

THREE ASSETS MODEL FOR PORTFOLIO SELECTION UNDER A
CONSTRAINED CONSUMPTION RATE PROCESS

by

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ABSTRACT

In this dissertation, we consider a particular case of an optimal consumption and portfolio selection problem for an infinitely lived investor whose consumption rate process is subject to downside constraint. We also suppose that the wealth dynamics is composed of three assets (i) riskless assets (ii) risky assets (iii) hedge assets.

We consider the investor's wealth process, interpreted in the sense of the Itô integral as

$$dX_t = \left[rX_t + \pi_t^s(\mu_t^s - r) + \pi_t^f(\mu_t^f - r) - c_t \right] dt + \pi_t^s \sigma_t^s dW_t^s + \pi_t^f \sigma_t^f dW_t^f$$

where $t \geq 0$ and $X_0 = x > 0$.

Our work aims to find the optimal policies which maximize the expected discount utility function given by

$$J(c_t) = E \left(\int_0^{\infty} e^{-\beta t} u(c_t) dt \right)$$

Subject to $c_t \geq R$, for all $t \geq 0$, for fixed R and $E \left[\int_0^{\infty} c_t H_t dt \right] \leq x$.

Furthermore, we obtain the optimal policies in an explicit form for the log utility function which is a special case ($\gamma \rightarrow 1$) of the general utility (CRRA) function, using the martingale method and applying the Legendre transform formula and the Feynman-kac formula. We derive some numerical results for the optimal policies and illustrated graphically.

DEDICATION

This dissertation is dedicated to everyone who helped me and guided me through the trials and tribulations of creating this manuscript. In particular, my family and close friends who stood by me throughout the time taken to complete this work.

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It has been a long journey for me from the childhood days in a rural hilly area of Nepal to the present day. Along the journey I have faced challenging and joyful moments. I have also met many people who have not only upheld faith in my abilities but also were able to inspire me to a greater goal. I dedicate this dissertation to my mother, father and brother Ganesh Thagunna; without their support it would have been impossible for me to continue my studies. I would like to thank my advisor, Dr. Zhijian Wu, for all of his help throughout the years. As a mentor, Dr. Wu is a great motivator. I truly consider it an honor to have been counseled by him.

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CHAPTER 1

INTRODUCTION

1.1. Development of Portfolio selection Theory

The *Mathematization* of the field of Finance has proceeded at a very rapid, sometimes explosive, pace during the last twenty years. Ever since the role of the so called equivalent martingale measure was noticed, and its significance understood in the context of both pricing and portfolio optimization problems, powerful techniques from stochastic analysis and stochastic control theory have been brought to bear on almost all aspects of Mathematical Finance: the study of arbitrage, hedging, pricing, consumption/Investment optimization, incomplete and/or constrained markets, equilibrium, differential information, the term structure of interest rates, transaction costs, and so on. At the same time, the development of sophisticated Analytical and Numerical Methods, based on partial differential equations and on their numerical solution, has helped to increase the relevance of these developments in the everyday practice of Finance.

The portfolio selection problem is a classic in finance literature. It was first developed by Markowitz (1952) and Tobin (1965) for a single period investment. The extension on this model made by Merton (1961,1971) provided the impetus behind a considerable volume of research investigating the multi period problem and extending it in new directions. The problem considered then was intertemporal portfolio selection where the investor invests dynamically and continuously so as to hedge against random changes in the opportunity set. This problem has often been considered in general equilibrium asset pricing models, where many investors act so as to maximize

their expected utility over consumption; see Lucas (1978), Breeden (1979) and Cox (1985). Samuelson (1969) presented an analogous problem in discrete time, but no closed form solution was given.

A mathematical tool well suited to handle dynamic portfolio optimization problem is stochastic dynamic programming. In the discrete time case, dynamic programming solves the optimization problem step by step working backward in time. In the continuous time case, the dynamic problem can be transformed into a static one, using the Bellman-Dreyfus fundamental equation of optimality. However, finding a closed form expression for the optimal consumption and investment allocations requires solving the Hamilton-Jacobi-Bellman (*HJB*) equation, a non linear partial differential equation (*PDE*). In most cases except some, it is difficult to find the closed form solution analytically or numerically. Merton (1971) solves the special case when the underlying security prices follow a geometric Brownian motion. In this case the problem can be simplified when the opportunity set is uncorrelated with asset returns and also when the investor has logarithmic utility. However, the assumption of log-normality of prices, or a constant opportunity set are very restrictive because the expected returns seem to vary through time; Lo, MacKinlay (1987). Recent research considers the portfolio problem with non constant expected returns, a problem which is more difficult to solve.

Kim and Omberg (1996) solve the portfolio problem analytically when the investor has Hyperbolic Absolute Risk Aversion (*HARA*) utility function, and invests in a constant risk free asset and a risky asset with a mean reverting diffusion process. Brennan, Lagnado and Schwartz (1997) assume that time variation in expected returns is driven by three state variables, short and long interest rates, and dividend yield on equity. They use a numerical scheme to show how dynamic portfolio allocation can significantly improve the investor's welfare, compared to one period investment. Zariphopoulou (1999) finds a closed form solution when the investor

maximizes terminal wealth, and invests in a bond and a risky security whose price process has non linear coefficients in the stock level.

In general, unless researchers focus on the state variable specification for which a closed form solution can be derived, the problem is highly intractable. Nevertheless, most of the research in portfolio selection with time varying expected returns considers the problem where the investor maximizes the terminal wealth for a single, fixed horizon. Although this problem is important when treated dynamically, it is not in general enough because it does not allow for intermediate consumption. Moreover, it is not consistent with microeconomic theory where consumption is inherent to the investor's behavior. Also, the results do not apply to an investor who is not only saving for retirement, but for a house or other purposes as well.

Campbell and Viceira (1999) consider the problem when an infinitely lived investor has utility over consumption and find an approximate solution to the discrete-time problem using Euler approximation. Harrison and Kreps (1979) introduced the martingale approach to price contingent claims. Karatzas (1986,1987), Pliska (1996) and Cox, Hang (1989) and (1991) applied the martingale approach and provided a closed form solution for the optimal portfolio when the underlying security prices follow a general diffusion process. The basic idea is to use the **No Arbitrage principle** for the complete market to separate the computation of optimal consumption rules and that of a trading strategy. In a first step, the optimal consumption is obtained by solving the first order conditions, essentially stating that the agent's marginal utility process at the optimum is proportional to an Arrow-Debreu state price density process. In the second step, the corresponding portfolio strategy is derived by means of the martingale representation theorem, assuming the completeness of the market. The second step is similar to option pricing where consumption plays the role of cash flows. This finding is a major breakthrough because it gives closed form solutions where dynamic programming does not.

An analytical solution is typically preferred because it is difficult to deduce an unbiased economic interpretation from the numerical solution and distinguish a general property which does not depend on the problem setting. However, in the case of the portfolio problem with an underlying Ito process, the theoretical formulas are often complex and hard to implement numerically. The optimal portfolio composition, where the security process follows a general diffusion process, can be expressed via the Clark-Ocone formula (Ocone and Karatzas 1991) which involves conditional expectations of variables expressed in Malliavin derivatives. Basak (1999) used the martingale technology to solve the consumption investment problem with short-selling constraints and labor income. Wachter (1999) derives a closed form solution to the portfolio choice problem when stock returns are predictable and when the investor has utility over consumption, which is the extension of Kim and Omberg(1996). Lakner and Nygren (2006) solved the portfolio optimization problem with both consumption and terminal wealth downside constraints using the gradient operator and Clark-Ocone formula in Malliavin calculus on a finite horizon. For an infinite horizon case, Sethi, Taksar and Presman (1992) solved a single investor's consumption and portfolio problem with positive subsistence and bankruptcy, maximizes the expected utility for the subsistence consumption only. Gong and Li (2006) studied the role of index bonds in a consumption and asset allocation model with real subsistence consumption using the dynamic programming method and obtained the optimal policies in the case of CRRA utility function. Shin, Lim and Choi (2007) solved a general optimal consumption and portfolio selection problem of an infinitely-lived investor whose consumption rate process is subject to a downside constraint i.e. her consumption rate is greater than or equal to some positive constant. They obtained an explicit form of the general portfolio problem using the martingale method and the Feynman-Kac formula. Their work is an extension of Karatzas (1986). The best reference for this dissertation is the work done by Shin et. al (2007)[1].

1.2. Problem Statement

By definition, the portfolio selection problem involves choosing the 'best' investment strategy, i.e. the one that maximizes the investor's objective subject to budget constraints. Generally, investors are faced with this problem when they want to determine the optimal allocation of wealth given an investment opportunity set. Suppose there are n different investment opportunities called securities whose returns at the end of the time period are random variables. The portfolio is a linear combination of these securities which has a positive market value. In this dissertation, we consider the optimal portfolio, hedging and consumption strategies for an agent subject to downside consumption constraints and budget constraints. We study a general portfolio selection problem of an infinitely lived investor, who may want to use the forward contract as a hedging instrument. Our work is an extension of Shin, Lim and Choi's (2007) work. We make same assumptions they made in their work. Beside that, we assume that the fluctuations in the derivative securities and risky security are independent, because the first one is caused by the interest rate variation where as the latter one is due to the change in the stock market.

Aside from consumption purchase, an investor also has the opportunity to invest in the capital market which consists of a riskless and a risky security. In addition an investor can make arrangements to be more secure by purchasing a forward contract. A forward contract is an agreement between two parties to sell or buy a risky asset at a future time at a specified price. We derive the properties of the optimal policies for the general utility function and obtain the explicit solution of the optimal portfolio, consumption and hedging policies. We solve the particular case of **CES** (Constant Elasticity of Substitution) utility function and derive the portfolio as well as consumption and hedging strategies for such class of utility functions. We obtain some numerical results for the log utility as a special case of the **CES** utility function. We

use a Legendre transformation to determine the duality case of the value function and we use martingale methods to define the equivalent measures. We solve the stochastic differential equations by using the Feynman-Kac formula which converts a stochastic differential equation in to an ordinary differential equation.

1.3. Aims and Organization

The main idea of this work is to compare alternative approaches to optimizing asset allocations for an investor in the case of the consumption, investment problem and we present a modified model of the work done by Yong Hyun Shin, Byung Hwa Lim, U Jin Choi (2007). Next, we compare these optimal policies with the results of Merton's classical model both analitically and numerically. The dissertation is organized as follows. In chapter two we present the preliminaries which include the relevant definitions and relevant examples. In Chapter three we describe the classical and the fundamental methods of portfolio selection for both discrete and continuous time horizons, including the classical portfolio selection by Harry Markowitz (1952) and Merton's (1969) original problem. We explain the dynamic portfolio selection method which includes the dynamic programming approach to solve Merton's original problem(1969). In chapter four we describe the financial market setup for an investor who invests on a riskless bond, a risky stock and a forward contract to hedge the risk that he might suffer due to the risky stock. We solve an optimization problem for the general utility function and we obtain the optimal policy in explicit form. In chapter five we solve the model for the **CES** (Constant Elasticity of Substitution) utility function and explicitly derive the numerical and theoritical results for the **log** utility function and illustrate graphically. Finally, chapter six will conclude this dissertation with a summary of the results and future directions for this research.

CHAPTER 2

PRELIMINARIES

2.1. Basic Assumptions, Definitions and Mathematical Preliminaries

The theory of stochastic processes is huge with large numbers of subtle concepts and powerful results. We shall see that by attempting to give precision to some key ideas in finance we can proceed very naturally to a number of core concepts in stochastic mathematics. The most common assumption regarding the probabilistic behavior of an asset price is that it is a geometric Brownian motion or a geometric Wiener process. Roughly speaking, Brownian motion is the limit of a random walk as both the step sizes and the time intervals converge to zero. A geometric Brownian motion is a limit of a random walk in which the step sizes are proportional to the price of the asset. In other words, assets modeled with geometric Brownian motion are based on percentage changes rather than absolute linear changes as would be the case if they are modeled on Brownian motion.

The following concepts provide details of the theory of stochastic processes with an indication of their financial interpretations.

DEFINITION 2.1. If Ω is a given set, then σ -**algebra** \mathcal{F} on Ω is a family of subsets of Ω satisfying following properties

- $\phi \in \mathcal{F}$
- $F \in \mathcal{F} \Rightarrow F^c \in \mathcal{F}$, where, $F^c = \Omega \setminus F$ is the complement of F in Ω
- $A_1, A_2, \dots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

The pair (Ω, \mathcal{F}) is called a measurable space. A probability measure \mathbf{P} on a measurable space (Ω, \mathcal{F}) is a function $\mathbf{P} : \mathcal{F} \rightarrow [0, 1]$ such that

- (1) $\mathbf{P}(\emptyset) = 0, \mathbf{P}(\Omega) = 1$
- (2) $A_1, A_2, \dots \in \mathcal{F}$ and $\{A_i\}_{i=1}^{\infty}$ is disjoint then

$$\mathbf{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbf{P}(A_i)$$

The triplet $(\Omega, \mathcal{F}, \mathbf{P})$ is called a **Probability space**. Furthermore, if $(\Omega, \mathcal{F}, \mathbf{P})$ is a given probability space, then a function $Y : \Omega \rightarrow R^n$ is called \mathcal{F} -**measurable** if $Y^{-1}(U) = \{\omega \in \Omega; Y(\omega) \in U\} \in \mathcal{F}$ for all open sets $U \in R^n$. If $X : \Omega \rightarrow R^n$ is any function, then the σ -**algebra** \mathcal{H}_X generated by X is the smallest σ -**algebra** on Ω containing all sets $X^{-1}(U); U \subset R^n$ open. It can be shown that $\mathcal{H}_X = \{X^{-1}(B); B \in \mathcal{B}\}$, where \mathcal{B} is the Borel σ -**algebra** on R^n . Clearly, X will be \mathcal{H}_X measurable and \mathcal{H}_X is the smallest σ -**algebra** with this property.

DEFINITION 2.2. Let $\{\mathcal{N}_t\}_{t \geq 0}$ be an increasing family of σ -**algebras** of subsets of Ω . A process $g(t, \omega) : [0, \infty) \times \Omega \rightarrow R^n$ is called \mathcal{N}_t -**adapted** if for each $\omega \in \Omega$ the function $\omega \rightarrow g(t, \omega)$ is \mathcal{N}_t -measurable.

REMARK 1. Let $\mathcal{V} = \mathcal{V}(S, T)$ be the class of functions $f(t, \omega) : [0, \infty) \times \Omega \rightarrow R^n$ such that

- (1) $(t, \omega) \rightarrow f(t, \omega)$ is $\mathcal{B} \times \mathcal{F}$ measurable, where \mathcal{B} is the Borel σ -**algebra** on $[0, \infty)$.
- (2) $f(t, \omega)$ is \mathcal{F}_t -adapted.
- (3) $E\left[\int_S^T f(t, \omega)^2 dt\right] < \infty$.

DEFINITION 2.3. A **filtration** on (Ω, \mathcal{F}) is a family $\mathcal{M} = \{\mathcal{M}_t\}_{t \geq 0}$ of σ **algebras** $\mathcal{M}_t \subset \mathcal{F}$ such that $0 \leq s < t \Rightarrow \mathcal{M}_s \subset \mathcal{M}_t$ (i.e. $\{\mathcal{M}_t\}$ is increasing).

DEFINITION 2.4. An n - dimensional stochastic process $\{M_t\}_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbf{P})$ is called a **martingale** with respect to a filtration $\{\mathcal{M}_t\}_{t \geq 0}$ (and with respect to \mathbf{P}) if

- (1) M_t is \mathcal{M}_t measurable for all t .
- (2) $E[|M_t|] < \infty$ is \mathcal{M}_t for all t , and
- (3) $E[M_t | \mathcal{M}_s] = M_s$ for all $t \geq s$.

DEFINITION 2.5. A stochastic process $\{X(t), t \geq 0\}$ is said to be a **standard Brownian motion process** if:

- (1) $X(0) = 0$;
- (2) $\{X(t), t \geq 0\}$ has stationary independent increments;
- (3) for every $t > 0$, $X(t)$ is normally distributed with mean 0 and variance t .

EXAMPLE 2.6. Let B_t be 1-dimensional Brownian motion and let $\sigma \in R$ be a constant, then by using above definition we can prove that

$M_t := \exp(\sigma B_t - \frac{1}{2}\sigma^2 t); t \geq 0$ is \mathcal{M}_t martingale.

Since,

$$\begin{aligned}
 E[M_s / \mathcal{M}_t] &= E[\exp(\sigma B_s - \frac{1}{2}\sigma^2 s) | \mathcal{M}_t] \text{ for all } s \geq t \\
 &= E[\exp(\sigma B_t - \frac{1}{2}\sigma^2 t) | \mathcal{M}_t] E[\exp(\sigma B_{s-t} - \frac{1}{2}\sigma^2(s-t)) / \mathcal{M}_t] \\
 &= M_t E[\exp(\sigma B_{s-t} - \frac{1}{2}\sigma^2(s-t)) | \mathcal{M}_t] \\
 &= M_t e^{-\frac{1}{2}\sigma^2(s-t)} E[\exp(\sigma B_{s-t})] \\
 &= M_t
 \end{aligned}$$

Furthermore, a stochastic process $\{X(t), t \geq 0\}$ is called a **Gaussian process (Wiener process)** if $X(t_1), X(t_2), \dots, X(t_n)$ has a multivariate normal distribution for all t_1, t_2, \dots, t_n . Since a multivariate normal distribution is completely determined by the marginal mean values and the covariance values, it follows that Brownian motion can also be defined as a Gaussian process having the following additional properties;

$$\text{Cov}(X(s), X(t)) = s \text{ if } s \leq t;$$

$$E[X(s)|X(t) = a] = as/t \text{ for } s < t$$

and

$$\text{Var}[X(s)|X(t) = a] = \frac{s(t-s)}{t}.$$

If $\{X(t), t \geq 0\}$ is a Brownian motion, then the process $\{Y(t), t \geq 0\}$, defined by $Y(t) = e^{X(t)}$ is called a **geometric Brownian motion** with mean

$$E[Y(t)] = e^{t/2} \text{ and variance } \text{Var}[Y(t)] = e^{2t} - e^t.$$

2.2. The Itô Formula

2.2.1. Itô process.

DEFINITION 2.7. Let B_t be 1-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbf{P})$. An Itô process (or stochastic integral) is a stochastic process X_t on $(\Omega, \mathcal{F}, \mathbf{P})$ of the form,

$$X_t = X_0 + \int_0^t \mu(s, \omega) ds + \int_0^t \sigma(s, \omega) dB_t \quad (2.2.1)$$

such that

$$P \left[\int_0^t \sigma(s, \omega)^2 ds < \infty \text{ for all } t \geq 0 \right] = 1$$

and

$$P \left[\int_0^t |\mu(s, \omega)| ds < \infty \text{ for all } t \geq 0 \right] = 1$$

If X_t is an Itô process of the form (2.2.1) it is sometimes written in the shorter differential form as $dX_t = \mu dt + \sigma dB_t$.

For example, $\frac{1}{2}B_t^2 = \int_0^t B_s ds + \int_0^t \frac{1}{2} ds$, is an Itô process (also called stochastic integral) because it is written as the sum of a dB_s and a ds integral. This family of integrals is stable under smooth maps.

2.2.2. Itô's Lemma.

DEFINITION 2.8. Let $F(X_t, t)$ be a twice differentiable function of t and of the random process X_t such that

$dX_t = \mu_t dt + \sigma_t dB_t$, $t \geq 0$ with well behaved drift and diffusion parameters, μ_t and σ_t . Then we can write Itô's formula as

$$\begin{aligned} dF_t &= \frac{\partial F}{\partial X_t} dX_t + \frac{\partial F}{\partial t} dt + \frac{1}{2} \frac{\partial^2 F}{\partial X_t^2} (dX_t)^2 \\ &= \frac{\partial F}{\partial X_t} (\mu_t dt + \sigma_t dB_t) + \frac{\partial F}{\partial t} dt + \frac{1}{2} \frac{\partial^2 F}{\partial X_t^2} \sigma_t^2 dt \end{aligned}$$

Hence,

$$dF_t = \left[\frac{\partial F}{\partial t} + \mu_t \frac{\partial F}{\partial X_t} + \frac{1}{2} \sigma_t^2 \frac{\partial^2 F}{\partial X_t^2} \right] dt + \frac{\partial F}{\partial X_t} \sigma_t dB_t \quad (2.2.2)$$

In situations where one has to apply Itô's formula, one will in general be given a SDE that derives the process X_t as

$$dX_t = \mu(X_t, t) dt + \sigma(X_t, t) dB_t, \quad t \geq 0$$

Thus, Itô's formula can be seen as a vehicle which takes the SDE of X_t and determines the SDE which corresponds to $F(X_t, t)$. Here Eq.(2.2.2) is the SDE corresponds to $F(X_t, t)$.

THEOREM 2.9. (The 1-dimensional Itô formula) Let X_t be an Itô process given by $dX_t = \mu_t dt + \sigma_t dB_t$.

Let $F(X_t, t) \in C^2([0, \infty) \times \mathbf{R})$ (i.e. F is twice continuously differentiable on $[0, \infty) \times \mathbf{R}$) then $Y_t = F(X_t, t)$ is again an Itô process and

$$dY_t = \frac{\partial F(X_t, t)}{\partial t} dt + \frac{\partial F(X_t, t)}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 F(X_t, t)}{\partial x^2} (dX_t)^2 \quad (2.2.3)$$

where $(dX_t)^2 = (dX_t) \cdot (dX_t)$ is computed accordingly to the rules

$$dB_t \cdot dB_t = dt, dt \cdot dt = dt \cdot dB_t = 0 \quad (2.2.4)$$

PROOF. The Eq.(2.2.2) can be written as,

$$F(X_t, t) = F(X_0, 0) + \int_0^t \left(\frac{\partial F}{\partial t} + \mu_t \frac{\partial F}{\partial X_t} + \frac{1}{2} \sigma_t^2 \frac{\partial^2 F}{\partial X_t^2} \right) dt + \int_0^t \frac{\partial F}{\partial X_t} \sigma_t dB_t \quad (2.2.5)$$

Here $\mu(t, \omega) = \mu_t$, $\sigma(t, \omega) = \sigma_t$

Eq.(2.2.5) is an Itô process in the sense of Eq.(2.2.1).

We may assume that F , $\frac{\partial F}{\partial t}$, $\frac{\partial F}{\partial x}$ and $\frac{\partial^2 F}{\partial x^2}$ are bounded, for if (2.2.5) is proved in the case we obtain the general case by approximating by C^2 functions F_n such that

F_n , $\frac{\partial F_n}{\partial t}$, $\frac{\partial F_n}{\partial x}$ and $\frac{\partial^2 F_n}{\partial x^2}$ are bounded for each n and converge uniformly on compact subsets of $[0, \infty) \times \mathbf{R}$ to F , $\frac{\partial F}{\partial t}$, $\frac{\partial F}{\partial x}$ and $\frac{\partial^2 F}{\partial x^2}$, respectively.

Moreover, we may assume that $\mu(t, \omega)$ and $\sigma(t, \omega)$ are elementary functions (\mathcal{F}_t measurable for all t). Using Taylor's theorem we get,

$$\begin{aligned} F(X_t, t) = & F(X_0, 0) + \sum_j \frac{\partial F}{\partial t} \Delta t_j + \sum_j \frac{\partial F}{\partial x} \Delta X_j + \frac{1}{2} \sum_j \frac{\partial^2 F}{\partial t^2} (\Delta t_j)^2 \\ & + \frac{1}{2} \sum_j \frac{\partial^2 F}{\partial x^2} (\Delta X_j)^2 + \sum_j \frac{\partial^2 F}{\partial t \partial x} (\Delta t_j) (\Delta X_j) + o(|\Delta t_j|^2| + |\Delta X_j|^2) \end{aligned}$$

where $\frac{\partial F}{\partial t}, \frac{\partial F}{\partial x}$ etc. are evaluated at points, $(X_j, t_j), \Delta t_j = t_{j+1} - t_j, \Delta X_j = X_{t_{j+1}} - X_{t_j},$

$$\Delta F(X_j, t_j) = F(X_{j+1}, t_{j+1}) - F(X_j, t_j)$$

$$\sum_j \frac{\partial F}{\partial t} \Delta t_j = \sum_j \frac{\partial F}{\partial t}(X_j, t_j) \Delta t_j \rightarrow \int_0^t \frac{\partial F}{\partial t}(X_t, t) dt \quad \text{in the sense of } L^2(P)$$

$$\sum_j \frac{\partial F}{\partial x} \Delta X_j = \sum_j \frac{\partial F}{\partial x}(X_j, t_j) \Delta X_j \rightarrow \int_0^t \frac{\partial F}{\partial x}(X_t, t) dX_t$$

Moreover, since μ and σ are elementary we get

$$\sum_j \frac{\partial^2 F}{\partial x^2} (\Delta X_j)^2 = \sum_j \frac{\partial^2 F}{\partial x^2} \mu_j^2 (\Delta t_j)^2 + 2 \sum_j \frac{\partial^2 F}{\partial x^2} \mu_j \sigma_j (\Delta B_j) (\Delta t_j) + \sum_j \frac{\partial^2 F}{\partial x^2} \sigma_j^2 (\Delta B_j)^2 \quad (2.2.6)$$

where, $\mu_j = \mu(\omega, t_j), \sigma_j = \sigma(\omega, t_j)$

The first two terms tend to 0 as $\Delta t_j \rightarrow 0$. We claim that the last term of (2.2.6)

tends to

$$\sum_j \frac{\partial^2 F}{\partial x^2} \sigma_j^2 (\Delta B_j)^2 \rightarrow \int_0^t \frac{\partial^2 F}{\partial x^2} \sigma^2 dt \quad \text{in } L^2(P), \text{ as } \Delta t_j \rightarrow 0.$$

To prove this, put $a(t) = \frac{\partial^2 F}{\partial x^2}(X_t, t) \sigma^2(\omega, t_j), a_j = a(t_j)$ and consider

$$E \left[\left(\sum_j a_j (\Delta B_j)^2 - \sum_j a_j \Delta t_j \right)^2 \right] = \sum_{i,j} E [a_i a_j ((\Delta B_i)^2 - \Delta t_i) ((\Delta B_j)^2 - \Delta t_j)].$$

If $i < j$ then $a_i a_j ((\Delta B_i)^2 - \Delta t_i)$ and $((\Delta B_j)^2 - \Delta t_j)$ are independent so the

terms vanish in this case, and similarly if $i > j$. So we get

$$\begin{aligned} \sum_j E [a_j^2 ((\Delta B_j)^2 - \Delta t_j)^2] &= \sum_j E [a_j^2] \cdot E [(\Delta B_i)^4 - 2(\Delta B_i)^2 \Delta t_j + (\Delta t_j)^2] \\ &= \sum_j E [a_j^2] \cdot (3(\Delta t_j)^2 - 2(\Delta t_j)^2 + (\Delta t_j)^2) \\ &= 2 \sum_j E [a_j^2] \cdot (\Delta t_j)^2 \rightarrow 0 \quad \text{as } \Delta t_j \rightarrow 0 \end{aligned}$$

Hence we have established that

$$\sum_j a_j(\Delta B_j)^2 \rightarrow \int_0^t a(t)dt \quad \text{in } L^2(P) \text{ as } \Delta t_j \rightarrow 0. \quad (2.2.7)$$

This conclusion is often expressed by the formula $dB_t^2 = dt$.

□

THEOREM 2.10. *(The General Itô formula)*

Let $dX(t) = \mu dt + \sigma dB(t)$ be an n -dimensional Itô process.

where,

$$X(t) = \begin{pmatrix} X_1(t) \\ \cdot \\ \cdot \\ X_n(t) \end{pmatrix}, \mu = \begin{pmatrix} \mu_1 \\ \cdot \\ \cdot \\ \mu_n \end{pmatrix}, \quad \sigma = \begin{pmatrix} \sigma_{11} & \cdot & \cdot & \cdot & \sigma_{1m} \\ \cdot & & & & \cdot \\ \sigma_{n1} & \cdot & \cdot & \cdot & \sigma_{nm} \end{pmatrix},$$

$$dB(t) = \begin{pmatrix} dB_1(t) \\ \cdot \\ \cdot \\ dB_m(t) \end{pmatrix}$$

Let $F(x, t) = (F_1(x, t), \dots, F_p(x, t))$ be a C^2 map from $[0, \infty) \times \mathbf{R}^n$ into \mathbf{R}^p ; then

the process

$Y(\omega, t) = F(X(t), t)$ is again an Itô process, given by

$$dY_k = \frac{\partial F_k(X, t)}{\partial t} dt + \sum_i \frac{\partial F_k(X, t)}{\partial x_i} dX_i + \frac{1}{2} \sum_{i,j} \frac{\partial^2 F_k(X, t)}{\partial x_i \partial x_j} dX_i dX_j \quad (2.2.8)$$

where $dB_i dB_j = \delta_{ij} dt$, $dB_i dt = dt dB_i = 0$

PROOF. The proof is similar to the 1-dimensional version. □

2.2.3. The Itô Integral. Consider the finite interval approximation of the stochastic differential equation, $X_k - X_{k-1} = \mu(X_{k-1}, k)h + \sigma(X_{k-1}, k)[B_k - B_{k-1}]$, $k = 1, \dots, n$ where, $[B_k - B_{k-1}]$ is a standard Wiener process with zero mean and variance h . Let

- (1) The random variables $\sigma(X_k, t)$ be non-anticipative, in the sense that they are independent of the future.
- (2) The random variables $\sigma(X_{t_k}, t)$ be non explosive i.e. $E\left[\int_0^T \sigma(X_{t_k}, t)^2 dt\right] < \infty$.

Then the Itô Integral $\int_0^T \sigma(X_{t_k}, t)dB_t$, is the mean square limit

$$\sum_{k=1}^n \sigma(X_{t_{k-1}}, t_k)[B_{t_k} - B_{t_{k-1}}] \rightarrow \int_0^T \sigma(X_{t_k}, t)dB_t, \text{ as } n \rightarrow \infty (h \rightarrow 0)$$

According to this definition, as the number of intervals goes to infinity and the length of each interval becomes infinitesimal, the sum will approach the random variable represented by the Ito integral. Clearly the definition makes sense only if such a limiting random variable exists. The assumption that $\sigma(X_k, t)$ is nonanticipating turns out to be a fundamental condition for the existence of such a limit.

EXAMPLE 2.11. Assume $B_0 = 0$ then $\int_0^t B_s dB_s = \frac{1}{2}B_t^2 - \frac{t}{2}$

and if we use the definition as

$$\begin{aligned}
\int_0^t B_s dB_s &= \lim_{n \rightarrow \infty} \sum_{k=1}^n B(X_{k-1}, k) [B_k - B_{k-1}] \\
&= \lim_{n \rightarrow \infty} \sum_{k=1}^n B_k B_{k-1} - B_{k-1}^2 \\
&= \lim_{n \rightarrow \infty} [B_1 B_0 + \dots + B_n B_{n-1} - B_0^2 - \dots - B_{n-1}^2] \\
&= \lim_{n \rightarrow \infty} \left[-\frac{1}{2} [(B_1 - B_0)^2 + \dots + (B_n - B_{n-1})^2] - \frac{B_0^2}{2} + \frac{B_n^2}{2} \right] \\
&= \lim_{n \rightarrow \infty} \left[\frac{B_n^2 - B_0^2}{2} - \frac{1}{2} \sum_{k=1}^n (\Delta B_k)^2 \right] \\
&= \int_0^t \frac{dB_s^2}{2} - \frac{t}{2} \\
&= \frac{1}{2} B_t^2 - \frac{t}{2}
\end{aligned}$$

Some properties of the Itô Integral

Let $f, g \in \mathcal{V}(0, T)$ and let $0 \leq S \leq U \leq T$ then,

- (1) $\int_S^T f dB_t = \int_S^U f dB_t + \int_U^T f dB_t$ for $a.a.\omega$
- (2) $\int_S^T (cf + g) dB_t = c \int_S^T f dB_t + \int_S^T g dB_t$ (c constant) for $a.a.\omega$
- (3) $E \left[\int_S^T f dB_t \right] = 0$
- (4) $\int_S^T f dB_t$ is \mathcal{F}_t measurable.

EXAMPLE 2.12. To find the Integral using the 1-dimensional Itô formula

Let $I = \int_0^t B_s^2 dB_s$

Choose $X_t = B_t^2$ and $F(x, t) = \frac{x^3}{3}$.

So we have $Y_t = F(B_t, t) = \frac{1}{3} B_t^3$

Then by the Itô formula

$$dY_t = \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial x} dB_t + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} (dB_t)^2 = B_t^2 dB_t + B_t dt$$

Hence

$$d\left(\frac{1}{3}B_t^3\right) = B_t^2 dB_t + B_t dt$$

or,

$$\int_0^t B_s^2 dB_s = \frac{1}{3}B_t^3 - tB_t.$$

2.2.4. Itô diffusions. A continuous stochastic process X_t which has finite first and second order moments is shown to follow the general SDE

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dB_t, \quad t \geq 0$$

Where $\mu(\cdot)$ and $\sigma(\cdot)$ are the drift and diffusion parameters. We now assume that the drift and diffusion parameters depends only on X_t . The SDE can be written as

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t, \quad t \geq 0. \text{ Processes that have this characteristic are}$$

called time homogeneous Itô diffusions.

The following results apply to these processes, whose instantaneous drift and diffusions are not dependent on t directly. The usual condition applies to these parameters in that they are not “too fast”.

The Markov property

Let X_t be an Itô diffusions satisfying the SDE

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dB_t, \quad t \geq 0,$$

Let $f(\cdot)$ be any bounded function, and suppose that the information set I_t contains all X_u until time t . We say X_t satisfies the *Markov property* if

$$E[f(X_{t+h})|I_t] = E[f(X_{t+h})|X_t], \quad h > 0 \text{ for all } t.$$

Generator of an Ito Diffusion

Let X_t be an Itô diffusion satisfying the SDE

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dB_t, \quad t \geq 0,$$

Let $f(X_t)$ be a twice differentiable function of X_t , and suppose that the process X_t has reached a particular value x_t at time t .

We let an operator \mathcal{A} be defined as the expected rate of change for $f(X_t)$:

$$\mathcal{A}f(X_t) = \lim_{\Delta \rightarrow 0} \frac{E[f(X_{t+\Delta})/f(x_t)] - f(x_t)}{\Delta}$$

Here \mathcal{A} is called the *generator* of the Itô diffusion X_t .

In the case when X_t is a mono-variant stochastic process, $dX_t = \mu(X_t)dt + \sigma(X_t)dB_t, t \geq 0$, the operator is given by

$$\mathcal{A}f = \mu_t \frac{\partial f}{\partial x} + \frac{1}{2} \sigma_t^2 \frac{\partial^2 f}{\partial x^2} \quad (2.2.9)$$

It is worthwhile to compare this with what Itô's lemma would give. Applying Itô's lemma to $f(X_t)$ we get

$$df(X_t) = \left[\mu_t \frac{\partial f}{\partial x} + \frac{1}{2} \sigma_t^2 \frac{\partial^2 f}{\partial x^2} \right] dt + \sigma_t \frac{\partial f}{\partial x} dB_t$$

Hence, the difference between the operator \mathcal{A} and the application of Itô's lemma occurs at two points

- (1) The dB_t terms in Itô's formula is replaced by its drift, which is zero.
- (2) Next, the remaining part of Itô's formula is divided by dt .

These two differences are consistent with the definition of \mathcal{A} . As defined above, \mathcal{A} calculates an expected rate of change starting from the immediate state x_t .

EXAMPLE 2.13. The n -dimensional Brownian motion is of course the solution of the stochastic differential equation

$$dX_t = dB_t \text{ i.e. we have } \mu = 0 \text{ and } \sigma = I_n, \text{ the } n\text{-dimensional}$$

identity matrix. So the generator of B_t is

$$\mathcal{A}f = \frac{1}{2} \sum \frac{\partial^2 f}{\partial x_i^2}; \quad f = f(x_1, \dots, x_n) \in C^2(R^n)$$

i.e. $\mathcal{A} = \frac{1}{2} \Delta$, where Δ is the Laplace operator.

EXAMPLE 2.14. Let B_t denote the 1-dimensional Brownian motion and let $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ be the solution of the stochastic differential equation

$$\begin{aligned} dX_1 &= dt; & X_1(0) &= t_0 \\ dX_2 &= dB_t; & X_2(0) &= x_0 \end{aligned}$$

i.e.

$$dX_t = \mu dt + \sigma dB_t; \quad X(0) = \begin{pmatrix} t_0 \\ X_0 \end{pmatrix}$$

with $\mu = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\sigma = \begin{pmatrix} 0 \\ \sigma \end{pmatrix}$. In other words, X_t may be regarded as the graph of a Brownian motion. The generator \mathcal{A} of X is given by the heat operator

$$\mathcal{A}f = \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}; \quad f = f(t, x) \in C^2(\mathbb{R}^n).$$

From now on we will, unless other stated, let $\mathcal{A} = \mathcal{A}_X$ denote the generator of the Itô diffusion X_t . We let $\mathcal{L} = \mathcal{L}_X$ denote the differential operator given by

$$\mathcal{L}f = \sum_i \mu_i(x) \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j} (\sigma \sigma')_{i,j}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

DEFINITION 2.15. The Dynkin Formula

Let $f \in C^2(\mathbb{R}^n)$. Suppose τ is a stopping time, $E^x[\tau] < \infty$. Then we define the Dynkin formula as the expected value of the form

$$E^x [f(X_\tau)] = f(x) + E^x \left[\int_0^\tau \mathcal{A}f(X_s) ds \right]$$

Where \mathcal{A} is the generator of the itô diffusion.

DEFINITION 2.16. Suppose \tilde{P} and P are two probability measures on a measurable space (Ω, \mathcal{F}) . \tilde{P} is said to be **absolutely continuous** with respect to P on the σ -algebra \mathcal{F} , and we write $\tilde{P} \ll P$, if for all $A \in \mathcal{F}$:

$$P[A] = 0 \quad \Rightarrow \quad \tilde{P}[A] = 0$$

If both $\tilde{P} \ll P$ and $P \ll \tilde{P}$ hold, we will say that \tilde{P} and P are **equivalent** and we can write $\tilde{P} \approx P$.

2.3. The Radon-Nikodym Derivative

Fix t and consider a normally distributed random variable z_t , such that $z_t \sim N(0, 1)$. Let $f(z_t)$ denote the density function of z_t the implied probability measure by P such that

$$dP(z_t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z_t^2} dz_t \quad (2.3.1)$$

Define a function,

$$\eta(z_t) = e^{\mu z_t - \frac{1}{2}\mu^2} \quad (2.3.2)$$

When we multiply $\eta(z_t)$ by $dP(z_t)$, we obtain a new probability. This can be seen from the following

$$\begin{aligned} [dP(z_t)] [\eta(z_t)] &= \frac{1}{\sqrt{2\pi}} e^{\mu z_t - \frac{1}{2}\mu^2 - \frac{1}{2}z_t^2} dz_t \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z_t - \mu)^2} \end{aligned}$$

We define a new probability measure $d\tilde{P}(z_t)$ as

$$d\tilde{P}(z_t) = [dP(z_t)] [\eta(z_t)]$$

or,

dividing both side by $dP(z_t)$ we obtain

$$\frac{d\tilde{P}(z_t)}{dP(z_t)} = \eta(z_t) \quad (2.3.3)$$

This expression can be regarded as a derivative. It reads as if the “*derivative*” of the measure \tilde{P} with respect to P is given by $\eta(z_t)$.

Such derivatives are called Radon-Nikodym derivatives and $\eta(z_t)$ can be regarded as the density of the probability measure \tilde{P} with respect to P . Thus the Radon-Nikodym derivative uses the resulting density of $\eta(z_t)$ to transform the mean of z_t by leaving its variance unchanged.

DEFINITION 2.17. Equivalent Martingale Measures

Given an interval dz_t , the probabilities P and \tilde{P} satisfy

$$\tilde{P}(dz_t) > 0 \quad \text{if and only if} \quad P(dz_t) > 0.$$

If the condition is satisfied, then $\eta(z_t)$ would exist, and we can always go back and forth between the two measures P and \tilde{P} using the relations

$$d\tilde{P}(z_t) = dP(z_t)\eta(z_t) \tag{2.3.4}$$

And

$$dP(z_t) = (\eta(z_t))^{-1}d\tilde{P}(z_t) \tag{2.3.5}$$

This means that, for all practical purposes, the two measures are equivalent.

Hence, an equivalent martingale measure \tilde{P} is a probability measure that is equivalent to P and such that, under \tilde{P} discounted prices are martingales. Such a measure is also called a *risk-neutral* measure.

LEMMA 2.18. Let \tilde{P} be a probability measure equivalent to P . An adapted process \tilde{M} is a \tilde{P} -martingale if and only if the process

$$\tilde{M}_t \cdot E \left[\frac{d\tilde{P}}{dP} \middle| \mathcal{F}_t \right], \quad t = 0, \dots, T$$

is a P -martingale.

PROOF. Let us denote

$$Z_t := E \left[\frac{d\tilde{P}}{dP} \middle| \mathcal{F}_t \right].$$

Observe that $\tilde{M}_t \in \mathcal{L}^1(\tilde{P})$ if and only if $\tilde{M}_t Z_t \in \mathcal{L}^1(P)$. Moreover, the process Z is $P - a.s.$ strictly positive by the equivalence of \tilde{P} and P . Hence

$$Z_t \cdot \tilde{E} \left[\tilde{M}_{t+1} \middle| \mathcal{F}_t \right] = E \left[\tilde{M}_{t+1} Z_{t+1} \middle| \mathcal{F}_t \right]$$

It follows that

$$\tilde{E} \left[\tilde{M}_{t+1} \middle| \mathcal{F}_t \right] = \tilde{M}_t \text{ if and only if } E \left[\tilde{M}_{t+1} Z_{t+1} \middle| \mathcal{F}_t \right] = \tilde{M}_t Z_t.$$

□

PROPOSITION 2.19. *Suppose that \tilde{P} and P are two probability measures on a measurable space (Ω, \mathcal{F}) and $\tilde{P} \ll P$ on \mathcal{F} with density η . If \mathcal{F}_0 is a σ -algebra contained in \mathcal{F} then $\tilde{P} \ll P$ on \mathcal{F}_0 and the corresponding density is given by*

$$\frac{d\tilde{P}}{dP} \Big|_{\mathcal{F}_0} = E [\eta | \mathcal{F}_0] \quad P - a.s.$$

PROOF. $\tilde{P} \ll P$ on \mathcal{F}_0 follows immediately from the definition of absolute continuity. Since η is the density on $\mathcal{F} \supseteq \mathcal{F}_0$, it follows for $A \in \mathcal{F}_0$ that

$$\tilde{P} [A] = \int_A \eta dP = \int_A E [\eta | \mathcal{F}_0] dP.$$

Therefore the \mathcal{F}_0 -measurable random variable $E [\eta | \mathcal{F}_0]$ must coincide with the density on \mathcal{F}_0 . □

2.4. The Girsanov Theorem

Let $X(t) \in R^n$ be an Itô process of the form

$$X_t = X_0 + \int_0^t \mu(s, \omega) ds + \int_0^t \sigma(s, \omega) dB_s; \quad t \leq T$$

Where, $B_t \in R^m$, $\mu(s, \omega) \in R^n$ and $\sigma(s, \omega) \in R^{m \times n}$

Suppose there exist adapted bounded processes $\theta(s, \omega)$ and $r(s, \omega)$ such that,

$$\mu(s, \omega) - r(s, \omega) = \theta(s, \omega)\sigma(s, \omega) \quad (2.4.1)$$

We define a martingale process

$$\eta_t = \exp \left\{ - \int_0^t \theta(s, \omega) dB_s - \frac{1}{2} \int_0^t \theta^2(s, \omega) ds \right\}; \quad t \leq T \quad (2.4.2)$$

The process η_t , $t \geq 0$ is the unique solution to $d\eta_t = \eta_t \theta_t dB_t$, $\eta_0 = 1$, which satisfies $E[\eta_t] = 1, \forall t \in [0, T]$.

Let \tilde{P} be the probability measure defined on (Ω, \mathcal{F}_T) by $\tilde{P}(A) = E_P(\mathbf{1}_A \eta_T)$.

Under \tilde{P} the process,

$$B_t^* = B_t + \int_0^t \theta(s, \omega) ds \quad (2.4.3)$$

is a Brownian motion process. In heuristic terms, this theorem says the following. If we are given a Wiener process B_t , then multiplying the probability distribution of this process by η_t we can obtain a new Wiener process B_t^* with probability distribution \tilde{P} . The main condition for performing such transformations is that η_t is a martingale with $E[\eta_t] = 1$.

Note:-

A sufficient condition that η_t be a martingale is the *Kazamaki condition*

$$E \left[\exp \left(\frac{1}{2} \int_0^t \theta(s, \omega) dB(s) \right) \right] < \infty \quad \text{for all } t \leq T.$$

This condition is implied by the following *Novikov condition*

$$E \left[\exp \left(\frac{1}{2} \int_0^t \theta^2(s, \omega) ds \right) \right] < \infty$$

2.5. Martingale Measure and an Arbitrage

Let $\pi = (\pi^0, \pi^1, \dots, \pi^d)$ represents the portfolio and we assume that the prices of d risky assets be $S^i, i \in \{1, 2, \dots, d\}$, then the wealth at time T is given by $x + \int_0^T \pi(s) dS(s)$, where x is an initial wealth and

$$\int_0^T \pi(s) dS(s) = \sum_{i=0}^d \int_0^T \pi^i(s) dS(s)$$

In order for the stochastic integrals involved to make sense, we need to impose the *adaptability condition* on π , μ and σ .

DEFINITION 2.20. The portfolio process $\pi(t)$ is called **self financing** if

$$\int_0^T \left[|\pi_0(s)r(s)S_0(s) + \sum_{i=1}^n \pi_i(s)\mu_i(s)| + \sum_{j=1}^m \left[\sum_{i=1}^n \pi_i(s)\sigma_{ij}(s) \right]^2 \right] ds < \infty \quad a.s.$$

and the value of wealth

$$dV(t) = \pi(t) \cdot dS(t), i.e.$$

$$V(t) = V(0) + \int_0^t \pi(s) \cdot dS(s); \quad \text{for } t \in [0, T].$$

DEFINITION 2.21. A pair (π, c) of portfolio/consumption processes which is self financing, is called **admissible** for the initial capital $x \geq 0$, if the wealth process is lower bounded, i.e.

$$X(t) \equiv X^{x,\pi,c}(t) \geq 0; \quad \forall 0 \leq t \leq T \quad a.s.$$

The class of such pair's is denoted by $\mathcal{A}(x)$.

DEFINITION 2.22. An admissible portfolio $\pi(t)$ is called an **arbitrage** if the corresponding value process $V^\pi(t)$ satisfies, $V^\pi(0) = 0, V^\pi(T) \geq 0$ a.s. and

$P[V^\pi(T)] > 0$. In other words, an arbitrage opportunity is a strategy that satisfies the following conditions

$$\pi_t \cdot S_t = \pi_0 \cdot S_0 + \int_0^t \pi_s dS_s \quad \forall t \leq T, P - a.s. \text{ with}$$

$$P(\pi_0 \cdot S_0 = 0) = 1$$

$$P(\pi_T \cdot S_T > 0) > 0$$

$$P(\pi_T \cdot S_T \geq 0) = 1$$

In other words, $\pi_T \cdot S_T$ is a positive random variable that has strictly positive expectation.

We work under the assumption that there are no opportunities for arbitrage. This assumption is often replaced by the following statement:

There exists a probability measure \tilde{P} that is equivalent to P and such that the vector of discounted prices $\frac{S_t}{S_0^t}$ is \tilde{P} -martingale.

DEFINITION 2.23. If the market has no arbitrage opportunities (NAO), then there exists a progressively measurable process $\theta : [0, T] \times \Omega \rightarrow R^d$,

called the **market price of risk (or relative risk)** process such that,

$$\mu(t) - r(t)1_n = \theta(t)\sigma(t), \quad 0 \leq t \leq T \quad a.s.$$

Conversely, if such a process $\theta(\cdot)$ exists and satisfies, in addition to be above requirements

$$\int_0^t \|\theta(t)\|^2 dt < \infty \quad a.s.$$

and,

$$E \left[\exp \left\{ - \int_0^T \theta(t) dB_t - \frac{1}{2} \int_0^T \|\theta(t)\|^2 dt \right\} \right] = 1,$$

then the market model (financial strategy) is arbitrage free.

DEFINITION 2.24. A positive semi martingale process.

$$H_0(t) \triangleq \frac{\eta_0(t)}{S_0(t)} = \zeta(t)\eta_0(t), \quad \text{where, } \zeta(t) = e^{-rt}, 0 \leq t \leq T$$

satisfies

$$E \left[H_0(T)X(T) + \int_0^T H_0(T)dc(t) \right] \leq x, \forall (\pi, c) \in \mathcal{A}(x), 0 \leq t \leq T \quad a.s. \quad (2.5.1)$$

is called a **state price density** process.

DEFINITION 2.25. A **contingent claim** is a random variable C on the underlying probability space $(\Omega, \mathcal{F}, \mathbf{P})$ such that

$$0 \leq C < \infty \quad P - a.s.$$

Claim C is called a **derivative** of the primary assets S^0, \dots, S^d if it is measurable with respect to the σ -field $\sigma(S^0, \dots, S^d)$ generated by the assets, i.e. if $C = f(S^0, \dots, S^d)$ for a measurable function f on R^{d+1} .

DEFINITION 2.26. We say that the claim C is **attainable** in the market $\{X(t)\}_{t \in [0, T]}$ if there exist an admissible portfolio $\pi(t)$ and a real number x such that

$$C = V_x^\pi(T) := x + \int_0^T \pi(t) dX(t) \quad a.s.$$

such that the normalized value process

$$\bar{V}^\pi(t) = x + \int_0^t \zeta(s) \sum_{i=0}^n \pi_i(s) \sigma_i(s) d\tilde{B}(s); \quad 0 \leq t \leq T \text{ is a } \tilde{P} \text{ martingale.}$$

If such $\pi(t)$ exists, we call it a replicating or hedging portfolio for C .

DEFINITION 2.27. The market $\{X(t)\}_{t \in [0, T]}$ is called **complete** if every C -claim is attainable, in other words, a claim C is attainable if there exists a real number x such that if we start with x as our initial fortune we can find an admissible portfolio $\pi(t)$ which generates a value $V_x^\pi(T)$ at time T which a.s. equals C

$$V_x^\pi(T, \omega) = C(\omega) \quad \text{for a.a. } \omega.$$

Next I am going to define the correspondance between the stochastic differential equation and the partial differential equations (PDE). In other words, in the next two sections, we present Kolmogorov's backward equation and the Feynman-Kac formula

to define a relation and transformation between the stochastic differential equations (SDE) and the partial differential equations (PDE).

2.6. Kolmogorov's Backward Equation

Let X_t be an Itô diffusion satisfying the SDE

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t \quad t \geq 0,$$

Let $f(X_t)$ be a twice differentiable bounded function of this process. Consider the expectation

$$\hat{f}(X^-, t) = E[f(X_t)|X^-], \quad \text{for all } t \geq 0.$$

Where $\hat{f}(X^-, t)$ represents the forecasted value, and X^- is the latest value observed before time t . Heuristically speaking, X^- is the immediate past. Then using the operator of Ito diffusion, we can characterize how $\hat{f}(X^-, t)$ may change over time. This evolution of the forecast is given by Kolmogorov's backward equation:

$$\begin{aligned} \frac{\partial \hat{f}}{\partial t} &= \mathcal{A} \hat{f}, \quad t > 0; x \in R^n \\ \hat{f}(0, x) &= f(x); \quad x \in R^n \end{aligned}$$

where the right hand side is to be interpreted as \mathcal{A} applied to the function $x \rightarrow \hat{f}(t, x)$.

Where

$$\mathcal{A} = \mu_t \frac{\partial}{\partial x} + \frac{1}{2} \sigma_t^2 \frac{\partial^2}{\partial x^2}$$

Kolmogorov's backward equation gives the first correspondence between an expectation of a stochastic process and a PDE.

THEOREM 2.28. The Kolmogorov backward PDE

Define $u(t, T, x) = E[f(X_T)|X_t = x] = E[f(X_T^{(t,x)})]$. Then $u(t, T, x)$ satisfies the backward Kolmogorov PDE

$$u_t + \mu(t, x)u_x + \frac{1}{2}\sigma^2(t, x)u_{xx} = 0,$$

$$u(T, T, x) = f(x)$$

(2.6.1)

Conversely, if u satisfies Eq.(2.6.1), then $u(t, T, x) = E[f(X_T)|X_t = x]$.

PROOF. Let u satisfies Eq.(2.6.1), then by Ito's formula

$$\begin{aligned} du(s, T, X_s) &= u_s ds + u_x dX_s + \frac{1}{2}u_{xx}dX_s dX_s \\ &= u_s ds + u_x[\mu(t, X_s)ds + \sigma(t, X_s)dB_s] + \frac{1}{2}u_{xx}\sigma^2(t, X_s)ds \\ &= [u_s + \mu(t, X_s)u_x + \frac{1}{2}\sigma^2(t, X_s)u_{xx}]ds + \sigma(t, X_s)u_x dB_s \\ &= \sigma(t, X_s)u_x dB_s \end{aligned}$$

We integrate the above equation,

$$\int_t^T du(s, T, X_s) = \int_t^T \sigma(t, X_s)u_x dB_s$$

Then,

$$u(T, T, X_T) - u(t, T, X_t) = \int_t^T \sigma(t, X_s)u_x dB_s$$

Taking expectation conditioned on $X_t = x$, we obtain

$$E[u(T, T, X_T) - u(t, T, X_t)|X_t = x] = E[\int_t^T \sigma(t, X_s)u_x dB_s|X_t = x] = 0$$

Thus

$$\begin{aligned} u(t, T, x) &= E[(T, T, X_T)|X_t = x] \\ &= E[f(X_T)|X_t = x]. \end{aligned}$$

Conversely, if $u(t, T, x) = E[f(X_T)|X_t = x]$, by a similar argument as before, u satisfies the Eq.(2.6.1). □

2.7. The Feynman-Kac Formula, Killing Diffusion

The Feynman-Kac formula is an extension of Kolmogorov's backward equation. The formula provides a PDE that corresponds to \hat{f} defined as

$$\hat{f}(t, x_t) = E \left[e^{-\int_0^t \beta(X_s) ds} f(X_t) \right]; \quad \text{for all } t \geq 0. \quad (2.7.1)$$

Let X_t be an Itô diffusion satisfying the SDE $dX_t = \mu(X_t)dt + \sigma(X_t)dB_t$; $t \geq 0$ under the equivalent martingale measure, then

$$\begin{aligned} \frac{\partial \hat{f}}{\partial t} &= \mathcal{A} \hat{f} - \beta \hat{f}; & t > 0, x \in R^n \\ \hat{f}(0, x) &= f(x); & x \in R^n. \end{aligned}$$

THEOREM 2.29. *Suppose that X_s satisfies the PDE $dX_s = \mu(s, X_s)ds + \sigma(s, X_s)dB_s$ and $\hat{f}(t, T, x) = E \left[e^{-\int_t^T \beta(s, X_s) ds} f(X_T) | X_t = x \right]$ if and only if $\hat{f}(t, T, x)$ satisfies the backward Kolmogorov PDE*

$$\begin{aligned} \hat{f}_t + \mu(t, x)\hat{f}_x + \frac{1}{2}\sigma^2(t, x)\hat{f}_{xx} - \beta\hat{f} &= 0 \\ \hat{f}(T, T, x) &= f(x). \end{aligned}$$

PROOF. Proof of this theorem is similar to that of backward Kolmogorov PDE theorem. □

REMARK 2. (*About killing a diffusion*)

We know that the generator of an Itô diffusion X_t given by $dX_t = \mu(X_t)dt + \sigma(X_t)dB_t$ is a partial differential operator of the form

$$\mathcal{L}f = \sum_i \mu_i(x) \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j} (\sigma\sigma')_{i,j}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}$$

It is natural to ask if one can also find processes whose generator has the form

$$\mathcal{L}f = \sum_i \mu_i(x) \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j} (\sigma\sigma')_{i,j}(x) \frac{\partial^2 f}{\partial x_i \partial x_j} - \beta f \quad (2.7.2)$$

where $\beta(x)$ is a bounded and continuous function.

If $\beta(x) \geq 0$, the answer is yes and the process \tilde{X}_t with generator (2.7.2) is obtained by killing X_t at a certain killing time ζ . By this we means that there exists a random time ζ such that if we put

$$\tilde{X}_t = X_t, \quad \text{if } t < \zeta,$$

and \tilde{X}_t undefined if $t \geq \zeta$, then \tilde{X}_t is also a strong Markov process and

$$E^x \left[f(\tilde{X}_t) \right] := E^x \left[f(X_t) \cdot \chi_{[0, \zeta)}(t) \right] = E^x \left[f(X_t) \cdot e^{-\int_0^t \beta(x) ds} \right] \quad (2.7.3)$$

for all bounded continuous functions f on R^n .

2.7.1. General case of the Feynman-Kac formula. Brownian motion can solve PDEs for us. Consider the following boundary value problem for fixed functions $\mu(t, x), \sigma(t, x), f(x)$ such that

$$\begin{aligned} \frac{\partial \hat{f}}{\partial t}(t, x) + \mu(t, x) \frac{\partial \hat{f}}{\partial x}(t, x) + \frac{1}{2} \sigma(t, x)^2 \frac{\partial^2 \hat{f}}{\partial x^2}(t, x) &= 0 \\ \hat{f}(t, x) &= f(x) \end{aligned}$$

There is a stochastic representation of the solution. Let $X = \{X_s : s \in [0, t]\}$ satisfy the SDE

$$dX_s = \mu(s, X_s) ds + \sigma(s, X_s) dB_s; \quad X_0 = x,$$

Applying itô's formula to $\hat{f}(t, X_s)$ and writing it in integrated form to get

$$\begin{aligned} \hat{f}(t, X_t) &= \hat{f}(0, X_0) + \int_0^t \left[\frac{\partial \hat{f}}{\partial t}(s, X_s) + \mu(s, X_s) \frac{\partial \hat{f}}{\partial x}(s, X_s) + \frac{1}{2} \sigma(s, X_s)^2 \frac{\partial^2 \hat{f}}{\partial x^2}(s, X_s) \right] ds \\ &\quad + \int_0^t \sigma(s, X_s) \frac{\partial \hat{f}}{\partial x}(s, X_s) dB_s, \end{aligned}$$

it follows that taking expectation and using the property of stochastic integrals with respect to Brownian motion, that their expectations are 0 and if \hat{f} solves the above PDE, then

$$\hat{f}(0, x) = E [f(X_t) / X_0 = x]$$

This is the Feynman-Kac formula, a fundamental connection between PDEs and SDEs. It shows that the solution to the PDE can be obtained by finding a conditional expectation and suggests a numerical technique for solving PDEs. It also can be used to find out properties of random processes X by solving certain PDEs.

2.8. Concepts from stochastic calculus and finance

Sample space Ω -

A set of financial outcomes or sample points such as the set of all possible three month U.S. treasury rates in one month.

σ -algebra \mathcal{F} -

A particular collection of financial events described as subsets of Ω such as the event that three month U.S. treasury rates will be above 10 per cent in one month.

Probability distribution-

A rule for assigning to each financial event, that is, to each member of a σ -algebra, the 'likelihood' of that event occurring. For example, the probability that the three month U.S. treasury rate will be 10 per cent in one month is 0.05.

Random variable-

A real-valued function on a σ -algebra of financial events describing the possible values of some future occurrence such as the foreign exchange rate between U.S. and Australia in three months.

Stochastic process-

A description of the probabilistic behavior of a financial asset (such as the price of a stock) or a financial variable (such as a foreign exchange rate) over future time. In other words, we have a probability distribution corresponding to each point in time belonging to a particular discrete or continuous set.

Filtration-

A mathematical description of the availability of information to a particular observer over time showing that increasing amounts of information are available as time progresses.

Adapted stochastic process-

A stochastic process with information (that is, with a filtration) that is available to a particular observer. Adaptedness is a criterion for determining whether a given stochastic process is actually ‘observed’ by this person.

Martingale-

A stochastic process for which past and current information gives “fair” information about future behavior. More precisely, it is a stochastic process in which the expected future value is conditional on its present value and past history is just its present value.

Markov process-

A stochastic process with the property that probabilistic knowledge about its behavior at some time in the future given its present value is not increased by knowing its past behavior.

Random walk-

A stochastic process made up of a sum of a number of independent steps. A simple symmetric random walk is formed by tossing a fair coin at each time $t = 1, 2, \dots$ and stepping up one unit or down one unit according as it is heads or tails.

Brownian motion or Wiener process-

It is the basic stochastic process B_t formed by taking finer and finer step sizes in a random walk.

Predictable process-

A mathematical model of trading strategy for buying and selling a particular financial asset that only uses information available up to and including the present time.

CHAPTER 3

Existing Models on the Portfolio selection

3.1. Introduction

The portfolio selection problem was originally set in a single period, where the investor seeks to maximize the expected return of his portfolio. Nicolas Bernoulli (18th century) demonstrated the limitation of the expected return rule by using the St.Petersburg paradox; see Ingersoll (1987) and Samuelson (1977). The expected-utility framework was generalized by Von Neumann and Morgenstern (1944), who extended the analysis to a general problem of decision making under uncertainty and developed a rigorous axiomatic approach for the expected utility by five axioms concerning individual behavior under uncertainty. However the problem of maximizing the expected utility of terminal wealth over a single period is not realistic. For a long time period, the market parameters change and lead to change in the investment opportunity set, so investors need to revise their asset allocation during the period of investment. This is the reason for considering dynamic portfolio selection through time.

Nevertheless, maximizing the terminal wealth dynamically through time is not a fully realistic problem either. Indeed Merton (1969) considers the problem for a household which is both an investor and a consumer. A household is also an investor because it invests a part of its savings in the available investment opportunities. In general the consumption and investment rules cannot be treated independently. This leads to the consideration of the intertemporal portfolio problem where the investor has utility with consumption. In the financial economic literature, this problem has

often been a starting point to solve general equilibrium models. The optimal consumption as determined at equilibrium implies the prices as well as the interest rates prevailing in this economy (Cox 1985).

In this chapter I do not add new things to the existing literature about optimal portfolio problems but put together different methods for solving this type of problem. Therefore this chapter would be the background for forthcoming topics.

3.2. Classical portfolio selection problem

3.2.1. The Markowitz Model. Harry Markowitz (1952) showed how to understand the trade-off between risk and returns dealing with the portfolio problem considering the mean-variance optimization. Although a subject to theoretical and empirical criticism, mean-variance analysis provides a basis for the derivation of the capital asset pricing model (CAPM), the Sharpe-Linter model, and the Black model. Moreover, mean variance analysis is fully consistent with expected utility maximization when investors have quadratic utility or when the asset returns are normally distributed.

Suppose there are n risky securities in the market. If μ denotes the vector of returns and Σ is the covariance matrix of these returns, then an investor chooses portfolio weights w to maximize his expected returns $E[R] = w^T \mu$, subject to a given variance level $w^T \Sigma w$. This problem is equivalent to minimizing the variance subject to a given expected returns (Merton 1972)

$$\left\{ \begin{array}{l} \min_w \quad \frac{1}{2} w^T \Sigma w \\ s.t. E[R] = w^T \mu \\ w^T \mathbf{1} = 1 \end{array} \right\} \quad (3.2.1)$$

, where $\mathbf{1}$ is the unit n -dimensional vector. Applying the Lagrange multiplier method, the system is equivalent to

$$\min_w \frac{1}{2} w^T \Sigma w + \lambda (E[R] - w^T \mu) + \gamma (w^T \mathbf{1} - 1) \quad (3.2.2)$$

where λ and γ are the Lagrange multipliers. Applying the first order conditions we get

$$\begin{aligned} \Sigma w &= \lambda \mu + \gamma \mathbf{1} \\ E[R] &= w^T \mu \\ w^T \mathbf{1} &= 1 \end{aligned} \quad (3.2.3)$$

For a given set of asset returns, the portfolios which satisfy the first order conditions make up the set of optimal portfolios. The efficient portfolios are those for which there are no other portfolios with the same or greater expected return and less variance. In general, the set of efficient portfolios is a subset of optimal portfolios. The feasible portfolios that minimize the variance given a level of expected returns are called frontier portfolios and they form a hyperbola in the standard deviation, expected return space. Markowitz (1987) solved the optimal portfolio problem for an efficient frontier with centre of the hyperbola $(0, A/C)$ and $E[R] = \frac{A}{C} \pm \sqrt{\frac{D}{C}} \sigma$ where

$$A = \mu^T \Sigma^{-1} \mathbf{1}, \quad B = \mu^T \Sigma^{-1} \mu, \quad C = \mathbf{1}^T \Sigma^{-1} \mathbf{1}, \quad D = BC - A^2,$$

and the optimal portfolio is

$$w = \frac{B \Sigma^{-1} \mathbf{1} - A \Sigma^{-1} \mu}{D} + \frac{C \Sigma^{-1} \mu - A \Sigma^{-1} \mathbf{1}}{D} E[R].$$

3.2.2. Single-Period Expected Utility Maximization. In this framework, the beginning of the investment period the investor chooses at a feasible portfolio

allocation which maximizes the expected value of a Von Neumann-Morgenstern utility function for the terminal wealth $u(W)$ which is assumed to be twice continuously, differentiable, strictly concave and an increasing function.

An investor with initial wealth W_0 , who has a subject of the returns of securities (R_1, \dots, R_n) chooses the portfolio weights $w_i, i = 1, 2, \dots, n$ that solves the following problem

$$\begin{aligned} & \max_{\{w_1, \dots, w_n\}} E [u (\sum_{i=1}^n w_i R_i W_0)] \\ & \text{subject to } \sum_{i=1}^n w_i = 1 \end{aligned}$$

If a riskless security with return r is added to the menu of available securities, then the problem becomes

$$\begin{aligned} & \max_{\{w_1, \dots, w_n\}} E [u \{ \sum_{i=1}^n w_i (R_i - r) + r \} W_0] \\ & \text{subject to } \sum_{i=1}^n w_i = 1 \end{aligned}$$

In the case when the utility function u is strictly concave and increasing the first order conditions is necessary and sufficient for a maximum. Then, the optimal portfolio is obtained by solving the following system of n equations,

$$E [(R_i - r) u \{ W_0 (1 + R^*) \}] = 0, \quad i = 1, \dots, n$$

Where

$$\sum_{i=1}^n w_i^* (R_i - r) + r, \quad (w_1^*, \dots, w_n^*)$$

is a solution to the maximized problem.

3.2.3. Multi-period setting. The main criticism of the one period model is that they do not allow for possible revisions of portfolio positions during the life of the investment. This requirement is critical if the underlying asset prices are highly volatile.

In continuous time models where the investor rebalances the portfolio continuously using the tools of stochastic control theory and stochastic calculus to obtain

explicit solutions where the required computational effort is less than in case of discrete time model. A more realistic scenario would be that the agent chooses to review his investment and consumption decision on a regular basis because either he has something else to do with his life or possibly because of transaction costs associated with the rebalancing. The discrete time case is more realistic but seldom yields closed form solutions, whereas the continuous time problem is mathematically more tractable but the question that remains: how realistic is it?

Briefly, the portfolio problem in continuous time, for an investor-consumer with initial wealth W_0 , consists of the following optimization problem

$$\begin{aligned} & \sup_{\varphi, c} E \left[\int_0^T u(c_t, t) dt + F(W_T) \right] \\ & \text{subject to } c_t \geq 0, W_t \geq 0, W_0 \geq 0 \\ & \text{and budget constraints.} \end{aligned}$$

Where the optimization is carried over all admissible consumption and investment strategies c and φ .

A standard assumption about capital market structure is the absence of transaction costs, taxes and problems with indivisibility of assets. Given these assumptions, an investor would prefer to revise his portfolio as often as they can at any time. In reality market frictions exist, so a well-defined equilibrium model should take these additional costs into account.

3.3. Continuous Time Finance

The first mathematical model of financial asset was prices developed by Bachelier (1900) for modeling stock exchange security prices. Since then, the main development in continuous time finance has been using stochastic processes driven by Brownian motion and sometimes combined with a jump process. Throughout this dissertation we deal with a financial market M consisting of $N + 1$ financial assets. One of these

assets is instantaneously risk free, and will be called a money market. The rest of the assets are risky and will be called stocks. These financial assets have continuous prices evolving continuously in time and driven by a d -dimensional Brownian motion with no or insignificant jumps. Our assumption that asset prices have no jumps is a significant one. It is tantamount to the assertion that there are no “surprises” in the market i.e. the price of stock at time t can be perfectly predicted from the knowledge of its price at times strictly prior to t .

As far as this dissertation goes, Brownian motion will be considered as a source of randomness in security prices. This section starts by setting the basic elements of continuous time finance. Then dynamic programming, Lagrange multiplier and martingale approaches to solve the basic portfolio problem are presented which would provide a view on the application to our model for the optimal-consumption-investment problem.

3.3.1. Market setup. Uncertainty in the financial market is modeled by a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ equipped with a filtration, that is a non decreasing family $\mathbf{F} = (\mathcal{F}_t)_{t \geq 0}$ of sub σ -field of \mathcal{F}

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F} \quad \text{for all } 0 \leq s < t < \infty$$

The filtered probability space $(\Omega, \mathcal{F}, \mathbf{P}, \mathbf{F})$ is assumed to satisfy the following conditions:

- \mathcal{F}_0 contains all \mathbf{P} -null sets of \mathbf{F}
- \mathbf{F} is right continuous, i.e. $\mathcal{F}_t = \mathcal{F}_{t+} \triangleq \bigcap_{s>t} \mathcal{F}_s$

A stochastic process $X = (X_t)_{t \geq 0}$ is a family of random variables defined on $(\Omega, \mathcal{F}, \mathbf{P}, \mathbf{F})$. The process X is said to be \mathbf{F} - adapted if X_t is \mathcal{F}_t - measurable for each t , that is X_t is known when \mathcal{F}_t is known at time t .

On the filtered probability space $(\Omega, \mathcal{F}, \mathbf{P}, \mathbf{F})$ consider a market that includes a bond with price $S^0(t)$ at time t and n risky assets with prices $S^1(t), \dots, S^n(t)$ following an itô process. The dynamics of the securities are

$$\begin{aligned} dS^0(t) &= S^0(t)r(t)dt, & S^0(0) &= 1 \\ dS^1(t) &= S^1(t) \left[\mu_i(t, S^i(t))dt + \sum_{j=1}^n \sigma_{ij}(t, S^i(t))dB_j(t) \right] \\ S^i(0) &= s_i; & 1 \leq i \leq n, & \text{ for all } 0 \leq t < \infty, \end{aligned}$$

where μ_i is the instantaneous conditional expected percentage change in price per unit time and σ_i^2 is the instantaneous conditional variance per unit time.

Here, $B(t) = (B_1(t), \dots, B_n(t))$ is an n -dimensional Brownian motion defined on $(\Omega, \mathcal{F}, \mathbf{P}, \mathbf{F})$ such that \mathcal{F}_t is the completion of the filtration $\sigma\{B(u) : 0 \leq u \leq t\}$. The interest rate $r(t)$, the mean rate of return $\mu(t) = (\mu_1(t), \dots, \mu_n(t))^T$ and the volatility $\sigma(t, S) = (\sigma_{ij}(t, S), 1 \leq i, j \leq n)$ are taken to be measurable, adapted and bounded processes. The dimension of the Brownian motion being equal to the number of risky assets ensures the completeness of the market.

If we assume that there exists an $\varepsilon > 0$ such that

$$\xi^T \sigma(t, \omega) \sigma(t, \omega)^T \xi \geq \varepsilon \|\xi\|_2^2; \quad \forall \xi \in R^n, (t, \omega) \in [0, \infty) \times \Omega$$

then the inverses of σ and σ^T exists and are bounded

$$\begin{aligned} \|\sigma(t, \omega)^{-1} \xi\|_2 &\leq \varepsilon^{-\frac{1}{2}} \|\xi\|_2 \\ \|\sigma^T(t, \omega)^{-1} \xi\|_2 &\leq \varepsilon^{-\frac{1}{2}} \|\xi\|_2, \quad \forall \xi \in R^n. \end{aligned}$$

In this case, the filtration \mathbf{F} is equivalently given as the completion of the filtration generated by the vector of prices S . Since the volatility matrix σ is non-singular, then it is possible to define a process θ called the *market price of risk* uniquely by

$$\theta(t, S) = \sigma(t, S)^{-1}(\mu(t, S) - r(t)1)$$

Where 1 is a n -dimensional unit vector. Let us introduce the strictly positive, \mathbf{P} martingale process η as

$$\eta_t = \exp \left\{ - \int_0^t \theta^T(S_u, u) dB(u) - \frac{1}{2} \int_0^t \|\theta^T(S_u, u)\|^2 du \right\}$$

such that the new probability measure \mathbf{P}^θ by setting

$$\frac{d\mathbf{P}^\theta}{d\mathbf{P}} \Big|_{\mathcal{F}_t} = \eta_t$$

Here,

η is called the Radon-Nikodym derivative which takes here the form of a Doleans exponential. Using Girsanov's theorem, the process

$$B_t^\theta = B_t + \int_0^t \theta(S_u, u) du$$

is a Brownian motion under \mathbf{P}^θ . Equivalently, the security i price process under \mathbf{P} is given by

$$dS^i(t) = S^i(t) \left[r(t)dt + \sum_{j=1}^n \sigma_{ij}(t, S^i(t)) dB_j^\theta(t) \right]$$

$$1 \leq i \leq n, \text{ for all } 0 \leq t < \infty$$

In this situation, where the market is complete, \mathbf{P}^θ is the unique equivalent risk-neutral or martingale measure. Harrison and Kreps (1979) show that the existence of an equivalent martingale measure is not only a sufficient condition but also a necessary condition for “free lunches” not to be available. For the finite discrete time horizon, the equivalence of no arbitrage opportunities and existence of an equivalent martingale measure holds, but breaks down when time horizon is infinite. Shirayaev (1999) established the concept of arbitrage in continuous time as the following:

“The condition of no arbitrage with bounded risk is equivalent to the existence of an equivalent martingale measure”.

3.4. Classical Merton's problem: Lifetime portfolio selection under uncertainty

Rober C. Merton (1969) derived the optimality equations for a multi asset problem when the rate of returns are generated by a Brownian motion process. A particular case was examined in detail for the two asset model with constant relative risk

aversion or iso-elastic marginal utility. He obtained the explicit solution for the case of constant absolute risk aversion. I am going to describe the major results from his original work and compare them with other successful methods.

3.4.1. The Budget Equation. In the usual continuous time model under certainty, the budget equation is a differential equation. However, when uncertainty is introduced by a random variable, the budget equation must be generalized to become a stochastic differential equation. Define,

$W(t)$ = total wealth at time t

$X_i(t)$ = price of i^{th} asset at time t , $i = 1, \dots, m$

$C(t)$ = consumption per unit time at time t

$\omega_i(t)$ = proportion of total wealth in the i^{th} asset at time t , ($i = 1, \dots, m$),

$$\sum_{i=1}^m \omega_i(t) \equiv 1$$

The budget equation in the discrete form can be written as

$$W(t) = \left[\sum_{i=1}^m \omega_i(t_0) \frac{X_i(t)}{X_i(t_0)} \right] \quad (3.4.1)$$

Let the process $Y_i(t)$ be generated by a Gaussian random walk as expressed by the stochastic difference equation,

$$Y_i(t) - Y_i(t_0) \equiv \Delta Y_i = \sigma_i Z_i(t) \sqrt{h}, \quad h = \Delta t \quad (3.4.2)$$

where each $Z_i(t)$ is an independent variate with a standard normal distribution for every t . Here σ_i^2 is the variance per unit time of the process Y_i and the mean of the increment ΔY_i is zero. The limit of the above process as $h \rightarrow 0$ (continuous time) can be expressed as

$$dY_i = \sigma_i Z_i(t) \sqrt{dt} \quad (3.4.3)$$

By applying the same limit process to the discrete time budget equation, we have

$$dW = \left[\sum_{i=1}^m \omega_i(t) \mu_i W(t) - C(t) \right] dt + \sum_{i=1}^m \omega_i(t) \sigma_i Z_i(t) W(t) \sqrt{dt} \quad (3.4.4)$$

The stochastic differential equation (3.4.4) is the generalization of the continuous time budget equation under uncertainty.

3.4.2. The Two asset Model. Define,

$\omega_1(t) \equiv \omega(t) =$ proportion invested in the risky assets;

$\omega_2(t) \equiv 1 - \omega(t) =$ proportion invested in the sure assets;

$\mu =$ expected rate of return on risky asset;

$r =$ return on the sure asset;

then the budget equation for the two asset model is given by

$$dW_t = W(t)[\omega(t)(\mu - r) + r]dt - c(t)dt + W(t)\omega(t)\sigma Z(t)\sqrt{dt} \quad (3.4.5)$$

The problem of choosing optimal portfolio selection and consumption rules is formulated as

$$\text{Max } E \left[\int_0^t e^{-\rho t} U(C(t))dt + F(W(T), T) \right] \quad (3.4.6)$$

subject to $c(t) \geq 0, W(t) \geq 0, W(t) \geq 0$

and budget constraints.

where $U(C)$ is assumed to be strictly concave utility function. $F(W(T), T)$ is to be specified “bequest valuation function”, which is assumed to be concave in $W(T)$.

3.4.2.1. *Explicit solution for CRRA utility.* Case (i)- finite time horizon

If the utility function is assumed to be of the form $U(C) = C^\gamma/\gamma, \gamma < 1$ and $\gamma \neq 0$ or $U(C) = \log C$ (the limiting form for $\gamma = 0$) where $-U''(C)C/U'(C) = 1 - \gamma = \delta$ is the Pratt’s measure of relative risk aversion. For the utility function yielding constant relative risk aversion (i.e., iso-elastic marginal utility), Merton solved the first order

conditions for the optimization problem (3.4.6) and obtained the following resulting decision rules for consumption and portfolio selection, $C^*(t)$ and $\omega^*(t)$.

$$C^*(t) = [b(t)]^{1/\gamma-1}W(t)$$

and

$$\omega^*(t) = \frac{(\mu-r)}{\sigma^2(1-\gamma)}W(t)$$

Where $b(t)$ satisfies the following differential equation,

$$b'(t) = \alpha b(t) - (1-\gamma)[b(t)]^{-\gamma/1-\gamma}$$

Subject to $b(T) = \varepsilon^{1-\gamma}$

and $\alpha = \rho - \gamma [(\mu - r)^2/2\sigma^2(1 - \gamma) + r]$.

Merton solved the differential equation and obtained $b(t)$ as

$$b(t) = \left\{ [1 + (\nu\varepsilon - 1)e^{\nu(t-T)}] / \nu \right\}^{1-\gamma} \text{ and } \nu = \alpha/(1 - \gamma).$$

Case (ii)- Infinite time horizon

Assuming that $\lim_{T \rightarrow \infty} F(W(T), T) = 0$, Merton solved the first order conditions for the optimization problem and obtained the following optimal solutions

$$C_\infty^*(t) = \left\{ \frac{\rho}{1-\gamma} - \gamma \left[\frac{(\mu-r)^2}{2\sigma^2(1-\gamma)^2} + \frac{r}{1-\gamma} \right] \right\} W(t) \quad (3.4.7)$$

and

$$\omega_\infty^*(t) = \frac{(\mu-r)}{\sigma^2(1-\gamma)}W(t) \quad (3.4.8)$$

3.5. The Dynamic Programming Approach

The dynamic principle or Bellman principle of optimality is “An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision” (Bellman, 1957).

Dynamic programming is a powerful technique in analyzing certain types of optimization problems. It has been used in many fields such as production and inventory planning and more generally in problems where an individual must make a sequence

of decisions in a specific order. This evolutionary character makes it natural to express the optimization recursively as a choice of one's current action on the basis of an already optimized action policy for the future.

To illustrate the idea of dynamic programming, consider the problem of finding the value of control variables u_1 and u_2 for period 1 and 2 with the state variables x_1 and x_2 , where x_2 is a function of x_1 and u_1 . Let the objective function to be maximized be

$$\varphi_1(x_1, u_1) + \varphi_2(x_2, u_2).$$

The dynamic programming consists of the following steps:

- To find the optimal control u_2 , maximize $\varphi_2(x_2, u_2)$ with respect to u_2 and obtain $u_2 = g_2(x_2)$ as a function of x_2
- The value function for period 2 is $V_2(x_2) = \varphi_2(x_2, g_2(x_2))$
- The optimal control u_1 solves the following problem

$$\begin{aligned} & \max_{u_1} \varphi_1(x_1, u_1) + V_2(x_2) \\ & \text{subject to } x_2 = f(x_1, u_1) \end{aligned}$$

Here $V_1(x_1)$ is the value function of the last optimization, is called **the Bellman equation**, and the Bellman equation is written as

$$\begin{aligned} V_1(x_1) &= \max_{u_1} \varphi_1(x_1, u_1) + V_2(x_2) \\ & \text{subject to } x_2 = f(x_1, u_1) \end{aligned}$$

For problems more than 2 periods, the Bellman equation becomes

$$V_t(x_t) = \max_{u_t} \varphi_t(x_t, u_t) + V_{t+1}(x_{t+1})$$

3.5.1. The Hamilton-Jacobi-Bellman Equation. Let $B_t = (B_1(t), \dots, B_d(t))$ in R^d denote the standard Brownian motion vector on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We fix the standard filtration $\mathbf{F} = (\mathcal{F}_t)_{t \geq 0}$ of B and the time horizon $[0, T]$ for some fixed time $T > 0$. We define the notation of functions and variables as following

- A set $\mathcal{A} \subset R^m$ of actions
- A set $\mathcal{Y} \subset R^k$ of states
- A set \mathcal{C} of \mathcal{A} valued adapted processes, called controls
- A drift function $g : \mathcal{A} \times \mathcal{Y} \rightarrow R^k$
- A diffusion function $h : \mathcal{A} \times \mathcal{Y} \rightarrow R^{k \times d}$
- A running reward function $f : \mathcal{A} \times \mathcal{Y} \times [0, T] \rightarrow R$
- A terminal reward function $F : \mathcal{Y} \rightarrow R$

A control c in \mathcal{C} given an initial state y in \mathcal{Y} if there is a unique Itô process Y^c valued in \mathcal{Y} with

$$dY_t^c = g(c_t, Y_t^c)dt + h(c_t, Y_t^c)dB_t; \quad Y_0^c = y$$

subject to some technical conditions on c, g and h .

Let $\mathcal{C}_a(y)$ denote the set of admissible controls given an initial state $y \in \mathcal{Y}$ and define a utility of any admissible control $c \in \mathcal{C}_a(y)$ as

$$V^c(y) = E \left[\int_0^T f(c_t, Y_t^c, t)dt + F(Y_T^c) \right]; \quad V^c(y) \in [-\infty, \infty]$$

For an initial state y in \mathcal{Y} , the optimal control problem is

$$V(y) = \sup_{c \in \mathcal{C}_a(y)} V^c(y)$$

With $V(y) = -\infty$ if there is no admissible control given for the initial state y . If there exists an admissible control c^* such that $V^{c^*}(y) = V(y)$, then c^* is an optimal control at y .

Let $V(y) = J(y, 0)$ for some function J in $\mathcal{Y} \times [0, T]$, that solves the non linear second order partial differential equation, which is also called the *Hamilton-Jacobi-Bellman(HJB)* equation. Here $J(y, t)$ is the optimal utility remaining at time t in state y .

The *HJB* equation of the stochastic control is

$$\sup_{c \in \mathcal{A}} \mathcal{D}^a J(y, t) + f(c, y, t) = 0, \quad (y, t) \in \mathcal{Y} \times [0, T]$$

(3.5.1)

where

$$\mathcal{D}^a J(y, t) = J_y(y, t)g(a, y) + J_t(y, t) + \frac{1}{2}tr [h(a, y)h(a, y)^T J_{yy}(y, t)] \quad (3.5.2)$$

with boundary condition

$$J(y, T) = F(y); \quad y \in \mathcal{Y}$$

Here \mathcal{D}^a is called the Dynkin operator over the variable y for a given control a .

3.6. Solving Merton's problem using Dynamic programming

Jinchun (2007) applied the dynamic programming approach to solve Merton's original problem introduced in (1969) and (1971) stated as follows.

Given an initial wealth W_0 , an investor who lives T years, continuously chooses the following optimization problem

$$\begin{aligned} & \sup_{\varphi, c} E \left[\int_0^T u(c_t, t) dt + F(W_T) \right] \\ & \text{subject to } c_t \geq 0, W_t \geq 0, W_0 \geq 0 \\ & \text{and budget constraints,} \end{aligned}$$

where the optimization is carried over all admissible consumption and investment strategies c and φ . Here $u(c_t, t)$ is the utility of consumption at time t , assumed to be increasing and concave with $u(0, t) = 0$ for each t , and $F(W)$ is a specified "bequest valuation function" assumed to be concave in W . As defined in the set up, consider a market that includes a bond with price $S^0(t)$ at time t , and n risky assets with prices $S^1(t), \dots, S^n(t)$ following an Itô process. The price per share $i, S^i(t)$ is generated by the Itô process

$$dS^i(t) = S^i(t) [\mu_i(t, S^i(t))dt + \sigma_i(t, S^i(t))dB_i(t)] \quad (3.6.1)$$

where μ_i is the instantaneous conditional is expected percentage change in price per unit time and σ_i^2 is the instantaneous conditional variance per unit time.

Here, $B(t) = (B_1(t), \dots, B_n(t))$ is an n -dimensional Brownian motion defined on $(\Omega, \mathcal{F}, \mathbf{P}, \mathbf{F})$ such that \mathcal{F}_t is the completion of the filtration $\sigma\{B(u) : 0 \leq u \leq t\}$. The interest rate $r(t)$, the mean rate of return $\mu(t) = (\mu_1(t), \dots, \mu_n(t))^T$ and the volatility $\sigma(t, S) = (\sigma_{ij}(t, S), 1 \leq i, j \leq n)$ are taken to be measurable, adapted and bounded processes. The dimension of the Brownian motion being equal to the number of risky assets ensures the completeness of the market.

The wealth of the investor at time t is then given by

$$W(t) = \sum_{i=0}^n H^i(t)S^i(t) - \int_0^t c(s)ds \quad (3.6.2)$$

where $H^i(t)S^i(t)$ represent the amount invested in asset i , $0 \leq i \leq n$, and $\int_0^t c(s)ds$ represents the total amount consumed up to time t . If the strategy H is self financing, then no money is added to the portfolio so that changes in the wealth derive only from changes in the asset prices, interest on the bond, and from consumption, so that

$$dW(t) = \sum_{i=0}^n H^i(t)dS^i(t) + (1 - H^i(t))dS^0(t) - c(t)dt. \quad (3.6.3)$$

Let us define the fraction of wealth φ_i invested in security i by $\varphi_i = \frac{H^i(t)S^i(t)}{W}$ such that $\sum_{i=0}^n \varphi_i = 1$.

Taking into account the riskless security and risky securities, the budget equation can be written as

$$dW(t) = \sum_{i=0}^n \varphi_t(t)(\mu_i - r)W(t)dt + (rW(t) - c(t))dt + \sum_{i=0}^n \varphi_t(t)\sigma_i W(t)dB_i(t). \quad (3.6.4)$$

In matrix notation

$$dW(t) = [W(t)(\varphi_t^T \lambda + r) - c(t)] dt + W(t)\varphi_t^T \sigma dB_t$$

where $\varphi_t = (\varphi_1(t), \dots, \varphi_n(t))$, $\lambda = (\mu_1 - r, \dots, \mu_n - r)^T$.

The primitives of the stochastic problem can be specified as follows:

- $\mathcal{A} = R_+ \times R^n$, with elements being (c, φ) representing consumption and the fraction of wealth invested in the risky assets,
- $\mathcal{Y} = R_+$ is the set of wealth W
- The set \mathcal{C} of \mathcal{A} valued adapted processes (c, φ) with values in R_+ called controls
- $g[(c, \varphi), W] = W(\varphi \cdot \lambda + r) - c$
- $h[(c, \varphi), W] = W\varphi^T \sigma$

with those specifications, the problem can be solved as follows:

- (1) The investor's indirect utility function for wealth is defined as

$$J(W, t) = \sup_{\varphi, c} E_t \left[\int_0^T u(c_t, t) dt + F(W_T) \right]$$

- (2) The HJB equation for Merton's problem is

$$\sup_{(c, \varphi) \in \mathcal{A}} \mathcal{D}^{(c, \varphi)} J(W, t) + u(c, t) = 0, \quad W > 0$$

where

$$\begin{aligned} \mathcal{D}^{(c, \varphi)} J(W, t) &= J_W(W, t) [W(\varphi^T \lambda + r) - c] + J_t(W, t) + \\ &\frac{W^2}{2} \varphi^T \sigma \sigma^T \varphi J_{WW}(W, t) \end{aligned}$$

with boundary conditions

$$J(W, T) = F(W), \quad W > 0$$

- (3) Applying the first order conditions to get

$$c^*(t) = u_c^{-1}(J_W, t), \quad \varphi^*(t) = -\frac{J_W}{J_{WW}W} (\sigma \sigma^T)^{-1} \lambda$$

$$\text{or,} \quad \varphi_t^*(t) = -\frac{J_W}{J_{WW}W} \sum_{j=1}^m \nu_{kj} (\mu_j - r); \quad k = 1, \dots, k-1,$$

where ν_{kj} is the common element of the k th row and j th column in the matrix $(\sigma \sigma^T)^{-1}$. The term $-\frac{J_W}{J_{WW}W}$ is called "Risk Tolerance".

- (4) Replacing the optimal controls in the HJB equation, yields a PDE in $J(W, t)$,

$$J_W(W, t) [W(\varphi^{*T} \lambda + r) - c] + J_t(W, t) + \frac{W^2}{2} \varphi^{*T} \sigma \sigma^T \varphi^* J_{WW}(W, t) + u(c^*, t) = 0 \quad (3.6.5)$$

with boundary condition $J(W, T) = F(W), \quad W > 0$

- (5) Solve the above PDE to solve for J , then obtain the closed form solutions for consumption and investment strategies.

Before solving the above partial differential equations, a utility function must be specified. However such a *PDE* raises formidable computational difficulties due to the non-linearity in W . Merton (1971) solved this *PDE* for the *HARA* class of utility functions;

$$u(c, t) = e^{-\rho t} \frac{1-\gamma}{\gamma} \left[\frac{ac}{1-\gamma} + b \right]^\gamma \quad (3.6.6)$$

And this PDE then has an explicit solution for one risky and one riskless asset, given by

$$J(W, t) = \frac{1-\gamma}{\gamma} e^{-\rho t} \left(\frac{(1-\gamma)[1 - e^{-(\rho-\gamma\nu)(T-t)/(1-\gamma)}]}{\rho - \gamma\nu} \right)^{1-\gamma} \left(\frac{W}{1-\gamma} + \frac{b}{ar} [1 - e^{-r(T-t)}] \right)^\gamma \quad (3.6.7)$$

where a and b are constants.

3.7. The Lagrange Multiplier Approach

As an alternative solution of the Bellman equations, Chow (1993) recommends the Lagrange multiplier method in optimal control problems involving stochastic differential equations. This approach has more computational advantages over dynamic programming, as it is an extension of the deterministic case. In the deterministic case, to solve the problem of a consumer maximizing a differentiable utility function of consumption subject to a budget constraint, one naturally applies the method of Lagrange multiplier to find the optimal demand function for the consumption good. In the stochastic case, this basic idea can be extended. The budget equation can be viewed as a constraint on wealth, which is a state variable, and both consumption and investment strategies which are control variables.

3.7.1. The Lagrange Method Applied to Merton's problem. Let us consider the consumer problem posed by Merton (1969) which consists of a model with one risky security whose price S_t follows a geometric Brownian motion and one riskless asset with constant return r . The dynamics of the asset are

$$\begin{aligned}\frac{dS_0}{S_0} &= r dt \\ \frac{dS_t}{S_t} &= \mu dt + \sigma dB_t\end{aligned}$$

Where dB_t is an increment of a Brownian motion and μ and σ are respectively the instantaneous constant return and the volatility of the risky security. The budget constraint is given by

$$dW_t = W_t[\varphi_t(\mu - r) + r]dt - c_t dt + W_t \varphi_t \sigma dB_t \quad (3.7.1)$$

where φ_t is the fraction of wealth invested in the risky security at time t and c_t is the consumption at time t . The problem is given by

$$\begin{aligned}\sup_{\varphi, c} E \left[\int_0^T u(c_t) dt \right] \\ \text{s.t.} \quad dW_t = W_t[\varphi_t(\mu - r) + r]dt - c_t dt + W_t \varphi_t \sigma dB_t\end{aligned} \quad (3.7.2)$$

The Lagrangean expression for this optimization problem is;

$$\mathcal{L} = \int_0^T E_t [u(c_t)dt - \lambda(t+dt) \{dW_t - W_t[\varphi_t(\mu - r) + r]dt + c_t dt - W_t \varphi_t \sigma dB_t\}] \quad (3.7.3)$$

where the conditional expectation E_t is used because when the control c_t and φ_t are determined, the information at time t including the value of W_t is given.

Setting the derivatives of \mathcal{L} with respect to c_t , φ_t and λ_t , to zero yields three equations to be solved for consumption, investment strategies and the Lagrange multiplier as a function of W_t . Note that $\lambda(t, W_t)$ is a stochastic process, so we have

$$d\lambda = \left[\frac{\partial \lambda}{\partial t} + \frac{\partial \lambda}{\partial W} [W_t[\varphi_t(\mu - r) + r] - c_t] + \frac{1}{2} \frac{\partial^2 \lambda}{\partial W^2} \varphi_t^2 \sigma^2 W_t^2 \right] dt + \frac{\partial \lambda}{\partial W} \varphi_t \sigma W_t dB_t \quad (3.7.4)$$

Chow (1997) shows the following steps involved in the further derivation. The first order conditions are

$$u'(c_t) - \lambda(t, W_t) = 0 \quad (3.7.5)$$

$$W_t \lambda(t, W_t) (\mu - r) + W_t^2 \frac{\partial \lambda(t, W_t)}{\partial W} \varphi_t \sigma^2 = 0 \quad (3.7.6)$$

Differentiating \mathcal{L} with respect to the state variable W_t , yields (after dropping the explicit dependence on W and t to lighten the presentation)

$$\varphi(\mu - r) + r] \lambda + [W(\varphi(\mu - r) + r + \varphi^2 \sigma^2) - c] \frac{\partial \lambda}{\partial W} + \frac{1}{2} \varphi^2 \sigma^2 W^2 \frac{\partial^2 \lambda}{\partial W^2} = 0 \quad (3.7.7)$$

After solving the last three equations, the optimal consumption and investment strategies can be obtained for unknowns c , φ and λ .

3.7.2. Comparison between Dynamic programming and the Lagrange multiplier method. The *HJB* equation for the Dynamic programming approaches, in the case of one risky asset and one riskless asset reduces to

$$\sup_{\varphi, c} u(c) + \frac{\partial J}{\partial t} + \frac{\partial J}{\partial W} [W(\varphi(\mu - r) + r) - c] + \frac{1}{2} \varphi^2 \sigma^2 W^2 \frac{\partial^2 J}{\partial W^2} = 0, \quad (3.7.8)$$

where J is the indirect utility. Differentiating the equation (3.7.8) with respect to wealth W we have

$$\varphi(\mu - r) + r] \frac{\partial \lambda}{\partial W} + [W(\varphi(\mu - r) + r + \varphi^2 \sigma^2) - c] \frac{\partial^2 \lambda}{\partial W^2} + \frac{1}{2} \varphi^2 \sigma^2 W^2 \frac{\partial^3 \lambda}{\partial W^3} = 0. \quad (3.7.9)$$

So by setting $J = \frac{\partial \lambda}{\partial W}$, equation (3.7.9) yields (3.7.7), that is, the dynamic programming equations imply those of the Lagrange multiplier.

3.8. The Martingale Approach

The main Idea

Harrison and Kreps (1979) introduced the martingale method to price contingent claims. This approach was applied by Karatas (1986) and (1987), Pliska (1986) and Cox and Hang (1989) to provide a closed form solution for the optimal portfolio when the underlying security prices follow a general diffusion process. The basic idea is to use the completeness and the arbitrage free property of the market to separate the computation of optimal consumption rules and that of a corresponding trading strategy. In the first step, the optimal consumption is obtained by solving the first-order conditions, essentially to state the Arrow-Debreu state price density process. In the second step, the corresponding portfolio strategy is derived by means of the martingale representation theorem while assuming the completeness of the market.

Methodology

Solving the portfolio problem either by dynamic programming or by the martingale approach yields in principle the same solution. However, the martingale approach requires additional technical assumptions about the consumption and trading strategies. Following Cox and Hang (1989) the following additional conditions must be satisfied by the consumption process c_t and the admissible trading strategy $(H^0(t), H(t))$ for each t in $[0, T]$.

- $\int_0^T |H^0(t)S^0(t)r(t) + H(t)^T S(t)\mu(t)|dt < \infty; \quad \mathbf{P}-a.s$
- $\int_0^T |H(t)^T S(t)\sigma(t)|^2 dt < \infty; \quad \mathbf{P}-a.s$
- $E \int_0^T |c_t^2| dt < \infty$
- $E \int_0^T |W_T^2| dt < \infty$

Let \mathcal{M} denote the set ordered pair of consumption and terminal wealth (c, W_T) satisfying the third condition. Given the assumptions in the continuous time setup, an equivalent martingale measure exists and is unique. Now consider an investor with

a utility function for consumption $u(x, t)$ and the function of the terminal wealth $F(x)$ and the initial wealth $W_0 > 0$.

We assume that $u(x, t)$ and $F(x)$ are strictly concave in x . The investor wants to solve the following optimization problem;

$$\sup_{\varphi, c} E \left[\int_0^T e^{-\rho t} u(c_t) dt + F(W_T) \right] \text{ Such that,}$$

$$dW_t = W_t[\varphi_t(\mu - r) + r]dt - c_t dt + W_t \varphi_t \sigma dB_t \quad (3.8.1)$$

Note that the objective function in the problem involves intertemporal consumption and terminal wealth only. The idea is to replace the constraints involving both intertemporal investment strategies and consumption by an equivalent constraint involving intertemporal consumption and terminal wealth only. Having done this, the problem can be solved using the Lagrange multiplier method.

The martingale method relies on the existence of a process for which the discounted security price process ϕ is martingale, i.e.

$$E[\phi_t S_t | F_s] = \phi_s S_s; \quad s \leq t.$$

The process ϕ is interpreted as a system of Arrow-Debreu prices. That is, the value of ϕ_t in each state gives the price per unit probability of a dollar in that state. The price of the asset is given by the sum of its payoffs in each state, multiplied by the price of dollar in the state, times the probability of the state occurring. The process ϕ is called state price density, and is equal to

$$\phi_t = \eta_t \zeta_t, \quad \zeta_t = e^{-rt},$$

where η_t is the Radon-Nikodym derivative defined in the previous chapter. For any random variable Z with $E_Q[Z] < \infty$, we have

$$E_Q[Z] = E[\eta Z];$$

Where Q is an equivalent martingale measure for the discounted security prices. Cox and Haung (1989) show that the budget constraints are equivalent to

$$E \left[\int_0^T \phi_t c_t dt + \phi_T W_T \right] = W_0. \quad (3.8.2)$$

This equivalent budget constraint states that the total cost of consumption, as given by the amount consumed in each state multiplied by the price of consumption in that state, must equal to investor's initial wealth. The control c^* solves the problem (3.8.2) if and only if there exist a scalar Lagrange multiplier $l > 0$ such that c^* solves the unconstrained problem

$$\sup_{(c, W_T) \in \mathcal{M}} E \left[\int_0^T [e^{-\rho t} u(c_t) - l \phi_t c_t] dt + F(W_T) - l \phi_T W_T \right],$$

with complementary condition

$$E \left[\int_0^T \phi_t c_t dt + \phi_T W_T \right] = W_0$$

The first order condition for the optimality c_t and W_T of are

$$\begin{aligned} e^{-\rho t} u_c(c_t, t) - l \phi_t &= 0 \\ F'(W_T) - l \phi_T &= 0 \end{aligned} \quad (3.8.3)$$

where u_c is the derivative of u with respect to c_t . The above equations are equivalent to

$$\begin{aligned} c_t^* &= I(e^{\rho t} l \phi_t, t) \\ W_T^* &= I_F(l \phi_T) \end{aligned} \quad (3.8.4)$$

where $I(\cdot, t)$ and $I_F(\cdot)$ are the inverse function of u_c and F respectively.

By solving for l in the complementary condition

$$E \left[\int_0^T \phi_t I(e^{\rho t} l \phi_t, t) dt + \phi_T I_F(l \phi_T) \right] = W_0 \quad (3.8.5)$$

and replacing the value of l in Eq.(3.8.3) an explicit solution to Merton's problem can be obtained.

Equation (3.8.1) gives the relationship between the initial wealth and future consumption. Cox and Hang (1989) show this relationship still holds and takes the form

$$W_t = \phi_t^{-1} E_t \left[\int_t^T \phi_s c_t ds + \phi_T W_T \right]$$

That is, the wealth at time t is the present discounted value of the future consumption, where state price deflator is the discount factor. When the optimal consumption rules are chosen, the wealth at time t is

$$W_t^* = \phi_t^{-1} E_t \left[\int_t^T \phi_s I(e^{\rho t} l \phi_t, t) ds + \phi_T I_F(l \phi_T) \right] \quad (3.8.6)$$

Now, in a complete market setting, any contingent payoff can be replicated by a dynamic trading strategy. In other words, at each time t , it is possible to take positions in the $n + 1$ assets so as to replicate the wealth W_t^* . This optimal trading strategy produces the wealth W_t^* that finances the optimal consumption c_t^* .

CHAPTER 4

THREE ASSETS MODEL

4.1. The financial market setup

We consider the continuous time financial market with an infinite horizon time interval $[0, \infty)$. We assume that there are one riskless asset with a constant interest rate r which satisfies,

$$\frac{dS_0}{S_0} = r dt, \quad (4.1.1)$$

one stock which is governed by an SDE

$$\frac{dS_t}{S_t} = \mu_t^s dt + \sigma_t^s dW_t^s, \quad (4.1.2)$$

and one forward contract which satisfies the SDE

$$\frac{dF_t}{F_t} = \mu_t^f dt + \sigma_t^f dW_t^f, \quad (4.1.3)$$

where μ_t^s, σ_t^s and μ_t^f, σ_t^f are the return and volatility of a stock S_t and a contract F_t . The financial market is driven by standard Brownian motions so we assume W_t^s, W_t^f are the standard Brownian motion on a Probability space (Ω, \mathcal{F}, P) endowed by an augmented filtration $\{\mathcal{F}_t\}_{t=0}^\infty$ generated by the Brownian motion.

DEFINITION 4.1. Market price of risk

Let the market price of risk due to stock be $\theta_t^s = \frac{\mu_t^s - r}{\sigma_t^s}$ and the market price of risk due to forward contract be $\theta_t^f = \frac{\mu_t^f - r}{\sigma_t^f}$. We define

$$\theta = \begin{bmatrix} \theta_t^s, \theta_t^f \end{bmatrix} = \begin{bmatrix} (\sigma_t^s)^{-1} [\mu_t^s - r], (\sigma_t^f)^{-1} [\mu_t^f - r] \end{bmatrix}$$

Hence θ is the market price of risk in the vector form.

DEFINITION 4.2. The exponential martingale

Let $Z_t \triangleq \exp[-\theta B'_t - \frac{1}{2}\theta\theta't]$

Here $B_t = \begin{bmatrix} W_t^s, W_t^f \end{bmatrix}$

So we have the exponential martingale as

$$Z_t \triangleq \exp\left(-\theta_t^s W_t^s - \theta_t^f W_t^f - \frac{1}{2}(\theta_t^s)^2 t - \frac{1}{2}(\theta_t^f)^2 t\right)$$

DEFINITION 4.3. State price density

We define the state price density as

$$H_t = \zeta_t Z_t = e^{\{-rt\}} Z_t$$

where, $\zeta_t = e^{\{-rt\}}$ is the discount process.

LEMMA 4.4. An arbitrage free financial model satisfies

$$E[\exp\left(-\theta B'_t - \frac{1}{2}\theta\theta't\right)] = 1.$$

PROOF. since Z_t is martingale then $E[Z_t] = E[Z_0] = E[1] = 1$, and hence the market model is arbitrage free. □

Let X_t be an investor's wealth process at time t . Let π_t^s and π_t^f be the proportion of the amount invested in stock and forward contract at time t and let C_t be the consumption rate process at time t . We assume that the process $\pi = (\pi_t^s, \pi_t^f)$ is \mathcal{F}_\square -measurable, adapted such that for all $t \geq 0$

$$\int_0^\infty (\pi_t^s)^2 + (\pi_t^f)^2 dt < \infty, a.s.$$

Furthermore the consumption rate process C_t is progressively measurable with respect to \mathcal{F}_t , $C_t > 0$ for all $t \geq 0$ and $\int_0^\infty C_s ds < \infty, a.s.$

4.1.1. Investor's wealth Dynamics. Let $X_0 = x > 0$ be the initial investment made by an investor at time t . An investor's wealth (percentage increment) at time t would be

$$\frac{dX_t}{X_t} = \int_0^t \pi_t^s \frac{dS_t}{S_t} + \int_0^t \pi_t^f \frac{dF_t}{F_t} + \int_0^t (1 - \pi_t^s - \pi_t^f) \frac{dS_0}{S_0} - \int_0^t \frac{dC_t}{X_t},$$

Applying equations (4.1.1), (4.1.2) and (4.1.3) we obtain the total wealth increment at time t as

$$dX_t = \pi_t^s (\mu_t^s dt + \sigma_t^s dW_t^s) + \pi_t^f (\mu_t^f dt + \sigma_t^f dW_t^f) + (X_t - \pi_t^s - \pi_t^f) r dt - C_t dt$$

or,

$$dX_t = [rX_t + \pi_t^s (\mu_t^s - r) + \pi_t^f (\mu_t^f - r) - C_t] dt + \pi_t^s \sigma_t^s dW_t^s + \pi_t^f \sigma_t^f dW_t^f \quad (4.1.4)$$

A consumption- portfolio (c, π) is called admissible if $X_t > 0$ for all $t \geq 0$. For a given $T > 0$ we define an equivalent martingale measure $\tilde{P}(A) = E[Z_0(T)\mathbf{1}_A]$ for all $A \in \mathcal{F}_t$. By Girsanov's theorem, we obtain the process,

$$\tilde{B}_t = B_t + \theta t; \quad 0 \leq t \leq T \quad (4.1.5)$$

i.e.

$$\begin{aligned} \tilde{B}_t &= [W_t^s, W_t^f] + [(\sigma_t^s)^{-1}(\mu_t^s - r), (\sigma_t^f)^{-1}(\mu_t^f - r)]t \\ &= [W_t^s + (\sigma_t^s)^{-1}(\mu_t^s - r)t, W_t^f + (\sigma_t^f)^{-1}(\mu_t^f - r)t] \end{aligned}$$

which is a standard Brownian motion under the new measure \tilde{P} . Then by using equation (4.1.5), the wealth process (4.1.4) can be reduced under the new probability measure \tilde{P} to

$$dX_t = [rX_t - C_t] dt + \pi_t^s \sigma_t^s d\tilde{W}_t^s + \pi_t^f \sigma_t^f d\tilde{W}_t^f, \quad (4.1.6)$$

where

$$[W_t^s, W_t^f] = [\tilde{W}_t^s - (\sigma_t^s)^{-1}(\mu_t^s - r)t, \tilde{W}_t^f - (\sigma_t^f)^{-1}(\mu_t^f - r)t]$$

LEMMA 4.5. \tilde{B}_t is a standard Brownian motion under new measure \tilde{P} .

PROOF. To show the process \tilde{B}_t under \tilde{P} is a standard Brownian process, it suffices to show that it has independent normally distributed increments with the correct variances. for this, it suffices to show that the joint MGF (Moment Generating Function) of the increments $\tilde{B}_t, \tilde{B}_{t_2} - \tilde{B}_{t_1}, \dots, \tilde{B}_{t_n} - \tilde{B}_{t_{n-1}}$ where $0 < t_1 < \dots < t_n$ is the same as that of n independent normally distributed random variables with mean 0 and variances $t_1, t_2 - t_1, \dots$

$$E_{\tilde{P}}[\exp\{\sum_{k=1}^n \alpha_k (\tilde{B}_{t_k} - \tilde{B}_{t_{k-1}})\}] = \prod_{k=1}^n \exp\{\alpha_k^2 (t_k - t_{k-1})\}$$

for $n=1$

$$\begin{aligned} E_{\tilde{P}}[\exp\{\alpha \tilde{B}_t\}] &= E_{\tilde{P}}[\exp\{\alpha B_t + \alpha \int_0^t \theta ds\}] \\ &= E_P[\exp\{\alpha B_t + \alpha \int_0^t \theta ds - \int_0^t \theta B_t - \frac{1}{2} \int_0^t \theta^2 ds\}] \\ &= E_P[\exp\{\int_0^t (\alpha - \theta) dB_t - \frac{1}{2} \int_0^t (\theta^2 - 2\alpha\theta) ds\}] \\ &= e^{\frac{\alpha^2 t}{2}} E_P[\exp\{\int_0^t (\alpha - \theta) dB_t - \frac{1}{2} \int_0^t (\alpha - \theta)^2 ds\}] \\ &= e^{\frac{\alpha^2 t}{2}} \end{aligned} \quad \square$$

Now we define a process $N_t = \int_0^t C_t H_t$; $0 \leq t < \infty$, a continuous local martingale under \tilde{P} , which is bounded below. By applying Fatou's lemma, N_t is a supermartingale under P i.e.

$$E[\int_0^t C_t H_t dt] \leq x \tag{4.1.7}$$

Equation (4.1.7) is called the budget constraint for the wealth process (4.1.6).

DEFINITION 4.6. **Utility function**

A function $u : [R, \infty) \rightarrow R$ is called a utility function if

- it is strictly increasing

- strictly concave
- continuously differentiable and satisfies

$$\lim_{c \downarrow R} u'(c) = u'(R^+) \text{ and } \lim_{c \uparrow R} u'(c) = 0$$

Note :

A function $u(x)$ is strictly concave if $\lim_{x \rightarrow 0} u'(x) = \infty$ and $\lim_{x \rightarrow \infty} u'(x) = 0$

4.2. Investor's optimization problem

Let the investor be endowed with initial wealth $x > 0$ in the beginning of the process. Let the portfolio and consumption rate process at time t be (π_t^s, π_t^f) and c_t . The investor's problem is to maximize her Von Neumann Morgenstern expected utility from consumption

$$J(x; c, \pi) \triangleq E\left[\int_0^\infty e^{-\beta t} u(c_t) dt\right]$$

subject to $c_t \geq R$, for all $t \geq 0$, for fixed R and $E\left[\int_0^\infty c_t H_t dt\right] \leq x$.

(4.2.1)

The first constraint is called the downside constraint and the second is the budget constraint. Here $\beta > 0$ is the subjective discount factor. The investor would require the consumption rate process c_t to be almost surely bounded below by a fixed level as the investor's minimum consumption requires.

Assumption-1

For the positive consumption rate process the investor's initial wealth has to be $x > \frac{R}{r}$.

Since, $X(t) = xe^{rt} - c_t$

i.e. $dX(t) = rxe^{rt} dt - c_t dt$

such that $dX(0) = rxdt - Rdt$; i.e. $\frac{dX(0)}{rx-R} = dt$

which yields

$$rx - R = e^{t+const} > 0.$$

Now we define the value function to our problem

$$V(x) = \sup_{(c,\pi) \in \mathcal{A}} J(x; c, \pi)$$

where \mathcal{A} is the set of all admissible pairs (c, π) such that $E[\int_0^\infty e^{-\beta t} u^-(c_t) dt] < \infty$

where $u^- \triangleq \max(-u, 0)$. For a Lagrange multiplier $\lambda > 0$, we define a dual value function

$$\tilde{V}(\lambda) \triangleq \sup_{(c,\pi) \in \mathcal{A}} \left[J(x; c, \pi) - \lambda E \left[\int_0^\infty c_t H_t dt \right] \right] \quad (4.2.2)$$

where the dual utility is defined from the Legendre transform of the concave function

$$\tilde{u}(y) = \sup_{c \geq R} (u(c) - cy).$$

Such that,

$$\begin{aligned} \tilde{V}(\lambda) &\triangleq \sup_{(c,\pi) \in \mathcal{A}} \left[E \left[\int_0^\infty e^{-\beta t} u(c_t) dt \right] - \lambda E \left[\int_0^\infty c_t H_t dt \right] \right] \\ &= E \left[\sup_{(c,\pi) \in \mathcal{A}} \left(\int_0^\infty (e^{-\beta t} u(c_t) - \lambda c_t H_t) dt \right) \right] \\ &= E \left[\sup_{(c,\pi) \in \mathcal{A}} \left(\int_0^\infty e^{-\beta t} (u(c_t) - \lambda e^{\beta t} c_t H_t) dt \right) \right] \\ &= E \left[\int_0^\infty e^{-\beta t} \sup_{(c,\pi) \in \mathcal{A}} (u(c_t) - \lambda e^{\beta t} c_t H_t) dt \right] \\ &= E \left[\int_0^\infty e^{-\beta t} \tilde{u}(\lambda e^{\beta t} H_t) dt \right] \end{aligned}$$

PROPOSITION 4.7. *For any $x > \frac{R}{r}$, an investor's optimal consumption process $c(t) = I(\lambda(x)H_t)$, where $\lambda(x)$ is chosen to satisfy $E \left[\int_0^\infty c_t H_t dt \right] = x$.*

PROOF. Let λ_c and λ_x denotes the Lagrange multipliers associated with the down-side constraint and budget constraints. Applying the first order conditions to the optimization problem we obtain,

$$u'(c_t) = \lambda_x H_t - \lambda_c \quad [*]$$

from the complementary slackness conditions

$\lambda_c(c_t - R) = 0, \lambda_c \geq 0$ and $c_t \geq 0$, we obtain

$$\lambda_c = [\lambda_x H_t - u'(R)]^+ \quad [**]$$

substituting $[**]$ in $[*]$, we obtain $c_t = (u')^{-1}(\lambda_x H_t)$. Case $x = \frac{R}{r}$ is trivial ($c_t = R$), thus for the rest of the paper we assume $x > \frac{R}{r}$, where $I(\cdot)$ is the inverse function of $u'(\cdot)$. \square

Hence we can write, $\tilde{u}(y) = [u(I(y)) - yI(y)]\chi_{\{0 < y \leq \tilde{y}\}} + [u(R) - Ry]\chi_{\{y \geq \tilde{y}\}}$

REMARK 3. *We can determine the optimal consumption for deriving the dual utility function $\tilde{u}(\cdot)$. This optimal consumption is*

$$c_t^* = \begin{cases} I(y_t); & \text{if } 0 < y_t \leq \tilde{y} \\ R; & \text{if } y_t \geq \tilde{y} \end{cases}$$

Applying the first order conditions to the dual utility function $\tilde{u}(\cdot)$ we obtain,

$$u'(c_t) - y_t = 0, \text{ if } 0 < y_t \leq \tilde{y}$$

i.e.

$$c_t = u^{-1}(y_t) = I(y_t)$$

furthermore for $y_t \geq \tilde{y}$, the optimal consumption rate is fixed i.e. $c_t = R$.

Hence the dual value function $\tilde{V}(\lambda)$ is given by

$$\begin{aligned} \tilde{V}(\lambda) &= E \left[\int_0^\infty e^{-\beta t} \tilde{u}(y_t^\lambda) dt \right] \\ &= E \left[\int_0^\infty e^{-\beta t} \{ [u(I(y_t^\lambda)) - y_t^\lambda I(y_t^\lambda)] \chi_{\{0 < y_t^\lambda \leq \tilde{y}\}} + [u(R) - Ry_t^\lambda] \chi_{\{y_t^\lambda \geq \tilde{y}\}} \} dt \right] \end{aligned} \quad (4.2.3)$$

where $y_t^\lambda = \lambda e^{\beta t} H_t$

We denote $y_t^\lambda = y_t = \lambda \exp\{(\beta - r - \frac{1}{2}\theta\theta')t - \theta B_t\}$

$$= \lambda \exp\left\{(\beta - r - \frac{1}{2}(\theta_t^s)^2 - \frac{1}{2}(\theta_t^f)^2)t - \theta_t^s W_t^s - \theta_t^f W_t^f\right\}$$

Here, $y_0^\lambda = \lambda$ is the initial value of y_t^λ . Applying Itô's formula, we obtain the *SDE*

$$dy_t = y_t\{(\beta - r)dt - \theta_t^s dW_t^s - \theta_t^f dW_t^f\}. \quad (4.2.4)$$

Now we consider the following problem:

$$\phi(t, y) = E^{y_t=y} \left[\int_0^\infty e^{-\beta t} \{ [u(I(y_t)) - y_t I(y_t)] \chi_{\{0 < y_t \leq \tilde{y}\}} + [u(R) - Ry_t] \chi_{\{y_t \geq \tilde{y}\}} \} dt \right] \quad (4.2.5)$$

The following remark will reduce the above *SDE* to a *PDE*. We apply the Feynman-Kac formula for the above *SDE*.

REMARK 4. *We apply the Feynman theorem (defined in ch.(2) for the function*

$\phi(t, y) = E^y[\int_0^t e^{-\beta s} u(y_s) ds]$ *as follows.*

$$\phi(t, y) = E^y[\int_0^\infty e^{-\beta s} u(y_s) ds] = E^y[e^{-\beta t} \int_t^\infty e^{\beta(t-s)} u(y_s) ds]$$

$$\text{Let } F(y_t) = \int_t^\infty e^{\beta(t-s)} u(y_s) ds,$$

so,

$$\phi(t, y) = E^y[e^{-\beta t} F(y_t)]$$

Applying the Feynman-Kac theorem $\phi_t = \mathcal{L}\phi - \beta\phi$

$$\begin{aligned} \text{Here, } \phi_t &= \frac{\partial}{\partial t} E^y[e^{-\beta t} F(y_t)] \\ &= E^y\left[\frac{\partial}{\partial t} e^{-\beta t} F(y_t)\right] \\ &= E^y[-\beta e^{-\beta t} F(y_t)] - e^{-\beta t} u(y_{t=s=0}) \\ &= -\beta\phi - e^{-\beta t} u(y) \end{aligned}$$

$$\text{i.e. } \mathcal{L}\phi + e^{-\beta t} u(y) = 0$$

Hence by using the Feynman-Kac formula we obtain the equivalent PDE from (4.2.5) as given by

$$\mathcal{L}\phi(t, y) + e^{-\beta t} \{u(I(y)) - yI(y)\} = 0; \text{ if } 0 < y \leq \tilde{y}$$

and

$$\mathcal{L}\phi(t, y) + e^{-\beta t}\{u(R) - Ry\} = 0; \text{ if } y \geq \tilde{y}, \quad (4.2.6)$$

where the partial differential operator

$$\mathcal{L} \triangleq \frac{\partial}{\partial t} + \mu \frac{\partial}{\partial y} + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial y^2}, \text{ such that } \mu = (\beta - r)y \text{ and } \sigma = -\theta y,$$

and hence,

$$\mathcal{L} \triangleq \frac{\partial}{\partial t} + (\beta - r)y \frac{\partial}{\partial y} + \frac{1}{2}\theta^2 y^2 \frac{\partial^2}{\partial y^2}.$$

To solve 4.2.6, at least for some solution, we consider the homogeneous equation first, i.e. $\mathcal{L}\phi(t, y) = 0$. We look for a bounded solution only.

(1) Assuming $\phi(t, y) = T(t)Y(y)$ solves $\mathcal{L}\phi(t, y) = 0$, i.e.

$$T'Y + (\beta - r)TY + \frac{1}{2}\theta^2 T y^2 Y'' = 0$$

$$\text{or, } -\frac{T'}{T} = \frac{(\beta - r)yY' + \frac{1}{2}\theta^2 y^2 Y''}{Y} = \gamma.$$

We solve first proportion as $T(t) = Ee^{-\gamma t}$. Since the solution must be bounded, we have $\gamma \geq 0$.

The second proportion can be written as

$$(\beta - r)yY' + \frac{1}{2}\theta^2 y^2 Y'' - \gamma Y = 0. \quad (4.2.7)$$

Let $Y = v(y) = y^n$ be a solution. Since $v'(y) = \frac{nv(y)}{y}$ and $v''(y) = \frac{n(n-1)v(y)}{y^2}$,

we obtain a quadratic equation of the form

$$\left\{\frac{1}{2}n(n-1)\theta^2 + (\beta - r)n - \gamma\right\}v(y) = 0$$

i.e.

$$\frac{1}{2}n^2\theta^2 + (\beta - r - \frac{1}{2}\theta^2)n - \gamma = 0.$$

Solving a quadratic equation for n , we obtain

$$n_1 = \frac{-(\beta - r - \frac{1}{2}\theta^2) + \sqrt{(\beta - r - \frac{1}{2}\theta^2)^2 + 2\theta^2\gamma}}{\theta^2},$$

$$n_2 = \frac{-(\beta - r - \frac{1}{2}\theta^2) - \sqrt{(\beta - r - \frac{1}{2}\theta^2)^2 + 2\theta^2\gamma}}{\theta^2},$$

Here, the roots are $n_1 > 1, n_2 < 0$, and the solution for (4.2.7) is,

$$v_h(y) = A_1 y^{n_1} + A_2 y^{n_2}, \text{ for some constants } A_1 \text{ and } A_2.$$

Applying the growth condition of $v(y)$ on $0 < y \leq \tilde{y}$, we must have $A_2 = 0$. Hence the homogeneous solution is, $\phi_\gamma(t, y) = E e^{-\gamma t} y^{n_1}, \gamma \geq 0$.

The homogeneous solution for (4.2.7) is, $v_h(y) = B_1 y^{n_1} + B_2 y^{n_2}$, on $y \geq \tilde{y}$. Applying the decay condition of $v(y)$ on $y \geq \tilde{y}$, we must have $B_1 = 0$. Hence the homogeneous solution is, $\phi_\nu(t, y) = D e^{-\nu t} y^{n_2}, \nu \geq 0$.

Secondly, we consider the non-homogeneous equation of (4.2.6).

(2) For $0 < y \leq \tilde{y}$, we apply the method of variation of parameter.

Let $\phi_p(t, y) = e^{-\gamma t} \{A_1(y) y^{n_1} + A_2(y) y^{n_2}\}$ be a particular solution of (4.2.6), which satisfies $\mathcal{L}\phi(t, y) + e^{-\beta t} \{uI(y) - yI(y)\} = 0$.

Here, $\frac{\partial \phi}{\partial t} = -\gamma e^{-\gamma t} (A_1 y^{n_1} + A_2 y^{n_2}), \frac{\partial \phi}{\partial y} = e^{-\gamma t} (n_1 A_1 y^{n_1-1} + n_2 A_2 y^{n_2-1})$, such that $A_1' y^{n_1} + A_2' y^{n_2} = 0$, and

$$\frac{\partial^2 \phi}{\partial y^2} = e^{-\gamma t} (A_1' n_1 y^{n_1-1} + A_1 n_1 (n_1 - 1) y^{n_1-2} + A_2' n_2 y^{n_2-1} + A_2 n_2 (n_2 - 1) y^{n_2-2})$$

Now the non-homogeneous equation

$$\mathcal{L}\phi(t, y) + e^{-\beta t} \{u(I(y)) - yI(y)\} = 0, \text{ we obtain,}$$

$$\begin{aligned} & e^{-\gamma t} [-\gamma A_1 y^{n_1} + (\beta - r) \{A_1' y^{n_1+1} + A_1 n_1 y^{n_1} + A_2' y^{n_2+1} + A_2 n_2 y^{n_2}\} + \\ & \frac{1}{2} \theta^2 \{A_1'' y^{n_1+2} \\ & + 2n_1 A_1' y^{n_1+1} + A_1 n_1 (n_1 - 1) y^{n_1} + A_2'' y^{n_2+2} + A_2 n_2 (n_2 - 1) y^{n_2} + \\ & 2n_2 A_2' y^{n_2+1}\}] + e^{-\beta t} \{u(I(y)) - yI(y)\} = 0. \end{aligned}$$

Since $\frac{1}{2} n^2 \theta^2 + (\beta - r - \frac{1}{2} \theta^2) n - \gamma = 0$., we can reduce it to,

$$\frac{1}{2} \theta^2 \{A_1' n_1 y^{n_1+1} + A_2' n_2 y^{n_2+1}\} = e^{-\beta t} \{yI(y) - uI(y)\}. \quad (4.2.8)$$

We find A_1 and A_2 by solving (4.2.8) and $A_1' y^{n_1} + A_2' y^{n_2} = 0$ as

$$A_1 = \frac{2e^{-(\beta-\gamma)t}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^y \frac{yI(y)-u(I(y))}{y^{n_1+1}} dy, \quad A_2 = -\frac{2e^{-(\beta-\gamma)t}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^y \frac{yI(y)-u(I(y))}{y^{n_2+1}} dy.$$

Hence,

$$\phi_p(t, y) = e^{-\beta t} \left\{ \frac{2y^{n_1}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^y \frac{yI(y)-u(I(y))}{y^{n_1+1}} dy - \frac{2y^{n_2}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^y \frac{yI(y)-u(I(y))}{y^{n_2+1}} dy \right\}.$$

(3) For $y \geq \tilde{y}$.

Let $\phi_p(t, y) = e^{-\gamma t} \{B_1(y)y^{n_1} + B_2(y)y^{n_2}\}$ be a solution of,
 $\mathcal{L}\phi(t, y) + e^{-\beta t} \{u(R) - yR\} = 0$.

We apply the method of variation of parameter as above and we obtained

B_1 and B_2 as

$$\begin{aligned} B_1 &= \frac{2e^{-(\beta-\gamma)t}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^{\infty} \frac{yR-u(R)}{y^{n_1+1}} dy, \quad B_2 = -\frac{2e^{-(\beta-\gamma)t}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^{\infty} \frac{yR-u(R)}{y^{n_2+1}} dy, \text{ and} \\ \phi_p(t, y) &= e^{-\beta t} \left\{ \frac{2y^{n_1}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^{\infty} \frac{yR-u(R)}{y^{n_1+1}} dy - \frac{2y^{n_2}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^{\infty} \frac{yR-u(R)}{y^{n_2+1}} dy \right\}. \\ &= e^{-\beta t} \left\{ \frac{2}{\theta^2(n_1-n_2)} \left[\frac{yR}{-n_1+1} + \frac{u(R)}{n_1} \right] - \frac{2}{\theta^2(n_1-n_2)} \left[\frac{yR}{-n_2+1} + \frac{u(R)}{n_2} \right] \right\} \\ &= \frac{e^{-\beta t}}{\theta^2(n_1-n_2)} \left\{ u(R) \left[\frac{1}{n_1} - \frac{1}{n_2} \right] + yR \left[\frac{1}{-n_1+1} - \frac{1}{-n_2+1} \right] \right\} \end{aligned}$$

Here, $n_1 n_2 = \frac{-2\gamma}{\theta^2}$ and $(-n_1 + 1)(-n_2 + 1) = -(\gamma + r - \beta)$.

Hence, $\phi_p(t, y) = e^{-\beta t} \left\{ \frac{u(R)}{\gamma} - \frac{Ry}{\gamma+r-\beta} \right\}$, and a complete solution for $v(y)$ in the case $y \geq \tilde{y}$ is given by

$$v(y) = B_2 y^{n_2} - \frac{R}{\gamma+r-\beta} y + \frac{u(R)}{\gamma}. \quad (4.2.9)$$

Finally, when we apply the smooth condition and continuous property at $y = \tilde{y}$, we get

$$A_1 \tilde{y}^{n_1} = B_2 \tilde{y}^{n_2} - \frac{R}{\gamma+r-\beta} \tilde{y} + \frac{u(R)}{\gamma}, \quad (4.2.10)$$

and

$$n_1 A_1 \tilde{y}^{n_1-1} = n_2 B_2 \tilde{y}^{n_2-1} - \frac{R}{\gamma+r-\beta}. \quad (4.2.11)$$

Solving above equations we obtain,

$$A_1 = \frac{\frac{R}{\gamma+r-\beta}(n_2-1) - \frac{n_2 u(R)}{\gamma \tilde{y}}}{(n_1-n_2) \tilde{y}^{n_1-1}} \text{ and } B_2 = \frac{\frac{R}{\gamma+r-\beta}(n_1-1) - \frac{n_1 u(R)}{\gamma \tilde{y}}}{(n_1-n_2) \tilde{y}^{n_2-1}}.$$

As a conclusion we have the following theorem, which provides a solution to the PDEs (4.2.6).

THEOREM 4.2.1. A solution to the PDE is of the form $\phi_1(t, y) = e^{-\beta t}v(y)$, where the function $v(y)$ is given by

$$\begin{aligned} v(y) &= A_1 y^{n_1} + \frac{2y^{n_1}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^y \frac{yI(y)-u(I(y))}{y^{n_1+1}} dy - \frac{2y^{n_2}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^y \frac{yI(y)-u(I(y))}{y^{n_2+1}} dy, \text{ if } 0 < y \leq \tilde{y} \\ &= B_2 y^{n_2} - \frac{R}{r} y + \frac{u(R)}{\beta} \quad \text{if } y \geq \tilde{y}, \end{aligned} \tag{4.2.12}$$

where

$$\begin{aligned} A_1 &= \frac{\frac{R}{r}(n_2-1) - \frac{n_2 u(R)}{\beta \tilde{y}}}{(n_1-n_2) \tilde{y}^{\tilde{n}_1-1}} \\ \text{and } B_2 &= \frac{\frac{R}{r}(n_1-1) - \frac{n_1 u(R)}{\beta \tilde{y}}}{(n_1-n_2) \tilde{y}^{\tilde{n}_2-1}}. \end{aligned}$$

REMARK 5. (Existence of dual value function $\tilde{V}(\lambda)$)

We can derive $\tilde{V}(\lambda)$ by using the above theorem and equation (4.2.12) from $\phi(t, y)$ at $t = 0$ and $y = \lambda$. So we can write $\tilde{V}(\lambda) = v(\lambda)$.

We now move to derive the value function of wealth from the dual value function $\tilde{V}(\lambda) = v(\lambda)$ via the following proposition.

PROPOSITION 4.8. If $\tilde{V}(\lambda)$ exists and is differentiable for $\lambda > 0$, then the value function can be derived as

$$V(x) = \inf_{\lambda > 0} (\tilde{V}(\lambda) + \lambda x)$$

We define the Legendre-Fenchel (LF) transform before the proof of the above proposition.

DEFINITION 4.9. Let $f : R^n \rightarrow R$ be a convex function, i.e. $\nabla^2 f > 0$, the Legendre transform of $f(x)$ is defined by the formula

$$L(f)(p) = \max_x \{px - f(x)\}.$$

We may denote the LF of f by f^* . The LF transform of $f^*(p)$ is

$$f^{**}(x) = \max_x \{px - f^*(p)\}$$

Property 1- Let $g(p) = L(f)(p)$ be the Legendre transform of $f(x)$ then $\nabla g(p) = x(p)$, where $x(p)$ is the solution to $p = (\nabla f)(x)$.

By the definition of LF transform, $g(p) = p \cdot x(p) - f(x(p))$.

Thus, we have,

$$\nabla g(p) = p \cdot \nabla(x)(p) + x(p) - (\nabla f)(x(p)) \cdot (\nabla x)(p) = p \cdot \nabla(x)(p) + x(p) - p \cdot \nabla(x)(p) = x(p).$$

Property 2- The Legendre transform is Involutive i.e. $f^{**}(x) = f(x)$.

Since, $g(p) = p \cdot x(p) - f(x(p))$

so, $L(g)(x) = x \cdot p(x) - g(p(x))$

$$= (\nabla g)(p(x)) \cdot p(x) - [p(x) \cdot \nabla g(p(x)) - f(\nabla g(p(x)))] = f(x).$$

We use definitions and these properties to prove the above proposition.

PROOF. $\tilde{V}(\lambda)$ exists and is differentiable by the definition of the Legendre transform of $V(x)$, defined as, $\tilde{V}(\lambda) = \sup_{x \in R} \{\lambda x - V(x)\}$. Taking the LF transform of \tilde{V} , we get $L(\tilde{V})(x) = L(L(V(x))) = \sup_{\lambda} \{\lambda x - \tilde{V}(\lambda)\}$. Since, $L(L(V(x))) = V(x)$ from the involution property of the LF transform, we deduce $V(x) = \sup_{\lambda=-q} \{-qx - \tilde{V}(\lambda = -q)\}$

$$= - \sup_{\lambda=-q} \{qx + \tilde{V}(q)\}$$

$$= \inf_{\lambda} \{\lambda x + \tilde{V}(\lambda)\} \tag{4.2.13}$$

□

EXAMPLE 4.10. Let $f(x) = e^{x-1}$. Let the Legendre transform of $f(x)$ be $g(y)$.

We obtain $g(y)$ using the above theorem as

$$g(y) = \max\{xy - e^{x-1}\}$$

Using first order condition for the optimal solution we obtain,

$$y = e^{x-1}$$

or,

$$x = 1 + \ln y$$

so,

$$g(y) = y(1 + \ln y) - y = y \ln y$$

Plots of the functions $f(x)$ and $g(y)$ are given in Fig(4.1) and Fig(4.2).

EXAMPLE 4.11. The Legendre transform of a two variable function $f(x, y) = 2x^2 + y^2$.

Let $z = f(x, y) = 2x^2 + y^2$

applying the definition of Legendre transform,

$$\begin{aligned} g(w) &= \max(w.z - f(z)) \\ &= \nabla_z(ux + vy - 2x^2 - y^2) \\ &= (u - 4x, v - 2y) \quad \text{which yields } u = 4x \text{ and } v = 2y. \end{aligned}$$

Hence, $g(u, v) = \frac{u^2}{8} + \frac{v^2}{4}$ where

graphs of $f(x, y)$ and $g(x, y)$ are given by Fig (4.3) and Fig (4.4).

Using Proposition (4.8) and applying the first order condition to the equation (4.2.13) we can deduce the wealth boundary as

$$\tilde{V}'(\lambda) + x = 0 \text{ or } x = -\tilde{V}'(\lambda)$$

FIGURE 4.1. The Graph of $f(x) = e^{x-1}$

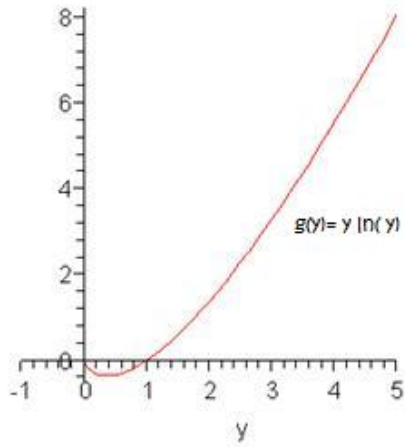
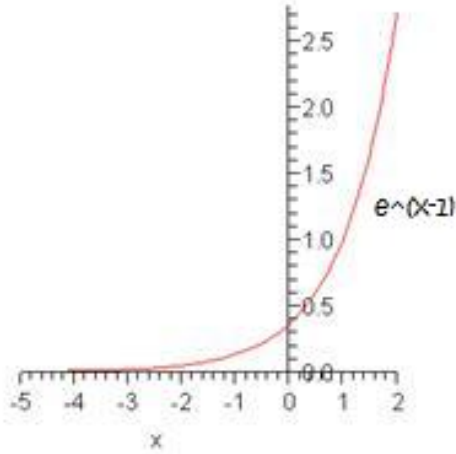


FIGURE 4.2. The Graph of $g(y) = y \ln y$

Applying Prop.(4.8) at $\lambda = \tilde{y}$, we obtain $\tilde{x} = -\tilde{V}'(\tilde{y})$, *i.e.*

$$\tilde{x} = -n_1 A_1 \tilde{y}^{n_1-1} = -n_2 B_2 \tilde{y}^{n_2-1} + \frac{R}{r} \quad (4.2.14)$$

FIGURE 4.3. The Graph of $f(x, y)$

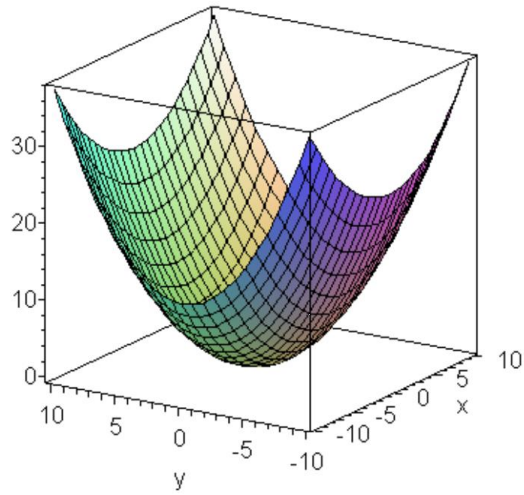
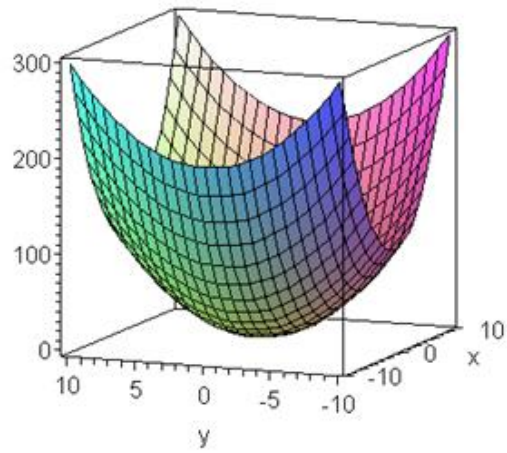


FIGURE 4.4. The Graph of $g(x, y)$

Now we can express the value function using above results in the following theorem.

THEOREM 4.12. *The value function is given by*

$$\begin{aligned}
V(x) &= B_2 \left(\frac{R-x}{B_2 n_2} \right)^{\frac{n_2}{n_2-1}} + \left(x - \frac{R}{r} \right) \left(\frac{R-x}{B_2 n_2} \right)^{\frac{1}{n_2-1}} + \frac{u(R)}{\beta}; \text{ if } R/r < x \leq \tilde{x} \\
&= A_1 (\lambda^*)^{n_1} + \frac{2(\lambda^*)^{n_1}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^{\lambda^*} \frac{yI(y)-u(I(y))}{y^{n_1+1}} dy - \frac{2(\lambda^*)^{n_2}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^{\lambda^*} \frac{yI(y)-u(I(y))}{y^{n_2+1}} dy + (\lambda^*)x; \\
&\hspace{15em} \text{if } x \geq \tilde{x}
\end{aligned} \tag{4.2.15}$$

where λ^* can be determined using the following equation

$$-n_1 A_1 (\lambda^*)^{n_1-1} - \frac{2n_1(\lambda^*)^{n_1-1}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^{\lambda^*} \frac{yI(y)-u(I(y))}{y^{n_1+1}} dy + \frac{2n_2(\lambda^*)^{n_2-1}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^{\lambda^*} \frac{yI(y)-u(I(y))}{y^{n_2+1}} dy = x \tag{4.2.16}$$

PROOF. From equation (4.2.14) we have $\tilde{x} = -n_2 B_2 \tilde{y}^{n_2-1} + \frac{R}{r}$, if $y \geq \tilde{y}$

solving for any y , we obtain $y = \left(\frac{R-x}{B_2 n_2} \right)^{\frac{1}{n_2-1}}$

Now the value function $V(x) = \inf_y \{yx + \tilde{V}(y)\} = \lambda x + \tilde{V}(\lambda)$ for some $\lambda = y$

hence, $V(x) = B_2 \lambda^{n_2} - \frac{R\lambda}{r} + \lambda x$.

Substituting $\lambda = y$ we obtain,

$$V(x) = B_2 \left(\frac{R-x}{B_2 n_2} \right)^{\frac{n_2}{n_2-1}} + \left(x - \frac{R}{r} \right) \left(\frac{R-x}{B_2 n_2} \right)^{\frac{1}{n_2-1}} + \frac{u(R)}{\beta}, \text{ if } R/r < x \leq \tilde{x}.$$

Furthermore, the value function $V(x) = \lambda^* x + \tilde{V}(\lambda^*)$ for some $\lambda^* = y$ when

$0 < y \leq \tilde{y}$, i.e.

$$\begin{aligned}
V(x) &= A_1 (\lambda^*)^{n_1} + \frac{2(\lambda^*)^{n_1}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^{\lambda^*} \frac{yI(y)-u(I(y))}{y^{n_1+1}} dy - \frac{2(\lambda^*)^{n_2}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^{\lambda^*} \frac{yI(y)-u(I(y))}{y^{n_2+1}} dy + \\
&(\lambda^*)x; \text{ if } x \geq \tilde{x}
\end{aligned}$$

where λ^* satisfies the equation $x = -\tilde{V}'(\lambda^*)$, i.e.

$$-n_1 A_1 (\lambda^*)^{n_1-1} - \frac{2n_1(\lambda^*)^{n_1-1}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^{\lambda^*} \frac{yI(y)-u(I(y))}{y^{n_1+1}} dy + \frac{2n_2(\lambda^*)^{n_2-1}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^{\lambda^*} \frac{yI(y)-u(I(y))}{y^{n_2+1}} dy = x.$$

□

REMARK 6. *It is easily seen that there is one-to-one correspondence between $\lambda^* \in (0, \tilde{y})$ and $x \in (\tilde{x}, \infty)$ in equation (4.2.14) because of the fact that as $\lambda^* \rightarrow 0, x \rightarrow \infty$ and as $\lambda^* = \tilde{y}, x = \tilde{x}$.*

REMARK 7. *For $R/r < x \leq \tilde{x}$, we also define an algebraic equation with respect to λ like equation (4.2.14). Let $\lambda^{**} \in [\tilde{y}, \infty)$ be any y satisfying*

$-n_2 B_2 y^{n_2-1} + \frac{R}{r} = x$, which yields

$$-n_2 B_2 (\lambda^{**})^{n_2-1} + \frac{R}{r} = x. \quad (4.2.17)$$

4.3. Optimal policies

Let $y_t^{\lambda^*}$ and $y_t^{\lambda^{**}}$ be the solutions to the SDE $dy_t = y_t\{(\beta - r)dt - \theta_t^s dW_t^s - \theta_t^f dW_t^f\}$, with initial values $y_0 = \lambda^*$ and $y_0 = \lambda^{**}$ respectively. In order to find the optimal policies we consider optimal wealth processes which are obtained by substituting $y_t^{\lambda^*}$ for λ^* and $y_t^{\lambda^{**}}$ for λ^{**} in equations (4.2.16) and (4.2.17).

Thus the wealth process is given by

$$X_t^* = -n_1 A_1 (y_t^{\lambda^*})^{n_1-1} - \frac{2n_1 (y_t^{\lambda^*})^{n_1-1}}{\theta^2 (n_1 - n_2)} \int_{\tilde{y}}^{\lambda^*} \frac{yI(z) - u(I(y))}{y^{n_1+1}} dy + \frac{2n_2 (y_t^{\lambda^*})^{n_2-1}}{\theta^2 (n_1 - n_2)} \int_{\tilde{y}}^{\lambda^*} \frac{yI(y) - u(I(y))}{y^{n_2+1}} dy \quad (4.3.1)$$

and,

$$X_t^{**} = -n_2 B_2 (y_t^{\lambda^{**}})^{n_2-1} + \frac{R}{r} \quad (4.3.2)$$

where, $\theta^2 = (\theta_t^s)^2 + (\theta_t^f)^2$

THEOREM 4.13. *The optimal policies are provided by (c^*, π^*) , where $\pi^* = \{\pi_t^s, \pi_t^f\}$, such that*

$$\begin{aligned}
c_t^* &= R; & \text{if } R/r < X_t \leq \tilde{x} \\
&= I(y_t^{\lambda^*}); & \text{if } X_t \geq \tilde{x}
\end{aligned}$$

with

$$\{\pi_t^{s^{**}}, \pi_t^{f^{**}}\} = \left\{ \frac{\theta_t^s}{\sigma_t^s} (n_2 - 1) \left(\frac{R}{r} - X_t \right), \frac{\theta_t^f}{\sigma_t^f} (n_2 - 1) \left(\frac{R}{r} - X_t \right) \right\} \text{ if } R/r < X_t \leq \tilde{x} \tag{4.3.3}$$

and

$$\{\pi_t^{s^*}, \pi_t^{f^*}\} = \left\{ \begin{aligned} & \frac{\theta_t^s}{\sigma_t^s} [n_1(n_1 - 1)A_1(y_t^{\lambda^*})^{n_1-1} + \frac{2}{\theta^2} \frac{y_t^{\lambda^*} I(y_t^{\lambda^*}) - u(I(y_t^{\lambda^*}))}{y_t^{\lambda^*}} + \\ & \frac{2n_1(n_1-1)(y_t^{\lambda^*})^{n_1-1}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y)-u(I(y))}{y^{n_1+1}} dy - \frac{2n_2(n_2-1)(y_t^{\lambda^*})^{n_2-1}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y)-u(I(y))}{y^{n_2+1}} dy], \\ & \frac{\theta_t^f}{\sigma_t^f} [n_1(n_1 - 1)A_1(y_t^{\lambda^*})^{n_1-1} + \frac{2}{\theta^2} \frac{y_t^{\lambda^*} I(y_t^{\lambda^*}) - u(I(y_t^{\lambda^*}))}{y_t^{\lambda^*}} + \\ & \frac{2n_1(n_1-1)(y_t^{\lambda^*})^{n_1-1}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y)-u(I(y))}{y^{n_1+1}} dy - \frac{2n_2(n_2-1)(y_t^{\lambda^*})^{n_2-1}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y)-u(I(y))}{y^{n_2+1}} dy] \end{aligned} \right\} \text{ if } X_t \geq \tilde{x} \tag{4.3.4}$$

PROOF. Since there is a one-to-one correspondence between $\lambda^* \in (0, \tilde{y})$ and $X_t \in (\tilde{x}, \infty)$ as well as between $\lambda^{**} \in [\tilde{y}, \infty)$ and $R/r < X_t \leq \tilde{x}$. So the optimal consumption process is previously determined from the remarks (4.14) and (4.15). Now we need to show that the optimal consumption and portfolio processes generate the optimal wealth processes defined in the above equations (4.3.1) and (4.3.2) and compare these with the wealth process we set up for the financial market and determine an optimal portfolio.

For $R/r < X_t \leq \tilde{x}$, we have the wealth process defined in (4.3.2) as

$$X_t^{**} = -n_2 B_2 (y_t^{\lambda^{**}})^{n_2-1} + \frac{R}{r};$$

Applying Itô's formula to this we obtain

$$dX_t^{**} = -n_2 B_2 (n_2 - 1) (y_t^{\lambda^{**}})^{n_2 - 2} (dy_t^{\lambda^{**}}) + \frac{1}{2} \left\{ -B_2 n_2 (n_2 - 1) (n_2 - 2) (y_t^{\lambda^{**}})^{n_2 - 3} \right\} (dy_t^{\lambda^{**}})^2 \quad (4.3.5)$$

here,

$$dy_t^{\lambda^{**}} = y_t^{\lambda^{**}} \{(\beta - r)dt - \theta dB_t\} = y_t^{\lambda^{**}} \{(\beta - r)dt - \theta_t^s dW_t^s - \theta_t^f dW_t^f\}$$

and,

$$(dy_t^{\lambda^{**}})^2 = \theta^2 (y_t^{\lambda^{**}})^2 dt = (y_t^{\lambda^{**}})^2 \left\{ (\theta_t^s)^2 dt + (\theta_t^f)^2 dt \right\}.$$

So,

$$dX_t^{**} = -B_2 n_2 (n_2 - 1) (y_t^{\lambda^{**}})^{n_2 - 1} \{(\beta - r)dt - \theta dB_t\} + \frac{1}{2} \left\{ -B_2 n_2 (n_2 - 1) (n_2 - 2) (y_t^{\lambda^{**}})^{n_2 - 1} \right\} \theta^2 dt \quad (4.3.6)$$

$$\begin{aligned} &= -B_2 n_2 \left(\frac{1}{2} \theta^2 n_2^2 + (\beta - r - \frac{1}{2} \theta^2) n_2 - \beta \right) (y_t^{\lambda^{**}})^{n_2 - 1} dt \\ &\quad + (B_2 \theta^2 n_2^2 - B_2 r n_2 - B_2 \theta^2 n_2) (y_t^{\lambda^{**}})^{n_2 - 1} dt + \theta (B_2 n_2 (n_2 - 1) (y_t^{\lambda^{**}})^{n_2 - 1}) dB_t \\ &= r \left(-n_2 B_2 (y_t^{\lambda^{**}})^{n_2 - 1} + \frac{R}{r} \right) dt - R dt + (\theta_t^s)^2 \left\{ B_2 n_2 (n_2 - 1) (y_t^{\lambda^{**}})^{n_2 - 1} \right\} dt \\ &\quad + (\theta_t^f)^2 \left\{ B_2 n_2 (n_2 - 1) (y_t^{\lambda^{**}})^{n_2 - 1} \right\} dt + \theta_t^s (B_2 n_2 (n_2 - 1) (y_t^{\lambda^{**}})^{n_2 - 1}) dW_t^s \\ &\quad + \theta_t^f (B_2 n_2 (n_2 - 1) (y_t^{\lambda^{**}})^{n_2 - 1}) dW_t^f \end{aligned}$$

$$\text{since } \frac{1}{2} \theta^2 n_2^2 + (\beta - r - \frac{1}{2} \theta^2) n_2 - \beta = 0$$

If we choose

$$\pi_t^{s**} = \frac{\theta_t^s}{\sigma_t^s} \left\{ B_2 n_2 (n_2 - 1) (y_t^{\lambda^{**}})^{n_2 - 1} \right\} \text{ and } \pi_t^{f**} = \frac{\theta_t^f}{\sigma_t^f} \left\{ B_2 n_2 (n_2 - 1) (y_t^{\lambda^{**}})^{n_2 - 1} \right\}$$

then we have,

$$dX_t^{**} = \left[r X_t^{**} + \pi_t^{s**} (\mu_t^s - r) + \pi_t^{f**} (\mu_t^f - r) - c_t^{**} \right] dt + \pi_t^{s**} \sigma_t^s dW_t^s + \pi_t^{f**} \sigma_t^f dW_t^f$$

which satisfies the market model of the wealth process for $R/r < X_t \leq \tilde{x}$.

We simplify,

$$B_2 n_2 (n_2 - 1) (y_t^{\lambda^{**}})^{n_2 - 1} = (n_2 - 1) (B_2 n_2 (y_t^{\lambda^{**}})^{n_2 - 1} = (n_2 - 1) \left(\frac{R}{r} - X_t\right)$$

Hence the portfolio is given by

$$\pi_t^{s**} = \frac{\theta_t^s}{\sigma_t^s} \left\{ (n_2 - 1) \left(\frac{R}{r} - X_t\right) \right\} \quad \text{and} \quad \pi_t^{f**} = \frac{\theta_t^f}{\sigma_t^f} \left\{ (n_2 - 1) \left(\frac{R}{r} - X_t\right) \right\}$$

Similarly for the case $X_t \geq \tilde{x}$, we apply Itô's formula to equation (4.3.1), we

obtain

$$\begin{aligned} dX_t^* &= [-n_1(n_1 - 1)A_1(y_t^{\lambda^*})^{n_1 - 2} - \frac{2n_1(n_1 - 1)(y_t^{\lambda^*})^{n_1 - 2}}{\theta^2(n_1 - n_2)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y) - u(I(y))}{y^{n_1 + 1}} dy \\ &\quad + \frac{2n_2(n_2 - 1)(y_t^{\lambda^*})^{n_2 - 2}}{\theta^2(n_1 - n_2)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y) - u(I(y))}{y^{n_2 + 1}} dy - \frac{2}{\theta^2} \frac{y_t^{\lambda^*} I(y_t^{\lambda^*}) - u(I(y_t^{\lambda^*}))}{(y_t^{\lambda^*})^2}] dy_t^{\lambda^*} \\ &\quad + \frac{1}{2} [-n_1(n_1 - 1)(n_1 - 2)A_1(y_t^{\lambda^*})^{n_1 - 3} \\ &\quad - \frac{2n_1(n_1 - 1)(n_1 - 2)(y_t^{\lambda^*})^{n_1 - 3}}{\theta^2(n_1 - n_2)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y) - u(I(y))}{y^{n_1 + 1}} dy \\ &\quad + \frac{2n_2(n_2 - 1)(n_2 - 2)(y_t^{\lambda^*})^{n_2 - 3}}{\theta^2(n_1 - n_2)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y) - u(I(y))}{y^{n_2 + 1}} dy \\ &\quad - \frac{2(n_1 + n_2 - 3)}{\theta^2} \frac{y_t^{\lambda^*} I(y_t^{\lambda^*}) - u(I(y_t^{\lambda^*}))}{(y_t^{\lambda^*})^3} - \frac{2}{\theta^2} \frac{I(y_t^{\lambda^*})}{(y_t^{\lambda^*})^2}] (dy_t^{\lambda^*})^2 \end{aligned}$$

here,

$$dy_t^{\lambda^*} = y_t^{\lambda^*} \{(\beta - r)dt - \theta dB_t\} = y_t^{\lambda^*} \{(\beta - r)dt - \theta_t^s dW_t^s - \theta_t^f dW_t^f\}$$

and,

$$(dy_t^{\lambda^*})^2 = \theta^2 (y_t^{\lambda^*})^2 dt = (y_t^{\lambda^*})^2 \left\{ (\theta_t^s)^2 dt + (\theta_t^f)^2 dt \right\}.$$

So,

$$\begin{aligned} dX_t^* &= [-n_1(n_1 - 1)A_1(y_t^{\lambda^*})^{n_1 - 1} - \frac{2n_1(n_1 - 1)(y_t^{\lambda^*})^{n_1 - 1}}{\theta^2(n_1 - n_2)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y) - u(I(y))}{y^{n_1 + 1}} dy \\ &\quad + \frac{2n_2(n_2 - 1)(y_t^{\lambda^*})^{n_2 - 1}}{\theta^2(n_1 - n_2)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y) - u(I(y))}{y^{n_2 + 1}} dy - \frac{2}{\theta^2} \frac{y_t^{\lambda^*} I(y_t^{\lambda^*}) - u(I(y_t^{\lambda^*}))}{(y_t^{\lambda^*})^2}] \{(\beta - r)dt - \\ &\quad \theta dB_t\} \\ &\quad + \frac{1}{2} [-n_1(n_1 - 1)(n_1 - 2)A_1(y_t^{\lambda^*})^{n_1 - 1} \\ &\quad - \frac{2n_1(n_1 - 1)(n_1 - 2)(y_t^{\lambda^*})^{n_1 - 1}}{\theta^2(n_1 - n_2)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y) - u(I(y))}{y^{n_1 + 1}} dy + \frac{2n_2(n_2 - 1)(n_2 - 2)(y_t^{\lambda^*})^{n_2 - 1}}{\theta^2(n_1 - n_2)} \end{aligned}$$

$$\begin{aligned}
& \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y)-u(I(y))}{y^{n_2+1}} dy - \frac{2(n_1+n_2-3)}{\theta^2} \frac{y_t^{\lambda^*} I(y_t^{\lambda^*})-u(I(y_t^{\lambda^*}))}{(y_t^{\lambda^*})} - \frac{2}{\theta^2} I(y_t^{\lambda^*})] \theta^2 dt \\
= & r \left[-n_1 A_1 (y_t^{\lambda^*})^{n_1-1} - \frac{2n_1 (y_t^{\lambda^*})^{n_1-1}}{\theta^2 (n_1-n_2)} \int_{\tilde{y}}^{\lambda^*} \frac{yI(y)-u(I(y))}{y^{n_1+1}} dy \right. \\
& + \frac{2n_2 (y_t^{\lambda^*})^{n_2-1}}{\theta^2 (n_1-n_2)} \int_{\tilde{y}}^{\lambda^*} \frac{yI(y)-u(I(y))}{y^{n_2+1}} dy] dt + \theta^2 [n_1 (n_1-1) A_1 (y_t^{\lambda^*})^{n_1-1} \\
& + \frac{2}{\theta^2} \frac{y_t^{\lambda^*} I(y_t^{\lambda^*})-u(I(y_t^{\lambda^*}))}{y_t^{\lambda^*}} + \frac{2n_1 (n_1-1) (y_t^{\lambda^*})^{n_1-1}}{\theta^2 (n_1-n_2)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y)-u(I(y))}{y^{n_1+1}} dy \\
& - \frac{2n_2 (n_2-1) (y_t^{\lambda^*})^{n_2-1}}{\theta^2 (n_1-n_2)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y)-u(I(y))}{y^{n_2+1}} dy] dt - I(y_t^{\lambda^*}) dt \\
& - n_1 A_1 \left[\frac{1}{2} \theta^2 n_1^2 + (\beta - r - \frac{1}{2} \theta^2) n_1 - \beta \right] (y_t^{\lambda^*})^{n_1-1} dt \\
& - \frac{2n_1 (y_t^{\lambda^*})^{n_1-1}}{\theta^2 (n_1-n_2)} \left[\frac{1}{2} \theta^2 n_1^2 + (\beta - r - \frac{1}{2} \theta^2) n_1 - \beta \right] \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y)-u(I(y))}{y^{n_2+1}} dy dt \\
& - \left(n_1 - n_2 - 1 + \frac{2(\beta-r)}{\theta^2} \right) \frac{y_t^{\lambda^*} I(y_t^{\lambda^*})-u(I(y_t^{\lambda^*}))}{y_t^{\lambda^*}} dt \\
& + \theta [n_1 (n_1-1) A_1 (y_t^{\lambda^*})^{n_1-1} + \frac{2n_1 (n_1-1) (y_t^{\lambda^*})^{n_1-1}}{\theta^2 (n_1-n_2)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y)-u(I(y))}{y^{n_1+1}} dy \\
& - \frac{2n_2 (n_2-1) (y_t^{\lambda^*})^{n_2-1}}{\theta^2 (n_1-n_2)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y)-u(I(y))}{y^{n_2+1}} dy + \frac{2}{\theta^2} \frac{y_t^{\lambda^*} I(y_t^{\lambda^*})-u(I(y_t^{\lambda^*}))}{(y_t^{\lambda^*})^2}] dB_t.
\end{aligned}$$

Since n_1 satisfies the Eq.(4.2.9), $\frac{1}{2}n^2\theta^2 + (\beta - r - \frac{1}{2}\theta^2)n - \beta = 0$, so we can simplify

it as

$$\begin{aligned}
= & r \left[-n_1 A_1 (y_t^{\lambda^*})^{n_1-1} - \frac{2n_1 (y_t^{\lambda^*})^{n_1-1}}{\theta^2 (n_1-n_2)} \int_{\tilde{y}}^{\lambda^*} \frac{yI(y)-u(I(y))}{y^{n_1+1}} dy + \frac{2n_2 (y_t^{\lambda^*})^{n_2-1}}{\theta^2 (n_1-n_2)} \int_{\tilde{y}}^{\lambda^*} \frac{yI(y)-u(I(y))}{y^{n_2+1}} dy \right] dt \\
& + \theta^2 [n_1 (n_1-1) A_1 (y_t^{\lambda^*})^{n_1-1} + \frac{2}{\theta^2} \frac{y_t^{\lambda^*} I(y_t^{\lambda^*})-u(I(y_t^{\lambda^*}))}{y_t^{\lambda^*}} \\
& + \frac{2n_1 (n_1-1) (y_t^{\lambda^*})^{n_1-1}}{\theta^2 (n_1-n_2)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y)-u(I(y))}{y^{n_1+1}} dy \\
& - \frac{2n_2 (n_2-1) (y_t^{\lambda^*})^{n_2-1}}{\theta^2 (n_1-n_2)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y)-u(I(y))}{y^{n_2+1}} dy] dt - I(y_t^{\lambda^*}) dt \\
& + \theta [n_1 (n_1-1) A_1 (y_t^{\lambda^*})^{n_1-1} + \frac{2n_1 (n_1-1) (y_t^{\lambda^*})^{n_1-1}}{\theta^2 (n_1-n_2)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y)-u(I(y))}{y^{n_1+1}} dy \\
& - \frac{2n_2 (n_2-1) (y_t^{\lambda^*})^{n_2-1}}{\theta^2 (n_1-n_2)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y)-u(I(y))}{y^{n_2+1}} dy + \frac{2}{\theta^2} \frac{y_t^{\lambda^*} I(y_t^{\lambda^*})-u(I(y_t^{\lambda^*}))}{(y_t^{\lambda^*})^2}] dB_t \\
= & r \left[-n_1 A_1 (y_t^{\lambda^*})^{n_1-1} - \frac{2n_1 (y_t^{\lambda^*})^{n_1-1}}{\theta^2 (n_1-n_2)} \int_{\tilde{y}}^{\lambda^*} \frac{yI(y)-u(I(y))}{y^{n_1+1}} dy + \frac{2n_2 (y_t^{\lambda^*})^{n_2-1}}{\theta^2 (n_1-n_2)} \int_{\tilde{y}}^{\lambda^*} \frac{yI(y)-u(I(y))}{y^{n_2+1}} dy \right] dt \\
& + (\theta^s)^2 [n_1 (n_1-1) A_1 (y_t^{\lambda^*})^{n_1-1} + \frac{2}{\theta^2} \frac{y_t^{\lambda^*} I(y_t^{\lambda^*})-u(I(y_t^{\lambda^*}))}{y_t^{\lambda^*}} \\
& + \frac{2n_1 (n_1-1) (y_t^{\lambda^*})^{n_1-1}}{\theta^2 (n_1-n_2)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y)-u(I(y))}{y^{n_1+1}} dy - \frac{2n_2 (n_2-1) (y_t^{\lambda^*})^{n_2-1}}{\theta^2 (n_1-n_2)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y)-u(I(y))}{y^{n_2+1}} dy] dt
\end{aligned}$$

$$\begin{aligned}
& + (\theta_t^f)^2 [n_1(n_1 - 1)A_1(y_t^{\lambda^*})^{n_1-1} + \frac{2}{\theta^2} \frac{y_t^{\lambda^*} I(y_t^{\lambda^*}) - u(I(y_t^{\lambda^*}))}{y_t^{\lambda^*}} \\
& + \frac{2n_1(n_1-1)(y_t^{\lambda^*})^{n_1-1}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y)-u(I(y))}{y^{n_1+1}} dy \\
& - \frac{2n_2(n_2-1)(y_t^{\lambda^*})^{n_2-1}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y)-u(I(y))}{y^{n_2+1}} dy] dt - I(y_t^{\lambda^*}) dt \\
& + \theta_t^s [n_1(n_1 - 1)A_1(y_t^{\lambda^*})^{n_1-1} + \frac{2n_1(n_1-1)(y_t^{\lambda^*})^{n_1-1}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y)-u(I(y))}{y^{n_1+1}} dy \\
& - \frac{2n_2(n_2-1)(y_t^{\lambda^*})^{n_2-1}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y)-u(I(y))}{y^{n_2+1}} dy + \frac{2}{\theta^2} \frac{y_t^{\lambda^*} I(y_t^{\lambda^*}) - u(I(y_t^{\lambda^*}))}{(y_t^{\lambda^*})^2}] dW_t^s \\
& + \theta_t^f [n_1(n_1 - 1)A_1(y_t^{\lambda^*})^{n_1-1} + \frac{2n_1(n_1-1)(y_t^{\lambda^*})^{n_1-1}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y)-u(I(y))}{y^{n_1+1}} dy \\
& - \frac{2n_2(n_2-1)(y_t^{\lambda^*})^{n_2-1}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y)-u(I(y))}{y^{n_2+1}} dy + \frac{2}{\theta^2} \frac{y_t^{\lambda^*} I(y_t^{\lambda^*}) - u(I(y_t^{\lambda^*}))}{(y_t^{\lambda^*})^2}] dW_t^f.
\end{aligned}$$

If we choose,

$$\begin{aligned}
\pi_t^{s*} &= \frac{\theta_t^s}{\sigma_t^s} [n_1(n_1 - 1)A_1(y_t^{\lambda^*})^{n_1-1} + \frac{2}{\theta^2} \frac{y_t^{\lambda^*} I(y_t^{\lambda^*}) - u(I(y_t^{\lambda^*}))}{y_t^{\lambda^*}} \\
& + \frac{2n_1(n_1-1)(y_t^{\lambda^*})^{n_1-1}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y)-u(I(y))}{y^{n_1+1}} dy \\
& - \frac{2n_2(n_2-1)(y_t^{\lambda^*})^{n_2-1}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y)-u(I(y))}{y^{n_2+1}} dy]
\end{aligned}$$

and,

$$\begin{aligned}
\pi_t^{f*} &= \frac{\theta_t^f}{\sigma_t^f} [n_1(n_1 - 1)A_1(y_t^{\lambda^*})^{n_1-1} + \frac{2}{\theta^2} \frac{y_t^{\lambda^*} I(y_t^{\lambda^*}) - u(I(y_t^{\lambda^*}))}{y_t^{\lambda^*}} \\
& + \frac{2n_1(n_1-1)(y_t^{\lambda^*})^{n_1-1}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y)-u(I(y))}{y^{n_1+1}} dy \\
& - \frac{2n_2(n_2-1)(y_t^{\lambda^*})^{n_2-1}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y)-u(I(y))}{y^{n_2+1}} dy]
\end{aligned}$$

then we have the wealth process,

$$dX_t^* = \left[rX_t^* + \pi_t^{s*}(\mu_t^s - r) + \pi_t^{f*}(\mu_t^f - r) - c_t^* \right] dt + \pi_t^{s*} \sigma_t^s dW_t^s + \pi_t^{f*} \sigma_t^f dW_t^f.$$

□

CHAPTER 5

A SOLUTION TO THE LOG UTILITY

DEFINITION 5.1. *CES Utility Function*

A Utility function of the form,

$u(x) \triangleq \frac{x^{1-\gamma}}{1-\gamma}$, $\gamma > 0$, $\gamma \neq 1$ is called the Constant Elasticity of Substitution Utility

function. It is a synonym for the Coefficient of relative risk aversion (*CRRA*) or Iso-elastic utility function. The elasticity of substitution between consumption at any two points in time is constant *i.e.* $\frac{1}{\gamma}$ and the elasticity of marginal utility is $-\gamma$. Here γ is the Coefficient of relative risk aversion.

DEFINITION 5.2. *Log utility function*

Log utility is the limiting function of *CES* utility at $\gamma \rightarrow 1$. We can derive the

Log utility from *CES* as follows

$$\lim_{\gamma \rightarrow 1} u(x) = \lim_{\gamma \rightarrow 1} \frac{x^{1-\gamma}}{1-\gamma}$$

Applying L-Hopital's rule

$$\lim_{\gamma \rightarrow 1} u'(x) = \lim_{\gamma \rightarrow 1} \frac{(1-\gamma)x^{-\gamma}}{1-\gamma} = \frac{1}{x}$$

So,

$u(x) \triangleq \ln(x)$ is a subclass of *CES* utility function

We first derive the value function and the Merton's constant for the general class of utility and then deduce the value function for the log utility class at $\gamma \rightarrow 1$.

Assumption-3

We assume the Merton's constant for the General class of utility function (CES)

as

$$K \triangleq r + \frac{\beta-r}{\gamma} + \frac{\gamma-1}{2\gamma^2}\theta^2 > 0$$

and the Merton's constant for the Log utility function

$$K \triangleq \beta$$

5.1. To derive the function $v(y)$ for the CES utility

Since the dual utility function $\tilde{u}(y)$ is defined by

$$\tilde{u}(y) = [u(I(y)) - yI(y)]\chi_{\{0 < y \leq \tilde{y}\}} + [u(R) - Ry]\chi_{\{y \geq \tilde{y}\}}$$

For,

$$u(y) \triangleq \frac{y^{1-\gamma}}{1-\gamma}, I(y) = y^{-\frac{1}{\gamma}}$$

and,

$$u(I(y)) - yI(y) = \frac{\gamma}{1-\gamma}y^{-\frac{1-\gamma}{\gamma}}$$

hence we have

$$\tilde{u}(y) = \left[\frac{\gamma}{1-\gamma}y^{-\frac{1-\gamma}{\gamma}}\right]\chi_{\{0 < y \leq \tilde{y}\}} + \left[\frac{R^{1-\gamma}}{1-\gamma} - Ry\right]\chi_{\{y \geq \tilde{y}\}}$$

5.1.1. To derive \tilde{y} . Applying the first order condition to the dual utility function

at $y = \tilde{y}$ as

$$\tilde{u}(y = \tilde{y}) = u(R) - R\tilde{y}$$

differentiating w.r.t. R

$$u'(R) - \tilde{y} = 0, \Leftrightarrow \tilde{y} = u'(R) = R^{-\gamma}$$

5.1.2. The function $v(y)$. From the Eq.(4.2.7) we define $v(y)$ as

$$\begin{aligned} v(y) &= A_1 y^{n_1} + \frac{2y^{n_1}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^y \frac{zI(z)-u(I(z))}{z^{n_1+1}} dz - \frac{2y^{n_2}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^y \frac{zI(z)-u(I(z))}{z^{n_2+1}} dz; \text{ if } 0 < y \leq \tilde{y} \\ &= B_2 y^{n_2} - \frac{R}{r}y + \frac{u(R)}{\beta}; \quad \text{if } y \geq \tilde{y} \end{aligned}$$

(5.1.1)

where n_1, n_2 are the roots of the homogeneous algebraic equation:

$$\frac{1}{2}n^2\theta^2 + (\beta - r - \frac{1}{2}\theta^2)n - \beta = 0$$

Here,

$$n_1 + n_2 = \frac{-2(\beta - r - \frac{1}{2}\theta^2)}{\theta^2}, \quad n_1 n_2 = \frac{-2\beta}{\theta^2} \quad \text{and} \quad zI(z) - u(I(z)) = \frac{-\gamma}{1-\gamma} z^{-\frac{1-\gamma}{\gamma}}.$$

For $0 < y \leq \tilde{y}$, $v(y)$ would be

$$\begin{aligned} v(y) &= A_1 y^{n_1} + \frac{2y^{n_1}}{\theta^2(n_1 - n_2)} \int_{\tilde{y}}^y \frac{-\gamma}{1-\gamma} z^{-\frac{1-\gamma}{\gamma}} dz - \frac{2y^{n_2}}{\theta^2(n_1 - n_2)} \int_{\tilde{y}}^y \frac{-\gamma}{1-\gamma} z^{-\frac{1-\gamma}{\gamma}} dz \\ &= A_1 y^{n_1} + \frac{\gamma}{1-\gamma} y^{-\frac{1-\gamma}{\gamma}} \frac{2}{\theta^2(n_1 - n_2)} \left[\frac{1}{\frac{1-\gamma}{\gamma} + n_1} - \frac{1}{\frac{1-\gamma}{\gamma} + n_2} \right] \\ &= A_1 y^{n_1} + \frac{\gamma}{1-\gamma} y^{-\frac{1-\gamma}{\gamma}} \left[\frac{-2}{\theta^2} \frac{1}{\left(\frac{1-\gamma}{\gamma} + n_1\right)\left(\frac{1-\gamma}{\gamma} + n_2\right)} \right] \end{aligned}$$

Here,

$$\begin{aligned} \frac{-\theta^2}{2} \left(\frac{1-\gamma}{\gamma} + n_1 \right) \left(\frac{1-\gamma}{\gamma} + n_2 \right) &= \frac{-\theta^2}{2} \left[\left(n_1 + \frac{1}{\gamma} - 1 \right) \left(n_2 + \frac{1}{\gamma} - 1 \right) \right] \\ &= \frac{-\theta^2}{2} \left[1 - \frac{2}{\gamma} + \frac{1}{\gamma^2} - (n_1 + n_2) + \frac{(n_1 + n_2)}{\gamma} + n_1 n_2 \right] \\ &= \frac{-\theta^2}{2} \left[1 - \frac{2}{\gamma} + \frac{1}{\gamma^2} + \frac{2(\beta - r - \frac{1}{2}\theta^2)}{\theta^2} - \frac{2(\beta - r - \frac{1}{2}\theta^2)}{\gamma\theta^2} - \frac{2\beta}{\theta^2} \right] \\ &= \frac{-\theta^2}{2} + \frac{\theta^2}{2} - \frac{\theta^2}{2\gamma^2} - \beta + r + \frac{\theta^2}{2} + \frac{(\beta - r - \frac{1}{2}\theta^2)}{\gamma} + \beta \\ &= r + \frac{\beta - r}{\gamma} + \frac{\gamma - 1}{2\gamma^2} \theta^2 - \beta + \beta \\ &= r + \frac{\beta - r}{\gamma} + \frac{\gamma - 1}{2\gamma^2} \theta^2 \\ &= K \quad (\text{A Merton's constant}) \end{aligned}$$

Hence,

$$\begin{aligned} v(y) &= A_1 y^{n_1} + \frac{1}{K} \frac{\gamma}{1-\gamma} y^{-\frac{1-\gamma}{\gamma}}, \quad \text{if } 0 < y \leq \tilde{y} \\ &= B_2 y^{n_2} - \frac{R}{r} y + \frac{u(R)}{\beta}, \quad \text{if } y \geq \tilde{y}. \end{aligned}$$

5.2. To derive the function $v(y)$ for the Log utility function

For the log utility function the dual boundary \tilde{y} is defined as

$$\tilde{y} = u'(R) = R^{-1}, \quad \text{and} \quad I(y) = R^{-1}.$$

So, the dual utility function for the log utility is defined as

$$\tilde{u}(y) = [-\ln(y) - 1] \chi_{\{0 < y \leq \tilde{y}\}} + [\ln(R) - Ry] \chi_{\{y \geq \tilde{y}\}}.$$

We find the Merton's constant for this class of utility function by letting $\gamma \rightarrow 1$,

$K \triangleq \beta$ (the discount factor)

So, for $0 < y \leq \tilde{y}$ the function $v(y)$

$$\begin{aligned} v(y) &= A_1 y^{n_1} + \frac{2y^{n_1}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^y \frac{1+\ln(z)}{z^{n_1+1}} dz - \frac{2y^{n_2}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^y \frac{(1+\ln(z))}{z^{n_2+1}} dz \\ &= A_1 y^{n_1} + \frac{2}{\theta^2(n_1-n_2)} (-\ln(y) - 1) \left[\frac{1}{n_1} - \frac{1}{n_2} \right] \\ &= A_1 y^{n_1} + \frac{2(\ln(y)+1)}{\theta^2 n_1 n_2}. \end{aligned}$$

Since $n_1 n_2 = \frac{-2\beta}{\theta^2}$

$$v(y) = A_1 y^{n_1} - \frac{1}{\beta} (\ln(y) + 1).$$

Hence,

$$\begin{aligned} v(y) &= A_1 y^{n_1} - \frac{1}{\beta} (\ln(y) + 1), \text{ if } 0 < y \leq \tilde{y} \\ &= B_2 y^{n_2} - \frac{R}{r} y + \frac{\ln(R)}{\beta}, \text{ if } y \geq \tilde{y}. \end{aligned}$$

5.2.1. To determine A_1 and B_2 . When we apply the smooth condition and continuous property of the function $v(y)$ at $y = \tilde{y}$, we get

$$A_1 \tilde{y}^{n_1} - \frac{1}{\beta} (\ln(\tilde{y}) + 1) = B_2 \tilde{y}^{n_2} - \frac{R}{r} \tilde{y} + \frac{\ln(R)}{\beta}, \quad (5.2.1)$$

and

$$n_1 A_1 \tilde{y}^{n_1-1} - \frac{1}{\beta \tilde{y}} = n_2 B_2 \tilde{y}^{n_2-1} - \frac{R}{r}. \quad (5.2.2)$$

For the log utility function $\tilde{y} = R^{-1}$, we can simplify the above equations as follows,

$$A_1 R^{-n_1} + \frac{1}{\beta} (\ln(R) - 1) = B_2 R^{-n_2} - \frac{1}{r} + \frac{\ln(R)}{\beta} \quad (5.2.3)$$

$$n_1 A_1 R^{-n_1+1} - \frac{R}{\beta} = n_2 B_2 R^{-n_2+1} - \frac{R}{r}. \quad (5.2.4)$$

Solving Eq.(5.2.3 and Eq.(5.2.4) A_1 and B_2 are given by

$$A_1 = \frac{(n_2-1)}{(n_1-n_2)} \left[\frac{\beta-r}{\beta r} \right] R^{n_1}, \quad B_2 = \frac{(n_1-1)}{(n_1-n_2)} \left[\frac{\beta-r}{\beta r} \right] R^{n_2}.$$

Now we define the wealth boundary as given by

$$\tilde{x} = -A_1 n_1 R^{1-n_1} + \frac{R}{\beta} = -B_2 n_2 R^{1-n_2} + \frac{R}{r}. \quad (5.2.5)$$

5.3. The value function $V(x)$

For $x \geq \tilde{x}$, from the remark of previous chapter, there is an one-to-one correspondence between $\lambda^* \in (0, \tilde{y})$ and $x \in (\tilde{x}, \infty)$. Furthermore, we can show that there is also one-to-one correspondence between $\lambda^{**} \in (\tilde{y}, \infty)$ and $x \in (\frac{R}{r}, \tilde{x})$. We can find λ^{**} by solving the equation

$$x = -B_2 n_2 (\lambda^{**})^{1-n_2} + \frac{R}{r} \text{ for } \lambda^{**}, \text{ where, } \lambda^{**} = \left(\frac{\frac{R}{r} - x}{B_2 n_2} \right)^{\frac{1}{n_2-1}} \text{ for } x \in \left(\frac{R}{r}, \tilde{x} \right).$$

So, by using proposition(4.8) we have the relation

$$V(x) = \inf_{\lambda > 0} (\tilde{V}(\lambda) + \lambda x) = B_2 (\lambda^{**})^{n_2} - \frac{R}{r} \lambda^{**} + \frac{u(R)}{\beta} + x \lambda^{**}$$

or,

$$V(x) = B_2 \left(\frac{\frac{R}{r} - x}{B_2 n_2} \right)^{\frac{n_2}{n_2-1}} + \left(x - \frac{R}{r} \right) \left(\frac{\frac{R}{r} - x}{B_2 n_2} \right)^{\frac{1}{n_2-1}} + \frac{u(R)}{\beta}, \text{ for } x \in \left(\frac{R}{r}, \tilde{x} \right)$$

(5.3.1)

Similarly, for $x \geq \tilde{x}$

$$\begin{aligned} V(x) &= A_1 (\lambda^*)^{n_1} + \frac{2(\lambda^*)^{n_1}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^{\lambda^*} \frac{yI(y)-u(I(y))}{y^{n_1+1}} dy - \frac{2(\lambda^*)^{n_2}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^{\lambda^*} \frac{yI(y)-u(I(y))}{y^{n_2+1}} dy + (\lambda^*)x \\ &= A_1 (\lambda^*)^{n_1} + \frac{1}{\beta} (-\ln(\lambda^*) - 1) + x \lambda^* \end{aligned}$$

(5.3.2)

where λ^* is determined from the following equation

$$-A_1 n_1 (\lambda^*)^{n_1-1} + \frac{1}{\beta \lambda^*} = x. \quad (5.3.3)$$

5.3.1. Optimal processes and optimal policies. (a)- **The Optimal wealth process -**

The optimal wealth process is defined as

$$X_t^* = -A_1 n_1 (y_t^{\lambda^*})^{n_1-1} + \frac{1}{\beta (y_t^{\lambda^*})}, \text{ if } X_t \geq \tilde{x} \quad (5.3.4)$$

and

$$X_t^{**} = -n_2 B_2 (y_t^{\lambda^{**}})^{n_2-1} + \frac{R}{r}, \text{ if } R/r < X_t \leq \tilde{x} \quad (5.3.5)$$

here,

$$dy_t^\lambda = y_t^\lambda \{(\beta - r)dt - \theta dB_t\} = y_t^\lambda \{(\beta - r)dt - \theta_t^s dW_t^s - \theta_t^f dW_t^f\}$$

and,

$$(dy_t^\lambda)^2 = \theta^2 (y_t^\lambda)^2 dt = (y_t^\lambda)^2 \left\{ (\theta_t^s)^2 dt + (\theta_t^f)^2 dt \right\}$$

(b)- **Consumption process** -

The optimal consumption process for the log utility function is given by

$$\begin{aligned} c_t^* &= R, \quad \text{if } R/r < X_t \leq \tilde{x} \\ &= (y_t^{\lambda^*})^{-1}, \quad \text{if } X_t \geq \tilde{x}. \end{aligned} \quad (5.3.6)$$

Here $\tilde{y} = (y_t^{\lambda^*})^{-1}$ is the boundary of the dual value function.

(c)- **The optimal portfolio process** - We derive the optimal portfolio process in both case as follows,

(i)- For $R/r < X_t \leq \tilde{x}$

We already obtained the optimal portfolios as

$$\pi_t^{s**} = \frac{\theta_t^s}{\sigma_t^s} \left\{ (n_2 - 1) \left(\frac{R}{r} - X_t \right) \right\}, \quad \pi_t^{f**} = \frac{\theta_t^f}{\sigma_t^f} \left\{ (n_2 - 1) \left(\frac{R}{r} - X_t \right) \right\}.$$

(ii)- For $X_t \geq \tilde{x}$, portfolio policies for the general utility case is given as

$$\begin{aligned} \pi_t^{s*} &= \frac{\theta_t^s}{\sigma_t^s} \left[n_1 (n_1 - 1) A_1 (y_t^{\lambda^*})^{n_1-1} + \frac{2}{\theta^2} \frac{y_t^{\lambda^*} I(y_t^{\lambda^*}) - u(I(y_t^{\lambda^*}))}{y_t^{\lambda^*}} \right. \\ &\quad + \frac{2n_1(n_1-1)(y_t^{\lambda^*})^{n_1-1}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y) - u(I(y))}{y^{n_1+1}} dy \\ &\quad \left. - \frac{2n_2(n_2-1)(y_t^{\lambda^*})^{n_2-1}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y) - u(I(y))}{y^{n_2+1}} dy \right] \end{aligned}$$

and,

$$\begin{aligned} \pi_t^{f*} &= \frac{\theta_t^f}{\sigma_t^f} \left[n_1 (n_1 - 1) A_1 (y_t^{\lambda^*})^{n_1-1} + \frac{2}{\theta^2} \frac{y_t^{\lambda^*} I(y_t^{\lambda^*}) - u(I(y_t^{\lambda^*}))}{y_t^{\lambda^*}} \right. \\ &\quad + \frac{2n_1(n_1-1)(y_t^{\lambda^*})^{n_1-1}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y) - u(I(y))}{y^{n_1+1}} dy \\ &\quad \left. - \frac{2n_2(n_2-1)(y_t^{\lambda^*})^{n_2-1}}{\theta^2(n_1-n_2)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{yI(y) - u(I(y))}{y^{n_2+1}} dy \right] \end{aligned}$$

For the Log utility function we find the portfolio policies by applying Itô's formula to the equation

$$X_t^* = -A_1 n_1 (y_t^{\lambda^*})^{n_1-1} + \frac{1}{\beta (y_t^{\lambda^*})} \quad (5.3.7)$$

We obtain,

$$\begin{aligned} dX_t^* &= [-n_1(n_1-1)A_1(y_t^{\lambda^*})^{n_1-2} - \frac{(y_t^{\lambda^*})^{-2}}{\beta}]y_t^{\lambda^*}\{(\beta-r)dt - \theta dB_t\} \\ &\quad + \frac{1}{2}[-n_1(n_1-1)(n_1-2)A_1(y_t^{\lambda^*})^{n_1-3} + \frac{2}{\beta}(y_t^{\lambda^*})^{-3}](y_t^{\lambda^*})^2\theta^2 dt \\ &= [-n_1(n_1-1)A_1(y_t^{\lambda^*})^{n_1-1} - \frac{(y_t^{\lambda^*})^{-2}}{\beta}]\{(\beta-r)dt - \theta dB_t\} \\ &\quad + \frac{1}{2}[-n_1(n_1-1)(n_1-2)A_1(y_t^{\lambda^*})^{n_1-1} + \frac{2}{\beta}(y_t^{\lambda^*})^{-1}]\theta^2 dt \\ &= r \left[-A_1 n_1 (y_t^{\lambda^*})^{n_1-1} + \frac{1}{\beta (y_t^{\lambda^*})} \right] dt + \theta \left[A_1 n_1 (n_1-1) (y_t^{\lambda^*})^{n_1-1} + \frac{1}{\beta (y_t^{\lambda^*})} \right] dB_t \\ &\quad - \frac{1}{(y_t^{\lambda^*})} dt + \theta^2 \left[A_1 n_1 (n_1-1) (y_t^{\lambda^*})^{n_1-1} + \frac{1}{\beta (y_t^{\lambda^*})} \right] dt \\ &\quad - A_1 n_1 (y_t^{\lambda^*})^{n_1-1} \left[\frac{1}{2} n_1^2 \theta^2 + (\beta - r - \frac{1}{2} \theta^2) n_1 - \beta \right]. \end{aligned}$$

Here,

$$\frac{1}{2} n_1^2 \theta^2 + (\beta - r - \frac{1}{2} \theta^2) n_1 - \beta = 0 \text{ such that}$$

$$\theta^2 = (\theta_t^s)^2 + (\theta_t^f)^2,$$

and

$$\theta dB_t = \theta_t^s dW_t^s + \theta_t^f dW_t^f.$$

So we have,

$$\begin{aligned} dX_t^* &= r \left[-A_1 n_1 (y_t^{\lambda^*})^{n_1-1} + \frac{1}{\beta (y_t^{\lambda^*})} \right] dt + \theta_t^s \left[A_1 n_1 (n_1-1) (y_t^{\lambda^*})^{n_1-1} + \frac{1}{\beta (y_t^{\lambda^*})} \right] dW_t^s \\ &\quad + \theta_t^f \left[A_1 n_1 (n_1-1) (y_t^{\lambda^*})^{n_1-1} + \frac{1}{\beta (y_t^{\lambda^*})} \right] dW_t^f - \frac{1}{(y_t^{\lambda^*})} dt \\ &\quad + (\theta_t^s)^2 \left[A_1 n_1 (n_1-1) (y_t^{\lambda^*})^{n_1-1} + \frac{1}{\beta (y_t^{\lambda^*})} \right] dt + (\theta_t^s)^2 \left[A_1 n_1 (n_1-1) (y_t^{\lambda^*})^{n_1-1} \right. \\ &\quad \left. + \frac{1}{\beta (y_t^{\lambda^*})} \right] dt. \end{aligned} \quad (5.3.8)$$

If we choose,

$$\pi_t^{s*} = \frac{\theta_t^s}{\sigma_t^s} \left[A_1 n_1 (n_1-1) (y_t^{\lambda^*})^{n_1-1} + \frac{1}{\beta (y_t^{\lambda^*})} \right]$$

and,

$$\pi_t^{f*} = \frac{\theta_t^f}{\sigma_t^f} \left[A_1 n_1 (n_1-1) (y_t^{\lambda^*})^{n_1-1} + \frac{1}{\beta (y_t^{\lambda^*})} \right]$$

we obtain the wealth process,

$$dX_t^* = \left[rX_t^* + \pi_t^{s*}(\mu_t^s - r) + \pi_t^{f*}(\mu_t^f - r) - c_t^* \right] dt + \pi_t^{s*}\sigma_t^s dW_t^s + \pi_t^{f*}\sigma_t^f dW_t^f \quad (5.3.9)$$

Hence the optimal portfolio process $\{\pi_t^{s*}, \pi_t^{f*}\}$ is given by $\{\pi_t^{s*}, \pi_t^{f*}\} =$

$$\left\{ \begin{array}{l} \frac{\theta_t^s}{\sigma_t^s} \left\{ (n_2 - 1) \left(\frac{R}{r} - X_t \right) \right\}, \frac{\theta_t^f}{\sigma_t^f} \left\{ (n_2 - 1) \left(\frac{R}{r} - X_t \right) \right\}, \text{if } R/r < X_t \leq \tilde{x} \\ \frac{\theta_t^s}{\sigma_t^s} \left[A_1 n_1 (n_1 - 1) (y_t^{\lambda^*})^{n_1 - 1} + \frac{1}{\beta (y_t^{\lambda^*})} \right], \frac{\theta_t^f}{\sigma_t^f} \left[A_1 n_1 (n_1 - 1) (y_t^{\lambda^*})^{n_1 - 1} + \frac{1}{\beta (y_t^{\lambda^*})} \right], \text{if } X_t \geq \tilde{x} \end{array} \right\}. \quad (5.3.10)$$

On the other hand if we define the portfolio $\pi_t^* = \pi_t^{s*} + \pi_t^{f*}$, as a risky investment including stocks and option, we can write π_t^* as

$$\pi_t^* = \left\{ \begin{array}{l} \frac{\theta_t^s}{\sigma_t^s} + \frac{\theta_t^f}{\sigma_t^f} \left\{ (n_2 - 1) \left(\frac{R}{r} - X_t \right) \right\}; \text{if } R/r < X_t \leq \tilde{x} \\ \frac{\theta_t^s}{\sigma_t^s} + \frac{\theta_t^f}{\sigma_t^f} \left[A_1 n_1 (n_1 - 1) (y_t^{\lambda^*})^{n_1 - 1} + \frac{1}{\beta (y_t^{\lambda^*})} \right]; \text{if } X_t \geq \tilde{x} \end{array} \right\}. \quad (5.3.11)$$

THEOREM 5.3. *The optimal portfolio for the hedge assets can be obtained by*

$$\pi_t^f = K \pi_t^s, \text{ for all } R/r < X < \infty,$$

$$\text{where, } K = \frac{\theta_t^s \sigma_t^f}{\theta_t^f \sigma_t^s}.$$

PROOF. From the previous theorems and proofs we have the following relations:

(i)- For $R/r < X_t \leq \tilde{x}$

$$\pi_t^{s**} = \frac{\theta_t^s}{\sigma_t^s} \left\{ (n_2 - 1) \left(\frac{R}{r} - X_t \right) \right\}, \pi_t^{f**} = \frac{\theta_t^f}{\sigma_t^f} \left\{ (n_2 - 1) \left(\frac{R}{r} - X_t \right) \right\}.$$

$$\text{i.e. } \frac{\pi_t^{s**}}{\pi_t^{f**}} = \frac{\theta_t^s}{\sigma_t^s} \cdot \frac{\sigma_t^f}{\theta_t^f}$$

(ii)- For $X_t \geq \tilde{x}$,

$$\pi_t^{s*} = \frac{\theta_t^s}{\sigma_t^s} \left[A_1 n_1 (n_1 - 1) (y_t^{\lambda^*})^{n_1 - 1} + \frac{1}{\beta (y_t^{\lambda^*})} \right]$$

and,

$$\pi_t^{f*} = \frac{\theta_t^f}{\sigma_t^f} [A_1 n_1 (n_1 - 1) (y_t^{\lambda^*})^{n_1 - 1} + \frac{1}{\beta (y_t^{\lambda^*})}]$$

i.e. $\frac{\pi_t^{s*}}{\pi_t^{f*}} = \frac{\theta_t^s}{\sigma_t^s} \cdot \frac{\sigma_t^f}{\theta_t^f}$.

Hence we conclude for all $R/r < X < \infty$,

$$\pi_t^f = \pi_t^s \cdot \frac{\theta_t^s \sigma_t^f}{\theta_t^f \sigma_t^s}.$$

□

5.4. Numerical Illustration

In this section we present two numerical results which strengthen the results we obtained in chapter 4. In the first example we show graphically the one-to-one correspondence between the domains of the value function and the dual value function. In the second example we obtain the numerical results and graphical plottings for the optimal portfolio and optimal consumption rate process with respect to the wealth function.

EXAMPLE 5.4. one-to-one correspondence

It is easily to see the one-to-one correspondence between $(0, \tilde{y})$ and (\tilde{x}, ∞) as well between (\tilde{y}, ∞) and $(R/r, \tilde{x})$ graphically.

Let $\beta = .07, r = .01, \mu = .05, R = 0.5, \sigma = 0.2, \tilde{y} = 2, \tilde{x} = 54.65603993$

Then, we have the optimal wealth function defined as

$$X_1 = -A_1 n_1 y^{n_1 - 1} + \frac{1}{\beta y}, \text{ if } X \geq \tilde{x} \text{ and}$$

$$X_2 = -n_2 B_2 y^{n_2 - 1} + \frac{R}{r}, \text{ if } R/r < X \leq \tilde{x}$$

Fig(5.1) and (5.2) shows clearly that when wealth approaches to \hat{x} , dual variable approaches to \hat{y} .

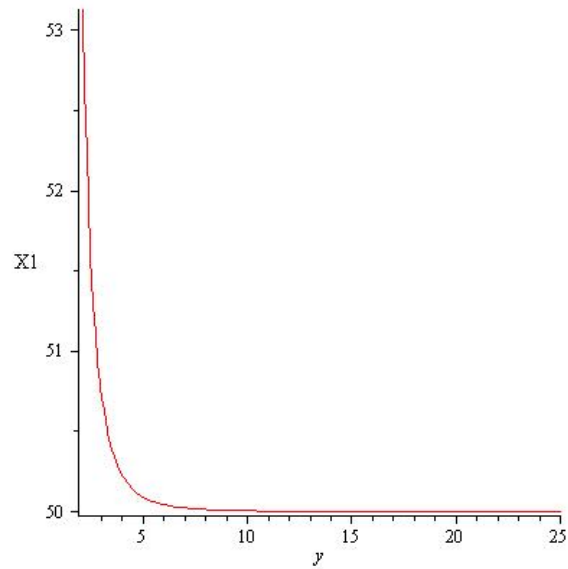


FIGURE 5.1. The Graph of X_1 vs $y(t)$

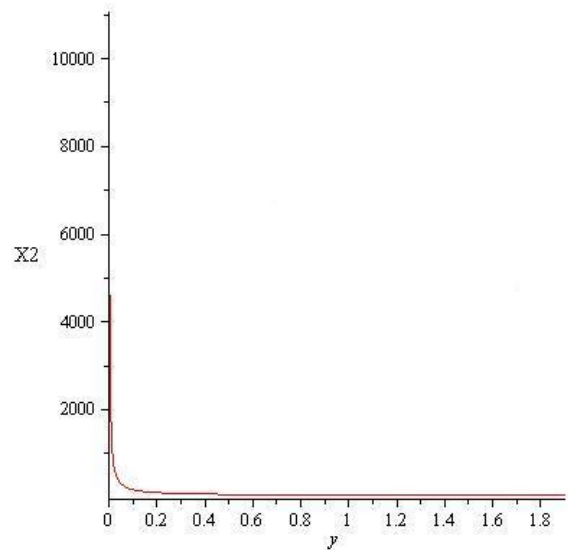


FIGURE 5.2. The Graph of X_2 vs $y(t)$

EXAMPLE 5.5. Optimal policies in a particular case.

Let us choose $\beta = .07, r = .01, \mu = .05, R = 0.5, \sigma = 0.2$

$$\theta = \frac{\mu - r}{\sigma} = 0.2$$

after some calculation we have

$$n_1 = 1.134196283, n_2 = -4.821696281, A_1 = -38.17058338, B_2 = 54.61627199$$

$$\tilde{x} = 53.82524221, \tilde{y} = 1/R = 2$$

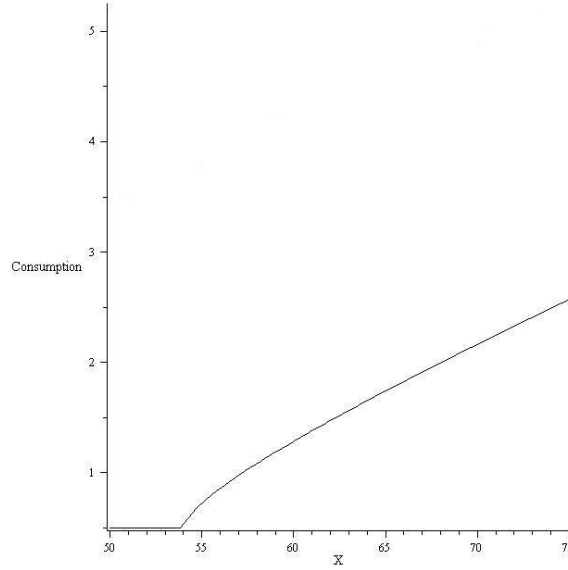


FIGURE 5.3. The Graph of wealth vs consumption

Fig(5.3) represents the Optimal consumption rate and Optimal wealth when $u(c) = \ln c$, is a subclass of the CES utility function for the case $\gamma \rightarrow 1$. It also gives the optimal consumption rate when the investor does have a downside consumption constraint with three assets model.

Similarly,

Fig(5.4) represents the optimal investment (portfolio) in the risky asset when the $u(c) = \ln c$. It also gives the optimal portfolio rate when the investor does have a downside consumption constraint with three assets model.

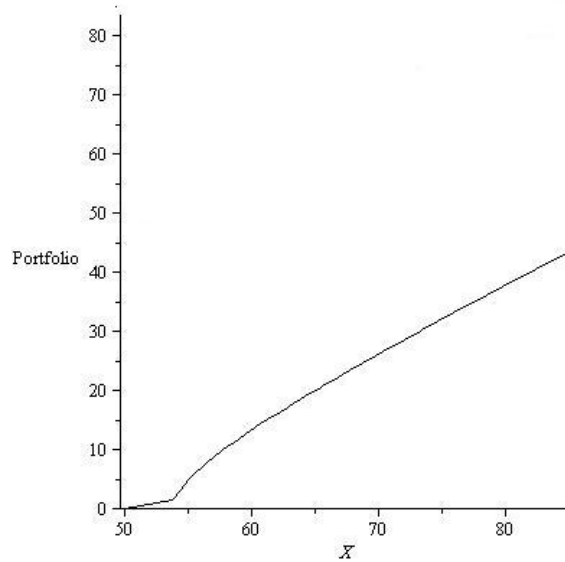


FIGURE 5.4. The Graph of wealth vs portfolio

EXAMPLE 5.6. If we choose the parameters as $\beta = .07, r = .01, \mu_t^s = .03, \mu_t^f = .02, R = 0.5, \sigma_t^s = 0.05, \sigma_t^f = 0.015$

Now we calculate the market price of risk in a vector form,

$$\theta = \begin{bmatrix} \theta_t^s \\ \theta_t^f \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.67 \end{bmatrix} \text{ such that } \theta^2 = \theta\theta'$$

i.e. $\theta = 0.78$

and $\sigma = 0.06$

Furthermore, we obtain

$$\frac{\theta_t^s}{\sigma_t^s} + \frac{\theta_t^f}{\sigma_t^f} = 52.66$$

$$n_1 = 1.026855$$

$$n_2 = -.2240937185$$

$$A_1 = -41.16367854$$

$$B_2 = 2.149308425$$

$$\tilde{x} = 50.20617666$$

$$\tilde{y} = 1/R = 2$$

$$x = 42.26913160 * y^{0.26855060} e^{-1} + 14.28571429/y$$

The portfolio in that case given by

$$52.66 * (n_2 - 1) * (R/r - x) = -3223.038759 + 64.46077519 * x, \text{ if } R/r < x \leq \tilde{x}$$

$$52.66 * ((A_1 * n_1 * (n_1 - 1) * y^{(n_1 - 1) + 1} / (\beta * y)) = -59.77647582 * y^{0.26855060} e^{-1} + 148.2857145/y, \text{ if } x \geq \tilde{x}.$$

We can see the plottings of both the consumption and the portfolio with respect to wealth in Fig.(5.5) and Fig.(5.6). It is quite clear from the graph that the consumption and portfolio process are non linear functions of wealth, which implies the stochasticity of the optimal processes.

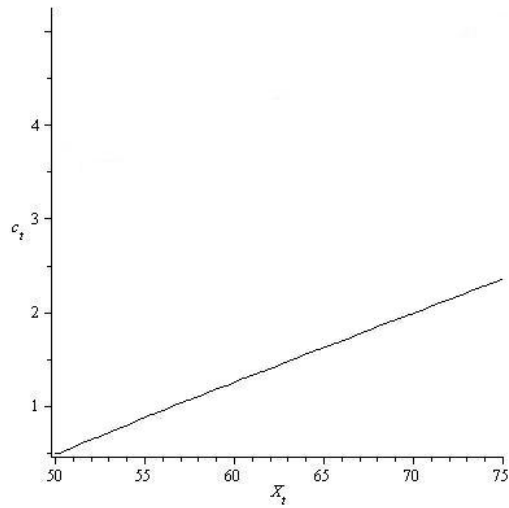


FIGURE 5.5. The Graph of X_t vs c_t

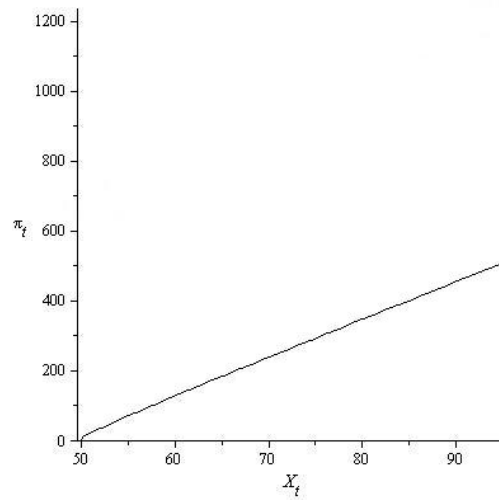


FIGURE 5.6. The Graph of X_t vs π_t

5.5. Analysis of results

Based on the above graphical and numerical outcome, we summarize results obtained as follows:

- Optimal consumption and portfolio are linear when $R/r < x \leq \tilde{x}$ and non-linear when $\tilde{x} \leq x < \infty$ with respect to $X(t)$.

- Risk averse investor has constant consumption until it reach at some wealth boundary.
- The non-linear function of portfolio and consumption with respect to wealth in Ex.(5.4) and Ex.(5.5) would make the wealth process dynamic and stochastic.
- Diverse portfolios maximize the wealth and minimize the risk.
- Portfolio diversification is an important hedging tool.

CHAPTER 6

SUMMARY, CONCLUSION AND FUTURE RESEARCH

We shall present a summary of the dissertation, emphasizing the important results of the research. We shall also discuss areas for improvement and directions for future research.

Summary and Conclusion

In this dissertation we have modified the existing models on portfolio theory and then developed a new mathematical model for an investor who is traditionally risk averse and who wants to be more secure with her risky assets by signing a forward contract to ease adverse situations. Like the existing portfolio methods in continuous horizons, we assume a complete market setup and let the investor's time horizon be infinite. We assume the financial market setup follows a stochastic differential equation driven by Brownian motion. In chapter four we solve a general optimal consumption and portfolio selection problem of an infinitely -lived investor whose consumption rate process is subject to a downside constraint. We derive explicit solutions, numerical and graphical results for the optimal portfolio problem in the case of a Log utility function.

In this dissertation we consider the financial market as a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ generated by a Brownian motion B_t . We assume that the portfolio process and the consumption rate process are measurable and adapted. We use the equivalent martingale measure and Girsanov's theorem to define the wealth process with a new probability measure $\tilde{\mathbf{P}}$. We define the dual function to obtain the value function of the optimization problem. We use the Feynman-Kac formula to obtain equivalent PDEs from the SDEs in Ito's formula. We solved the ODEs by using the

separation of variables method and obtain the dual value function. We then used the Legendre transform formula to obtain the value function. Then we obtained the optimal policies that generate the optimal value function.

In chapter five we derive the optimal portfolio and consumption process for the Log utility function for which we obtain an explicit solution. We obtain numerical solutions for the process in this case and illustrate graphically. The best reference for this dissertation is the work done by Shin et. al (2007)[1].

Areas of Improvement

We have assumed that the shocks in the financial market follow Brownian motion and that the financial market has a budget constraint consisting of one riskless, one risky and one hedge asset. Forward contracts are the best tools to hedge market risk in a complete market, but in our model, we have yet to verify that our model is better than recent existing model without forward contracts. Furthermore it would be interesting to investigate how the market model behaves when the financial assets follow a jump diffusion process. This investigation would generalize such efforts as we have made of our current work. The solutions and explicit formulas in our work are longer and complicated. We believe that perhaps we might incorporate some additional assumptions that might then reduce solutions into simpler and easy forms.

Future Research

The model developed in this work can be expanded to bring together other consideration that would enable models obtain to be more practical and flexible. Some areas of further enhancement are those below.

It would be interesting to investigate and prove that the forward contract is the best hedging tool compared to other financial hedging tools. Also, an extension can perhaps be made to the market setup in which risky assets follow Brownian motion with Jump diffusion. In fact recent research has indicated that stock market volatility

follows jump diffusion rather than just Brownian motion. Practical applications of the results are not simple and these could be extended by using algorithms and programming where the user can enter the system parameters and then run the algorithms. We can extend our work for the customer who derives utility from consumption and other variables like reward and labour. We can further assume that this agent has finite time horizon like a fixed retirement date to deal with that such an agent could buy insurance policy instead of a forward contract.

Another area for future research would be to consider these problems when all budget equation parameters follow stochastic processes instead being constants. For example, the discount process, the interest rate process, expected return from risky assets and variance can be assumed to follow a stochastic motion process. The goal is to determine and evaluate the optimal portfolio and optimal consumption policies.

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