Comments on the Non-Commutative Description of Classical Gravity

A. Stern – University of Alabama

et al.

Citation of published version:

Bimonte, G., Musto, R., Stern, A., Vitale, P. (1998): Comments on the Non-Commutative Description of Classical Gravity. *Physics Letters B,* 441(1-4).
DOI: https://doi.org/10.1016/S0370-2693(98)01200-3

26 November 1998

PHYSICS LETTERS B

Physics Letters B 441 (1998) 69–76

Comments on the non-commutative description of classical gravity

G. Bimonte [a,1], R. Musto [a,2], A. Stern [b,3], P. Vitale [a,4]

[a] *Dipartimento di Scienze Fisiche, Università di Napoli Federico II and INFN Mostra d'Oltremare, Pad.19, I-80125 Napoli, Italy*
[b] *Department of Physics, University of Alabama, Tuscaloosa, AL 35487, USA*

Received 29 April 1998; revised 19 September 1998
Editor: L. Alvarez-Gaumé

Abstract

We find a one-parameter family of Lagrangian descriptions for classical general relativity in terms of tetrads which are not c-numbers. Rather, they obey exotic commutation relations. These noncommutative properties drop out in the metric sector of the theory, where the Christoffel symbols and the Riemann tensor are ordinary commuting objects and they are given by the usual expression in terms of the metric tensor. Although the metric tensor is not a c-number, we argue that all measurements one can make in this theory are associated with c-numbers, and thus that the common invariant sector of our one-parameter family of deformed gauge theories (for the case of zero torsion) is physically equivalent to Einstein's general relativity. © 1998 Published by Elsevier Science B.V. All rights reserved.

It is well known that $3+1$ gravity admits a gauge theory description [1]. In this description, the connection one forms correspond to the tetrads and spin connections, while the dynamics is given by the Palatini action. The gauge group is the Poincaré group, although the action is only invariant under local Lorentz transformations.

In a couple of recent papers [2,3] we obtained a generalization of the gauge theory description of general relativity where the gauge group is replaced by a q-gauge group [4], while space-time remains a usual commutative manifold. This result was first achieved in three space-time dimensions [2] using a deformed Chern–Simons action [5] and a quantum group, that we denote by $ISO_q(2,1)$, which is a particular deformation of the Poincaré group in $2+1$ dimensions [6]. In the resulting description of $2+1$ gravity, although the underlying space-time is associated with an ordinary manifold, the dreibeins and spin-connections are not c-numbers, but instead, obey nontrivial braiding relations. In four space-time dimensions [3], one uses a deformed Palatini action and the quantum group $ISO_q(3,1)$ by which we denote the $3+1$ generalization of $ISO_q(2,1)$ [7]. Once again, space-time is described by an ordinary manifold, while the connections obey nontrivial braiding relations. There is, in fact, a one-parameter family of such theories, parameterized by q, and the usual (undeformed) theory is obtained in the limit

[1] E-mail: bimonte@napoli.infn.it
[2] E-mail: musto@napoli.infn.it
[3] E-mail: astern@ua1vm.ua.edu
[4] E-mail: vitale@napoli.infn.it

PII: S0370-2693(98)01200-3

$q \to 1$. The metric tensor, which is needed for recovering Einstein's theory, can be constructed as a suitable bilinear in the tetrads, and it reduces to the usual expression when $q \to 1$. A remarkable feature of our description (in the absence of torsion) is that the entire one-parameter family of descriptions (including the $q = 1$ case) has a common metric sector. That is, *all descriptions lead to the same classical dynamics - that given by the Einstein equations*. In this regard, the components of the metric tensor $g_{\mu\nu}$ commute among themselves and the field equations they satisfy are formally identical to those of the ordinary theory associated with $q = 1$. Remarkably, the relevant fields of the metric theory, such as Christoffel symbols and the Riemann tensor, turn out to be ordinary commuting fields given by the usual expressions in term of $g_{\mu\nu}$.

In this letter, after giving a concise review of our formulation of gravity, we shall present new results concerning two important issues that were not addressed in our previous papers. First, we shall show that for q real or equal to a phase there exists a set of reality conditions for our non-commuting tetrads and spin-connections which are consistent with their commutation properties, and ensure that the space-time metric is real. Second, we shall argue that the equivalence of our deformed gauge theory with classical general relativity goes beyond the formal arguments sketched above. By 'formal', we are alluding to the fact that our deformed metric tensor does not commute with all the connections of the theory and hence is not a c-number. One may therefore question its physical relevance, since the metric is necessary for defining distances. Since the Einstein-Hilbert action $\mathscr{S}_{EH}$ contains $g_{\mu\nu}$, one might conclude that $\mathscr{S}_{EH}$ too is not a c-number, causing new obstacles, say if one were to attempt a path integral quantization. We will show below that despite the fact that the metric is noncommuting, the ratio of *any* two quantities with the units of 'length' can be made to be in the center of the algebra, and hence any actual distance measurement one makes in this theory can be associated with a c-number. Moreover, once we allow dimensionful quantities (say, for example, the analogue of Newton's constant) to have non trivial commutation properties, we can argue that *all* possible dimensionless ratios, and hence *all* possible measurements, are associated with c-numbers, and thus are physically meaningful. Concerning the action integral, although it is not a c-number, we find that it has homogeneous commutation relations. By that, we mean that all terms appearing in the integrand have the same commutation relations. (We should point out that this, in general, is not guaranteed when dealing with theories of noncommuting fields.) $\mathscr{S}_{EH}$ can then be made to be central in the algebra of fields, when one allows for the dimensionful coefficient, here the analogue of Newton's constant, to be noncommuting. The commutation properties required for this purpose are precisely those which are needed to make dimensionless ratios central elements.

It should be stressed that the equivalence with the usual metric theory holds, not only for pure gravity, but also in the presence of matter, provided there are no sources for torsion. Upon applying the canonical formalism to the above system in any number of dimensions, one obtains a one-parameter family of Hamiltonian descriptions for gravity. [3] They may or may not lead to inequivalent quantum theories. This is currently under investigation, and will not concern us in this letter.

The starting point of the construction is a particular deformation of the Poincaré group, first found in [7], where other inhomogeneous groups are also treated. This deformation is different from others already known in the literature [8]. The technique used in [7] to derive the quantum inhomogenous groups $ISO_q(N)$ together with their bicovariant differential calculus is based on a projection from the multiparametric $SO_{q,r=1}(N+2)$, where $r = 1$ corresponds to the minimal deformations, or twistings, with diagonal R matrix.

The non-commutative algebra of functions which defines the quantum group can be described very simply in terms of a Lorentz vector z_a and a Lorentz matrix ℓ_{ab}, whose elements, instead of being c-numbers, have the following non-trivial commutation relations

$$z^a \, \ell^b_{\ c} = q^{\Delta(b)} \, \ell^b_{\ c} \, z^a \,, \tag{1}$$

where $\Delta(1) = -1, \Delta(2) = \Delta(3) = 0, \Delta(4) = 1,$ while all other commutation relations are trivial. The

Lorentz metric tensor is taken to be the following off-diagonal matrix:

$$\eta = \begin{pmatrix} & & & 1 \\ & 1 & & \\ & & 1 & \\ 1 & & & \end{pmatrix}. \quad (2)$$

Then the commutation relations (1) are consistent with the Lorentz constraints

$$\ell_{ab}\ell_c^{\ b} = \ell_{ba}\ell_c^{\ b} = \eta_{ac}, \quad \det\|\ell_{ab}\| = 1, \quad (3)$$

due to the identity $\eta_{ab} = q^{\Delta(a)+\Delta(b)}\eta_{ab}$. $ISO_q(3,1)$ thus contains the undeformed Lorentz group. The algebra of functions generated by $\{\ell_{ab}, z_a\}$ is a non-commutative Hopf algebra, that is, a quantum group. A rigorous proof, together with the explicit construction of the co-structures and a bicovariant differential calculus, may be found in [7].

We need now to formulate the reality properties of the group elements. Thus we introduce a $*$-involution on the q-Poincaré group with the usual property for the conjugation of a product, $(\alpha\beta)^* = \beta^*\alpha^*$. We shall demand that the commutation relations (1) be preserved under conjugation, with z^a real (i.e., $(z^a)^* = z^a$) and $\ell_a^{\ b}$ defining the Lorentz group. We find that this is possible for two cases:

i) q is a phase. Then the commutation relations (1) are preserved with $(z^a)^* = z^a$ and $(\ell_a^{\ b})^* = \ell_a^{\ b}$.

ii) q is real. The commutation relations (1) are preserved with $(z^a)^* = z^a$ and $(\ell_a^{\ b})^*$ proportional to $\ell_{ab} = \ell_a^{\ d}\eta_{db}$. It remains to ensure that the matrices $\ell_a^{\ b}$ still define the Lorentz group. For this we can choose the proportionality constants according to:

$$\left(\ell_a^{\ 1}\right)^* = \ell_a^{\ 4}, \quad \left(\ell_a^{\ 2}\right)^* = \ell_a^{\ 2},$$
$$\left(\ell_a^{\ 3}\right)^* = -\ell_a^{\ 3}, \quad \left(\ell_a^{\ 4}\right)^* = \ell_a^{\ 1}. \quad (4)$$

Although $\ell_a^{\ b}$ are not real, real matrices $\ell_{\tilde a}^{\ b}$ can be constructed according to

$$\ell_{\tilde a}^{\ 1} = \frac{1}{\sqrt 2}\left(\ell_a^{\ 2} + i\ell_a^{\ 3}\right), \quad \ell_{\tilde a}^{\ 2} = \frac{1}{\sqrt 2}\left(\ell_a^{\ 1} + \ell_a^{\ 4}\right)$$
$$\ell_{\tilde a}^{\ 3} = \frac{i}{\sqrt 2}\left(\ell_a^{\ 1} - \ell_a^{\ 4}\right), \quad \ell_{\tilde a}^{\ 4} = \frac{1}{\sqrt 2}\left(\ell_a^{\ 2} - i\ell_a^{\ 3}\right), \quad (5)$$

and it can be checked that these matrices satisfy the Lorentz conditions (3), e.g. $\tilde\ell_{ab}\tilde\ell_c^{\ b} = \tilde\ell_{ba}\tilde\ell_c^{\ b} = \eta_{ac}$.

A set of (left-)invariant Maurer-Cartan forms on $ISO_q(3,1)$ can be constructed according to:

$$\omega^{ab} = \left(\ell^{-1}d\ell\right)^{ab}, \quad e^a = \left(\ell^{-1}dz\right)^a. \quad (6)$$

(Notice that the Lorentz constraints (3) imply that $(\ell^{-1})_a^{\ b} = \ell_a^{\ b}$). The differential d, in these formulae, satisfies all the usual properties of the undeformed case, like $d^2 = 0$ and the Leibnitz rule. See Ref. [3]. The commutation properties of (the components of) the Maurer-Cartan one-forms follow easily from Eqs. (1):

$$\omega_\mu^{ab}\omega_\nu^{cd} = \omega_\nu^{cd}\omega_\mu^{ab},$$
$$e_\mu^a\omega_\nu^{bc} = q^{\Delta(b)+\Delta(c)}\,\omega_\nu^{bc}e_\mu^a,$$
$$e_\mu^a e_\nu^b = q^{\Delta(b)-\Delta(a)}\,e_\nu^b e_\mu^a. \quad (7)$$

As for their reality properties, for i) q equal to a phase, e^a and ω^{ab} are real, while for ii) q real, one finds:

$$\omega^{ab\,*} = \omega_{ab}, \quad e^{a\,*} = q^{-\Delta(a)}e_a. \quad (8)$$

Under infinitesimal (right) Poincaré transformations, the transformation law of the Maurer-Cartan forms has the standard form:

$$\delta\omega^{ab} = d\tau^{ab} + \omega_c^{\ a}\,\tau^{cb} - \omega_c^{\ b}\,\tau^{ca},$$
$$\delta e^c = d\rho^c + \omega_b^{\ c}\,\rho^b - \tau^{cb}\,e_b. \quad (9)$$

but consistency with Eqs. (7) implies that the gauge parameters ρ^a and τ^{ab} must be non-commuting elements such that:

$$\rho^a\,\omega^{bc} = q^{\Delta(b)+\Delta(c)}\,\omega^{bc}\,\rho^a,$$
$$\rho^a\,e^b = q^{\Delta(b)-\Delta(a)}\,e^b\,\rho^a$$
$$\tau^{ab}\,e^c = q^{-\Delta(a)-\Delta(b)}\,e^c\,\tau^{ab},$$
$$\tau^{ab}\omega^{cd} = \omega^{cd}\tau^{ab}. \quad (10)$$

When e^a and ω^{ab} are given by (6), they satisfy a set of Maurer-Cartan equations $\mathcal{R}^{ab} = \mathcal{T}^a = 0$, where the curvature $\mathcal{R}^{ab}$ and the torsion and $\mathcal{T}^a$ have the usual expression

$$\mathcal{R}^{ab} = d\omega^{ab} + \omega_c^{\ a}\wedge\omega^{cb},$$
$$\mathcal{T}^a = de^a + \omega_b^{\ a}\wedge e^b. \quad (11)$$

Even though Eqs. (6), (9), and (11) all look identical to the standard expressions, one should keep in mind that they involve non-commuting quantities and so the ordering is crucial in all of them.

The passage to gauge theory is now achieved upon relaxing the flatness conditions Eq. (6) on the forms e^a and ω^{ab}, while keeping Eqs. (7)–(11), and pulling them back from the quantum group to space-time. One thus ends up having a non-commuting set of tetrad and spin-connection one-forms defined on space-time. Next one writes down a locally Lorentz invariant action:

$$\mathscr{S} = \frac{1}{4}\int_M q^{-\Delta(d)}\epsilon_{abcd}\mathscr{R}^{ab}\wedge e^c\wedge e^d\,, \tag{12}$$

M is a four manifold and ϵ_{abcd} is the ordinary, totally antisymmetric tensor with $\epsilon_{1234}=1$. The expression (12) differs from that of the undeformed case by the $q^{-\Delta(d)}$ factor. Note that this factor can be written differently using the identity

$$q^{\Delta(a)+\Delta(b)+\Delta(c)+\Delta(d)}\,\epsilon_{abcd} = \epsilon_{abcd}\,. \tag{13}$$

As in the undeformed case, the action is invariant under the full set of local Poincaré transformations (9), provided we impose the torsion to be zero upon making the variations. The expression (12) also differs from that of the undeformed case due to the fact that it is not a c-number. On the other hand, it has definite commutation properties, i.e. each term in the sum has the same commutation relations with the connection one forms.

The equations of motion obtained from varying the tetrads have the usual form, i.e.

$$\epsilon_{abcd}\mathscr{R}^{ab}\wedge e^c = 0\,, \tag{14}$$

while varying ω^{ab} gives

$$\epsilon_{abcd}\mathscr{T}^c\wedge e^d\, q^{-\Delta(d)} = 0\,. \tag{15}$$

To make a connection with Einstein gravity, we need to introduce the space-time metric $\mathrm{g}_{\mu\nu}$ on M. As in the undeformed case, it has to be a real bilinear in the tetrads which is symmetric in the space-time indices and invariant under local Lorentz transformations. It should also reduce to the usual expression in the limit $q\to 1$. These requirements uniquely fix $\mathrm{g}_{\mu\nu}$ to be

$$\mathrm{g}_{\mu\nu} = q^{\Delta(a)}\,\eta_{ab}\,e^a_\mu e^b_\nu\,, \tag{16}$$

Using Eqs. (7) we see that $\mathrm{g}_{\mu\nu}$ is symmetric, although the tensor elements are not c-numbers since

$$\mathrm{g}_{\mu\nu}\,\omega^{ab}_\rho = q^{2\Delta(a)+2\Delta(b)}\,\omega^{ab}_\rho\,\mathrm{g}_{\mu\nu}\,,$$
$$\mathrm{g}_{\mu\nu}\,e^a_\rho = q^{2\Delta(a)}\,e^a_\rho\,\mathrm{g}_{\mu\nu}\,. \tag{17}$$

The components of $\mathrm{g}_{\mu\nu}$ do however commute with themselves. Finally, one can check that the $\mathrm{g}_{\mu\nu}$ are real, for both hermiticity structures i) and ii) given above.

The inverse e^μ_a of the tetrads e^a_μ can be defined if we enlarge our algebra by a new element e^{-1} such that:

$$\mathrm{e}^{-1}e^a_\mu = q^{-4\Delta(a)}e^a_\mu\,\mathrm{e}^{-1}\,, \tag{18}$$

$$\mathrm{e}^{-1}\omega^{ab}_\mu = q^{-4(\Delta(a)+\Delta(b))}\omega^{ab}_\mu\,\mathrm{e}^{-1}\,, \tag{19}$$

and $\mathrm{e}^{-1}\mathrm{e}=1$. Here e is the determinant: $\mathrm{e}=\epsilon^{\mu\nu\rho\sigma}e^1_\mu e^2_\nu e^3_\rho e^4_\sigma$, which is consistent because its left hand side commutes with everything, due to Eqs. (19). Moreover, one can check that $\mathrm{e}^{-1}\mathrm{e}=\mathrm{e}\mathrm{e}^{-1}$. The inverse of the tetrads can now be written:

$$e^\mu_a = \frac{1}{3!}\,\hat{\epsilon}_{abcd}\,\epsilon^{\mu\nu\rho\sigma}e^b_\nu e^c_\rho e^d_\sigma\,\mathrm{e}^{-1}\,, \tag{20}$$

where the totally q-antisymmetric tensor $\hat{\epsilon}_{abcd}$ is defined such that

$$\hat{\epsilon}_{abcd}\,e^a\wedge e^b\wedge e^c\wedge e^d$$
$$= e^1\wedge e^2\wedge e^3\wedge e^4 \quad \text{no sum on } a,b,c,d \tag{21}$$

The solution to this equation can be expressed by

$$\hat{\epsilon}_{abcd} = q^{3\Delta(a)+2\Delta(b)+\Delta(c)+3}\,\epsilon_{abcd}\,. \tag{22}$$

It is easy to prove that the inverse of the tetrads (20) have the usual properties:

$$e^a_\mu e^\mu_b = e^\mu_b e^a_\mu = \delta^a_b\,,\quad e^a_\mu e^\nu_a = e^\nu_a e^a_\mu = \delta^\nu_\mu\,. \tag{23}$$

By using the inverse of the tetrads, one can now prove that Eq. (15) implies the vanishing of the torsion. Details can be found in [3].

The Christoffel symbols $\Gamma^\sigma_{\mu\nu}$ are defined in the same way as in the undeformed case, i.e. by demanding that the covariant derivative of the tetrads vanishes,

$$0 = \mathrm{D}_\mu e^b_\nu = \partial_\mu e^b_\nu + \omega^{bc}_\mu e_{\nu c} - \Gamma^\sigma_{\mu\nu}e^b_\sigma\,. \tag{24}$$

The difference with the undeformed case is that we cannot switch the order of the fields arbitrarily. To eliminate the spin-connection from (24) we now multiply on the left by $q^{\Delta(a)}\eta_{ab} e^a_\rho$, and proceed as in the undeformed case. We can then isolate $\Gamma^{\sigma}_{\mu\nu}$ according to

$$2\, q^{\Delta(a)}\eta_{ab} e^a_\rho e^b_\sigma \Gamma^{\sigma}_{\mu\nu} = q^{\Delta(a)}\eta_{ab}\Big[e^a_\rho\big(\partial_\mu e^b_\nu + \partial_\nu e^b_\mu\big) + e^a_\nu\big(\partial_\mu e^b_\rho - \partial_\rho e^b_\mu\big) + e^a_\mu\big(\partial_\nu e^b_\rho - \partial_\rho e^b_\nu\big)\Big] \tag{25}$$

or

$$2 g_{\rho\sigma} \Gamma^{\sigma}_{\mu\nu} = \partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\nu\mu} . \tag{26}$$

To solve this equation we need the inverse of the metric $g^{\mu\nu}$. The expression

$$g^{\mu\nu} = q^{\Delta(a)}\eta^{ab} e^\mu_a e^\nu_b , \tag{27}$$

does the job as it can be checked that $g^{\mu\rho} g_{\rho\nu} = g_{\nu\rho} g^{\rho\mu} = \delta^\mu_\nu$. Notice that unlike in the usual Einstein Cartan theory $g^{\mu\nu}\eta_{ab} e^b_\nu = q^{\Delta(a)} e^\mu_a$. We are now able to solve Eq. (26). Upon multiplying it by $g^{\tau\rho}$ on each side, we get the usual expression for the Christoffel symbols in terms of the metric tensor and its inverse. It may be verified, using these expressions, that the Christoffel symbols commute with everything and thus, even if written in terms of non-commuting quantities, they can be interpreted as being ordinary numbers.

The covariant derivative operator ∇_μ defined by the Christoffel symbols is compatible with the metric $g_{\mu\nu}$, i.e. $\nabla_\mu g_{\nu\rho} = 0$. This is clear because our Christoffel symbols have the standard expression in terms of the space-time metric $g_{\mu\nu}$, but it also follows from Eq. (24)

$$\nabla_\mu g_{\nu\rho} = D_\mu g_{\nu\rho} = D_\mu\big(q^{\Delta(a)}\eta_{ab} e^a_\nu e^b_\rho \big) = 0 . \tag{28}$$

We now construct the Riemann tensor. It is defined as in the undeformed theory:

$$R_{\mu\nu\rho}{}^{\sigma} v_\sigma = \big(D_\mu D_\nu - D_\nu D_\mu \big) v_\rho , \tag{29}$$

where v_μ is a vector. It follows from (24) that it has the standard expression in terms of the Christoffel symbols (and thus in terms of the space-time metric and its inverse) and therefore its components commute with everything. (This is also true for the Ricci tensor $R_{\mu\nu} = R_{\mu\sigma\nu}{}^{\sigma}$, of course, but not for $R_{\mu\nu\rho\tau}$ as the lowering of the upper index of the Riemann tensor implies contraction with $g_{\sigma\tau}$ which is not in the center of the algebra). The relation among the Riemann tensor and the curvature of the spin connection follows from Eq. (24):

$$\begin{aligned} e^a_\sigma R_{\mu\nu\rho}{}^{\sigma} v^\rho &= e^a_\sigma \big(D_\nu D_\mu - D_\mu D_\nu \big) v^\sigma \\ &= \big(\mathscr{D}_\nu \mathscr{D}_\mu - \mathscr{D}_\mu \mathscr{D}_\nu \big) e^a_\sigma v^\sigma \\ &= -\mathscr{R}^{ac}_{\mu\nu} \eta_{bc} e^b_\sigma v^\sigma , \end{aligned} \tag{30}$$

$\mathscr{D}_\nu e^a_\sigma = \partial_\nu e^a_\sigma + \omega_\nu^{ab} e_{\sigma b}$, with $\mathscr{R}^{ab}_{\mu\nu}$ being the space-time components of $\mathscr{R}^{ab}$. As v^μ is arbitrary, it follows from the above equation that:

$$R_{\mu\nu\rho}{}^{\tau} = -\mathscr{R}^{ac}_{\mu\nu} \eta_{bc} e^b_\rho e^\tau_a . \tag{31}$$

Using this equation it can be checked directly that the components of the Riemann tensor commute with everything, as pointed out earlier. Our Riemann tensor has the usual symmetry properties:

$$R_{\mu\nu\rho}{}^{\sigma} = -R_{\nu\mu\rho}{}^{\sigma} , \quad R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho} , \qquad R_{[\mu\nu\rho]}{}^{\sigma} = 0 . \tag{32}$$

The first of these equations is obvious. The proof of the other two can be found in Ref. [3].

We now show that the action (12) becomes equal to the *undeformed* Einstein-Hilbert action $\mathscr{S}_{EH}$, once the spin connection is eliminated using its equations of motion, namely the zero torsion condition. As in the undeformed case, first we rewrite (12) in a form analogous to Palatini action, and then show that the latter reduces to the *undeformed* Einstein-Hilbert action, once the spin-connection is eliminated from it. Consider thus the following deformation of the Palatini action:

$$\mathscr{S} = \frac{1}{2}\int_M d^4x\, q^{\Delta(a)-3} e\, e^\mu_a e^\nu_b \mathscr{R}^{ab}_{\mu\nu} . \tag{33}$$

To see that it coincides with (12), we use the identity:

$$q^{\Delta(a)-\Delta(b)-6}\, \hat{\epsilon}^{abcd} e^\mu_a e^\nu_b e = -\epsilon^{\mu\nu\lambda\sigma} e^c_\lambda e^d_\sigma . \tag{34}$$

The result (33) then follows after multiplying both sides of this equation on the left by $-1/8\, q^{-2\Delta(f)-\Delta(g)-3}\, \hat{\epsilon}_{fgcd} \mathscr{R}^{fg}_{\mu\nu}$ and using the identity

$$\hat{\epsilon}_{fgcd} \hat{\epsilon}^{abcd} = -2 q^6 \big(\delta^a_f \delta^b_g - q^{\Delta(f)-\Delta(g)} \delta^a_g \delta^b_f \big) ,$$

along with (22).

To show that Eq. (33) in turn is equal to the undeformed Einstein-Hilbert action, we eliminate the spin connection via its equation of motion. This amounts to expressing $\mathscr{R}^{ab}_{\mu\nu}$ in terms of the Riemann tensor by inverting Eq. (31) and then plugging the result in Eq. (33). We have:

$$q^{\Delta(a)} e^{\mu}_{a} e^{\nu}_{b} \mathscr{R}^{ab}_{\mu\nu} = -q^{\Delta(a)} \mathrm{R}_{\mu\nu\rho}{}^{\tau} e^{\mu}_{a} e^{\nu}_{b} e^{a}_{\tau} e^{b\rho}$$

$$= -q^{\Delta(b)} \mathrm{R}_{\mu\nu\rho}{}^{\mu} e^{\nu}_{b} e^{b\rho} = \mathrm{R}_{\nu\mu\rho}{}^{\mu} \mathrm{g}^{\nu\rho} = \mathrm{R}, \qquad (35)$$

where we have made use of (23). Moreover we get, after a short calculation [3]:

$$\mathrm{g} \equiv \det \| \mathrm{g}_{\mu\nu} \|$$

$$= \frac{1}{4!} \epsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \epsilon^{\nu_1 \nu_2 \nu_3 \nu_4} \mathrm{g}_{\mu_1\nu_1} \mathrm{g}_{\mu_2\nu_2} \mathrm{g}_{\mu_3\nu_3} \mathrm{g}_{\mu_4\nu_4}$$

$$= q^{-6} \mathrm{e}^2, \qquad (36)$$

Putting together (35) and (36) we see that the q-Palatini action (33) becomes equal to:

$$\mathscr{S}_{EH} = \frac{1}{2} \int_M d^4x \sqrt{-\mathrm{g}}\, \mathrm{R}, \qquad (37)$$

which is the *undeformed* Einstein-Hilbert action. It is obviously real, since $\mathrm{g}_{\mu\nu}$ are real. Moreover, since the components of $\mathrm{g}_{\mu\nu}$ and its inverse all commute among themselves, it is clear that the equations of motion of the metric theory will be equal to those of the undeformed Einstein theory in vacuum. One can obtain the same result starting directly from Eq. (14) and using (31).

We now address the issue of the physical interpretation of our construction. As we have seen above, in our theory the components of the metric tensor $\mathrm{g}_{\mu\nu}$, even though they commute amongst themselves, are not c-numbers, as their commuting properties with the tetrads and the spin-connections are nontrivial. This raises the doubt that our theory, although formally resembling ordinary general relativity, may in fact not be *physically* equivalent to ordinary general relativity, already at the classical level. We remarked that the Christoffel symbols $\Gamma^{\mu}_{\nu\rho}$ are c-numbers and, consequently, so is the Riemann tensor, which encodes most of the geometric information on the space-time manifold. Moreover, we can write the geodesic equation for a test particle moving in a gravitational field, which in its parametric form only involves the Christoffel symbols:

$$\frac{d^2 x^{\mu}}{d\sigma^2} + \Gamma^{\mu}_{\nu\rho} \frac{dx^{\nu}}{d\sigma} \frac{dx^{\rho}}{d\sigma} = F^{\mu}(\sigma),$$

where we have included an arbitrary force $F^{\mu}(\sigma)$ which transforms under diffeomorphisms like $\frac{dx^{\mu}}{d\sigma}$.

But is this enough to conclude the equivalence of our theory with the standard metric theory? After all, the invariant ‘‘distance’’ l between any two points of space-time, measured along some path $\mathscr{C}$ connecting them:

$$l_{\mathscr{C}} = \int_{\mathscr{C}} ds \qquad (38)$$

is an observable quantity. However, ds constructed from our metric tensor is not a c-number, and hence its physical meaning is unclear. A closer inspection is therefore necessary. In this regard, we first remark that when we measure a dimensionful quantity (like length) what we are actually doing is comparing it with a standard unit (like a meter). So we need not require that all dimensionful quantities be c-numbers. Rather, what really matters is that the *ratio* of any two quantities carrying the same units is a c-number. As we shall see below, this is indeed possible in our description of gravity.

To be definite, let us consider a self-gravitating system of spinless (electrically) charged point-particles moving in a four dimensional space-time. In the metric formulation of gravity it is described by the action (we take $c=1$):

$$S = \frac{1}{4\pi G} \mathscr{S}_{EH}$$

$$- \sum_{\alpha} \left\{ m_{\alpha} \int ds^{(\alpha)} + \tilde{e}_{\alpha} \int A_{\mu}(x^{(\alpha)})\, dx^{(\alpha)\mu} \right\}$$

$$- \frac{1}{16\pi} \int d^4x \sqrt{-\mathrm{g}}\, F_{\mu\nu} F^{\mu\nu}, \qquad (39)$$

where G is Newton's constant, m_{α} are the masses of the particles, A_{μ} is the potential associated with the electromagnetic (e.m.) field strength $F_{\mu\nu}$ and $\tilde{e}_{\alpha}$ are the electric charges of the particles. Using the parameters of this model, we can construct a number of quantities having the dimension of a length, which play the rôle of natural units for this system. For example, we could define:

$$l_\alpha^{(1)} = G m_\alpha\,, \quad l_\alpha^{(2)} = \frac{\tilde{e}_\alpha^2}{m_\alpha}\,. \tag{40}$$

If we also use Planck's constant, we can of course introduce the Planck's length:

$$l^{(P)} = \sqrt{G\hbar}\,. \tag{41}$$

What we shall prove below is that in order for the equations of motions for the particles and fields to be consistent, the coupling constants in the action (39) cannot in general be taken as c-numbers. (In general, the classical fields need not be c-numbers either.) Although, at first sight, this appears to be a problem, it is in fact a blessing, because the different length units shown above acquire just the correct commutation properties needed to make the ratios $l_{\mathscr{C}}/l_\alpha^{(i)}$ c-numbers.

In order to see this, let us go back to the action (39). The integral in the first term, i.e. the Einstein-Hilbert action $\mathscr{S}_{EH}$, is not a c-number. Even though it commutes with $\mathrm{g}_{\mu\nu}$, it fails to commute with the tetrads and the spin-connections, since:

$$\mathscr{S}_{EH}\,\omega_\mu^{ab} = q^{2\Delta(a)+2\Delta(b)}\omega_\mu^{ab}\,\mathscr{S}_{EH}\,,$$
$$\mathscr{S}_{EH}\,e_\mu^a = q^{2\Delta(a)}e_\mu^a\,\mathscr{S}_{EH}\,. \tag{42}$$

Similarly, the integral in the second term, giving the interaction between the particles and the gravitational field, fails to commute with the tetrads and the spin connections. On the contrary, at least in four dimensions, we can consistently keep the e.m. potential and the electric charges as c-numbers.

If we now insist that the action be a c-number, which seems desirable if one is to quantize our system using path-integral techniques, we are forced to give G and m_α nontrivial commutation relations. (Alternatively, if the action were not a c-number, $\hbar$ could not be a c-number.) It is easy to verify that the following commutators do the job:

$$G\,\omega_\mu^{ab} = q^{2(\Delta(a)+\Delta(b))}\omega_\mu^{ab}\,G\,, \quad G\,e_\mu^a = q^{2\Delta(a)}e_\mu^a\,G\,,$$
$$G\,m_\alpha = m_\alpha\,G\,,$$
$$m_\alpha\,\omega_\mu^{ab} = q^{-\Delta(a)-\Delta(b)}\omega_\mu^{ab}\,m_\alpha\,,$$
$$m_\alpha\,e_\mu^a = q^{-\Delta(a)}e_\mu^a\,m_\alpha\,, \tag{43}$$

while the electric charges and fields commute with everything. We remark that this choice guarantees that G and m_α commute with $\mathrm{g}_{\mu\nu}$, so that when deriving the equations of motion from (39), no ordering problem is encountered and one gets the equations of motion of Einstein gravity. The above commutation relations ensure that the equations of motion are consistent. Here for simplicity, let us set all the electric charges to zero and write down Einstein's equations:

$$\mathrm{R}_{\mu\nu} - \frac{1}{2}\mathrm{g}_{\mu\nu}\mathrm{R} = 8\pi G\,T_{\mu\nu}^{(part)}\,, \tag{44}$$

where

$$T^{(part)\mu\nu}(x^0,\bar{x}) = \sum_\alpha \frac{m_\alpha}{\sqrt{-\mathrm{g}}}\frac{dx^{(\alpha)\mu}}{d\tau}\frac{dx^{(\alpha)\nu}}{dx^0}\delta^{(3)}(\bar{x}-\bar{x}^{(\alpha)}(x^0))\,. \tag{45}$$

In the above equation $\bar{x}$ denotes the ''space''-coordinates x^i and we think of the particle trajectories $\bar{x}^{(\alpha)}(x^0)$ as parameterized by the ''time'' coordinate x^0. Thus

$$\frac{d}{d\tau} = \left\{\mathrm{g}_{\mu\nu}\frac{dx^{(\alpha)\mu}}{dx^0}\frac{dx^{(\alpha)\mu}}{dx^0}\right\}^{-1/2}\frac{d}{dx^0}\,.$$

We see that if the masses and Newton's constant had been c-numbers the right hand side of (44) would not have been a c-number (this can be seen commuting it with, say the tetrads), contrary to the left hand side of (44). On the other hand, using the commutation properties (43) solves this difficulty. Analogous results hold for all the equations of motion.

Moreover, one can easily check that the reference lengths Eqs. (40)–(41) have commutation properties such that the ratios $l_{\mathscr{C}}/l_\alpha^{(i)}$ commute with everything:

$$\left(\frac{l_{\mathscr{C}}}{l_\alpha^{(i)}}\right)\omega_\mu^{ab} = \omega_\mu^{ab}\left(\frac{l_{\mathscr{C}}}{l_\alpha^{(i)}}\right), \quad \left(\frac{l_{\mathscr{C}}}{l_\alpha^{(i)}}\right)e_\mu^a = e_\mu^a\left(\frac{l_{\mathscr{C}}}{l_\alpha^{(i)}}\right), \tag{46}$$

and thus are c-numbers, as promised earlier. (Here we assume the existence of inverse lengths $1/l_\alpha^{(i)}$.) The same is true, of course, for the ratio of any two masses. What about the strength of the gravitational interaction? In this regard, the meaningful thing to do is to compare the gravitational force between any two particles with the electric force between them and thus consider the dimensionless ratios: $\frac{G m_\alpha m_\beta}{\tilde{e}_\alpha \tilde{e}_\beta}$. It easily follows from Eqs. (43) that this quantity commutes with everything.

We point out that these conclusions do not depend on the combination of the parameters that one chooses to construct the unit of measure for the lengths or the strength of the gravitational interaction. Consider for example the former. Keeping into account the units of G, $\tilde{e}_\alpha$ and $\hbar$ (we remind the reader that we are assuming $c=1$ and thus a length has the same dimension as a time), it is easy to check that a monomial of the form:

$$\mathscr{P} = G^p m^q \tilde{e}^{2r} \hbar^s \tag{47}$$

will have the dimensions of a length if and only if:

$$p+r+s=1\,, \quad -p+q+r+s=0\,. \tag{48}$$

Consider now commuting $\mathscr{P}$ with, say, the tetrads. One finds:

$$\mathscr{P}\, e^a_\mu = q^{(2p-q)\Delta(a)} e^a_\mu\, \mathscr{P} \tag{49}$$

Subtracting the second of Eqs. (48) from the first, we see that $2p-q=1$ and thus the above commutator can be rewritten as:

$$\mathscr{P}\, e^a_\mu = q^{\Delta(a)} e^a_\mu\, \mathscr{P}\,. \tag{50}$$

Repeating the same computation for the commutator of $\mathscr{P}$ with the spin-connections would give:

$$\mathscr{P}\, \omega^{ab}_\mu = q^{\Delta(a)+\Delta(b)} \omega^{ab}_\mu\, \mathscr{P}\,. \tag{51}$$

Thus the ratios $l_{\mathscr{C}}/\mathscr{P}$ are c-numbers for any choice of $\mathscr{P}$. An analogous result holds for any monomial $\mathscr{P}$ with the units of G.

In view of the above considerations, we can now claim the complete *physical* equivalence of our theory with Einstein gravity.

From the above results we may conclude that if we just consider the theory constructed in terms of the space-time metric $g_{\mu\nu}$, ignoring the underlying gauge formulation, our theory is completely equivalent to Einstein's theory. No trace of the non-commutative structure existing in the gauge formulation of the theory can be found at the metric level. Though the metric does not commute with the connection components, all the physical objects constructed out of it, e. g. the Christoffel symbols together with the Riemann, Ricci and Einstein tensors, are c-numbers. Thus it appears that, at the level of *classical* general relativity we can choose whatever representative of the one parameter family of q-gauge theories (not only the well known $q=1$ theory) without changing the physics we are describing. That is, we have discovered a non-commutative structure in general relativity which is hidden, even in the presence of matter, provided there are no sources of torsion.

R.M. acknowledges E.E.C. contract ERBFMRX-CT96-0045 and a grant of the Italian Ministry of Education and Scientific Research. A.S. was supported in part by the U.S. Department of Energy under contract number DE-FG05-84ER40141. A.S. would like to thank the members of the theory group at the University of Naples for their warm hospitality while this work was completed.

References

[1] R. Utiyama, Phys. Rev. 101 (1956) 597; T.W.B. Kibble, J. Math. Phys. 2 (1961) 212; F.W. Hehl, P. van der Heyde, G.D. Kerlich, J.M. Nester, Rev. Mod. Phys. 48 (1976) 393.
[2] G. Bimonte, R. Musto, A. Stern, P. Vitale, 2+1 Einstein Gravity as a Deformed Chern-Simons Theory, to appear in Int. J. Mod. Phys. A.
[3] G. Bimonte, R. Musto, A. Stern, P. Vitale, Nucl. Phys. B 525 (1998) 483.
[4] L. Castellani, Mod. Phys. Lett. A 9 (1994) 2835; Phys. Lett. B 292 (1992) 93; B 327 (1994) 22.
[5] G. Bimonte, R. Musto, A. Stern, P. Vitale, Phys. Lett. B 406 (1997) 205.
[6] L. Castellani, Comm. Math. Phys. 171 (1995) 383.
[7] P. Aschieri, L. Castellani, Lett. Math. Phys. 36 (1996) 197; Int. J. Mod. Phys. A 11 (1996) 4513.
[8] O. Ogievetskii, W.B. Schmidke, J. Wess, B. Zumino, Commun. Math. Phys. 150 (1992) 495; L. Lukierski, A. Nowicki, H. Ruegg, Phys. Lett. B 293 (1992) 344; S. Majid, H. Ruegg, Phys. Lett. B 334 (1994) 348; A. Stern, I. Yakushin, J. Math. Phys. 37 (1996) 2053.