_{l=1}^{N} r_{l1}\right)^3} \begin{bmatrix} \left(B_1 - \sum_{l \neq 1} r_{l1}\right)^2 & \left(B_1 - \sum_{l \neq 1} r_{l1}\right) r_{11} & \dots & \left(B_1 - \sum_{l \neq 1} r_{l1}\right) r_{11} \\ \left(B_1 - \sum_{l \neq 1} r_{l1}\right) r_{11} & r_{11}^2 & \dots & r_{11}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \left(B_1 - \sum_{l \neq 1} r_{l1}\right) r_{11} & r_{11}^2 & \dots & r_{11}^2 \end{bmatrix}$$

We define a matrix J such that $J\frac{2A_1 D_{11}^{\lambda_1}}{\left(B_1 - \sum_{l=1}^{N}\right)^3} = H$. Since $\frac{2A_1 D_{11}^{\lambda_1}}{\left(B_1 - \sum_{l=1}^{N}\right)^3} > 0$, showing that $M_{\beta,\gamma,p}(H) = 0$ is equivalent to showing that $M_{\beta,\gamma,p}(J) = 0$.

Now, we consider four cases. The first case is $\beta_1 = \{1, s\}$, where s is any row except 1, and $\gamma_1 = \{1, t\}$, where t is any column except 1. It is possible, but not necessary, that $s = t$. The submatrix formed is

$$J_{\beta_1,\gamma_1,2} = \begin{bmatrix} \left(B_1 - \sum_{l \neq 1} r_{l1}\right)^2 & \left(B_1 - \sum_{l \neq 1} r_{l1}\right) r_{11} \\ \left(B_1 - \sum_{l \neq 1} r_{l1}\right) r_{11} & r_{11}^2 \end{bmatrix}.$$

So, $M_{\beta_1,\gamma_1,2}(J) = \left(B_1 - \sum_{l \neq 1} r_{l1}\right)^2 r_{11}^2 - \left(B_1 - \sum_{l \neq 1} r_{l1}\right)^2 r_{11}^2 = 0.$

The second case is $\beta_2 = \{1, s\}$, where s is any row except 1, and $\gamma_2 = \{t, u\}$, where t and u are any two distinct columns different from 1. It is possible, but not necessary, that $s = t$ or $s = u$. The submatrix formed is

$$J_{\beta_2,\gamma_2,2} = \begin{bmatrix} \left(B_1 - \sum_{l \neq 1} r_{l1}\right) r_{11} & \left(B_1 - \sum_{l \neq 1} r_{l1}\right) r_{11} \\ r_{11}^2 & r_{11}^2 \end{bmatrix}.$$

So, $M_{\beta_2,\gamma_2,2}(J) = \left(B_1 - \sum_{l \neq 1} r_{l1}\right)^2 r_{11}^3 - \left(B_1 - \sum_{l \neq 1} r_{l1}\right)^2 r_{11}^3 = 0.$

The third case is $\beta_3 = \{s, t\}$, where s and t are any two distinct rows different from 1, and $\gamma_3 = \{1, u\}$, where u is any column different from 1. It is possible, but not necessary, that $s = u$ or $t = u$. The submatrix formed is

$$J_{\beta_3,\gamma_3,2} = \begin{bmatrix} \left(B_1 - \sum_{l \neq 1} r_{l1}\right) r_{11} & r_{11}^2 \\ \left(B_1 - \sum_{l \neq 1} r_{l1}\right) r_{11} & r_{11}^2 \end{bmatrix}.$$

So, $M_{\beta_3,\gamma_3,2}(J) = \left(B_1 - \sum_{l\neq 1} r_{l1}\right)^2 r_{11}^2 - \left(B_1 - \sum_{l\neq 1} r_{l1}\right)^2 r_{11}^2 = 0.$

The fourth case is $\beta_4 = \{s,t\}$, where s and t are any two distinct rows different from 1, and $\gamma_4 = \{u,v\}$, where u and v are any two distinct columns different from 1. It is possible, but not necessary, that $s = u$, $s = v$, $t = u$, or $t = v$. The submatrix formed is

$$J_{\beta_4,\gamma_4,2} = \begin{bmatrix} r_{11}^2 & r_{11}^2 \\ r_{11}^2 & r_{11}^2 \end{bmatrix}.$$

So, $M_{\beta_4,\gamma_4,2}(J) = r_{11}^4 - r_{11}^4 = 0$. ■

Following the same steps we did in Section 3.1, we will now show that any p x p, $p \geq 2$ submatrix of H, the Hessian matrix of $h_{11}(\bar{r}_1)$, has a determinant of 0. This is so we can show that H is positive semidefinite.

Lemma 9 *For $p > 2$, any β, a set of p distinct rows, and any γ, a set of p distinct columns, $M_{\beta,\gamma,p}(H) = 0$.*

Proof Again, we consider matrix J such that $J\frac{2A_1 D_{11}^{\lambda_1}}{\left(B_1 - \sum_{l=1}^{N}\right)^3} = H$ because showing that $M_{\beta,\gamma,p}(J) = 0$ is equivalent to showing that $M_{\beta,\gamma,p}(H) = 0$. We use induction to show that for any β, any γ, and $p > 2$, $M_{\beta,\gamma,p}(J) = 0$. Lemma 8 states that for any β and any γ, $M_{\beta,\gamma,2}(J) = 0$. Now we assume that for any β and any γ, $M_{\beta,\gamma,n}(J) = 0$.

Now, consider the minors $M_{\beta,\gamma,n+1}(J)$. Since for any β and any γ, $M_{\beta,\gamma,n}(J) = 0$, $M_{\beta,\gamma,n+1}(H) = \det J_{\beta,\gamma,n+1} = \sum_{u=1}^{n} (j_{uv}(-1)^{u+v} \cdot 0) = 0$ for any β and any γ. Since $M_{\beta,\gamma,p}(J) = 0$ for every β and γ and $\frac{2A_1 D_{11}^{\lambda_1}}{\left(B_1 - \sum_{l=1}^{N} r_{li}\right)^3} > 0$ we know that $M_{\beta,\gamma,p}(H) = 0$ for every β and γ. ■

We again use the results from Bazaraa et al. (2006) and Chong and Żak (2001) to show that $h_{ik}(\bar{r}_k)$ is convex over an open set S.

Lemma 10 *For $i = 1, ..., N$ and $k = 1, ..., K$, the function $h_{ik}(\bar{r}_k) = \frac{A_k D_{ik}^{\lambda_k} r_{ik}^2}{B_k - \sum_{l=1}^{N} r_{l1}}$ is a convex function over $S_k' = \{\bar{r}_k | B_1 - \sum_{l=1}^{N} r_{li} > 0,\ \bar{r}_k \geq 0\}$.*

Proof Let S_k be the open set $\{\bar{r}_k | B_1 - \sum_{l=1}^{N} r_{li} > 0\}$. The fraction $\frac{2A_1 D_{11}^{\lambda_1}}{\left(B_1 - \sum_{l=1}^{N} r_{li}\right)^3}$ is always positive over the set S_k. The set $S'_k = \{\bar{r}_k | B_1 - \sum_{l=1}^{N} r_{li} > 0,\ \bar{r}_k \geq 0\}$ is a subset of S_k. We consider the matrix J, where $\frac{2A_1 D_{11}^{\lambda_1}}{\left(B_1 - \sum_{l=1}^{N}\right)^3} J = H$. If J is positive semidefinite then the Hessian matrix, H, of $h_{ik}(\bar{r}_k)$ is also.

The first principal minors of J are the elements of the diagonal of J: $J_{ii} = \left(B_k - \sum_{l \neq i} r_{lk}\right)^2$ and $J_{uu} = r_{ik}^2,\ u \neq i$. Both J_{ii} and J_{uu} are nonnegative for all values of $\bar{r}_k$. Thus, they are nonnegative on S_k. We have shown, in Lemmas 8 and 9, that for any β, any γ, and $p \geq 2$, $M_{\beta,\gamma,p}(J) = 0$ over the set of all real values of $\bar{r}_k$. Thus, every principal minor is nonnegative on the set S_k. The matrix J, and thus H, is positive semidefinite for values in S_k and S'_k which implies that $h_{ik}(\bar{r}_k)$ is convex over S_k and S'_k. ■

We have shown that $h_{ik}(\bar{w}_k)$ is a convex function over the set $\{\bar{r}_k | B_1 - \sum_{l=1}^{N} r_{li} > 0,\ \bar{r}_k \geq 0\}$, allowing us to now show that the objective function of RCAP, $\sum_{k=1}^{K} h_k(\bar{r}_k)$, is convex over the set $\{\bar{r}_k | B_1 - \sum_{l=1}^{N} r_{li} > 0,\ \bar{r}_k \geq 0,\ \forall k = 1, ..., K\}$.

Theorem 11 *The function* $\sum_{k=1}^{K} h_k(\bar{r}_k)$ *is jointly convex over the set* $S' = \{\bar{r}_k | B_1 - \sum_{l=1}^{N} r_{li} > 0,\ \bar{r}_k \geq 0,\ \forall k = 1, ..., K\}$ *and* $S = \{\bar{r}_k | B_1 - \sum_{l=1}^{N} r_{li} > 0,\ \forall k = 1, ..., K\}$.

Proof We again use the result from Hillier and Lieberman (2001) that the sum of convex functions is a convex function. Since $h_{ik}(\bar{r}_k)$ is convex over the set $S'_k = \{\bar{r}_k | B_1 - \sum_{l=1}^{N} r_{li} > 0,\ \bar{r}_k \geq 0\}$ for every market k, $h_k(\bar{r}_k) = \sum_{i=1}^{N} h_{ik}(\bar{r}_k)$ is convex over S'_k. Thus, $\sum_{k=1}^{K} h_k(\bar{r}_k)$ is a sum of convex functions and is convex over the set S'. Note that for the same reasons $\sum_{k=1}^{K} h_k(\bar{r}_k)$ is jointly convex over the set S. ■

Besides being convex, the function $h_k(\bar{r}_k)$ has some other nice properties. We show the relationship between the number of workers from market k allocated to facility i and the labor costs of the different facilities.

Theorem 12 *The function* $h_k(\bar{r}_k) = \sum_{i=1}^{N} \frac{A_k D_{ik}^{\lambda_k} r_{ik}^2}{B_k - \sum_{l=1}^{N} r_{lk}}$ *is strictly increasing in* r_{ik}.

Proof Again, Anton (1998), Definition 5.1.1, states that a function, f is strictly increasing in x if $x < \omega$ implies that $f(x) < f(\omega)$.

Again, we consider the ith term of the sum of the function, $h_{ik}(\bar{r}_k)$. Let $r_{ik} < \rho_{ik}$ and $r_{il} = \rho_{il}$, $\forall l \neq k$. Then

$$\begin{aligned}
&h_{ik}(\bar{r}_k) - h_{ik}(\bar{\rho}_k) \\
=&A_k D_{ik}^{\lambda_k} \frac{r_{ik}^2}{B_k - \sum_{l=1}^K r_{il}} - A_k D_{ik}^{\lambda_k} \frac{\rho_{ik}^2}{B_k - \sum_{l=1}^K \rho_{il}} \\
=&A_k D_{ik}^{\lambda_k} \frac{r_{ik}^2 \left(B_k - \sum_{l=1}^K \rho_{il}\right) - \rho_{ik}^2 \left(B_k - \sum_{l=1}^K r_{il}\right)}{\left(B_k - \sum_{l=1}^K r_{il}\right)\left(B_k - \sum_{l=1}^K \rho_{il}\right)} \\
=&A_k D_{ik}^{\lambda_k} \frac{\left(B_k - \sum_{l=1}^K r_{il}\right)(r_{ik}^2 - \rho_{ik}^2)}{\left(B_k - \sum_{l=1}^K r_{il}\right)\left(B_k - \sum_{l=1}^K \rho_{il}\right)} \qquad < 0
\end{aligned}$$

By assumption 5, $B_k - \sum_{l=1}^K r_{il} > 0$. Since $r_{ik} < \rho_{ik}$, we know that $r_{ik}^2 - \rho_{ik}^2 < 0$. Thus, $h_{ik}(\bar{r}_k) - h_{ik}(\bar{\rho}_k) < 0$. So, $h_{ik}(\bar{r}_k) < h_{ik}(\bar{\rho}_k)$. Since this is true for a generic facility i, it is true for every facility. Thus, $h_k(\bar{r}_k) < h_k(\bar{\rho}_k)$. ■

Now, we show that $h_k(\bar{r}_k)$ is strictly increasing in r_{il}, $l = 1, ..., K$, not just r_{ik}.

Theorem 13 *The function* $h_k(\bar{r}_k) = \sum_{i=1}^N \frac{A_k D_{ik}^{\lambda_k} r_{ik}^2}{B_k - \sum_{l=1}^N r_{lk}}$ *is strictly increasing in* r_{il}, $l \neq k$.

Proof Again, we use Anton's definition. We consider the ith term of the sum of the function, $h_{ik}(\bar{r}_k)$. Let $r_{il} < \rho_{il}$ and $r_{im} = \rho_{im}$, $\forall m \neq l$. Then

$$\begin{aligned}
&h_{ik}(\bar{r}_k) - h_{ik}(\bar{\rho}_k) \\
=&A_k D_{ik}^{\lambda_k} \frac{r_{ik}^2}{B_k - \sum_{m \neq l} r_{im} - r_{il}} - A_k D_{ik}^{\lambda_k} \frac{\rho_{ik}^2}{B_k - \sum_{m \neq l} \rho_{im} - \rho_{il}} \\
=&A_k D_{ik}^{\lambda_k} \frac{r_{ik}^2 \left(B_k - \sum_{m \neq l} \rho_{im} - \rho_{il}\right) - \rho_{ik}^2 \left(B_k - \sum_{m \neq l} r_{im} - r_{il}\right)}{\left(B_k - \sum_{m=1}^K r_{im}\right)\left(B_k - \sum_{m=1}^K \rho_{im}\right)} \\
=&A_k D_{ik}^{\lambda_k} \frac{(r_{ik}^2 - \rho_{ik}^2)\left(B_k - \sum_{m \neq l} \rho_{im}\right) + r_{ik}^2 (r_{il} - \rho_{il})}{\left(B_k - \sum_{m=1}^K r_{im}\right)\left(B_k - \sum_{m=1}^K \rho_{im}\right)} \\
=&A_k D_{ik}^{\lambda_k} \frac{r_{ik}^2 (r_{il} - \rho_{il})}{\left(B_k - \sum_{m=1}^K r_{im}\right)\left(B_k - \sum_{m=1}^K \rho_{im}\right)}
\end{aligned}$$

By assumption 5, $B_k - \sum_{l=1}^{K} r_{il} > 0$. Thus, the denominator is positive. Since $r_{il} < \rho_{il}$ the numerator is negative. Thus, $h_{ik}(\bar{r}_k) - h_{ik}(\bar{\rho}_k) < 0$. So, $h_{ik}(\bar{r}_k) < h_{ik}(\bar{\rho}_k)$. Since this is true for a generic facility i, it is true for every facility. Thus $h_k(\bar{r}_k) < h_k(\bar{\rho}_k)$. ■

Theorems 12 and 13 show that the objective function of RCAP increases if r_{ik}, the allocation of workers from any market, k, to any facility, i, is increased.

Before we discuss solving the RCAP, we disqualify most of the feasible solutions as possible optimal solutions.

Theorem 14 *The optimal solution to RCAP is on the boundary of the feasible region where each function of constraint set 7,* $\sum_{k=1}^{K} r_{ik} \geq R_i, \ \forall i = 1, ..., N,$ *is satisfied as an equality.*

Proof We show this using a proof by contradiction. Let the optimal solution to RCAP be $\mathbf{r}^* = (r^*_{ik})$ where $\sum_{k=1}^{K} r^*_{ik} = R_i, \ \forall i \neq j$ and $\sum_{k=1}^{K} r^*_{jk} = \hat{R}_j > R_j$. Also, let $r_{jl} > 0$. Suppose that r_{jl} is lowered such that $\sum_{k=1}^{K} r^*_{jk} > R_j$ still and r_{jk} is unchanged for every market $k \neq l$. This solution is still feasible. Since we have shown in Theorems 12 and 13 that the objective function is strictly increasing in $r_{ik}, \ \forall i = 1, ..., N, \ k = 1, ..., K$, we know that lowering r_{jl} decreases the objective function. Thus, $\mathbf{r}^*$ is not an optimal solution. So, there is a contradiction. This result implies that if there is a solution where a facility, j, is allocated more workers than they require then there is a market, say l, for which $r_{jl} > 0$ can be lowered while decreasing the optimal solution. Thus, the objective function may be minimized by doing this until every facility is allocated exactly the number of workers required. ■

Now we are prepared to discuss the solution to the master problem of the convex allocation formulation of the multiple facility, multiple market WSP problem. Since RCAP is a nonlinear program the Karush-Kuhn-Tucker necessary conditions must be met by any optimal solution. Bazaraa et al. (2006), in Theorem 4.2.13, present the KKT conditions that are to be satisfied. The KKT conditions hold for problems having a standard form. We rewrite RCAP in standard form as follows:

RCAP

$$\begin{aligned}
\min \quad h(\mathbf{r}) = & \sum_{k=1}^{K} h_k(\bar{r}_k) \\
\text{s.t.} \quad a_i(\tilde{r}_i) = & \; R_i - \sum_{k=1}^{K} r_{ik} \leq 0, \; \forall i \in \{1, ...N\} \\
p_{ik}(r_{ik}) = & \quad -r_{ik} \leq 0, \; \forall i \in \{1, ..., N\}, k \in \{1, ..., K\} \\
& \quad r_{ik} \in \Omega, \; \forall i \in \{1, ..., N\}, \; k \in \{1, ..., K\}.
\end{aligned}$$

Before presenting these conditions, it is useful to show that RCAP has the characteristics necessary for the KKT conditions to hold. One of these characteristics is that the problem is minimized over a nonempty open set. RCAP is optimized over $\Omega = \{r_{ik} | \sum_{i=1}^{N} r_{ik} < B_k\}$ which is a nonempty open set.

Supposing that $\hat{\mathbf{r}}$ solves RCAP, another condition is that h be differentiable at $\hat{\mathbf{r}}$ and that $a_i, \; \forall i = 1, ..., N$ and $p_{ik}, \; \forall i = 1, ..., N, \; k = 1, ..., K$ be differentiable and continuous at $\hat{\mathbf{r}}$. The functions $h(\mathbf{r}), \; a_i(\tilde{r}_i), \; i = 1, ..., N$ and $p_{ik}(r_{ik}), \; i = 1, ..., N, \; k = 1, ..., K$ are each continuous and differentiable at every point in the feasible region. Thus, this condition is met for any feasible solution.

It is also necessary that the active constraints be linearly independent, for a given solution $\hat{\mathbf{r}}$. We show that this is always true for a feasible solution.

Theorem 15 *Let $\Phi = \{i | a_i(\tilde{\hat{r}}_i) = 0\}$ and $\Upsilon = \{(i, k) | p_{ik}(\hat{r}_{ik}) = 0\}$, where $\hat{\mathbf{r}}$ is a feasible solution to RCAP. The gradients $\nabla a_i(\tilde{\hat{r}}_i)$ and $\nabla p_{ik}(\hat{r}_{ik})$ are linearly independent.*

Proof Since

$$\frac{\partial a_i(\tilde{\hat{r}}_i)}{\partial r_{ik}} = -1, \; \forall k = 1, ..., K.$$

and

$$\frac{\partial a_i(\tilde{\hat{r}}_i)}{\partial r_{lk}} = 0, \; \forall l \neq i, \; k = 1, ..., K,$$

then any two constraints $i, j \in \Phi$ have linearly independent gradients. In fact, every component equal to -1 in $\nabla a_i(\tilde{\hat{r}}_1)$ equals 0 in $\nabla a_j(\tilde{\hat{r}}_j)$.

Thus, we need only consider a single arbitrary $i \in \Phi$. We show that $\nabla a_i(\tilde{\hat{r}}_i)$ is linearly independent of $\nabla p_{ik}(\hat{r}_{ik}),\ \forall (i,k) \in \Upsilon$. In doing so we show that it is true for every $i \in \Phi$. We show this by contradiction. Remember that we assume that $R_i > 0$. Let υ be a subset of Υ such that $\nabla a_i(\tilde{\hat{r}}_i)$ is linearly dependent with $\nabla p_{ik}(\hat{r}_{ik}),\ \forall (i,k) \in \upsilon$. Again, we use the fact that

$$\frac{\partial a_i(\tilde{\hat{r}}_i)}{\partial r_{ik}} = -1,\ \forall k = 1, ..., K.$$

and

$$\frac{\partial a_i(\tilde{\hat{r}}_i)}{\partial r_{lk}} = 0,\ \forall l \neq i,\ k = 1, ..., K.$$

Thus, in order to have a set of vectors, υ, that is linearly dependent with $\nabla a_i(\tilde{\hat{r}}_i)$ then $\nabla p_{ik}(\hat{r}_{ik}),\ k = 1, ..., K$ must be in *upsilon*. This implies that $p_{ik}(\hat{r}_{ik}),\ k = 1, ..., K$ are active constraints or that $-r_{ik} = 0,\ \forall k = 1, ..., K$. This implies that $-\sum_{k=1}^{K} r_{ik} = 0$. Thus, $R_i - \sum_{k=1}^{K} r_{ik} = R_i \neq 0$. But that means that $i \notin \Phi$ which is a contradiction. Thus, for a feasible solution to RCAP the gradients of the active constraints are linearly independent.
■

Now, we examine the KKT conditions, as they apply to the master problem of the convex allocation formulation. Let $\hat{\mathbf{r}}$ be a feasible solution to RCAP. The necessary conditions for $\hat{\mathbf{r}}$ to be an optimal solution to RCAP is that there exist scalars $\mu_i,\ i = 1, ..., N$ and $\nu_{ik},\ i - 1, ..., N,\ k = 1, ..., K$ such that

$$\nabla h(\hat{\mathbf{r}}) + \sum_{i=1}^{N} \mu_i \nabla a_i(\tilde{\hat{r}}_i) + \sum_{i=1}^{N} \sum_{k=1}^{K} \nu_{ik} \nabla p_{ik}(\hat{r}_{ik}) = \vec{0} \tag{8}$$

$$\mu_i a_i(\tilde{\hat{r}}_i) = 0,\ i = 1, ..., N \tag{9}$$

$$-\nu_{ik} \hat{r}_{ik} = 0,\ i = 1, ..., N,\ k = 1, ..., K \tag{10}$$

$$\mu_i \geq 0,\ i = 1, ..., N \tag{11}$$

$$\nu_{ik} \geq 0,\ i = 1, ..., N,\ k = 1, ..., K, \tag{12}$$

where $\vec{0}$ is a column vector in which every component is 0.

Bazaraa et al. (2006), Theorem 4.2.16, show that $\hat{\mathbf{r}}$ satisfying the KKT conditions is a global optimal solution to RCAP. This is due to the nice properties of the objective function and constraints. In particular, $h(\hat{\mathbf{r}})$ is convex at every feasible point. Also, $a_i(\tilde{r}_i),\ i = 1, ...N$ along with $-r_{ik},\ i = 1, ...N,\ k = 1, , ..., K$ are linear and, thus, quasiconvex at every feasible point.

In Theorem 14 we showed that the optimal solution to RCAP satisfies constraint set 7 as equalities. This implies that $a_i(\tilde{\hat{r}}_i) = 0,\ i = 1, ..., N$, and so condition 9 is met for all $i = 1, ..., N$. The values of the scalars $\mu_{ik},\ i - 1, ..., N$ do not affect this condition.

Using Lemma 6, condition 8 can be written as

$$
\begin{aligned}
(A_k D_{ik}^{\lambda_k} &+ \mu_i + \nu_{ik}) r_{ik}^2 + 2(A_k D_{ik}^{\lambda_k} + \mu_i + \nu_{ik}) r_{ik} \sum_{j \neq i} r_{jk} \\
&- \sum_{j \neq i} (A_k D_{jk}^{\lambda_k} - \mu_i - \nu_{ik}) r_{jk}^2 - 2B_k (A_k D_{ik}^{\lambda_k} + \mu_i + \nu_{ik}) r_{ik} \\
&- 2B_k (\mu_i + \nu_{ik}) \sum_{j \neq i} r_{jk} + B_k^2 (\mu_i + \nu_{ik}).
\end{aligned}
$$

This forms a system of quadratic equations.

Because systems of quadratic equations are not easily solved, using the KKT conditions directly to solve for the variables $r_{ik},\ i = 1, ..., N,\ k = 1, ..., K$ is difficult. Bazaraa et al. (2006), pages 603-612, present Wolfe's reduced gradient method for solving problems having a nonlinear objective function and linear constraints. They also discuss the convergence of the algorithm. Most importantly, they show that a solution to the problem that satisfies the stopping condition of the algorithm also satisfies the KKT conditions. Because RCAP is minimizing a convex nonlinear function over a linear constraint set, we use the Reduced Gradient (RG) method to solve for $r_{ik},\ i = 1, ..., N,\ k = 1, ..., K$. Any solution satisfying the stopping conditions of the algorithm will be an optimal solution to RCAP.

Bazaraa et al. (2006) describe the algorithm using matrices and linear algebra techniques. We are able to simplify it for our purposes. Much of the simplification is because the coefficients of each variable $r_{ik},\ i = 1, ..., N,\ k = 1, ..., K$ is 1 or 0 in each of the constraint functions. Because solutions to RCAP must be in Ω, we define

$$\ddot{h}_{ik}(\bar{r}_k) = \begin{cases} \frac{A_k D_{ik}^{\lambda_k} r_{ik}^2}{B_k - \sum_{l=1}^{N} r_{lk}} & \text{if } B_k - \sum_{l=1}^{N} r_{lk} > 0 \\ +\infty & \text{if } B_k - \sum_{l=1}^{N} r_{lk} \leq 0 \end{cases}$$

In the initial step of the algorithm we assign values to each $r_{ik} \geq 0$ such that $\sum_{k=1}^{K} r_{ik} = R_i, \ \forall i = 1, .., N$ and $\sum_{i=1}^{N} r_{ik} = B_k, \ \forall k = 1, ..., K$. In other words, the point chosen is feasible and is on the boundary of the feasible region. The first step of every iteration is to find the gradient and subsequent reduced gradient of $h(\mathbf{r})$. We always choose one variable r_{ik} from the set $\{r_{ik}, \ k = 1, ..., k\}$ for each facility i to be in the basis. This is so that the coefficients of the basis variables from constraint set 7 form an identity matrix. Based on the reduced gradient we determine if there is an improving direction. If there is one then the second step is to determine the optimal step size given the improving direction. If there is not an improving direction then the algorithm has found the optimal solution. Once the optimal solution is found, the last step is to determine the wage paid by facility i to workers from market k, $w_{ik}(\bar{r})$.

We adapt the RG algorithm to solve RCAP as follows:

Step 0

1. Set $c = 1$.
2. Set $r_{ik}^{(1)} = R_i \frac{B_k}{\sum_{l=1}^{K} B_l}, \ \forall i = 1, ..., N, \ k = 1, ..., K$.
3. Go to Step 1.

Step 1

1. Let I be a set of indices such that exactly one index, for each $i = 1, ..., N$ – the index of the largest $r_{ik} \in \{r_{i1}, ...r_{N1}\}$ – is chosen.
   I $= \{(i, k) | r_{ik}^{(c)} \geq r_{ij}^{(c)}, \ \forall j = 1, ..., K\}$.

2. For each $i = 1, ..., N$, let $b(i)$ be market k corresponding with facility i such that $(i, k) \in \mathrm{I}$.

3. Set

$$\begin{aligned}\nabla_{ik} h(\mathbf{r}^{(c)}) =& \frac{\partial h(\mathbf{r}^{(c)})}{\partial r_{ik}} \\ =& \frac{A_k}{(B_k - \sum_{j=1}^{N} r_{jk}^{(c)})^2} \left[2D_{ik}^{\lambda_k} \left(B_k - \sum_{j=1}^{N} r_{jk}^{(c)} \right) r_{ik}^{(c)} + \sum_{j=1}^{N} D_{jk}^{\lambda_k} (r_{jk}^{(c)})^2 \right].\end{aligned}$$

4. Set $p_{ik} = \nabla_{ik} h(\mathbf{r}^{(c)}) - \nabla_{i,b(i)} h(\mathbf{r}^{(c)})$.

5. Set $d_{ik} = \begin{cases} -p_{ik} & \text{if } (i,k) \notin \mathrm{I} \text{ and } p_{ik} \leq 0 \\ -r_{ik}^{(c)} p_{ik} & \text{if } (i,k) \notin \mathrm{I} \text{ and } p_{ik} > 0 \\ -\sum_{l \neq k} d_{il} & \text{if } (i,k) \in \mathrm{I} \end{cases}$.

6. If there is any (i, k) such that $d_{ik} > 0$ then go to Step 2.

7. If $d_{ik} = 0,\ \forall i = 1, ..., N,\ k = 1, ..., K$ then stop; $\mathbf{r}^{(c)}$ is a KKT point and is the optimal solution to RCAP. Go to Step 3.

Step 2

1. Set $\mu_{ik} = \begin{cases} -\frac{r_{ik}^{(c)}}{d_{ik}} & \text{if } d_{ik} < 0 \\ +\infty & \text{if } d_{ik} \geq 0 \end{cases}$.

2. Set $\hat{\mu} = \{\mu_{ik} | \mu_{ik} \leq \mu_{jl}, \forall j = 1, ..., N,\ l = 1, ..., K\}$.

3. Set $\ddot{h}_{ik}(\bar{r}_k) = \begin{cases} \frac{A_k D_{ik}^{\lambda_k} r_{ik}^2}{B_k - \sum_{l=1}^{N} r_{lk}} & \text{if } B_k - \sum_{l=1}^{N} r_{lk} > 0 \\ +\infty & \text{if } B_k - \sum_{l=1}^{N} r_{lk} \leq 0 \end{cases}$.

4. Solve

$$\begin{aligned} \min & \sum_{i=1}^{N} \sum_{k=1}^{K} h_{ik}(\bar{r}_k^{(c)} + \mu \bar{d}_k) \\ \text{s.t.} \quad & \mu \leq \ \hat{\mu} \\ & \mu \geq \ 0 \end{aligned}$$

using a golden section line search.

5. Set $r_{ik}^{(c+1)} = r_{ik}^{(c)} + \mu d_{ik}$.

6. Set $c = c + 1$.

7. Go to Step 1.

Step 3

Set $w_{ik}(\bar{r}) = \frac{A_k D_{ik}^{\lambda_k} r_{ik}^{(c)}}{B_k - \sum_{l=1}^{N} r_{lk}^{(c)}}\ \forall i = 1, ..., N,\ k = 1, ..., K.$

The entire algorithm is coded in C++. The run time of the algorithm is less than one minute for problems as large as twenty facilities and twenty markets. The C++ code is provided in the appendix.

For WSP-SM, every facility pays a non-zero positive wage to workers from the market. We now investigate the circumstances under which an internal facility chooses not to compete in a market in the multiple market WSP. First, consider a WSP problem with $K > 1$ markets and a single facility. We do so using the reformulated model RCAP.

Theorem 16 *For a single facility RCAP problem, the facility hires workers from every market, $r_{1k}^* > 0, k = 1, ..., K$.*

Proof In the case of a single facility $h(\mathbf{r}) = \sum_{k=1}^{K} \frac{A_k D_{1k}^{\lambda_k} r_{1k}^2}{B_k - r_{1k}}$. The derivative of $h(\mathbf{r})$ with respect to r_{1k} is $\frac{\partial h}{\partial r_{1k}} = \frac{A_k D_{1k}^{\lambda_k}}{(B_k - r_{1k})^2}(2B_k r_{1k} - r_{1k}^2)$. The constraint, $a_1(\tilde{r}_1)$ is the same as in the multiple facility instance.

We show this result by contradiction using the KKT conditions. Assume that $r_{11}^* = 0$ and $r_{1k}^* > 0,\ k = 2, ..., K$. We can assign $r_{11}^* = 0$ without loss of generality because the indexing of markets is arbitrary. From equation 8, we know that $\nabla h(\mathbf{r}^*) + \sum_{i=1}^{N} \mu_i \nabla a_i(\bar{r}_i^*) + \sum_{i=1}^{N} \sum_{k=1}^{K} \nu_{ik} \nabla p_{ik}(r_{ik}^*) = \vec{0}$. Since, $r_{1k}^* > 0,\ k = 2, ..., K$ we know that $\nu_{ik} = 0,\ k = 2, ..., K$ in order to satisfy equation 10. For market 1, ν_{11} can take on any positive value and satisfy the equations 10 and 12. We have shown that the optimal solution to RCAP is always found where $a_1(\tilde{r}_1^*) \geq R_1$ is satisfied as an equality. Thus, μ_1 can take on any positive value in order to satisfy the equations 9 and 11. Thus, equation 8 results in the following set of equations

$$0 - \mu_1 - \nu_{11} = 0 \tag{13}$$

$$\frac{A_k D_{1k}^{\lambda_k}}{(B_k - r_{1k})^2}(2B_k r_{1k} - r_{1k}^2) - \mu_1 = 0,\ k = 2, ..., K \tag{14}$$

From equation 14 we know that $\mu_1 = \frac{A_k D_{1k}^{\lambda_k}}{(B_k - r_{1k})^2}(2B_k r_{1k} - r_{1k}^2) > 0,\ k = 2, ..., K$. From equation 13 we know that $-\nu_{11} = \mu_1 > 0$. This implies that $\nu_{11} < 0$. This is a contradiction. So it is not optimal that $r_{11}^* = 0$. ∎

Even though we have shown that the optimal solution to a single facility RCAP problem satisfies $r_{1k}^* > 0, k = 1, ..., K$, it is possible that the optimal number of workers that the facility hires from some facility k takes on a very small positive value. It is obvious that in practice only a whole number of workers can be hired from a market. We do not restrict our model to choosing integer values because of the difficulty this would introduce. Thus, the RCAP model allows for a facility hiring an extremely small fraction of a worker from a market, but in practice the facility would not hire any workers from that market.

Next we investigate the RCAP when there are $N > 1$ internal facilities. We show the circumstances under which each of the internal facilities choose not to hire any workers from one of the markets.

Theorem 17 *In the optimal solution to the RCAP problem for every market $k = 1, ..., K$ there is at least one facility i such that $r_{ik}^* > 0$. In other words, the optimal solution includes workers from each market being hired by at least one of the facilities.*

Proof Remember that the objective function is $h(\mathbf{r}) = \displaystyle\sum_{i=1}^{N}\sum_{k=1}^{K} \frac{A_k D_{ik}^{\lambda_k} r_{ik}^2}{B_k - \sum_{l=1}^{N} r_{lk}}$. The derivative of $h(\mathbf{r})$ with respect to r_{ik} is $\frac{\partial h}{\partial r_{ik}} = \frac{A_k D_{ik}^{\lambda_k}}{(B_k - \sum_{l=1}^{N} r_{lk})^2}(2r_{ik}(B_k - \sum_{j \neq i} r_{jk}) - r_{1k}^2) + \sum_{j \neq i} \frac{A_k D_{jk}^{\lambda_k}}{(B_k - \sum_{l=1}^{N} r_{lk})^2} r_{jk}^2$. The derivative of the ith constraint with respect to r_{ik} is $\frac{\partial a_i(\tilde{r}_i)}{\partial r_{ik}} = -1,\ i = 1, ..., N,\ k = 1, ..., K$.

We show this result by contradiction using the KKT conditions. Assume that $r_{i1}^* = 0,\ i = 1, ..., N$ and $r_{ik}^* > 0,\ i = 1, ..., N,\ k = 2, ..., K$ is the optimal solution to RCAP. We can assign

$r^*_{i1} = 0,\ i = 1, ..., N$ without loss of generality because the indexing of markets is arbitrary. We use a similar argument here as we did to prove Theorem 16. Using equations 9–12 we know that $\mu_i \geq 0,\ i = 1, ..., N,\ \nu_{i1} \geq 0,\ i = 1, ..., N,$ and $\nu_{ik} = 0,\ i = 1, ..., N,\ k = 2, ..., K$. Using equation 8 we know that the following equations are satisfied by the optimal solution:

$$0 - \mu_i - \nu_{i1} = 0 \quad i = 1, ..., N \tag{15}$$

$$\frac{A_k D_{ik}^{\lambda_k}}{(B_k - \sum_{l=1}^N r_{lk})^2}(2r_{ik}(B_k - \sum_{j \neq i} r_{jk}) - r_{ik}^2) + \sum_{j \neq i} \frac{A_k D_{jk}^{\lambda_k}}{(B_k - \sum_{l=1}^N r_{lk})^2} r_{jk}^2 - \mu_i = 0 \quad i = 1, ..., N,\ k = 2, ..., K \tag{16}$$

From equation 16 we know that for any market $k \neq 1$

$$\mu_i = \frac{A_k D_{ik}^{\lambda_k}}{(B_k - \sum_{l=1}^N r_{lk})^2}(2r_{ik}(B_k - \sum_{j \neq i} r_{jk}) - r_{ik}^2) + \sum_{j \neq i} \frac{A_k D_{jk}^{\lambda_k}}{(B_k - \sum_{l=1}^N r_{lk})^2} r_{jk}^2 > 0,\ i = 1, ..., N.$$

From equation 15 we know that $-\nu_{i1} = \mu_i > 0,\ i = 1, ..., N$. This implies that $\nu_{i1} < 0,\ i = 1, ..., N$. This is a contradiction. So it is not optimal that $r^*_{i1} = 0,\ i = 1, ..., N$. Thus, $r^*_{i1} > 0$ for some facility i. ■

Having shown that $r^*_{i1} = 0,\ i = 1, ..., N$ cannot be the optimal answer to RCAP, we also show the circumstances under which only one of the internal facility chooses to not compete in one of the markets.

Theorem 18 *In RCAP it is optimal for facility i to hire no one from market k only if*

$$\sum_{j \neq i} \frac{A_k D_{jk}^{\lambda_k}}{(B_k - \sum_{l=1}^N r_{lk})^2} r_{jk}^2 \geq \frac{A_m D_{im}^{\lambda_m}}{(B_m - \sum_{l=1}^N r_{lm})^2}(2r_{im}(B_m - \sum_{n \neq i} r_{nm}) - r_{im}^2) + \sum_{n \neq i} \frac{A_m D_{nm}^{\lambda_m}}{(B_m - \sum_{l=1}^N r_{lm})^2} r_{nm}^2,\ m = 1, ..., k-1, k+1, ..., K.$$

Proof Suppose that $r^*_{11} = 0$ and all other $r^*_{ik} > 0$ is the optimal solution to RCAP. We can assign $r^*_{11} = 0$ without loss of generality because the indexing of facilities and markets is arbitrary. We again use the KKT conditions for a solution to RCAP being optimal. Using equations 9–12 we know that $\mu_i \geq 0,\ i = 1, ..., N,\ \nu_{11} \geq 0$, and every other $\nu_{ik} = 0$. Using equation 8 we know that the following equations are satisfied by the optimal solution:

$$\sum_{j \neq 1} \frac{A_1 D_{j1}^{\lambda_1}}{(B_1 - \sum_{l=1}^{N} r_{l1})^2} r_{j1}^2 - \mu_1 - \nu_{11} = 0 \tag{17}$$

$$\frac{A_1 D_{i1}^{\lambda_1}}{(B_1 - \sum_{l=1}^{N} r_{l1})^2} (2r_{i1}(B_1 - \sum_{j \neq 1,\ j \neq i} r_{j1}) - r_{i1}^2) + \sum_{j \neq 1,\ j \neq i} \frac{A_1 D_{j1}^{\lambda_1}}{(B_1 - \sum_{l=1}^{N} r_{l1})^2} r_{j1}^2 - \mu_i = 0 \quad i = 2, ..., N \tag{18}$$

$$\frac{A_k D_{ik}^{\lambda_k}}{(B_k - \sum_{l=1}^{N} r_{lk})^2} (2r_{ik}(B_k - \sum_{j \neq i} r_{jk}) - r_{ik}^2) + \sum_{j \neq i} \frac{A_k D_{jk}^{\lambda_k}}{(B_k - \sum_{l=1}^{N} r_{lk})^2} r_{jk}^2 - \mu_i = 0 \quad i = 1, ..., N.\ k = 2, ..., K \tag{19}$$

Equation 17 implies that

$$\nu_{11} = \sum_{j \neq 1} \frac{A_1 D_{j1}^{\lambda_1}}{(B_1 - \sum_{l=1}^{N} r_{l1})^2} r_{j1}^2 - \mu_1.$$

The KKT condition that $\nu_{11} \geq 0$ implies that an optimal solution satisfies

$$\mu_1 \leq \sum_{j \neq 1} \frac{A_1 D_{j1}^{\lambda_1}}{(B_1 - \sum_{l=1}^{N} r_{l1})^2} r_{j1}^2.$$

From equation 19 we know that

$$\frac{A_k D_{1k}^{\lambda_k}}{(B_k - \sum_{l=1}^{N} r_{lk})^2} (2r_{1k}(B_k - \sum_{j \neq 1} r_{jk}) - r_{1k}^2) + \sum_{j \neq 1} \frac{A_k D_{jk}^{\lambda_k}}{(B_k - \sum_{l=1}^{N} r_{lk})^2} r_{jk}^2 = \mu_1,\ k = 2, ..., K.$$

This implies that

$$\frac{A_k D_{1k}^{\lambda_k}}{(B_k - \sum_{l=1}^{N} r_{lk})^2} (2r_{1k}(B_k - \sum_{j \neq 1} r_{jk}) - r_{1k}^2) + \sum_{j \neq 1} \frac{A_k D_{jk}^{\lambda_k}}{(B_k - \sum_{l=1}^{N} r_{lk})^2} r_{jk}^2 \leq \sum_{j \neq 1} \frac{A_1 D_{j1}^{\lambda_1}}{(B_1 - \sum_{l=1}^{N} r_{l1})^2} r_{j1}^2,\ k = 2, ..., K.\ \blacksquare$$

Table 1 shows the different problems that we have presented above and the solution methodology developed for each one.

Now that we have modeled the workforce supply problem and developed an optimal solution procedure, we discuss the benefits of a company's facilities making coordinated labor decisions as opposed to each facility doing so independently.

Table 1: Table of WSP Models and Solution Procedures

Problem	**Solution**
Single Facility, Single Market	Closed-form Solution
Multiple Facility, Single Market	Closed-form Solution
Single Facility, Multiple Market	KKT Conditions Necessary and Sufficient, Reduced Gradient Algorithm after Converting to RCAP
Multiple Facility, Multiple Market	Convert to RCAP, KKT Conditions Necessary and Sufficient, Reduced Gradient Algorithm

3.4 Coordinated Versus Uncoordinated Labor Decisions

None of the modeling and results we have shown above is useful unless it is beneficial for a company to coordinate its facilities labor decisions. First, we consider the situation where every internal facility competes for workers from a single labor market. In Section 3.2 we showed that the optimal wage to be paid by any facility i when the internal facilities are coordinating their labor decisions is $w_{i1}(\bar{r}) = \frac{A_1 D_{i1}^{\lambda_1} R_i}{B_1 - \sum_{l=1}^N R_l}$. The optimal objective function value of WSP-SM is $\sum_{i=1}^{N} \frac{A_1 D_{i1}^{\lambda_1} R_i^2}{B_1 - \sum_{l=1}^N R_l}$.

In order to compare the optimal solution to WSP-SM to the labor costs that a company would incur if it does not coordinate its labor decisions, we develop the single market uncoordinated workforce supply problem (UWSP-SM). UWSP-SM can be written as

$$\sum_{i=1}^{N} \min_{\hat{w}_i} \{ f_{i1}(\bar{w}_1) | g_i(\bar{w}_1) = R_i \},$$

where $\hat{w}_i = (w_{i1}, ..., w_{iK})$ is the ith row of the matrix $\mathbf{W}$. It is solved by sequentially solving single facility WSP-SM problems using the following algorithm.

Step 0

1. Set $w_{i1} = 0, \ i = 1, ..., N$.
2. Set $TC = 0$

3. Set $i = 1$.

4. Go to Step 1.

Step 1

1. Set $w_{i1} = \frac{(A_1 + \sum_{j \neq i} \frac{w_{i1}}{D_{i1}^{\lambda_1}}) D_{i1}^{\lambda_1} R_i}{B_1 - R_i}$.

2. If $i < N$, then set $i = i + 1$ and repeat Step 1.

3. If $i = N$, then go to Step 2.

Step 2

1. Set $TC' = \displaystyle\sum_{i=1}^{N} B_1 \frac{\frac{w_{i1}^2}{D_{i1}^{\lambda_1}}}{\sum_{l=1}^{N} \frac{w_{i1}}{D_{i1}^{\lambda_1}} + A_1}$.

2. Set $\Delta = TC' - TC$.

3. If $\Delta > \alpha$ then set $TC = TC'$, set i=1, and go to Step 1.

4. If $\Delta \leq \alpha$ then stop. TC' is the final value to UWSP-SM and w_{i1} is the final wage to be paid by facility i to workers from the market.

The algorithm meets the stopping criteria that $\Delta \leq \alpha$ if and only if the total cost of UWSP-SM converges to some value as the iterations are performed. Before showing that this is indeed the case we present some useful lemmas.

Lemma 11 $\frac{\prod_{i=1}^{N} R_i}{\prod_{i=1}^{N} B_1 - R_i} < 1$.

Proof We show that $\frac{\prod_{i=1}^{N} R_i}{\prod_{i=1}^{N} B_1 - R_i} < 1$ using two cases. The first case is that $0 < \frac{R_i}{B_1 - R_i} < 1,\ i - 1, ..., N$. For any two numbers $0 < a < 1$ and $0 < b < 1$, $0 < ab < 1$. In this case, every component of the product is between zero and one. Thus, $\frac{\prod_{i=1}^{N} R_i}{\prod_{i=1}^{N} B_1 - R_i} < 1$.

The second case is that $\frac{R_i}{B_1 - R_i} > 1$ for some facility i. This means that $R_i > \frac{B_1}{2}$. Since $\displaystyle\sum_{j=1}^{N} R_j < B_1$ by assumption, the number of workers hired by all other facilities $j \neq i$, is less than $\frac{B_1}{2}$. Thus, $\frac{R_j}{B_1-R_j} < 1, j = 1, ...i-1, i+1, ..., N$. We consider the product $\frac{R_i}{B_1-R_i}\frac{R_j}{B_1-R_j} = \frac{R_j}{B_1-R_i}\frac{R_i}{B_1-R_j}$. Since $R_i+R_j < B_1$ we know that $B_1-R_j > R_i$ and $B_1-R_i > R_j$. Thus, $\frac{R_j}{B_1-R_i} < 1$ and $\frac{R_i}{B_1-R_j} < 1$. Every other component of the product is less than one. In this case, $\frac{\prod_{i=1}^N R_i}{\prod_{i=1}^N B_1-R_i} < 1$ also. ∎

Given this lemma, we are able to show that an infinite series involving $\frac{\prod_{i=1}^N R_i}{\prod_{i=1}^N B_1-R_i} < 1$ converges.

Lemma 12 *The infinite sum* $\sum_{n=0}^{\infty}\left(\prod_{i=1}^{N}\frac{R_i}{B_1-R_i}\right)^n$ *converges and is equal to* $\frac{\prod_{i=1}^N (B_1-R_i)}{\prod_{i=1}^N (B_1-R_i)+\prod_{i=1}^N R_i}$.

Proof The infinite sum $\sum_{n=0}^{\infty}\left(\prod_{i=1}^{N}\frac{R_i}{B_1-R_i}\right)^n$ is a geometric series. It will converge if $\prod_{i=1}^N \frac{R_i}{B_1-R_i} < 1$. In Lemma 11 we show that this is the case. Thus, we know that the sum of the series is

$$
\begin{aligned}
&\frac{1}{1-\prod_{i=1}^N \frac{R_i}{B_1-R_i}} \\
=&\frac{1}{\frac{\prod_{i=1}^N (B_1-R_i)-\prod_{i=1}^N R_i}{\prod_{i=1}^N (B_1-R_i)}} \\
=&\frac{\prod_{i=1}^N (B_1-R_i)}{\prod_{i=1}^N (B_1-R_i)-\prod_{i=1}^N R_i}. \blacksquare
\end{aligned}
$$

Theorem 19 *The total cost of UWSP-SM converges to a unique solution as the number of iterations of the algorithm goes to infinity.*

Proof After the first iteration of the UWSP-SM algorithm

$$\begin{aligned}
\frac{w_{11}}{D_{11}^{\lambda_1}} =& A_1 \frac{R_1}{B_1 - R_1} \\
\frac{w_{21}}{D_{21}^{\lambda_1}} =& (A_1 + \frac{w_{11}}{D_{11}^{\lambda_1}}) \frac{R_2}{B_1 - R_2} \\
=& A_1 \frac{R_1 R_2}{(B_1 - R_1)(B_1 - R_2)} \\
=& A_1 \sum_{j=1}^{2} \frac{\prod_{l=j}^{2} R_l}{\prod_{l=j}^{2} (B_1 - R_l)} \\
\vdots& \\
\frac{w_{N1}}{D_{N1}^{\lambda_1}} =& A_1 \sum_{j=1}^{N} \frac{\prod_{l=j}^{N} R_l}{\prod_{l=j}^{N} (B_1 - R_l)}
\end{aligned}$$

After the nth iteration of the UWSP-SM algorithm

$$\frac{w_{i1}}{D_{i1}^{\lambda_1}} = A_1 \sum_{j=1}^{i} \prod_{l=j}^{i} \frac{R_l}{B_1 - R_l} \left[1 + \sum_{j=1}^{N} \prod_{l=j}^{N} \frac{R_l}{B_1 - R_l} \left(\sum_{m=0}^{n-1} \left(\prod_{l=1}^{N} \frac{R_l}{B_1 - R_l} \right)^m \right) \right].$$

Thus, $\frac{w_{i1}}{D_{i1}^{\lambda_1}}$ has a finite value if and only if $\sum_{m=0}^{\infty} \left(\prod_{l=1}^{N} \frac{R_l}{B_1 - R_l} \right)^m$ converges to a finite value. In Lemma 12 we showed that the value of this infinite sum is finite.

Because $\sum_{m=0}^{\infty} \left(\prod_{l=1}^{N} \frac{R_l}{B_1 - R_l} \right)^m$ converges to a unique finite value, we know that $\frac{w_{i1}}{D_{i1}^{\lambda_1}}, i = 1, ..., N$ has a unique solution. ■

Given that the UWSP-SM algorithm converges, we now discuss the solutions to this problem.

Theorem 20 *The unique solution to UWSP-SM is* $w_{i1} = \frac{A_1 D_{i1}^{\lambda_1} R_i}{B_1 - \sum_{l=1}^{N} R_l}$, $i = 1, .., N$.

Proof At each step of the algorithm w_{i1} is chosen such that $B_1 \frac{\frac{w_{i1}}{D_{i1}^{\lambda_1}}}{A_1' + \frac{w_{i1}}{D_{i1}^{\lambda_1}}} = R_i$, where $A_1' = A_1 + \sum_{j \neq i} \frac{w_{j1}}{D_{j1}^{\lambda_1}}$. Since we have shown that the algorithm converges we know that as the number of iterations of the UWSP-SM algorithm goes to infinity then $B_1 \frac{\frac{w_{i1}}{D_{i1}^{\lambda_1}}}{A_1' + \frac{w_{i1}}{D_{i1}^{\lambda_1}}} = R_i$ for all $i = 1, ..., N$ simultaneously. In Theorem 8 we have shown that the unique solution to this system of equations is $w_{i1} = \frac{A_1 D_{i1}^{\lambda_1} R_i}{B_1 - \sum_{l=1}^{N} R_l}$, $i = 1, .., N$. ■

Now we examine the uncoordinated revised convex allocation problem where there are $K > 1$ markets from which workers are hired. In order to compare the optimal solution of RCAP to the labor costs that a company would incur if it does not coordinate its labor decisions, we develop the uncoordinated revised convex allocation problem (URCAP). URCAP can be written as

$$\sum_{i=1}^{N} \min_{\tilde{r}_i} \{ \sum_{k=1}^{K} h_{ik}(\bar{r}_k) | a_i(\tilde{r}_i) = R_i \},$$

where $\tilde{r}_i = (r_{i1}, ..., r_{iK})$ is the ith row of the matrix $\mathbf{r}$. It is solved by sequentially solving single facility RCAP problems. The algorithm to solve URCAP is similar to the UWSP-SM algorithm. It is useful to define $\acute{h}_{ik}(\bar{r}_k) = \frac{(A_k + \sum_{i=1}^{N} \frac{w_{ik}}{D_{ik}^{\lambda_k}}) D_{ik}^{\lambda_k} r_{ik}^2}{B_k - r_{ik}}$.

Step 0

1. Set w_{ik} and $r_{ik} = 0,\ i = 1, ..., N,\ k = 1, ..., K$.
2. Set $TC = 0$
3. Set $i = 1$.
4. Go to Step 1.

Step 1

1. Solve a single facility RCAP problem

$$\begin{aligned} \min \sum_{k=1}^{K} \acute{h}_{ik}(\bar{r}_k) & \\ \text{s.t.} \quad \sum_{k=1}^{K} r_{ik} & \geq R_i \\ r_{ik} & \geq 0, \forall k \in \{1, ..., K\}. \end{aligned}$$

2. Set $w_{ik}(\bar{r}) = \frac{A_k D_{ik}^{\lambda_k} r_{ik}}{B_k - \sum_{l=1}^{N} r_{lk}}\ \forall k = 1, ..., K$
3. If $i < N$, then set $i = i + 1$ and repeat Step 1.

Table 2: 2X2 RCAP Example - Market Parameters

	k	
	1	**2**
A	6000	7000
B	2000	1000
λ	1.7	1.7

Table 3: 2X2 RCAP Example - Facility Parameters

		R	$\mathbf{D_{i1}}$	$\mathbf{D_{i2}}$
i	**1**	1200	10	5
	2	200	16	8

4. If $i = N$, then go to Step 2.

Step 2

1. Set $TC' = \sum_{i=1}^{N} \sum_{k=1}^{K} h_{ik}(\bar{r}_k)$.
2. Set $\Delta = TC' - TC$.
3. If $\Delta > \alpha$ then set $TC = TC'$, set i=1, and go to Step 1.
4. If $\Delta \leq \alpha$ then stop. TC' is the final value to UWSP-SM and r_{ik} is the final number of workers hired from market k to work at facility i.

We are unable to analytically show that the URCAP algorithm converges. Neither can we analytically compare solutions of URCAP to the optimal solution of RCAP. However, we can numerically solve the algorithm for different problem instances. In doing so, we have yet to find a problem instance where the algorithm did not converge. We provide two examples that illustrate these points.

The first example is a 2-facility, 2-market problem instance. The parameters pertaining to the two markets are shown in table 2. The facilities' workforce requirements and distance matrix is shown in table 3.

Table 4: 2X2 RCAP Example - Optimal Solution

r	**1**	**2**
1	679.495	520.505
2	132.912	67.088

Table 5: 2X2 RCAP Example - Uncoordinated Solution

r	**1**	**2**
1	692.884	507.116
2	107.038	92.962

The optimal solution to the coordinated RCAP problem is shown in table 4. The corresponding objective function value is $200,413,221.56.

Now we compare this optimal solution to the uncoordinated solution. The point where the algorithm converges is dependent on the sequence or labeling of the facilities. Table 5 shows the solution to the 2-facility, 2-market uncoordinated problem. The total labor costs of this solution are $201,306,286.83. The difference is $893,065.27. The labor costs are 0.45% higher if the facilities do not coordinate.

Table 6: 5X8 RCAP Example - Market Parameters

	k							
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
A	14000	22500	39000	47500	55500	43000	63500	35000
B	700	6000	3800	1000	7000	2800	3500	5000
λ	1.3	1.1	1.2	1.4	1.6	1.8	1.7	1.5

Table 7: 5X8 RCAP Example - Facility Parameters

		R	$\mathbf{D_{i1}}$	$\mathbf{D_{i2}}$	$\mathbf{D_{i3}}$	$\mathbf{D_{i4}}$	$\mathbf{D_{i5}}$	$\mathbf{D_{i6}}$	$\mathbf{D_{i7}}$	$\mathbf{D_{i8}}$
	1	1500	10	15	8	2	28	7	21	8
	2	2500	20	5	18	30	4	26	9	12
i	**3**	1300	6	27	7	3	5	6	24	25
	4	2300	29	5	25	9	5	10	8	22
	5	1700	3	30	3	23	11	7	20	10

The other example we show in this dissertation is a 5-facility, 8-market problem instance. The markets' parameters are shown in table 6. The facilities' workforce requirements and distance matrix is shown in table 7.

Table 8: 5X8 RCAP Example - Optimal Solution

r	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	54.453	332.852	289.363	266.642	31.257	125.052	22.834	377.545
2	24.709	1245.684	119.950	6.629	764.170	12.720	103.753	222.385
3	87.212	143.964	290.296	128.346	428.539	145.447	16.134	60.061
4	16.010	1309.223	84.374	37.352	555.422	73.568	131.156	92.896
5	223.991	133.725	829.778	7.683	124.995	113.178	22.566	244.083

Table 9: 5X8 RCAP Example - Uncoordinated Solution

r	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	61.248	387.822	296.663	250.632	30.937	112.207	19.823	340.670
2	29.708	1273.628	131.677	7.793	737.966	12.229	93.742	213.257
3	95.999	179.761	296.894	132.602	400.934	125.911	13.482	54.417
4	19.431	1317.677	93.263	42.480	547.261	70.265	118.819	90.804
5	218.360	175.998	811.538	9.336	128.013	105.257	20.173	231.326

The optimal solution to the coordinated RCAP problem is shown in Table 8. The corresponding objective function value is \$593,755,965.73.

Now we compare this optimal solution to the uncoordinated solution. The point where the algorithm converges is dependent on the sequence or labeling of the facilities. Table 9 shows the solution to the 5-facility, 8-market uncoordinated problem. The total labor costs of this solution are \$599,515,232.23. The difference is \$5,759,266.50. The labor costs are 0.97% higher if the facilities do not coordinate.

These two examples demonstrate that a company making labor decisions for multiple facilities located in the same geographic region benefits from coordinating these labor decisions. The labor costs incurred by the company when the facilities do not coordinate is a small percentage higher than the optimal solution, however the dollar amount is significant.

We have presented the workforce supply problem, its mathematical model, and a solution procedure. We have also shown the usefulness of a company considering this problem because labor costs are lower than if each of the company's facilities make labor decisions independently. However, there are some aspects of the real world that not captured by the our presentation of the problem. We address some of those aspects in the next section.

3.5 Workforce Supply Models for Future Research

The model of the workforce supply problem we presented in this chapter is a general model for making workforce supply decisions under the assumptions listed above. However, there are some problems that real-world managers face that are not addressed by this model. We briefly discuss some of these problems and how the general model can be altered to address them, but we do not attempt to find solutions to these problems at this point in our research.

The general model for the workforce supply problem allows a facility to pay a different amount to workers from different markets. In other words, the wage paid by facility i to some market k, w_{ik}, is not necessarily equal to the wage paid to another market l, w_{il}. This is fine if the workforce of the facility is not unionized. However, a facility with a unionized workforce typically pays every worker the same wage regardless of the labor market where they live. The manager of a facility with a unionized workforce still wants to minimize labor costs despite having to pay every worker the same amount. In this case, the workforce supply problem must be reformulated so that the wage paid to workers from any single market is equal to the wage paid to workers from every other market. We call this problem the workforce supply problem with uniform wages (WSP-UW). One way WSP can be altered to address WSP-UW is to add the constraint $w_{ik} = w_{i1},\ i = 1, ..., N,\ k = 2, ..., K$. We know that the optimal solution to WSP-UW will be greater than or equal to the optimal solution to WSP because the feasible region of WSP-UW is either smaller or the same size as that of WSP. The other way to reformulate the WSP into WSP-UW is to use w_i to indicate the wage paid by facility i to workers from every market and model it as follows:

WSP-UW

$$\min \sum_{i=1}^{N} \sum_{k=1}^{K} B_k \frac{\frac{w_i^2}{D_{ik}^{\lambda_k}}}{\sum_{l=1}^{N} \frac{w_l}{D_{lk}^{\lambda_k}} + A_k}$$

$$\text{s.t.} \quad \sum_{k=1}^{K} B_k \frac{\frac{w_i}{D_{ik}^{\lambda_k}}}{\sum_{l=1}^{N} \frac{w_l}{D_{lk}^{\lambda_k}} + A_k} \geq R_i, \forall i \in \{1, ..., N\}$$

$$w_i \geq 0, \ \forall i \in \{1, ..., N\}.$$

Formulating the WSP-UW model in this way is more convenient because it can be reformulated into market subproblems and a convex allocation master problem similar to the way in which WSP was reformulated into CAP-SM and CAP-MP.

Another difficulty that exists in the real-world that is not addressed by the general model of the workforce supply problem presented above is that the wage paid to workers is often limited in some way. On one hand, a government may set a minimum wage requirement. On the other hand, a company may have a policy that no worker is paid more than a maximum wage. These circumstances may even occur simultaneously. In the case of a mandated minimum wage, w_{min}, WSP is altered by adding the constraint $w_{ik} \geq w_{min}, \ i = 1, ..., N, \ k = 1, ..., K$. If w_{max} is the highest wage to be paid to any worker then WSP is changed by adding the constraint $w_{ik} \leq w_{max}, \ i = 1, ..., N, \ k = 1, ..., K$. Adding these constraints to WSP prevents the problem from being reformulated into CAP-SM and CAP-MP.

We have developed a model for the WSP problem and provided optimal solution procedures. Thus, we can use this basis for solving other problems. In Chapter 4 we consider the problem of a company that is locating a number of facilities in a region. The company considers the labor costs in making the facility location decisions. The labor costs are modeled in the same way in the workforce supply and facility location problem as they are in WSP.

4 WORKFORCE SUPPLY AND FACILITY LOCATION PROBLEM

In the workforce supply and facility location problem a company must determine where to locate a collection of facilities that form at least part of a supply chain and that compete with each other as well as other existing facilities for workers from a common set of labor markets. We refer to all the company facilities as "internal" facilities. The locations for the new internal facilities are chosen from a given set of potential sites. The objective is to minimize the total labor costs and site costs incurred by all of the facilities.

The company must choose (1) the location of the new facilities and (2) the number of workers to hire from each labor market. There are "external" facilities that also seek to hire from the same group of qualified workers in the different labor markets. The locations and wages of the external facilities are known and fixed. The labor costs incurred by any one internal facility depends on the amount of competition it faces from the external facilities and the other internal facilities. Each new facility incurs a purchase and preparation cost depending on the site chosen for its location.

There are other assumptions made in order to formulate mathematical models for this problem. The locations of these markets and their associated populations of potential workers are assumed to be known. We approximate the distance between any worker in a labor market and a facility by the distance between the center of the labor market and the facility. The same is done for the distance between the labor markets and the potential facility locations.

The following notation and assumptions are used to formulate a model for the workforce supply and facility location (WSFL) problem.

Parameters

K	=	number of labor markets
N	=	number of internal facilities
P	=	number of potential sites for the internal facilities
B_k	=	number of workers at labor market k, $k = 1, ..., K$
R_i	=	number of workers required by internal facility i, $i = 1, ..., N$
D_{kp}	=	distance between potential site p and market k, for $k = 1, ..., K$ and $p = 1, ..., P$
λ_k	=	attraction parameter for market k, $k = 1, ..., K$
S_{ip}	=	annual fixed costs of locating internal facility i at potential site p, for $i = 1, ..., N$ and $p = 1, ..., P$

Decision Variables

r_{ikp} = the number of workers from market k allocated to work at facility i when it is located at potential location p, for $i = 1, ..., N$, $k = 1, ..., K$, and $p = 1, ..., P$

$$y_{ip} = \begin{cases} 1 & \text{if facility } i \text{ is located at potential location } p \\ 0 & \text{otherwise} \end{cases},$$

for $i = 1, ..., N$ and $p = 1, ..., P$

Vectors and Matrices

$\mathbf{r}$ = matrix of the number of workers from each market allocated to work at each internal facility at every potential location;

$= (r_{ikp})$

$\bar{r}_k$ = kth submatrix of matrix $\mathbf{r}$;

$$= \begin{bmatrix} r_{1k1} & \dots & r_{Nk1} \\ \vdots & \ddots & \vdots \\ r_{1kP} & \dots & r_{NkP} \end{bmatrix}$$

Assumptions

1. There are more potential sites than there are facilities to be located, $P > N$.

2. There exists competition from external facilities at every market, $A_k > 0,\ k = 1, ..., K$.

3. The distance between any potential facility location and any labor market is strictly positive, $D_{kp} > 0,\ p = 1, ..., P, k = 1, ..., K$.

4. The attraction parameter is always positive for every market, $\lambda_k > 0,\ k = 1, ..., K$.

5. Each internal facility requires at least one worker, $R_i \geq 1,\ i = 1, ..., N$.

6. The total number of workers required by all company facilities is strictly less than the total number of available workers, $\sum_{i=1}^{N} R_i < \sum_{k=1}^{K} B_k$.

The WSFLP is an extension to the WSP. In addition to the assumptions listed above, many of the assumptions made for the WSP are made for the WSFLP. This includes the assumptions about how workers choose between employers and the external competition. For more details on these assumptions see Chapter 3. Given the assumptions above and those from Chapter 3, it is known that for a given matrix $\mathbf{r}$, the wage paid by facility i at site p to workers from market k is

$$w_{ikp}(\bar{r}_k) = \frac{A_k D_{kp}^{\lambda_k} r_{ikp}}{B_k - \sum_{q=1}^{P} \sum_{l=1}^{N} r_{lkq}}.$$

The same labor relationships that exist between internal and external facilities and between different internal facilities in the WSP problem exist in the WSFL problem. However, in the WSFL problem there are other labor relationships based on the distance between the internal facilities and the labor markets. If the decision of where to place a new internal facility i is moved to a different site such that the distance between labor market k and facility i is increased, the number of workers at labor market k attracted to the new internal facility i is decreased, while the number of workers at labor market k attracted to a different internal facility $j \neq i$ is increased. The cost of facility i located at site p attracting workers from market k depends on the number of workers that each facility attracts from that same market, $r_{ikp},\ i = 1, ..., N,\ p = 1, ..., P$. This cost is modeled in WSFL similar to the way we modeled the labor costs in WSP. It is

$$h_{ikp}(\bar{r}_k) = \frac{A_k D_{kp}^{\lambda_k} r_{ikp}^2}{B_k - \sum_{q=1}^{P} \sum_{l=1}^{N} r_{lkq}}.$$

The annual cost of locating facility i at site p is

$$y_{ip} S_{ip}.$$

It is dependent on which of the potential site locations is chosen for the facility's location.

Given this notation and these assumptions, we present the general model of the workforce supply and facility location problem (WSFL).

WSFLP-CAP

$$\min \quad \sum_{i=1}^{N} \sum_{k=1}^{K} \sum_{p=1}^{P} h_{ikp}(\bar{r}_k) + \sum_{i=1}^{N} \sum_{p=1}^{P} y_{ip} S_{ip}$$

$$\text{s.t.} \quad \sum_{k=1}^{K} r_{ikp} = y_{ip} R_i,\ \forall i \in \{1, ..., N\},\ \forall p \in \{1, ..., P\} \tag{20}$$

$$\sum_{i=1}^{N} y_{ip} \leq 1, \qquad \forall p \in \{1, ..., P\} \tag{21}$$

$$\sum_{p=1}^{P} y_{ip} = 1, \qquad \forall i \in \{1, ..., N\} \tag{22}$$

$$r_{ik} \geq 0, \qquad \forall i \in \{1, ..., N\},\ \forall k \in \{1, ..., K\} \tag{23}$$

$$y_{ip} \in \{0, 1\}, \forall i \in \{1, ..., N\},\ \forall p \in \{1, ..., P\}. \tag{24}$$

This model is a mixed-integer nonlinear program. In Theorem 10 we have shown that $h_{ik}(\bar{r}_k)$ is a convex function in $r_{ik}, i = 1, ..., N$. The function $h_{ikp}(\bar{r}_k)$ is of the same form and, is also convex in $r_{ikp}, i = 1, ..., N,\ p = 1, ..., P$. The function $s_{ip}(y_{ip})$ is a discontinuous function. The constraint functions 21, 22, and 24 are all discontinuous functions. The constraint function 23 is a linear function.

We now present some solution procedures for WSFLP-CAP. In this dissertation we simply outline the procedures. We do not provide analysis or testing of the different approaches. That is future work that will follow this dissertation.

4.1 Solution Procedures

Mixed-integer nonlinear programming is a widely studied field. The difficulty in solving the programs to optimality lies in both the nonlinearity of some of the functions and the discontinuity of some of decision variables. Many optimal and heuristic algorithms have been developed for different formulations. We present two heuristic algorithms and one optimal solution procedure for WSFLP-CAP. Before we do so, it is important to point out that if we take the locations of the new internal facilities as a given then the problem simplifies into the RCAP problem and can be solved using the reduced gradient algorithm presented in Chapter 3.

4.1.1 Assignment Decomposition Heuristic

The first solution procedure for WSFLP-CAP we present is a heuristic. First, we estimate the labor costs of each new internal facility at each of the potential sites. Then we decide where to locate each new internal facility by solving an assignment problem. The assignment problem is a well known problem and is easily solved using the simplex algorithm or other specialized algorithms. Given the locations for the new facilities the reduced gradient (RG) algorithm for solving RCAP is used to minimize the labor costs. More formally stated, the algorithm is as follows:

Assignment Decomposition Heuristic

Step 0

1. Set $i = 1$.
2. Set $p = 1$.

Step 1

1. Set $y_{ip} = 1$ and $y_{iq} = 0,\ \forall q \neq p$.
2. Set $D_{ik} = \sum_{p=1}^{P} y_{ip} D_{pk},\ k = 1, ..., K$.
3. Using $D_{ik},\ k = 1, ..., K$, Solve a single facility RCAP for facility i located at site p using the RG algorithm.
4. Set $LC_k = \frac{A_k D_{ik}^{\lambda_k} r_{ik}^2}{B_k - r_{ik}}$.
5. Set $S'_{ip} = S_{ip} + \sum_{k=1}^{K} LC_k$.
6. If $p < P$ then set $p = p + 1$. Repeat Step 1.
7. If $p = P$ then set $p = 1$. If $i < N$ then set $i = i + 1$ and repeat Step 1. If $i = N$ then go to Step 2.

Step 2

1. Solve the assignment problem

$$\begin{aligned} \min \quad & \sum_{i=1}^{N} \sum_{p=1}^{P} y_{ip} S'_{ip} \\ \text{s.t.} \quad & \sum_{i=1}^{N} y_{ip} \leq 1, \qquad \forall p \in \{1, ..., P\} \\ & \sum_{p=1}^{P} y_{ip} = 1, \qquad \forall i \in \{1, ..., N\} \\ & y_{ip} \in \{0, 1\},\ \forall i \in \{1, ..., N\},\ \forall p \in \{1, ..., P\}. \end{aligned}$$

2. Let y^*_{ip} be the optimal solution to the assignment problem.
3. Go to Step 3.

Step 3

1. Set $D_{ik} = \sum_{p=1}^{P} y^*_{ip} D_{pk},\ i = 1, ..., N\ k = 1, ..., K$.

2. Using these distances, solve the RCAP problem for each of the internal facilities using the RG algorithm.

This algorithm requires that the RG algorithm for solving RCAP be run $N \cdot P + 1$ times. We compare this to a total enumeration approach. Total enumeration requires that the labor costs be calculated for every possible combination of location decisions for the internal facilities. There are $\binom{P}{N}$ possible combinations of facility location decisions. In most cases, the number of RG algorithm runs required in the assignment problem heuristic is much less than the number of runs required in total enumeration.

This algorithm always results in a feasible solution. It also provides us with both an upper and a lower bound for the optimal solution to WSFLP-CAP. The lower bound is found by adding the optimal objective function value of a RCAP problem involving only the facilities with fixed locations to the optimal objective function value to the assignment problem in Step 2. This is a lower bound because the labor costs do not take into account the competition from the other internal facilities. The solution to the assignment problem heuristic algorithm is an upper bound.

It is possible that the algorithm finds the optimal solution to WSFLP-CAP. This occurs if and only if the locations chosen for the new internal facilities in Step 2, y_{ip}^*, are the optimal locations for the facilities.

In this algorithm we took advantage of the fact that WSFLP-CAP can be reduced to RCAP by fixing the values for y_{ip} $i = 1, ..., N,\ p = 1, ..., P$. We also use this to develop a tabu search heuristic algorithm for solving WSFLP-CAP.

4.1.2 Tabu Search Heuristic

WSFLP-CAP is a combinatorial optimization problem. For a given set of location decisions, we can minimize the labor costs by solving the resulting RCAP. Thus, we develop a tabu search technique for solving WSFLP-CAP. The tabu search technique was first introduced by Glover (1986). According to Gendreau (2003), since its introduction tabu search has been applied to problems in over a hundred papers. He also argues that this heuristic is one of the best solution procedures in terms of effectiveness.

Tabu search is a local search solution procedure. It starts with one or more initial solutions. Given the initial solutions, neighbor solutions are investigated in hopes of finding improved objective function values. The best solution among the neighbors is always chosen, even if the solution is not as good as the current one. However, in order to prevent the algorithm from cycling all solutions that are visited become "tabu" for a number of iterations. In other words, the algorithm cannot return to previous solutions for at least a number of iterations. Thus, the span of the search done by the algorithm depends on the number of iterations a neighborhood move is tabu. The main objective of the algorithm is to search most of the solution space in order to, hopefully, find an optimal solution.

Rather than present a step by step tabu search algorithm we address the major parts of the algorithm that we propose using in future work. Gendreau (2003) provides an excellent overview of the most important and effective components of a tabu search heuristic.

Search Space and Neighborhood Structure: Gendreau (2003) says that the most critical aspect of a tabu search algorithm is the search space and neighborhood structure. For the workforce supply and facility location problem the search space is the feasible set of location decisions. Each facility is assigned to exactly one site and every facility has either one or zero facilities assigned to it.

We propose a neighborhood structure here, but recognize that testing the algorithm may lead us to alter it. Given a solution $\mathcal{S} = y_{ip}\ i = 1, ..., N,\ p = 1, ..., P$ the neighbors to $\mathcal{S}$ are all solutions where some facility i is moved from site p to site q. In other words, we set $y_{ip} = 0$ and $y_{iq} = 1,\ q \neq p$ for some i. If site q is unoccupied then no other changes need to be made. We call this a "move". On the other hand, if site q is occupied by some other facility j then we set $y_{jq} = 0$ and $y_{jp} = 1$. We call this a "swap".

Tabu List: At each iteration of the tabu search algorithm some facility i is moved from site p to site $q \neq p$. Then moving facility i back to site p is disallowed for some number of iterations, T. Thus, at each iteration there are T moves that are disallowed. These jobs form the "tabu list". This number of iterations that a move is tabu is an important parameter. If it is too low then cycling is likely. However, storing the tabu moves in memory can be expensive if it is too large. Thus, it is important to test different levels of this parameter in our future work.

We make storing tabu moves less expensive in terms of memory by minimizing the amount of information that is stored. We do not need to save the entire solution at each iteration. Given an initial solution, we can determine the set of location decisions at each iteration by indicating only the order of moves. At each iteration we store only the facility that is moved and the site from which it is moved. This requires the storage of only two integer values.

Aspiration Criteria: In order to prevent the tabu list from prohibiting a move to a solution that is desirable, we allow the algorithm to perform a tabu move if the solution found by making that move is better the best-known solution. In order to allow this, we store the value of the best solution that has been found at any iteration.

Termination Criteria: We plan to stop the algorithm after some number of iterations without finding an improving solution. In order to determine this parameter, testing different values will be done as part of the future work.

Intensification: Assuming that the algorithm is at solution $\mathcal{S}$, an iteration of the intensification consists the following steps:

1. Determine the neighbors of $\mathcal{S}$.
2. Calculate the labor costs for each of the neighbors of $\mathcal{S}$.
3. Calculate the site costs for each of the neighbors of $\mathcal{S}$.

4. Calculate the total costs for each of the neighbors of $\mathcal{S}$. Suppose that moving facility i from site p to site q results in the neighbor with the lowest total cost.

5. If moving facility i from site p to site q is tabu then it is accepted only if it is lower than the best-known solution. Otherwise, the neighbor with the next lowest total cost is considered.

6. If moving facility i from site p to site q is allowed then accept that solution. This move is made whether or not the resulting solution is lower than the total costs of $\mathcal{S}$. If site q is occupied by a facility j then move facility j from site q to site p. Update $\mathcal{S}$ and add the move to the tabu list.

Diversification: We want to be sure that we search as much of the solution space as possible. Of course, searching the entire space is possible but the goal of the algorithm is to find a good solution without having to do so. Thus, we allow our algorithm to diversify the solutions for searching only occasionally. We perform a diversification step if the intensification process fails to find an improving solution for a number of iterations. This number will be determined through experimentation. The diversification step consists of choosing a new solution $\mathcal{S}'$ that is not in the neighborhood of $\mathcal{S}$. $\mathcal{S}'$ is determined in the same way as our initial solution, but is different than the initial solution.

This fully describes the components of the tabu search heuristic to be used to solve WSFLP-CAP in our future work. We have now presented two heuristics solution procedures. Now, we present an algorithm that solves WSFLP-CAP to optimality.

4.1.3 Branch and Bound Algorithm

One typical application of the branch and bound algorithm as it applies to integer problems is to relax the integrality of the problem. In particular, we are concerned with binary decision variables, y_j. Solving the relaxed problem results in a lower bound on the optimal solution to the integer problem. If the optimal solution to the relaxed problem is integer then that solution is optimal for the integer problem. For a variable, y_j that has a non-integer solution to the relaxed problem, two problems can be considered. One where $y_j = 0$ and the other where $y_j = 1$. Then the problems can be compared. Assuming that there are many binary decision variables, then in each branch of the algorithm some of the variables are set to be either 1 or 0 and the integrality of the other variables is relaxed. More and more of the variables are set equal to 1 or 0 as the algorithm progresses until an integer solution is found. Any time an integer solution is found then all the branches where the solution to a relaxed problem is higher than the solution to the integer solution can be abandoned because the relaxed problem is a lower bound on any integer solution that might be found from that branch.

If the nonlinear program that results from relaxing the integrality of the variables $y_{ip},\ i = 1, ..., N,\ p = 1, ..., P$ can be solved to optimality then a branch and bound algorithm can be used to solve the workforce supply and facility location problem. We have already noted that $h_{ikp}(\bar{r}_k)$ is convex in $r_{ikp},\ \forall i = 1, ..., N,\ \forall p = 1, ..., P$. If we allow $y_{ip},\ i = 1, ..., N,\ p = 1, ..., P$ to take on any value between 0 and 1 then $s_{ip}(y_{ip})$ is a linear function in y_{ip}. In this case, the objective function is convex in $r_{ikp},\ \forall i = 1, ..., N,\ \forall p = 1, ..., P$ and $y_{ip},\ \forall i = 1, ..., N,\ \forall p = 1, ..., P$ because the sum of convex functions is convex. Also, when the integrality of $y_{ip},\ i = 1, ..., N,\ p = 1, ..., P$ is relaxed, constraints 21, 22, and 23 are linear constraints. These need not be altered in any way. However, equation 24 is replaced by two linear equations: $y_{ip} \geq 0,\ i = 1, ..., N,\ p = 1, ..., P$ and $y_{ip} \leq 1,\ i = 1, ..., N,\ p = 1, ..., P$

Given that the objective function is convex and the constraints form a convex set it is known from Bazaraa et al. (2006), Theorem 4.2.16 that a solution satisfying the KKT conditions is optimal. In Chapter 3 we use a reduced gradient algorithm to solve RCAP. We plan to develop a reduced gradient algorithm that solves the relaxed workforce supply and facility location (RWSFLP). Solutions to the reduced gradient algorithm are guaranteed to satisfy the KKT conditions. Thus, the RG algorithm results in an optimal solution to

RWSFLP.

Note that in the description of the algorithm we refer to an "integer solution" to RWSFLP to be one where $y_{ip},\ i = 1, ..., N,\ p = 1, ..., P$ are integers. We call this an integer solution even though it is unlikely that $r_{ik},\ i = 1, ..., N,\ k = 1, ..., K$ are integer.

For our version of the branch and bound algorithm we assume that the facilities are indexed in order of the size of their required workforce. In other words, we assume that $R_1 > R_2 > ... > R_{N-1} > R_N$. This is an idea that we hope will result in reducing the time spent in solving the algorithm. However, we plan to experiment with this and other indexing techniques.

Rather than provide a step by step algorithm, we present the aspects that define the branch and bound algorithm.

Initial Node: In the initial node we solve a RWSFLP where the integrality $y_{ip},\ i = 1, ..., N,\ p = 1, ..., P$ is relaxed. This node is considered to be level 0 of the tree.

Branching: At any level n of the tree, the node with the lowest objective function value to RWSFLP that does not have an integer solution and has not already been branched into the next level. This is a "depth-first" approach. Testing can be performed on the branching technique once the algorithm is coded to see if other techniques might work better.

Suppose that we choose to branch off of node m at level n. We say that at node m the integer value of $y^m_{ip}, i = 1, ..., n,\ p = 1, ..., P$ has been determined. We formulate $P - n$ nodes branching from node m, and each one assumes that $y_{ip} = y^m_{ip}, i = 1, ..., n,\ p = 1, ..., P$. In each of the $P - n$ nodes facility $n + 1$ is assigned to exactly one of the unoccupied sites. For example, suppose that facility $n + 1$ is assigned to site p in a node at level $n + 1$. This implies that $y_{n+1,p} = 1$ and $y_{n+1,q} = 0,\ \forall q \neq p$. One might consider this to be branching on the constraint 22 because the constraint for facility $n + 1$ is satisfied as an equation at each of the nodes at level $n + 1$. Since we index the facilities according to the size of their required workforce, each of the facilities, $i = 1, ..., n$, with a larger required workforce than facility $n + 1$ are already assigned to a particular site. Figure 4 demonstrates this branching technique.

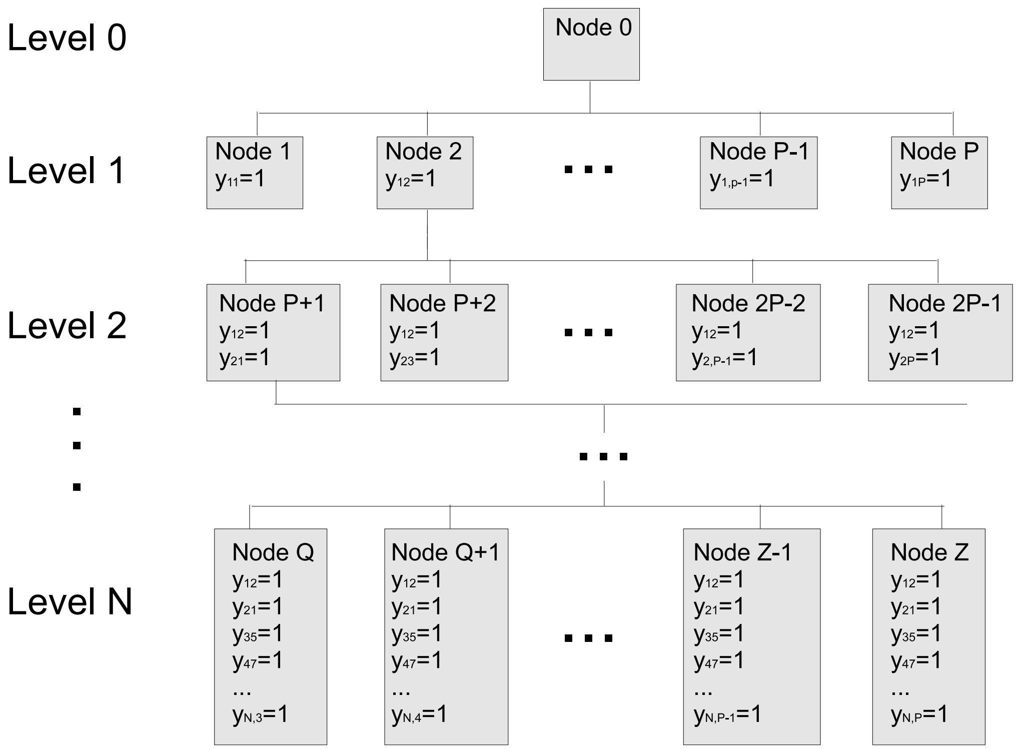

Figure 4: Branching Technique Example

Incumbent Solution: Recall that the Assignment Decomposition Heuristic provides a lower bound on WSFLP. Before starting the branch and bound algorithm, we solve the Assignment Decomposition Heuristic and use the resulting solution as the initial incumbent solution and lower bound. After that point, if an integer solution is found the objective function value is compared to that of the incumbent solution. If the incumbent solution's objective function value is lower then it remains as incumbent solution. Otherwise, it is replaced by the new integer solution.

Bounding: A node at any level of the tree is either used to branch into the next level or it is "fathomed". A node is fathomed if the objective function value of the RWSFLP at that node is higher than the incumbent integer solution. This is true whether or not the solution at the node is integer. This branching is allowed because the lowest integer solution that could be found by using that node to branch into lower levels of the tree has a solution no lower than the solution to RWSFLP at the node.

Forming New Branches: If at some level of the tree it is found that every node is either fathomed or has an integer solution then no further branching at the level can be done. In this case, the algorithm returns to level 1 and systematically looks for a node on which branching needs to be done. In other words, it is possible that when one branch is exhausted there is another branch that has not yet been eliminated from consideration that needs to be explored for a better solution or in order to confirm that the incumbent solution is the optimal solution.

Stopping Criteria: The algorithm continues until there are no nodes which can be used to branch into lower levels of the tree. This occurs if every node examined in the algorithm has either been used to branch, has been fathomed, or has an integer solution. At this point we can be sure that the incumbent solution is the optimal solution to WSFLP-CAP.

The areas for future work on this new problem, WSFLP, are plentiful and promising. We have outlined three solution procedures. These procedures need to be coded and tested. We will then be able to compare the algorithms in terms of quality of the solutions obtained and the computation time required. This development of the algorithms will allow us to present this new problem with solution procedures as an important contribution to the literature.

5 CONTRIBUTIONS

Both workforce supply and facility location decisions affect the immediate and long-term success of a firm. This is especially true since labor costs, a significant portion of the total operating costs of a facility, are largely dependent on the facility's location. Both costs need to be considered when making these critical decisions. These decisions are even more vital to the success of a company when the labor resources in a region are limited. Good facility location and labor decisions of multiple coordinated facilities are even more important and difficult to make when the various facilities are being located simultaneously. Facilities located near each other compete for labor resources. This competition increases the wage that must be paid to workers.

The major contribution of this work is to introduce two problems that are new to the literature. These are the Workforce Supply Problem and the Workforce Supply and Facility Location Problem. The Workforce Supply Problem is the only problem we know of that seeks to minimize the labor costs of coordinated facilities located in the same region. The decision maker decides how many workers each facility will hire from each of the labor markets in the region. These decisions determine the wage, and thus the labor costs, that must be paid to be able to hire the workers. An optimal solution procedure is shown for problems where workers are hired from multiple labor markets. Also a closed-form solution is shown for problems where all workers are hired from the same market. Using these analytical results we point out important managerial insights.

We are able to show the benefit of the development and solution procedure for the Workforce Supply Problem. We show that when there are more than one labor market from which workers are hired, the savings of the company are significant if the different internal facilities coordinate their labor decisions rather than do so independently. It is also interesting that when all workers are hired from a single labor market the labor decisions and costs are the same whether or not the facilities coordinate their labor decisions. In addition to these important contributions, we map out some direction for future work. In particular, we present some additional real-world labor concerns that are not addressed by the WSP.

The Workforce Supply and Facility Location Problem which is a version of the facility location problem where the primary tradeoff is between wages paid to workers and the cost of transporting goods. This is the first problem that allows for a company and its supply chain to minimize its cost by making coordinated facility location and labor decisions. Specifically, the company chooses locations for at least some of its facilities from among a set of potential sites. It must also decide how many workers from each labor market should be hired to work at each facility. The goal of the company is to minimize its total costs. The total costs consist of labor costs and fixed site costs. The fixed site costs are the annualized site purchase and preparation costs along with any other fixed costs associated with a site. One of these fixed costs that can be included is transportation costs. While many facility location models in the literature consider fixed site purchase and preparation costs and transportation costs, we do not know of any facility location models that deal with the labors costs in as direct a way as we do.

We outline two heuristic algorithms and one optimal solution procedure for solving the WSFLP. In all three cases we plan to do future work on the problem by implementing and testing the algorithms. We plan to compare the results of the three procedures in order to find the most efficient solution procedure.

Our models of the WSF and WSFL problems are the first to use an attraction function, common to the competitive location literature, to model how workers decide between potential employers. Specifically, we assume that workers are attracted to employers according to Huff's gravity model of attraction. Since the attraction function is so important to the validity of the mathematical models, our future work may include gathering workforce data from some employers. This data would be used to either validate the use of Huff's gravity model or gain some insight into a more appropriate attraction model for workers.

REFERENCES

Aikens, C. H. 1985. Facility location models for distribution planning. *European Journal of Operational Research* **22**(3) 263–279.

Anton, Howard. 1998. *Calculus : a new horizon*. Wiley, New York; Chichester.

Avriel, M., W.E. Diewert, S. Schaible, I. Zang. 1988. *Generalized concavity*. Plenum Press, New York.

Bartik, Timothy J. 1985. Business location decisions in the united states: Estimates of the effects of unionization, taxes, and other characteristics of states. *Journal of Business & Economic Statistics* **3**(1) 14–22.

Bastian, Lisa. 2002. Putting rural on the map. *Area Development Site and Facility Planning* **37**(12) 22–26.

Bazaraa, Mokhtar S., Hanif D. Sherali, C. M. Shetty. 2006. *Nonlinear Programming: Theory and Algorithms*. 3rd ed. Wiley, Hoboken, New Jersey.

Brandeau, Margaret L., Samuel S. Chiu. 1989. An overview of representative problems in location research. *Management Science* **35**(6) 645–674.

Bussieck, Michael R., Armin Pruessner. 2003. Mixed-integer nonlinear programming.

Chong, Edwin K. P., Stanislaw H. Żak. 2001. *An Introduction to Optimization*. 2nd ed. Wiley, New York, New York.

Chopra, Sunil, Peter Meindl. 2007. *Supply Chain Management: Strategy, Planning & Operation*. 3rd ed. Pearson Prentice Hall, Upper Saddle River, New Jersey.

Daskin, Mark S. 2008. What you should know about location modeling. *Naval Research Logistics* **55**(4) 283–294.

Drezner, Tammy. 1994. Optimal continuous location of a retail facility, facility attractiveness, and market share: An interactive model. *Journal of Retailing* **70**(1) 49–64.

Drezner, Tammy. 1998. Location of multiple retail facilities with limited budget constraints – in continuous space. *Journal of Retailing and Consumer Services* **5**(3) 173–184.

Drezner, Tammy, Zvi Drezner. 1996. Competitive facilities: Market share and location with. *Journal of Regional Science* **36**(1) 1.

Drezner, Tammy, Zvi Drezner. 1997. Replacing continuous demand with discrete demand in a competitive location model. *Naval Research Logistics* **44**(1) 81–95.

Drezner, Tammy, Zvi Drezner. 2002. Validating the gravity-based competitive location model using inferred attractiveness. *Annals of Operations Research* **111**(1) 227–237.

Drezner, Tammy, Zvi Drezner. 2004. Finding the optimal solution to the huff based competitive location model. *Computational Management Science* **1**(2) 193–208.

Drezner, Tammy, Zvi Drezner, H. A. Eiselt. 1996. Consistent and inconsistent rules in competitive facility choice. *The Journal of the Operational Research Society* **47**(12) 1494–1503.

Drezner, Tammy, Zvi Drezner, Said Salhi. 2002. Solving the multiple competitive facilities location problem. *European Journal of Operational Research* **142**(1) 138–151.

Drezner, Zvi, ed. 1995. *Facility Location: A Survey of Applications and Methods*. Springer, New York.

Drezner, Zvi, George O. Wesolowsky, Tammy Drezner. 1998. On the logit approach to competitive facility location. *Journal of Regional Science* **38**(2) 313–327.

Fernández, José, Blas Pelegrín, Frank Plastria, Boglárka Tóth. 2007. Solving a huff-like competitive location and design model for profit maximization in the plane. *European Journal of Operational Research* **179**(3) 1274–1287.

Flaherty, Nick. 1998. The net advantage. *Corporate Location* **March 1** 22–29.

Floudas, Christodoulos A. 1995. *Nonlinear and Mixed-Integer Optimization*. Oxford University Press, New York, New York.

Francis, Richard L., Jr. Leon F. McGinnis, John A. White. 1992. *Facility Layout and Location: An analytical Approach*. 2nd ed. Prentice Hall International Series in Industrial and Systems Engineering, Prentice Hall, Upper Saddle River, NJ.

Friedberg, Stephen H., Arnold J. Insel, Lawrence E. Spence. 1997. *Linear Algebra*. 3rd ed. Prentice Hall, Upper Saddle River, NJ.

Gendreau, Michel. 2003. An introduction to tabu search. Fred Glover, Gary A. Kochenberger, eds., *Handbook of Metaheuristics*, chap. 2. Kluwer's International Series, Kluwer Academic Publishers, 37–54.

Glover, Fred. 1986. Future paths for integer programming and links to artificial intelligence. *Computers and Operations Research* **13**(5) 533–549.

Hakimi, S. L. 1986. p-median theorems for competitive locations. *Annals of Operations Research* **6**(4) 75–98.

Hillier, Frederick S., Gerald J. Lieberman. 2001. *Introduction to Operations Research*. 7th ed. McGraw-Hill, New York, NY.

Hotelling, Harold. 1929. Stability in competition. *The Economic Journal* **39**(153) 41–57.

Huff, David L. 1964. Defining and estimating a trading area. *Journal of Marketing* **28**(3) 34.

Huff, David L. 1966. A programmed solution for approximating an optimum retail location. *Land Economics* **42**(3) 293.

Manning, Alan. 2003. The real thin theory: monopsony in modern labour markets. *Labour Economics* **10**(2) 105–131.

McGarvey, Ronald G., Tom M. Cavalier. 2005. Constrained location of competitive facilities in the plane. *Computers & Operations Research* **32**(2) 359–378.

Nakanishi, Masao, Lee G. Cooper. 1974. Parameter estimation for a multiplicative competitive interaction model–least squares approach. *Journal of Marketing Research* **11**(3) 303–311.

Newman, Robert J., Dennis H. Sullivan. 1988. Econometric analysis of business tax impacts on industrial location: What do we know, and how do we know it? *Journal of Urban Economics* **23**(2) 215–234.

Owen, S. H., M. S. Daskin. 1998. Strategic facility location: A review. *European Journal of Operational Research* **111**(3) 423–447.

Peeters, Peter, Frank Plastria. 1998. Discretization results for the huff and pareto-huff competitive location models on networks. *TOP* **6**(2) 247–260.

Plastria, Frank. 2001. Static competitive facility location: An overview of optimisation approaches. *European Journal of Operational Research* **129**(3) 461–470.

Rynes, Sara L., Barry Gerhart, Kathleen A. Minette. 2004. The importance of pay in employee motivation: Discrepancies between what people say and what they do. *Human Resource Management* **43**(4) 381–394.

Shiode, Shogo, Zvi Drezner. 2003. A competitive facility location problem on a tree network with stochastic weights. *European Journal of Operational Research* **149**(1) 47–52.

Suárez-Vega, R., D. Santos-Peñate, P. Dorta-González. 2004a. Discretization and resolution of the $(r|X|p)$-medianoid problem involving quality criteria. *TOP* **12**(1) 111–133.

Suárez-Vega, Rafael, Dolores R. Santos-Peñate, Pablo Dorta-González. 2004b. Competitive multifacility location on networks: the $(r|X|p)$-medianoid problem. *Journal of Regional Science* **44**(3) 569–588.

Suárez-Vega, Rafael, Dolores R. Santos-Peñate, Pablo Dorta-González. 2007. The follower location problem with attraction thresholds. *Papers in Regional Science* **86**(1) 123–137.

Swamidass, Paul M. 1990. A comparison of the plant location strategies of foreign and domestic manufacturers in the u.s. *Journal of International Business Studies* **21**(2) 301–317.

Tong, Hsin-Min, C. K. Walter. 1980. An empirical study of plant location decisions of foreign manufacturing investors in the united states. *Columbia Journal of World Business* **15**(1) 66–73.

Ulgado, Francis M. 1996. Location characteristics of manufacturing investments in the u.s.: a comparison of american and foreign-based firms. *Management International Review* **v36**(n1) p7(20).

van den Berg, Gerard J., Cees Gorter. 1997. Job search and commuting time. *Journal of Business & Economic Statistics* **15**(2) 269–281.

Weber, Alfred. 1909. *Uber den Standort der Industrien.* J.C.B. Mohr, Tbingen.

APPENDIX

The purpose of this appendix is to display the C++ code used to solve the reformulated capacity allocation problem (RCAP). The algorithm is a reduced gradient algorithm. The code is as follows:

```
#include <iostream>
#include <vector>
#include <cmath>
#include <float.h>
#include <fstream>
#include <cstdlib>

using namespace std;

ifstream DataFile;
ofstream SolutionFile;

const int K=5; //Number of Markets
const int N=5; //Number of Facilities

struct MarketInfo
{
        double B; //Market Populations
        double A; //Market Competition
        double Lambda;
};

class FacilityInfo
{
public:
        int R; //Facility Worker Requirement
        int BasisElement; //Market associated with the facility
                        //chosen to form the basis
        int DetermineBasis(int j);
        //Precondition: R and B defined. Current r values
      //have been determined.
        //Postcondition: The market with the largest value r_ik
      //for each facility, i, has been chosen.
};

MarketInfo market[K];
FacilityInfo facility[N];
```

```
class AllocationInfo
{
public:
        double Dist; //distance between a particular
                   //facilty and market
        double r; // number of workers allocated from a
                 //particular market to a particular facility
        double h; // the (i,k)th term of the objective function
        double Grad_h; //Gradient of h with respect to a
                     //particular facility and market
        double RedGrad; //Reduced Gradient of h with respect to a
                       //particular facility and market
        double dir; //Improving direction for a particular
                   //facility and market
        double Wage;
        double CalculateInitialSolution(int j,
        int l, double TotalPop);
        //Precondition: A, B and distance defined.
        //Postcondition: Initial solution for market i
        //and facility k has been calculated and returned.
        double CalculateGradient(double A_k, double B_k,
       int j, int l, double L_k);
        //Precondition: A, B and distance defined.
        //Current r values have been determined.
            //Dist and r arrays are constant in market.
        //Postcondition: Gradient for market i and facility
        //k has been calculated and returned.
        double CalculateRedGradient(int j, int l);
        //Precondition: Gradient has been calculated for
        //every facility/market pair
            //Basis variables have been determined
        //Postcondition: Reduced Gradient for market i
          //and facility k has been calculated and returned.
        double CalculateNonBasisDirection(int j, int l);
        //Precondition: Reduced gradient has been
         //calculated for every facility/market pair
        //Postcondition: Direction calculated for every
         //facility/market pair that is not in the basis
        double CalculateBasisDirection(int j);
        //Precondition: Reduced gradient has been
       //calculated for every facility/market pair
        //Direction has been calculated for every
       //facility/market pair that is not in the basis
        //Postcondition: Direction calculated for every
```

```
        //facility/market pair that is in the basis
};

AllocationInfo alloc[N][K];
double ObjectiveValue;
double NewObjectiveValue;
void InitialObjectiveValue();
//Precondition: A, B, Dist, and r have all been assigned values.
//Postcondition: The objective value of the initial solution
    //is calculated and reported.
void CalculateNewAllocation();
//Precondition: Direction has been calculated for
    //every facility/market pair
//Postcondition: mu_max has been calculated and used in
    //golden section line search to find a new and
//improved allocation. The objective value for the
    //solution has been determined.
double CalculateObjectiveValue(double r_ik[][K]);
//Precondtion: A, B, Dist all have values.
//It will calculate the objective function value
//for any allocation.
//PostCondition: The objective value has been determined.
void OutputOptimalSolution();
//Precondition: The stopping criteria has been met and
    //the final allocation and objective function
    //value has been found.
//Postcondition: The optimal solution is output to both
    //screen and file.
void Input();
//Precondition: The number of markets and facilities in the
    //problem must be entered manually in the code.
//Postcondition: B, A, Lambda, R, and Dist are assigned values.
void Output();
//Precondition: The algorithm has converged to an optimal answer.
//Postcondition: r, wage, and objective function are output
   //to a text file.

int main()
{
        Input();

        cout.setf(ios::fixed,ios::floatfield);
        cout.precision(9);

        int i; //counter on facilities
```

```
  int k; //counter on markets
  double TotalPopulation=0;
  double max_dir=100; //the largest of the calculated
                     //directions at each iteration
  int IterationCount=0;
  double Stop=0.1;

  for (k=0; k<K; k++)
  {
          TotalPopulation=TotalPopulation+market[k].B;
  };


  for (i=0; i<N; i++)
          {
                  for (k=0; k<K; k++)
                  {
                          alloc[i][k].r=alloc[i][k].
  CalculateInitialSolution(i,k,TotalPopulation);
                  };
          };

  InitialObjectiveValue();

  while (max_dir>=Stop)
  {

          for (i=0; i<N; i++)
          {
                  for (k=0; k<K; k++)
                  {
                          alloc[i][k].Grad_h=alloc[i][k].
CalculateGradient(market[k].A, market[k].B,
i, k, market[k].Lambda);
                  };
          };

          for (i=0; i<N; i++)
          {
                  facility[i].BasisElement=facility[i].
DetermineBasis(i);
          };

          for (i=0; i<N; i++)
          {
```

```
                for (k=0; k<K; k++)
                {
                        alloc[i][k].RedGrad=alloc[i][k].
CalculateRedGradient(i,k);
                };
        };

        for (i=0; i<N; i++)
        {
                for (k=0; k<K; k++)
                {
                        if (k==facility[i].BasisElement)
                                alloc[i][k].dir=0;
                        else
                        {
                                alloc[i][k].dir=
 alloc[i][k].CalculateNonBasisDirection(i,k);
                        }
                };
        };

        for (i=0; i<N; i++)
        {
                for (k=0; k<K; k++)
                {
                        if (k==facility[i].BasisElement)
                        {
                                alloc[i][k].dir=
     alloc[i][k].
  CalculateBasisDirection(i);
                        }
                };
        };
        max_dir=0.0;

        for (i=0; i<N; i++)
        {
                for (k=0; k<K; k++)
                {
                        if (fabs(alloc[i][k].dir)>max_dir)
                                max_dir=
    fabs(alloc[i][k].dir);
                };
        };
        cout << "The maximum direction this
```

```
 iteration is " << max_dir << ".\n";

        if (max_dir>=Stop)
        {
                CalculateNewAllocation();

                cout << "The new objective function
    value is " << ObjectiveValue
    << " in iteration " << IterationCount
    <<".\n";
                IterationCount++;
        }
        else
        {
                OutputOptimalSolution();
        }
    }


    cout << "The optimal answer was found
  in iteration " << IterationCount << ".\n";

    Output();


return 0;
}
void Input()
{

    char Indicator;
    double Value;
    int Size;
    int s;
    int t;

    DataFile.open("5X5Problem-CanBeZero.txt");
    if(DataFile.fail())
    {
        cout << "Input file opening failed.\n";
        exit(1);
    }

    DataFile >> Indicator;
    while (Indicator != 'B')
```

```
    {
            DataFile >> Indicator;
    }

    for (t=0; t<K; t++)
    {
            DataFile >> Value;
            market[t].B=Value;
    };

    DataFile.seekg(0,std::ios::beg);
  //reset read pos to beginning
DataFile.clear(); //clear eofbit

    DataFile >> Indicator;
    while (Indicator != 'A')
    {
            DataFile >> Indicator;
    }

    for (t=0; t<K; t++)
    {
            DataFile >> Value;
            market[t].A=Value;
    };

    DataFile.seekg(0,std::ios::beg);
  //reset read pos to beginning
DataFile.clear(); //clear eofbit

    DataFile >> Indicator;
    while (Indicator != 'L')
    {
            DataFile >> Indicator;
    }

    for (t=0; t<K; t++)
    {
            DataFile >> Value;
            market[t].Lambda=Value;
    };

    DataFile.seekg(0,std::ios::beg);
   //reset read pos to beginning
DataFile.clear(); //clear eofbit
```

```
        DataFile >> Indicator;
        while (Indicator != 'R')
        {
                DataFile >> Indicator;
        }

        for (s=0; s<N; s++)
        {
                DataFile >> Size;
                facility[s].R=Size;
        };

        DataFile.seekg(0,std::ios::beg);
        //reset read pos to beginning
    DataFile.clear(); //clear eofbit

        DataFile >> Indicator;
        while (Indicator != 'D')
        {
                DataFile >> Indicator;
        }

        for (s=0; s<N; s++)
        {
                for (t=0; t<K; t++)
                {
                        DataFile >> Value;
                        alloc[s][t].Dist=Value;
                };
        };

        DataFile.close();

        return;
}

double AllocationInfo::
   CalculateInitialSolution(int j, int l, double TotalPop)
{
        double r_ik;

        r_ik=facility[j].R*market[l].B/TotalPop;

        return r_ik;
```

```
}
void InitialObjectiveValue()
{
        int s;
        int t;
        int u;
        int v;
        double ObjValue=0;
        double TotalMarketAlloc[K];
        double ExcessPop[K];

        for (v=0; v<K; v++)
        {
                TotalMarketAlloc[v]=0;
                for (u=0; u<N; u++)
                {
                        TotalMarketAlloc[v]=
     TotalMarketAlloc[v]+alloc[u][v].r;
                };
                ExcessPop[v]=market[v].B-TotalMarketAlloc[v];
        };


        for (s=0; s<N; s++)
        {
                for (t=0; t<K; t++)
                {
                        alloc[s][t].h=market[t].A*
     pow(alloc[s][t].Dist,market[t].Lambda)
     *pow(alloc[s][t].r,2.0)/ExcessPop[t];
                        ObjValue=ObjValue+alloc[s][t].h;
                };
        };

        ObjectiveValue=ObjValue;
        cout << "The objective value for the
    initial solution is " << ObjectiveValue
    << ".\n";
        return;
}

double AllocationInfo::
CalculateGradient(double A_k, double B_k,
   int j, int l, double L_k)
{
```

```
        double Gradient_ik;
        int s;
        double TotalMarketAllocation=0;
        double TotalSqrdTimesDist=0;

        for (s=0; s<N; s++)
        {
                TotalMarketAllocation=
      TotalMarketAllocation+alloc[s][l].r;
        };

        for (s=0; s<N; s++)
        {
                TotalSqrdTimesDist=TotalSqrdTimesDist+
     pow(alloc[s][l].Dist,L_k)*pow(alloc[s][l].r,2.0);
        };

        Gradient_ik=A_k*(2*alloc[j][l].r*
     pow(alloc[j][l].Dist,L_k)*
     (B_k-TotalMarketAllocation)+
     TotalSqrdTimesDist)/
     (pow((B_k-TotalMarketAllocation),2.0));

        return Gradient_ik;
}

int FacilityInfo::DetermineBasis(int l)
{
        int s;
        double max=0;
        int basis=-1;

        for (s=0; s<K; s++)
        {
                if (alloc[l][s].r>max)
                {
                        max = alloc[l][s].r;
                        basis=s;
                }
        };

        return basis;
}

double AllocationInfo::CalculateRedGradient(int j, int l)
```

```
{
        double RedGradient_ik;
        int basis;

        basis = facility[j].BasisElement;

        RedGradient_ik=alloc[j][l].Grad_h-alloc[j][basis].Grad_h;

        return RedGradient_ik;
}
double AllocationInfo::CalculateNonBasisDirection(int j, int l)
{
        double Direction_ik;

        if (alloc[j][l].RedGrad<=0)
                Direction_ik=-alloc[j][l].RedGrad;
        else if (alloc[j][l].RedGrad>0)
                Direction_ik=-alloc[j][l].RedGrad*alloc[j][l].r;

        return Direction_ik;
}
double AllocationInfo::CalculateBasisDirection(int j)
{
        double Direction_ik=0;
        int basis;
        int s;

        basis=facility[j].BasisElement;

        for (s=0; s<K; s++)
        {
                if (s==basis)
                        Direction_ik=Direction_ik+0;
                else
                        Direction_ik=Direction_ik-alloc[j][s].dir;
        };

        return Direction_ik;
}
void CalculateNewAllocation()
{
        int s;
        int t;
        double mu_ik[N][K];
        double mu_max=DBL_MAX;
```

```
double MuUpperBound;
double MuLowerBound=0;
double BoundDifference;
double AcceptanceValue=0.0000000001;
double LowerSolution[N][K];
double UpperSolution[N][K];
double LowerFunctionEval;
double UpperFunctionEval;
double MinEval;
int NotInfinity=0;
const double UpperSolutionRatio=0.6180;
const double LowerSolutionRatio=0.3820;
int Count=0;
double ZeroDirection=-0.0001;

for (s=0; s<N; s++)
{
        for (t=0; t<K; t++)
        {
                if (alloc[s][t].dir<ZeroDirection)
                        mu_ik[s][t]=
-alloc[s][t].r/alloc[s][t].dir;
                else
                        mu_ik[s][t]=DBL_MAX;
        };

};

for (s=0; s<N; s++)
{
        for (t=0; t<K; t++)
        {
                if (mu_ik[s][t]<=mu_max)
                        mu_max=mu_ik[s][t];
        };
};

MuUpperBound=mu_max;
BoundDifference=mu_max;

for (s=0; s<N; s++)
{
        for (t=0; t<K; t++)
        {
                UpperSolution[s][t]=alloc[s][t].r+
```

```
(MuLowerBound+UpperSolutionRatio*
BoundDifference)*alloc[s][t].dir;
                        LowerSolution[s][t]=alloc[s][t].r+
(MuLowerBound+LowerSolutionRatio*
BoundDifference)*alloc[s][t].dir;
              };
    };

    UpperFunctionEval=CalculateObjectiveValue(UpperSolution);
    LowerFunctionEval=CalculateObjectiveValue(LowerSolution);

    while (NotInfinity==0)
    {
              if (LowerFunctionEval<DBL_MAX)
              {
                        NotInfinity=1;
              }

              else
              {
                        MuUpperBound=MuLowerBound+
LowerSolutionRatio*BoundDifference;
                        BoundDifference=MuUpperBound-MuLowerBound;

                        for (s=0; s<N; s++)
                        {
                                  for (t=0; t<K; t++)
                                  {
                                            LowerSolution[s][t]=
alloc[s][t].r+(MuLowerBound+
LowerSolutionRatio*BoundDifference)*
alloc[s][t].dir;
                                  };
                        };

                        LowerFunctionEval=
CalculateObjectiveValue(LowerSolution);
                        cout << "The new lower bound
is " << MuLowerBound <<".\n";
                        cout << "The new function
evaluation is " << LowerFunctionEval
<< ".\n";

                        NotInfinity=0;
              }
```

```
    }

    while (BoundDifference>=AcceptanceValue)
    {
            MinEval=UpperFunctionEval;
            if (UpperFunctionEval>LowerFunctionEval)
                    MinEval=LowerFunctionEval;

            if (UpperFunctionEval==LowerFunctionEval)
            {
                    MuUpperBound=MuLowerBound+
UpperSolutionRatio*BoundDifference;
                    MuLowerBound=MuLowerBound+
LowerSolutionRatio*BoundDifference;
            }

            else if (MinEval==LowerFunctionEval)
                    MuUpperBound=MuLowerBound+
UpperSolutionRatio*BoundDifference;
            else if (MinEval==UpperFunctionEval)
                    MuLowerBound=MuLowerBound+
LowerSolutionRatio*BoundDifference;

            BoundDifference=MuUpperBound-MuLowerBound;

            if (BoundDifference >= AcceptanceValue)
            {
                    for (s=0; s<N; s++)
                    {
                            for (t=0; t<K; t++)
                            {
                                    UpperSolution[s][t]=
alloc[s][t].r+(MuLowerBound+
UpperSolutionRatio*BoundDifference)
*alloc[s][t].dir;
                                    LowerSolution[s][t]=
alloc[s][t].r+(MuLowerBound+
LowerSolutionRatio*BoundDifference)
*alloc[s][t].dir;
                            };
                    };

                    UpperFunctionEval=
CalculateObjectiveValue(UpperSolution);
                    LowerFunctionEval=
```

```
    CalculateObjectiveValue(LowerSolution);
                        Count++;
                        }
                else
                {
   << Count << ".\n";
                        for (s=0; s<N; s++)
                        {
                                for (t=0; t<K; t++)
                                {
                                        alloc[s][t].r=
   alloc[s][t].r+MuLowerBound*alloc[s][t].dir;
                                };
                        };
                }
        }

        ObjectiveValue=CalculateObjectiveValue(LowerSolution);

        return;
}

double CalculateObjectiveValue(double r_ik[][K])
{
        int s;
        int t;
        int u;
        int v;
        double ObjValue=0;
        double TotalMarketAlloc[K];
        double ExcessPop[K];
        double h_ik[N][K];

        for (v=0; v<K; v++)
        {
                TotalMarketAlloc[v]=0;
                for (u=0; u<N; u++)
                {
                        TotalMarketAlloc[v]=
    TotalMarketAlloc[v]+r_ik[u][v];
                };
                ExcessPop[v]=market[v].B-TotalMarketAlloc[v];
        };

        for (s=0; s<N; s++)
```

```
	{
		for (t=0; t<K; t++)
		{
			if (ExcessPop[t]<=0)
				h_ik[s][t]=DBL_MAX;
			else
				h_ik[s][t]=market[t].A*
	pow(alloc[s][t].Dist,market[t].Lambda)*
	pow(r_ik[s][t],2.0)/ExcessPop[t];

			ObjValue=ObjValue+h_ik[s][t];
		};
	};

	return ObjValue;
}

void OutputOptimalSolution()
{
	double Wage[N][K];
	double TotalMarketAlloc[K];
	double ExcessPop[K];
	int s;
	int t;
	int u;
	int v;

	for (v=0; v<K; v++)
	{
		TotalMarketAlloc[v]=0;
		for (u=0; u<N; u++)
		{
			TotalMarketAlloc[v]=
   TotalMarketAlloc[v]+alloc[u][v].r;
		};
		ExcessPop[v]=market[v].B-TotalMarketAlloc[v];
	};

	for (s=0; s<N; s++)
	{
		for (t=0; t<K; t++)
		{
			cout << "The optimal allocation of workers
   from market " << t << " to facility "
   << s << " is " << alloc[s][t].r << ".\n";
```

```
                        Wage[s][t]=market[t].A*
    pow(alloc[s][t].Dist,market[t].Lambda)*
    alloc[s][t].r/ExcessPop[t];
                        alloc[s][t].Wage=Wage[s][t];
                        cout << "The optimal wage paid to workers
    from market " << t << " by facility "
    << s << " is " << Wage[s][t] << ".\n";
                };
        };
        cout << "The minimum cost is " << ObjectiveValue << ".\n";
        return;
}
void Output()
{
        int s;
        int t;

        SolutionFile.setf(ios::fixed,ios::floatfield);
        SolutionFile.precision(9);

        SolutionFile.open("5X5Problem-CanBeZero-Solution3.dat");
        if(SolutionFile.fail())
        {
                cout << "Output file opening failed.\n";
                exit(1);
        }

        SolutionFile << "r   ";

        for (s=0; s<N; s++)
        {
                if (s>0)
                        SolutionFile << "     ";
                for (t=0; t<K; t++)
                {
                        SolutionFile << alloc[s][t].r <<", ";
                };
                SolutionFile << endl;
        };
        SolutionFile << endl << endl;

        SolutionFile << "w   ";

        for (s=0; s<N; s++)
        {
```

```
                if (s>0)
                        SolutionFile << "    ";
                for (t=0; t<K; t++)
                {
                        SolutionFile << alloc[s][t].Wage <<", ";
                };
                SolutionFile << endl;
        };
        SolutionFile << endl << endl;

        SolutionFile << "Objective Value"
    << endl << ObjectiveValue;

        return;
}
```