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# Integrability of the Wess–Zumino–Witten model as a non-ultralocal theory

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## Abstract

We consider the 2-dimensional Wess–Zumino–Witten (WZW) model in the canonical formalism introduced by Rajeev, Sparano and Vitale [Int J. Mod. Phys. A 31 (1994) 5469]. Using an  $r$ - $s$  matrix approach to non-ultralocal field theories we find the Poisson algebra of monodromy matrices and of conserved quantities with a new, non-dynamical,  $r$  matrix.

## 1. Introduction

In the last decades a large class of 2-dimensional solvable models in field theory has been well established both at the classical and at the quantum level, via the inverse scattering method; for a comprehensive review see [1] and references therein. The distinguishing feature of such solvable models is the existence of a couple of matrices (a Lax pair), which are functions of the fields and of a spectral parameter, and which linearize the equations of motion. Such matrices satisfy a Poisson algebra which is ultralocal, which means that it doesn't contain derivatives of the delta function. It is this property which ensures the existence of a well defined Poisson bracket for monodromy matrices, the Jacobi identity for such an algebra being insured by the Yang–Baxter equation for an  $r$ -matrix.

Ultralocality also implies that the  $r$  matrix is non-dynamical (i.e., it doesn't depend on space–time coordinates). As a result an infinite subset of conserved quantities of the theory is in involution.

Unfortunately interesting theories like the WZW model are non-ultralocal. As was shown in [5], non-ultralocality introduces discontinuous functions in the Poisson brackets of monodromy matrices, making it difficult to have a well defined algebra of conserved quantities; also, the Jacobi identity does not follow anymore from a Yang–Baxter equation. Nonetheless many attempts have been made to solve theories of this kind and solutions have been proposed for different models [4–16].

In this paper we follow the approach contained in [4] where an extended Yang–Baxter type algebra for the monodromy matrices is established and conditions for the conserved quantities to be in involution are given.

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## 2. Canonical formalism for the WZW model and associated linear system

In [2] a new canonical formalism for the WZW model was given. In this section we first sketch the conventional description, then summarize the results found in [2] which will be used in the paper, and introduce the Lax pair for our theory.

The action which describes the model is

$$S = \frac{1}{4\chi^2} \int d^2x \operatorname{Tr}(\partial_\mu g \partial_\nu g^{-1}) \eta^{\mu\nu} \sqrt{-\det(\eta)} + n\Gamma, \quad (1)$$

where  $g : R^{1,1} \rightarrow SU(2)$  is a map from the two-dimensional Minkowski space to the simple compact Lie group  $SU(2)$ ,  $\eta$  is the Minkowskian metric,  $\Gamma$  is the WZW action

$$\Gamma = \frac{1}{24\pi} \int_B d^3y \epsilon^{ijk} \operatorname{Tr} g^{-1} \partial_i g g^{-1} \partial_j g g^{-1} \partial_k g \quad (2)$$

and  $B$  is a 3-dimensional manifold whose boundary is the space-time. The coefficient of the WZW action is defined as

$$\rho = \frac{n\chi^2}{4\pi}. \quad (3)$$

The equations of motion are

$$\partial^\mu (\partial_\mu g g^{-1}) - \rho \epsilon^{\mu\nu} \partial_\mu g g^{-1} \partial_\nu g g^{-1} = 0. \quad (4)$$

The crucial property of this model is that it can be formulated entirely in terms of the current  $\partial_\mu g g^{-1}$ . The equation of motion (4) is equivalent to the following pair of first order equations,

$$\frac{\partial I}{\partial t} = \frac{\partial J}{\partial x} + \rho [I, J], \quad (5)$$

$$\frac{\partial J}{\partial t} = \frac{\partial I}{\partial x} + [I, J]. \quad (6)$$

The second equation is the integrability condition for the existence of  $g : R^{1,1} \rightarrow SU(2)$  satisfying

$$I = \frac{\partial g}{\partial t} g^{-1}, \quad (7)$$

$$J = \frac{\partial g}{\partial x} g^{-1}. \quad (8)$$

$I$  and  $J$  are traceless antihermitian matrices valued in the Lie algebra of  $SU(2)$

$$I = I_\alpha(x) \frac{i}{2} \sigma^\alpha, \quad J = J_\alpha(x) \frac{i}{2} \sigma^\alpha, \quad (9)$$

where  $\sigma^\alpha$  are the Pauli matrices. If we also impose the boundary condition

$$\lim_{x \rightarrow -\infty} g(x) = 1, \quad (10)$$

the solution for  $g$  is unique. Then Eq. (5) guarantees that  $g$  satisfies the equation of motion (4).

Though (5) can be obtained from an action principle, the Hamiltonian formalism is more suitable for our purposes; the Hamiltonian and Poisson brackets which give (5) are then

$$H_1 = -\frac{1}{2\chi^2} \int \operatorname{tr}(I^2 + J^2) dx, \quad (11)$$

$$\frac{1}{2\chi^2} \{I_\alpha(x), I_\beta(y)\}_1 = \epsilon_{\alpha\beta\gamma} J_\gamma(x) \delta(x-y) + \rho \epsilon_{\alpha\beta\gamma} J_\gamma \delta(x-y), \quad (12)$$

$$\frac{1}{2\chi^2} \{I_\alpha(x), J_\beta(y)\}_1 = \epsilon_{\alpha\beta\gamma} J_\gamma(x) \delta(x-y) - \delta_{\alpha\beta} \delta'(x-y), \quad (13)$$

$$\frac{1}{2\chi^2} \{J_\alpha(x), J_\beta(y)\}_1 = 0, \quad (14)$$

where  $\epsilon_{\alpha\beta\gamma}$  are the structure constants of  $SU(2)$ .  $I_\alpha$ ,  $J_\alpha$  defined by (9) are square integrable functions on the space-time; the square integrability is a condition on how quickly the currents must decay to zero at infinity. It is needed for finiteness of energy (see (11)).

As can be easily checked, the Poisson algebra given above is the semidirect sum of an abelian algebra and a Kac-Moody algebra associated to  $SU(2)$ .

The same equations of motion can be obtained from an alternative Hamiltonian formalism [2], where the new Hamiltonian and Poisson brackets depend on an extra parameter  $\tau$ , the limit  $\tau \rightarrow 0$  being the conventional formalism. The alternative Hamiltonian and Poisson brackets are, respectively,

$$H = -\frac{1}{2\chi^2(1-\tau^2)^2} \int \operatorname{tr}(I^2 + J^2) dx, \quad (15)$$

$$\frac{1}{2\chi^2}\{I_\alpha(x), I_\beta(y)\} = (1 - \tau^2)\epsilon_{\alpha\beta\gamma}I_\gamma(x)\delta(x - y) + a(1 - \tau^2)\epsilon_{\alpha\beta\gamma}J_\gamma(x)\delta(x - y), \quad (16)$$

$$\frac{1}{2\chi^2}\{I_\alpha(x), J_\beta(y)\} = (1 - \tau^2)\epsilon_{\alpha\beta\gamma}J_\gamma(x)\delta(x - y) - (1 - \tau^2)^2\delta_{\alpha\beta}\delta'(x - y) + (1 - \tau^2)\epsilon\epsilon_{\alpha\beta\gamma}I_\gamma(x)\delta(x - y), \quad (17)$$

$$\frac{1}{2\chi^2}\{J_\alpha(x), J_\beta(y)\} = \tau^2(1 - \tau^2)\epsilon_{\alpha\beta\gamma}I_\gamma(x)\delta(x - y) + (1 - \tau^2)\mu\epsilon_{\alpha\beta\gamma}J_\gamma(x)\delta(x - y), \quad (18)$$

where  $a, \mu, \epsilon$  are real parameters depending on  $\tau$  as we will state below;  $\tau$  can be chosen either real or imaginary as both the Hamiltonian and the Poisson brackets only depend on  $\tau^2$ . Let us call this algebra  $C_2$ . It can be verified that we get the correct equations of motion, Eq. (5), if we pose

$$\rho = \frac{a - \epsilon}{1 - \tau^2}. \quad (19)$$

The Poisson algebra  $C_2$  can be rewritten in a much simpler way if we perform the change of variables:

$$I = 2\chi^2(1 - \tau^2)(\alpha L + \beta R), \\ J = 2\chi^2(1 - \tau^2)(\gamma L + \delta R). \quad (20)$$

with  $L$  and  $R$  generators of two commuting Kac-Moody algebras,

$$\{L_\alpha(x), L_\beta(y)\}_2 = \epsilon_{\alpha\beta\gamma}L_\gamma(x)\delta(x - y) + \frac{k}{2\pi}\delta_{\alpha\beta}\delta'(x - y), \quad (21)$$

$$\{R_\alpha(x), R_\beta(y)\}_2 = \epsilon_{\alpha\beta\gamma}R_\gamma(x)\delta(x - y) - \frac{\bar{k}}{2\pi}\delta_{\alpha\beta}\delta'(x - y), \quad (22)$$

$$\{L_\alpha(x), R_\beta(y)\}_2 = 0, \quad (23)$$

and  $k, \bar{k}$  are a pair of constants. Note that the change of variables to  $L$  and  $R$  is singular if  $\tau = 0$ . It is now straightforward (although quite tedious) to show that this algebra goes over to  $C_2$  under the above change of variables, if we choose

$$\alpha = 1 - \rho\tau, \quad \beta = 1 + \rho\tau, \quad (24)$$

$$\gamma = \tau(\rho\tau - 1), \quad \delta = \tau(\rho\tau + 1), \quad (25)$$

$$\epsilon = \mu = \rho\tau^2, \quad a = \rho, \quad (26)$$

$$k = \frac{\pi}{2\chi^2\tau(1 - \rho\tau)^2}, \quad \bar{k} = \frac{\pi}{2\chi^2\tau(1 + \rho\tau)^2}. \quad (27)$$

Thus our current algebra  $C_2$  is isomorphic to a direct sum of two commuting Kac-Moody algebras, when  $\tau$  is real. We can conclude that the alternative canonical formalism for the WZW model here sketched is equivalent to the conventional one but has the advantage of exhibiting a Poisson algebra which is a sum of two Kac-Moody algebras,  $\widehat{LSU}(2) \oplus \widehat{LSU}(2)$ , if  $\tau$  is real; if we choose  $\tau$  to be imaginary it can be checked that  $C_2$  is instead the Kac-Moody algebra associated to  $SL(2, C)$ ,  $\widehat{LSL}(2, C)$  (this is due to the fact that  $I$  and  $J$  become complex combinations of the  $\widehat{LSU}(2) \oplus \widehat{LSU}(2)$  generators, when  $\tau$  is imaginary). We will assume from now on  $\tau$  real, the procedure being identical for  $\tau$  imaginary.

The WZW model admits an associated linear system [1]

$$\partial_x\psi(x, t, \lambda) = A(x, t, \lambda)\psi(x, t, \lambda), \quad (28)$$

$$\partial_t\psi(x, t, \lambda) = M(x, t, \lambda)\psi(x, t, \lambda), \quad (29)$$

where  $(A, M)$  is the so-called Lax pair,

$$A(x, t, \lambda) = \frac{1}{2}I\left\{\frac{1 - \rho}{1 - \lambda} - \frac{1 + \rho}{1 + \lambda}\right\} + \frac{1}{2}J\left\{\frac{1 - \rho}{1 - \lambda} + \frac{1 + \rho}{1 + \lambda}\right\}, \quad (30)$$

$$M(x, t, \lambda) = \frac{1}{2}I\left\{\frac{1 - \rho}{1 - \lambda} + \frac{1 + \rho}{1 + \lambda}\right\} + \frac{1}{2}J\left\{\frac{1 - \rho}{1 - \lambda} - \frac{1 + \rho}{1 + \lambda}\right\}. \quad (31)$$

Because of their dependence on the currents,  $A$  and  $M$  are traceless matrices valued in the Lie algebra of  $SU(2)$  if the spectral parameter  $\lambda$  is real, or in the Lie algebra of  $SL(2, C)$  if we choose  $\lambda$  to be complex. It is convenient to rewrite the Lax pair in terms of the new basis that we have chosen for the current algebra,

$$L(x, t) = \frac{\tau I - J}{4\chi^2\tau(1 - \tau^2)(1 - \rho\tau)}, \\ R(x, t) = \frac{\tau I + J}{4\chi^2\tau(1 - \tau^2)(1 + \rho\tau)}, \quad (32)$$

so that

$$A(x, t, \lambda) = a(\lambda)L(x, t) + b(\lambda)R(x, t), \quad (33)$$

$$M(x, t, \lambda) = c(\lambda)L(x, t) + d(\lambda)R(x, t), \quad (34)$$

with

$$\begin{aligned} a(\lambda) &= \chi^2(1 - \tau^2)(1 - \rho\tau) \\ &\times \left[ \frac{1 - \rho}{1 - \lambda}(1 - \tau) - \frac{1 + \rho}{1 + \lambda}(1 + \tau) \right], \\ b(\lambda) &= \chi^2(1 - \tau^2)(1 + \rho\tau) \\ &\times \left[ \frac{1 - \rho}{1 - \lambda}(1 + \tau) - \frac{1 + \rho}{1 + \lambda}(1 - \tau) \right], \\ c(\lambda) &= \chi^2(1 - \tau^2)(1 - \rho\tau) \\ &\times \left[ \frac{1 - \rho}{1 - \lambda}(1 - \tau) + \frac{1 + \rho}{1 + \lambda}(1 + \tau) \right], \\ d(\lambda) &= \chi^2(1 - \tau^2)(1 + \rho\tau) \\ &\times \left[ \frac{1 - \rho}{1 - \lambda}(1 + \tau) + \frac{1 + \rho}{1 + \lambda}(1 - \tau) \right]. \end{aligned} \quad (35)$$

The compatibility condition of Eqs. (28) and (29) for any value of  $\lambda$ , which reads

$$\partial_t A - \partial_x M + [A, M] = 0, \quad (36)$$

implies the equations of motion, Eqs. (5) and (6). Relation (36) is also known as zero curvature condition for the connection  $(M, A)$ .

We define the monodromy matrix  $T(x, y, \lambda)$  (the  $t$  dependence being understood from now on), as a particular solution of (28), (29):

$$\partial_x T(x, y, \lambda) = A(x, t, \lambda)T(x, y, \lambda), \quad (37)$$

$$\begin{aligned} \partial_t T(x, y, \lambda) &= M(x, t, \lambda)T(x, y, \lambda) \\ &- T(x, y, \lambda)M(y, t, \lambda), \end{aligned} \quad (38)$$

with

$$T(x, x, \lambda) = 1, \quad T(y, x, \lambda) = T^{-1}(x, y, \lambda),$$

$$T(x, y, \lambda)T(y, z, \lambda) = T(x, z, \lambda).$$

We have then

$$T(x, y; \lambda) = \text{P exp} \int_y^x A(x', \lambda) dx', \quad (39)$$

where P denotes path ordering. The infinite volume limit of  $T(x, y, \lambda)$

$$T(\infty, -\infty; \lambda) \equiv T(\lambda) = \text{P exp} \int_{-\infty}^{\infty} A(x, \lambda) dx, \quad (40)$$

is a conserved quantity for any value of  $\lambda$ . In fact from (38) we have

$$\begin{aligned} \partial_t T(\infty, -\infty; \lambda) \\ = \partial_t \left[ M(\infty, \lambda)T(\lambda) - T(\lambda)M(-\infty, \lambda) \right], \end{aligned} \quad (41)$$

which is zero, because

$$\lim_{x \rightarrow \pm\infty} I(x) = \lim_{x \rightarrow \pm\infty} J(x) = 0,$$

and  $M$  a linear function of the currents  $I, J$ . Note that for periodic boundary conditions only the trace of  $T(\lambda)$  is conserved.

From the definition of  $T(\lambda)$  we note that  $T(\lambda)$  (as a function of  $\lambda$ ) is an element of the loop group of  $SL(2, \mathbb{C})$ ; in fact  $A(\lambda = \pm\infty) = 0$  thus implying

$$T(\lambda = -\infty) = T(\lambda = \infty) = 1.$$

Also it satisfies

$$T^\dagger(\lambda)T(\bar{\lambda}) = 1. \quad (42)$$

The last equation is easily proven observing that  $A^\dagger(x, \lambda) = -A(x, \bar{\lambda})$ . If we choose the spectral parameter to be real, (42) implies that  $T(\lambda)$  be an element of the loop group of  $SU(2)$ .

Because  $T(\lambda)$  is an ordered exponential of  $A(x, \lambda)$ , we only need the Poisson algebra of the  $A$ 's to determine the Poisson bracket of the monodromy matrices. From (33), (34), and the Poisson brackets of the Kac–Moody generators, we get

$$\begin{aligned} \{A^a(x, \lambda), A^b(y, \mu)\} \\ = \varepsilon_{abc} \left( g(\lambda, \mu)A^c(x, \lambda) + g(\mu, \lambda)A^c(y, \mu) \right) \\ \times \delta(x - y) - f(\lambda, \mu)\delta_{ab}\delta'(x - y), \end{aligned} \quad (43)$$

where

$$g(\lambda, \mu) = a(\mu)b(\mu) \frac{a(\lambda) - b(\lambda)}{a(\lambda)b(\mu) - a(\mu)b(\lambda)}, \quad (44)$$

$$g(\mu, \lambda) = -a(\lambda)b(\lambda) \frac{a(\mu) - b(\mu)}{a(\lambda)b(\mu) - a(\mu)b(\lambda)}, \quad (45)$$

$$f(\lambda, \mu) = b(\lambda)b(\mu)\frac{\bar{k}}{2\pi} - a(\lambda)a(\mu)\frac{k}{2\pi}. \quad (46)$$

We pose

$$\hat{g}(\lambda, \mu) = g(\lambda, \mu)C, \quad \hat{g}(\mu, \lambda) = g(\mu, \lambda)C,$$

$$\hat{f}(\lambda, \mu) = f(\lambda, \mu)C,$$

where  $C = \frac{1}{4}\sigma_a \otimes \sigma^a$ . Defining then  $r(\lambda, \mu)$  and  $s(\lambda, \mu)$  as the skew-symmetric and the symmetric part of  $\hat{g}(\lambda, \mu)$ , respectively,

$$r(\lambda, \mu) = \frac{1}{2}(\hat{g}(\lambda, \mu) - \hat{g}(\mu, \lambda)), \quad (47)$$

$$s(\lambda, \mu) = \frac{1}{2}(\hat{g}(\lambda, \mu) + \hat{g}(\mu, \lambda)), \quad (48)$$

and observing that

$$\hat{f}(\lambda, \mu) = 2s(\lambda, \mu) \quad (49)$$

we can rewrite (43) as

$$\begin{aligned} &\{A_1(x, \lambda), A_2(y, \mu)\} \\ &= \left( [r(\lambda, \mu), A_1(x, \lambda) + A_2(x, \mu)] \right. \\ &\quad \left. - [s(\lambda, \mu), A_1(x, \lambda) - A_2(x, \mu)] \right) \delta(x - y) \\ &\quad - 2s(\lambda, \mu) \delta'(x - y), \end{aligned} \quad (50)$$

where  $A = A^a \frac{i}{2} \sigma_a$ ,  $A_1 = A \otimes 1$ ,  $A_2 = 1 \otimes A$ . The matrices  $r, s$ , as explicit functions of  $\lambda$  and  $\mu$ , are

$$\begin{aligned} r &= 2\chi^2 \frac{(1 - \tau^2)}{\lambda - \mu} \\ &\times \left\{ \frac{\lambda^2(1 - \rho^2\tau^2) - 2\lambda\rho(1 - \tau^2) + \rho^2 - \tau^2}{\lambda^2 - 1} \right. \\ &\quad \left. + \frac{\mu^2(1 - \rho^2\tau^2) - 2\mu\rho(1 - \tau^2) + \rho^2 - \tau^2}{\mu^2 - 1} \right\} C, \end{aligned} \quad (51)$$

$$\begin{aligned} s &= 2\chi^2 \frac{(1 - \tau^2)}{\lambda - \mu} \\ &\times \left\{ \frac{\lambda^2(1 - \rho^2\tau^2) - 2\lambda\rho(1 - \tau^2) + \rho^2 - \tau^2}{\lambda^2 - 1} \right. \\ &\quad \left. - \frac{\mu^2(1 - \rho^2\tau^2) - 2\mu\rho(1 - \tau^2) + \rho^2 - \tau^2}{\mu^2 - 1} \right\} C. \end{aligned} \quad (52)$$

In the limit  $\rho = 0, \tau = 0$ , which is the chiral model in the usual formalism,<sup>4</sup> the  $r$  and  $s$  matrices given above reduce to the ones found in [6] by general arguments on Poisson brackets related to Kac–Moody algebras. Models which are described by a Poisson algebra containing derivatives of the delta function are called non-ultralocal [1]. In general they also exhibit a space–time dependence for  $r$  and  $s$  matrices [4,5], the general form of the Poisson algebra thus being

$$\begin{aligned} \{A_1(x, \lambda), A_2(y, \mu)\} &= \left( \partial_x r(x, \lambda, \mu) \right. \\ &\quad \left. + [r(x, \lambda, \mu), A_1(x, \lambda) + A_2(x, \mu)] \right. \\ &\quad \left. - [s(x, \lambda, \mu), A_1(x, \lambda) - A_2(x, \mu)] \right) \delta(x - y) \\ &\quad - (s(x, \lambda, \mu) + s(y, \lambda, \mu)) \delta'(x - y). \end{aligned} \quad (53)$$

In principle there could be higher derivatives of the delta function. Note however that in our case, though the system is non-ultralocal, the  $r$  and  $s$  matrices are non-dynamical. The Poisson bracket (50) has a remarkable property: it is well defined for any value of  $\tau$ , even for the singular value  $\tau = 0$ , where the Kac–Moody generators,  $L, R$ , are not independent functions of  $I, J$ .

### 3. Poisson algebra of monodromy matrices and equal-point limits

Once the Poisson algebra of the currents  $A(x, \lambda)$  has been proven to be of the form (53), which is the standard form for integrable, non-ultralocal models, the Poisson bracket for monodromy matrices is determined and the equal-point limit can be defined through a regularization procedure. We recall here the main steps for completeness (see [5,4], for a detailed description).

The Poisson algebra for monodromy matrices obtained from the algebra (50) is

<sup>4</sup> Note that for  $\rho = 0, \tau \neq 0$  we have a family of principal chiral models depending on an extra parameter  $\tau$  [3].

$$\begin{aligned}
& \{T_1(x, y, \lambda), T_2(x', y', \mu)\} \\
&= T_1(x, x_0, \lambda) T_2(x', x_0, \mu) \left( r(\lambda, \mu) \right. \\
&+ \epsilon(x - x') s(\lambda, \mu) \left. \right) T_1(x_0, y, \lambda) T_2(x_0, y', \mu) \\
&- T_1(x, y_0, \lambda) T_2(x', y_0, \mu) \left( r(\lambda, \mu) \right. \\
&+ \epsilon(y - y') s(\lambda, \mu) \left. \right) T_1(y_0, y, \lambda) T_2(y_0, y', \mu), \quad (54)
\end{aligned}$$

where  $\epsilon(x) = \text{sign}(x)$ ,  $x_0 = \min(x, x')$ ,  $y_0 = \max(y, y')$ . We recall that the Poisson brackets (54) have been found under the explicit assumption that  $x, x', y, y'$  be all different; also this algebra shows a discontinuity of amplitude  $2s$ , in the equal-point limits  $x = x', y = y'$ .

The Jacobi identity for the Poisson brackets (54)

$$\begin{aligned}
& \{T_1(x, y, \lambda), \{T_2(x', y', \mu), T_3(x'', y'', \nu)\}\} \\
&+ \text{cycl. perm.} = 0 \quad (55)
\end{aligned}$$

results in an equation for the  $r$  and  $s$  matrices:

$$\begin{aligned}
& [(r - s)_{12}(\lambda, \mu), (r + s)_{13}(\lambda, \nu)] \\
&+ [(r + s)_{12}(\lambda, \mu), (r + s)_{23}(\mu, \nu)] \\
&+ [(r + s)_{13}(\lambda, \nu), (r + s)_{23}(\mu, \nu)] = 0. \quad (56)
\end{aligned}$$

If  $r$  and  $s$  depend on space–time variables, Eq. (56) will also contain terms involving the Poisson bracket of  $r$  and  $s$  with  $A$ . As can be seen, (56) reduces to the usual Yang–Baxter equation for the  $r$  matrix when  $s$  is zero. This is the ultralocal case.

It can be checked that our  $r$  and  $s$  matrices given in (51) and (52) satisfy condition (56), which can also be rewritten as an equation for the matrix  $g$ . Noting that

$$\begin{aligned}
& \hat{g}(\lambda, \mu) = r(\lambda, \mu) + s(\lambda, \mu), \\
& -\hat{g}(\mu, \lambda) = r(\lambda, \mu) - s(\lambda, \mu), \quad (57)
\end{aligned}$$

Eq. (56) becomes then

$$\begin{aligned}
& -[\hat{g}_{12}(\mu, \lambda), \hat{g}_{13}(\lambda, \nu)] + [\hat{g}_{12}(\lambda, \mu), \hat{g}_{23}(\mu, \nu)] \\
&+ [\hat{g}_{13}(\lambda, \nu), \hat{g}_{23}(\mu, \nu)] = 0. \quad (58)
\end{aligned}$$

It can be verified, as a consistency check, that the same condition is obtained from the Jacobi identity of the current algebra (50).

The equal points limits  $x = x', y = y'$ , which we need to define the Poisson brackets of conserved quantities  $T(\infty, -\infty, \lambda)$ , is defined in [5,4] through a symmetric limit procedure to be

$$\begin{aligned}
& \{T_1(x, y, \lambda), T_2(x, y, \mu)\} \\
&= [r(\lambda, \mu), T_1(x, y, \lambda) T_2(x, y, \mu)]. \quad (59)
\end{aligned}$$

Note that this Poisson bracket has the same form as the Poisson bracket obtained for ultralocal models but it satisfies the Jacobi identity only through the symmetric limit. If we check the Jacobi identity directly on (59) we find a Yang–Baxter equation for the  $r$ -matrix given in (51), which is not satisfied. For this reason we say that (59) satisfies the Jacobi identity only in a weak sense.

In the infinite volume limit, Eq. (59) reads

$$\{T_1(\lambda), T_2(\mu)\} = [r(\lambda, \mu), T_1(\lambda) T_2(\mu)], \quad (60)$$

and the conserved quantities,  $\text{Tr } T(\lambda)$ , are in involution, being  $\text{Tr}(A \otimes B) = \text{Tr } A \cdot \text{Tr } B$ , so that

$$\{\text{Tr } T(\lambda), \text{Tr } T(\mu)\} = \text{Tr } \{T_1(\lambda), T_2(\mu)\}, \quad (61)$$

which is zero because of (60), being the trace of a commutator. Note that the Poisson algebra

$$\{\text{Tr } T(\lambda), \text{Tr } T(\mu)\} = 0 \quad (62)$$

satisfies the Jacobi identity strongly (which is trivially true for zero Poisson brackets).

#### 4. Conclusions

We have found a three-parameter family of non-ultralocal integrable models (the parameters being  $\tau$ ,  $\rho$ , and the coupling constant  $\chi$ ). The Poisson algebra of the monodromy matrices can be rewritten in terms of  $r$  and  $s$  matrices which are independent on space–time variables and satisfy an extended Yang–Baxter equation. We also exhibit the conserved quantities of the theory which are in involution with respect to such a Poisson structure.

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