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Deformed Chern–Simons theories

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Abstract

We construct a Chern-Simons action for q -deformed gauge theory which is a simple and straightforward generalization of the usual one. Space-time continues to be an ordinary (commuting) manifold, while the gauge potentials and the field strengths become q -commuting fields. Our approach, which is explicitly carried out for the case of 'minimal' deformations, has the advantage of leading naturally to a consistent Hamiltonian structure that has essentially all of the features of the undeformed case. For example, using the new Poisson brackets, the constraints form a closed algebra and generate q -deformed gauge transformations. © 1997 Elsevier Science B.V.

1. Introduction

Chern-Simons (CS) theory has played a unique rôle in unifying different, previously unrelated, physical problems and mathematical ideas in $1+1$ and $2+1$ dimensions. CS theory gives a natural description of anyonic excitations of condensed matter systems [1] constrained to two space dimensions, and it provides an important key to understanding and classifying the intriguing physics of quantum Hall fluids [2]. Also, unique to three dimensions is the possibility of expressing quantum gravity as a pure CS theory written in terms of dreibeins and spin-connections transforming locally under the Poincaré group [3,4].

Here our motivation is concerned with the rôle the CS action provides in describing mathematical and

physical problems usually formulated in two dimensions, which was first discussed by E. Witten in Ref. [5]. By showing that quantum CS theory provides a framework for understanding Jones polynomials of knots in three dimensions, he also shed new light on conformal theories in $1+1$ dimensions [6]. In making the correspondence between the two and three dimensional theories one can consider CS theory on a manifold with a boundary, such as a disk $D \times \mathbf{R}^1$, \mathbf{R}^1 being the time line. Then all states in the bulk can be gauged away and one is left with a family of conformal states, called edge states, on the boundary. An extensive literature has been devoted to the study of edge states [7]. In particular, the edge states of quantum Hall devices at fractional fillings, which have been analyzed from different theoretical points of view [8], provide a unique experimental laboratory for probing non-fermi liquids. Here we recall two theoretical aspects of edge states. First, they define a Kac-Moody algebra which is easily derived from the Hamiltonian

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formulation of the theory. Second, the essential tool for building affine algebras starting from conformal field theory [9], namely the Fubini-Veneziano vertex operator [10], has a natural realization in terms of the Wilson line for CS theory [11].

While this picture relating $2+1$ and $1+1$ physics, which we briefly sketched, has been greatly developed and clarified, new ideas have emerged in $1+1$ dimensions whose corresponding rôle (if any) in $2+1$ dimensions remains unclear. In this regard, q -deformed affine Lie algebras associated with quantum groups [12] have been formulated for the entire non-exceptional series and a construction in terms of anyonic q -deformed oscillators has been given, at least for the unitary and symplectic cases [13]. Furthermore, q -deformed affine Lie algebras, enter in different aspects of $1+1$ integrable models [14,15]. In particular, they appear as the minimal symmetry needed to determine the S -matrix up to an overall scalar factor. Such algebraic structures naturally arise [16] when conformal models are driven off criticality by an appropriate perturbation preserving the integrability [17] of the theory. Here q is related to the value of the coupling appearing in the generalized vertex operators.

Without expanding further on different aspects of these developments, it seems fair to say that a complete picture is still lacking, and therefore it may be useful to look at them from the new perspective of a $2+1$ dimensional theory. It is with this in mind, that we construct a CS action for q -deformed gauge theory as a simple and straightforward generalization of the undeformed case.

We note that in [18] deformed Chern characters are constructed for $SU_q(2)$, based on a different differential calculus on quantum groups [19]. The resulting CS action differs from ours, in many respects. Theirs requires a trilinear metric, and extra conditions placed on the quantum group metric, neither of which are needed in our approach. Furthermore, our approach leads naturally to a consistent Hamiltonian structure that has all the features present in the ordinary case. Analogous to the undeformed case, first class constraints appear in the formalism which generate the q -deformed gauge transformations and form a closed algebra. Our construction of the q -deformed CS action, with its natural Hamiltonian structure, paves the way to an analysis of q -CS edge states, which we will discuss in a separate paper [20]. It is also very sim-

ple in our framework to discuss q -deformed general relativity in $2+1$ dimensions [21], where we desire a departure from the undeformed theory ($q=1$) at Planck scale curvatures.

The paper is organized as follows. In Section 1, we briefly review the structure of q -deformed gauge theories mainly along the line of Ref. [22]. In Section 2, we show that a q -CS action can be defined in the case of a *minimal* deformation. The requirement that the theory be minimally deformed is sufficient (and perhaps necessary, as well) to ensure the gauge invariance of the CS action. (Minimality is also needed for the closure of gauge transformations.) In Section 3, it is shown that the solutions of the equations of motion are, as expected, flat connections and a number of useful properties are exhibited, among them the relation with q -deformed Pontryagin densities. Finally, the last section is devoted to the construction of a consistent Hamiltonian formalism. By introducing deformed Poisson brackets for the components of the connection we show that the field strengths, smeared with Lie algebra valued functions generate gauge transformations and lead to a closed algebra. Details of calculations are given in the appendix. We close with brief final remarks.

2. q -deformed gauge theories

In this section we briefly review the mathematical setting which is needed to introduce gauge field theories whose infinitesimal gauge symmetry is associated with a quantum Lie algebra.

Recently there have been various proposals for such q -deformed theories [22–29]. In some of them [26] the structure of space-time is made noncommutative, which is especially relevant for applications to gravity. In other theories [22,25,27] the structure of space-time stays commutative and a bicovariant differential calculus [30–33] is needed in order to define the quantum Lie algebra. In [28] an $SU_q(2)$ gauge theory is proposed that is based on a definition of a quantum Lie algebra which doesn't need a bicovariant differential calculus, while [29] contains a proposal of an $SU_q(2)$ gauge theory on the lattice. We will follow in the paper the approach of Castellani, because it seems to us closer to usual gauge theory and differential calculus on classical Lie groups.

Let us first recall the definition of a quantum Lie algebra and its connection to differential calculus on quantum groups, as described in [33]. Starting from the definition of a quantum group G_q as the noncommutative algebra of functions on the Lie group G , $G_q \equiv \text{Fun}_q(G)$, a bimodule of left (right) invariant forms for G_q is constructed, in the same way as the bimodule of left (right) invariant forms is constructed for classical Lie groups. Such a bimodule inherits the noncommutative nature of the product in $\text{Fun}_q(G)$

$$R_{ef}^{ab} M_c^e M_d^f = M_f^b M_e^a R_{cd}^{ef} \tag{2.1}$$

(M is an element of G in its defining representation) so that the usual definition of exterior product for one-forms, $\theta^i \wedge \theta^j = \theta^i \otimes \theta^j - \theta^j \otimes \theta^i$, is replaced on q -groups by

$$\theta^i \wedge \theta^j = \theta^i \otimes \theta^j - \Lambda_{kl}^{ij} \theta^k \otimes \theta^l, \tag{2.2}$$

where Λ is the braiding matrix. Following the analogy with the differential calculus on classical Lie groups, the algebra of left invariant vector fields, which is dual to the algebra of left invariant one-forms, can be obtained,⁵ with q -commutation relations

$$T_i T_j - \Lambda_{ij}^{kl} T_k T_l \equiv [T_i, T_j] = C_{ij}^k T_k. \tag{2.3}$$

It is this algebra which is called a quantum Lie algebra. In the limit $q \rightarrow 1$, $\Lambda_{ij}^{kl} \rightarrow \delta_i^k \delta_j^l$ and T_i become the generators of the classical Lie algebra. C_{ij}^k are q -structure constants, which in general are not antisymmetric in the lower two indices except in the limit $q \rightarrow 1$. (In principle, the q -Lie algebra might be bigger than the classical Lie algebra, its dimension being equal to the dimension of the bimodule.)

In order to define a bicovariant calculus the braiding matrix Λ and the structure constants have to satisfy the following relations [33]:

$$\Lambda_{kl}^{ij} \Lambda_{sp}^{lm} \Lambda_{qu}^{ks} = \Lambda_{kl}^{jm} \Lambda_{qs}^{ik} \Lambda_{up}^{sl} \tag{2.4}$$

(Yang Baxter equation),

$$C_{mi}^r C_{rj}^n - \Lambda_{ij}^{kl} C_{mk}^r C_{rl}^n = C_{ij}^k C_{mk}^n \tag{2.5}$$

(q -Jacobi),

⁵In the same way we can introduce right invariant objects. The differential calculus which is at the basis of the deformed gauge theory we are going to consider is bicovariant; bicovariance requires that left and right actions of the q -group on the bimodule commute.

$$\Lambda_{mk}^{ir} \Lambda_{nl}^{ks} C_{rs}^j = \Lambda_{kl}^{ij} C_{mn}^k, \tag{2.6}$$

$$\Lambda_{ri}^{jq} \Lambda_{kl}^{si} C_{ps}^r + \Lambda_{pi}^{jq} C_{kl}^i = C_{is}^j \Lambda_{rl}^{sq} \Lambda_{pk}^{ir} + C_{ri}^q \Lambda_{pk}^{jr}. \tag{2.7}$$

The first condition is the quantum Yang Baxter equation; the second is the Jacobi identity for the algebra (2.3), while the last equations are trivial in the limit $q \rightarrow 1$.

Following [22], the gauge potential is assumed to be a q -Lie algebra valued one-form $A \equiv A_\mu^i T_i dx^\mu$ (we will often write A^i to mean the one-form $A^i = A_\mu^i dx^\mu$). In this approach the deformation occurs solely in the fiber and thus the A_μ^i are taken to be q -fields subject to nontrivial commutation relations. Space-time, instead, remains an ordinary manifold so that dx^μ are ordinary space-time differentials commuting with A_μ^i . The exterior product of one-forms on the space-time manifold is deformed in the same way as the exterior product of invariant forms on the group manifold (2.2) and, for general groups, one has [34]

$$A^i \wedge A^j = -Z_{kl}^{ij} A^k \wedge A^l, \tag{2.8}$$

where Z is a matrix of ordinary c -numbers which depends on the group. The undeformed case obviously corresponds to the choice $Z_{kl}^{ij} = \delta_i^j \delta_k^l$ for any group. It is determined in general by insisting that $Z_{kl}^{ij} + \delta_k^j \delta_l^i$ is proportional to a projection operator, so that there are no further restrictions on $A^i \wedge A^j$. In the rest of this section and the beginning of the next we shall consider the general deformation of $U(n)$. [We shall later specialize to a particular type of deformation known as *minimal*.] For the general deformation of $U(n)$, the matrix Z has a simple expression in terms of the braiding matrix Λ [22]:

$$A^i \wedge A^j = -\frac{1}{r^2 + r^{-2}} (\Lambda + \Lambda^{-1})_{kl}^{ij} A^k \wedge A^l, \tag{2.9}$$

where r is a deformation parameter; we are assuming multiparametric deformations as considered in [35]. The braiding matrix Λ will depend in general on a set of parameters q_i and on r . The number of independent parameters depends on the group (to make contact with the $U_q(2)$ gauge theory of [22] one has to remember that in this case there is only one parameter). The commutators of the q -fields A_μ^i follow from Eq. (2.9), after one factorizes the coefficients $dx^\mu \wedge dx^\nu$:

$$A^i_{[\mu} A^j_{\nu]} = -\frac{1}{r^2 + r^{-2}} (\Lambda + \Lambda^{-1})^{ij}_{kl} A^k_{[\mu} A^l_{\nu]}. \quad (2.10)$$

From now on we will omit the symbol \wedge for product of forms. The deformed gauge transformations are assumed to be of the usual form

$$\delta_\epsilon A = -d\epsilon - A\epsilon + \epsilon A, \quad (2.11)$$

where $\epsilon \equiv \epsilon^i T_i$. The gauge parameters ϵ^i are now q -numbers and are assumed to have the following commutation rules with the gauge fields:

$$\epsilon^i A^j = \Lambda^{ij}_{mn} A^m \epsilon^n. \quad (2.12)$$

The commutation relations for A^i with $d\epsilon^j$ and dA^i with ϵ^j can be obtained by taking the exterior derivative of the above equation and imposing that the terms containing dA^i and ϵ^j cancel separately. The field strength is defined in the usual way

$$F \equiv \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu = dA + A^2, \quad (2.13)$$

where $A^2 = A^i A^j T_i T_j$. F is an element of the deformed Lie-algebra [22] and under a gauge transformation (2.11) it transforms as:

$$\delta_\epsilon F = \epsilon F - F\epsilon. \quad (2.14)$$

In [22] it is shown that the q -Lagrangian $\mathcal{L} = \langle F_{\mu\nu}, F^{\mu\nu} \rangle_q$ is invariant under the transformation (2.11), if the q -deformed scalar product on the quantum Lie algebra $\langle \cdot, \cdot \rangle_q$ obeys the following invariance condition

$$\Lambda^{nj}_{rs} \langle [T_m, T_n], T_j \rangle_q + \langle T_m, [T_r, T_s] \rangle_q = 0, \quad (2.15)$$

which generalizes the invariance property of the Killing metric on a Lie algebra. Upon introducing the matrix $g_{ij} = \langle T_i, T_j \rangle_q$, the above equation can also be written as

$$\Lambda^{nj}_{rs} C^i_{mn} g_{ij} + C^j_{rs} g_{mj} = 0. \quad (2.16)$$

It is interesting to notice that the deformed metric g_{ij} is not symmetric, in general: $g_{ij} \neq g_{ji}$. We will show that this invariance condition is necessary, but not sufficient to construct a CS Lagrangian with q -symmetry. In Section 3, we will make an additional assumption on the theory whereby a certain combination of g_{ij} and Λ^{nj}_{rs} is nondegenerate.

3. The Chern–Simons Lagrangian density

In this section we will try to construct a deformed CS Lagrangian density in the framework of the deformed gauge theories sketched in the previous section. So, we search for a Lagrangian density \mathcal{L}_{CS} such that:

- 1) \mathcal{L}_{CS} is a three-form that changes by a total derivative under an infinitesimal gauge transformation (2.11);
- 2) $d\mathcal{L}_{CS} = \langle F, F \rangle_q$;
- 3) The equations of motion for the q -CS action are the zero curvature conditions $F = 0$.

Inspired by the classical formula, we make the following ansatz for the deformed CS Lagrangian density

$$\mathcal{L}_{CS} = \langle dA + \beta A^2, A \rangle_q, \quad (3.1)$$

where β is a factor to be determined (we shall omit writing, from now on, the subscript q).

We start by checking 1) whether, for any choice of β , the variation of Eq. (3.1) under an infinitesimal q -gauge transformation (2.11) is a total derivative. When one performs such a transformation in Eq. (3.1) two types of terms arise, those containing ϵ and those containing $d\epsilon$. We shall collect them separately and accordingly split $\delta_\epsilon \mathcal{L}_{CS}$ as

$$\delta_\epsilon \mathcal{L}_{CS} = \delta \mathcal{L}_1 + \delta \mathcal{L}_2, \quad (3.2)$$

with the ϵ dependent terms in $\delta \mathcal{L}_1$ and the $d\epsilon$ dependent terms in $\delta \mathcal{L}_2$. An easy computation gives

$$\begin{aligned} \delta \mathcal{L}_1 = & \langle dA, \epsilon A - A\epsilon \rangle + \langle -dA\epsilon + \epsilon dA, A \rangle \\ & + \beta \langle \epsilon A^2 - A^2 \epsilon, A \rangle + \beta \langle A^2, -A\epsilon + \epsilon A \rangle, \end{aligned} \quad (3.3)$$

and

$$\begin{aligned} \delta \mathcal{L}_2 = & -\langle dA, d\epsilon \rangle + (1 - \beta) \langle d\epsilon A + A d\epsilon, A \rangle \\ & - \beta \langle A^2, d\epsilon \rangle. \end{aligned} \quad (3.4)$$

$\delta \mathcal{L}_1$ vanishes, by virtue of the invariance condition for the scalar product (2.15). In fact, using the commutation rules Eqs. (2.3), (2.12) the sum of the first two terms of the right hand side of Eq. (3.3) can be rewritten as

$$\begin{aligned} & \langle dA, \epsilon A - A\epsilon \rangle + \langle -dA\epsilon + \epsilon dA, A \rangle \\ & = -dA^i A^m \epsilon^n \{ \langle T_i, [T_m, T_n] \rangle + \Lambda^{jk}_{mn} \langle [T_i, T_j], T_k \rangle \}, \end{aligned} \quad (3.5)$$

which is zero because of Eq. (2.15). As for the sum of the third and fourth terms in the right hand side of Eq. (3.3) it can be shown to vanish using the following commutation relations, proven in [22]:

$$\epsilon^i (A^2)^j = \Lambda_{mn}^{ij} (A^2)^m \epsilon^n. \quad (3.6)$$

Using this, we get

$$\begin{aligned} & \langle \epsilon A^2 - A^2 \epsilon, A \rangle + \langle A^2, -A\epsilon + \epsilon A \rangle \\ &= -(A^2)^i A^j \epsilon^k \{ \Lambda_{jk}^{im} \langle [T_i, T_l], T_m \rangle + \langle T_i, [T_j, T_k] \rangle \} \\ &= 0. \end{aligned} \quad (3.7)$$

We are thus left with $\delta \mathcal{L}_2$. From Eq. (3.4) we see that, among the three terms appearing in it, only the first one is a total derivative; in order to see whether the remaining two vanish we first rewrite them in a convenient way. Using the commutation properties of $d\epsilon$ with A^i , the second term can be rewritten as

$$\begin{aligned} \langle d\epsilon A + A d\epsilon, A \rangle &= -A^i A^j d\epsilon^k \Lambda_{jk}^{qm} \langle [T_i, T_q], T_m \rangle \\ &= A^i A^j d\epsilon^k \langle T_i, [T_j, T_k] \rangle. \end{aligned} \quad (3.8)$$

Using instead the explicit expression for A^2 [22]

$$(A^2)^i = \frac{1}{(2+r^2+r^{-2})} A^j A^k [C_{jk}^i - (\Lambda^{-1})_{jk}^{lm} C_{lm}^i] \quad (3.9)$$

we get for the third term of Eq. (3.4)

$$\begin{aligned} \langle A^2, d\epsilon \rangle &= \frac{1}{(2+r^2+r^{-2})} A^i A^j d\epsilon^k \\ &\times \{ \langle [T_i, T_j], T_k \rangle - (\Lambda^{-1})_{ij}^{ml} \langle [T_m, T_l], T_k \rangle \}. \end{aligned} \quad (3.10)$$

Summing Eq. (3.8) to Eq. (3.10) we get

$$\begin{aligned} \delta \mathcal{L}_{CS} &= -\langle dA, d\epsilon \rangle \\ &+ A^i A^j d\epsilon^k \left\{ (1-\beta) \langle T_i, [T_j, T_k] \rangle - \frac{\beta}{(2+r^2+r^{-2})} \right. \\ &\times \left. \langle \langle [T_i, T_j], T_k \rangle - (\Lambda^{-1})_{ij}^{ml} \langle [T_m, T_l], T_k \rangle \right\}. \end{aligned} \quad (3.11)$$

In order for $\delta_\epsilon \mathcal{L}_{CS}$ to be a total derivative, the expression between the curly brackets must vanish. However, it does not vanish in general, as we have checked

that it is different from zero for the case of $U_q(2)$, using the explicit formulae in [22]. On the other hand, it can be proven to be zero if we make a further assumption on the theory; namely, that the deformation be *minimal*, which means that

$$\Lambda_{kl}^{ij} \Lambda_{mn}^{kl} = \delta_m^i \delta_n^j. \quad (3.12)$$

In this case, the anticommutation relations Eq. (2.8) can be derived directly from Eq. (2.2) [setting $\theta^i = A^i$] by multiplying both sides by Λ_{ij}^{mn} (and summing over i and j), so that we have

$$A^m A^n = -\Lambda_{ij}^{mn} A^i A^j, \quad \text{for minimal deformations.} \quad (3.13)$$

Consistency with Eq. (2.9) then implies $r^2 = 1$. The simple commutation relations above allow us to write A^2 in a form analogous to the undeformed case:

$$A^2 = \frac{1}{2} A^i A^j [T_i, T_j]. \quad (3.14)$$

In the *minimal* case, one can also prove a number of further simplifying relations. For example, by multiplying Eq. (2.3) by Λ on the right, one finds that the q -structure constants C_{ij}^k are Λ -antisymmetric [22]

$$C_{ij}^k = -\Lambda_{ij}^{rs} C_{rs}^k, \quad C_{ij}^k = -(\Lambda^{-1})_{ij}^{rs} C_{rs}^k. \quad (3.15)$$

Moreover, multiplying Eq. (2.15) by Λ^{-1} and using Eq. (3.15), one gets the following usual expression for the invariance condition of the inner product:

$$\langle [T_i, T_j], T_k \rangle = \langle T_i, [T_j, T_k] \rangle. \quad (3.16)$$

Using Eq. (3.15) and Eq. (3.16) into Eq. (3.11) one finds

$$\delta \mathcal{L}_{CS} = -\langle dA, d\epsilon \rangle + (2-3\beta) \langle A^2, d\epsilon \rangle. \quad (3.17)$$

The second term in the r.h.s. is zero if $\beta = 2/3$ and this is the value we choose. With this choice, our deformed CS Lagrangian density has the same expression as the undeformed one; this resemblance is only formal, as the new Lagrangian is written in terms of a deformed scalar product, and the gauge fields are noncommuting.

Even though the above proof of the gauge invariance of \mathcal{L}_{CS} has been given for the case of $U_q(N)$, we point out that all the formulas exhibited here for *minimal* B, C, D groups, $U_q(N)$ continue to hold for *minimal* B, C, D groups,

as well as their inhomogeneous partners. In particular the commutation rule for one-forms Eq. (3.13) is independent of the particular class of algebras, when $\Lambda^2 = 1$, since it follows directly from Eq. (2.2), which holds in general. From now on whenever we refer to a q -gauge theory, we shall assume that it is minimally deformed, unless otherwise specified. Nontrivial (in general multiparametric) minimal deformations with the corresponding q -differential calculus are known to exist for all groups $U(N)$ with $N > 2$ and for groups of the B, C, D type (see [23,35]) together with their inhomogeneous associates.

4. Properties of the q -CS term

Among properties 1)–3) listed at the beginning of the previous section, we verified the first one for minimal deformations, using our deformed CS Lagrangian density Eq. (3.1) with $\beta = 2/3$. We next show that the properties 2) and 3) are also satisfied. We begin with 3) which deals with the equations of motion.

Consider a three-dimensional manifold M and let S_{CS} be the action obtained by integrating \mathcal{L}_{CS} on M :

$$S_{CS} = \int_M \langle dA + \frac{2}{3}A^2, A \rangle. \tag{4.1}$$

In order to compute the equations of motion, we take a variation of Eq. (4.1):

$$\begin{aligned} \delta S_{CS} &= \int_M \delta \langle dA + \frac{2}{3}A^2, A \rangle \\ &= \int_M (\langle \delta A, dA \rangle + \langle dA, \delta A \rangle) \\ &\quad + \frac{2}{3} \int_M (\langle \delta A^2, A \rangle + \langle A^2, \delta A \rangle) \end{aligned} \tag{4.2}$$

(where an integration by parts has been performed). To proceed further we need the commutation relations for δA and A . Consider now the case of a variation corresponding to a gauge transformation (2.11). One can check by means of a direct computation that for a minimal deformation:

$$\delta_\epsilon A^i A^j = -\Lambda_{kl}^{ij} A^k \delta_\epsilon A^l. \tag{4.3}$$

We assume that analogous commutation relations hold for arbitrary variations of A (the commutation relations for δA with dA can be obtained by taking an exterior derivative of the above equation and assuming that the terms containing $d\delta A$ and dA vanish separately). With the help of Eq. (4.3), the terms between the first pair of parenthesis in the right hand side of Eq. (4.2) can be rewritten as

$$dA^i \delta A^j (\Lambda_{ij}^{kl} \langle T_k, T_l \rangle + \langle T_i, T_j \rangle). \tag{4.4}$$

The terms between the second pair of parenthesis in Eq. (4.2) multiplied by $\frac{2}{3}$ are equal to

$$\begin{aligned} &(\frac{1}{3} \delta A^i A^j A^k + \frac{1}{3} A^i \delta A^j A^k) \langle [T_i, T_j], T_k \rangle + \frac{2}{3} \langle A^2, \delta A \rangle \\ &= (\frac{1}{2} \delta A^i A^j A^k + \frac{1}{6} A^i \delta A^j A^k) \langle [T_i, T_j], T_k \rangle \\ &\quad + \frac{2}{3} \langle A^2, \delta A \rangle \\ &= \langle \delta A, A^2 \rangle + \frac{1}{6} A^i \delta A^j A^k \langle T_i, [T_j, T_k] \rangle + \frac{2}{3} \langle A^2, \delta A \rangle \\ &= \langle \delta A, A^2 \rangle + \langle A^2, \delta A \rangle \\ &= (A^2)^i \delta A^j (\Lambda_{ij}^{kl} \langle T_k, T_l \rangle + \langle T_i, T_j \rangle), \end{aligned} \tag{4.5}$$

where we used $C_{ij}^k A^i \delta A^j = C_{ij}^k \delta A^i A^j$. Finally, substituting Eq. (4.4) and Eq. (4.5) into Eq. (4.2) we get

$$\delta S_{CS} = \int_M F^i \delta A^j H_{ij}, \tag{4.6}$$

where H_{ij} is the matrix

$$H_{ij} = (\Lambda_{ij}^{kl} \langle T_k, T_l \rangle + \langle T_i, T_j \rangle). \tag{4.7}$$

If the matrix H_{ij} is nondegenerate we obtain the desired equation of motion

$$F = 0. \tag{4.8}$$

Our proof of Prop. 3) thus requires H_{ij} , constructed from the q -group metric, and the braiding matrix to be nondegenerate. This condition is in addition to the minimality assumption made earlier on the theory. We must therefore search for quantum groups satisfying both of these conditions. With regard to the nondegeneracy requirement, we note that a nonsingular metric does not exist for all classical groups; So we certainly don't expect that H_{ij} (which reduces to twice the classical metric when $q \rightarrow 1$) be nondegenerate for all quantum groups. We do know of an example of a minimal theory with a nondegenerate H_{ij} . It

has $ISO_q(2, 1)$ for the quantum gauge symmetry, and therefore is relevant for the CS formulation of gravity in 2+1 dimensions. We intend to discuss it in a future article [21].

We next prove that $\langle F, F \rangle$ is a closed 4-form,

$$d\langle F, F \rangle = 0, \tag{4.9}$$

and then finally that it is exact, as stated in Prop. 2). We use the Bianchi identity

$$DF \equiv dF + AF - FA = 0, \tag{4.10}$$

which follows from the associativity of the deformed wedge product [30,33]. Thus we have

$$\begin{aligned} d\langle F, F \rangle &= \langle DF, F \rangle + \langle F, DF \rangle \\ &= \langle AF - FA, F \rangle - \langle F, AF - FA \rangle \\ &= -\langle AF - FA, F \rangle - \langle F, AF - FA \rangle. \end{aligned} \tag{4.11}$$

The (ordinary) commutator between F and A can be obtained by assuming that they are the same as in the $F = 0$ case [22]. Thus, the A, dA commutation relations are the same as the A, A^2 ones and, by means of a computation completely analogous to that leading to Eq. (3.6), one can prove that

$$A^i F^j = \Lambda_{kl}^{ij} F^k A^l. \tag{4.12}$$

Using the above commutators and (3.16) in Eq. (4.11) gives

$$\begin{aligned} d\langle F, F \rangle &= F^i A^j F^k (\langle [T_i, T_j], T_k \rangle - \langle T_i, [T_j, T_k] \rangle) \\ &= 0. \end{aligned} \tag{4.13}$$

Being closed, $\langle F, F \rangle$ is locally exact. In fact it is equal to the exterior derivative of the CS Lagrangian density; we have

$$\begin{aligned} d\mathcal{L}_{CS} &= \langle dA, dA \rangle + \frac{2}{3} (\langle dAA - AdA, A \rangle + \langle A^2, dA \rangle) \\ &= \langle dA, dA \rangle + \left(\frac{1}{2} dA^i A^j A^k - \frac{1}{6} A^i dA^j A^k \right) \\ &\quad \times \langle [T_i, T_j], T_k \rangle + \frac{2}{3} \langle A^2, dA \rangle \\ &= \langle dA, dA \rangle + \frac{1}{2} dA^i A^j A^k \langle T_i, [T_j, T_k] \rangle \\ &\quad - \frac{1}{6} A^i A^r dA^s \Lambda_{rs}^{jk} \langle [T_i, T_j], T_k \rangle + \frac{2}{3} \langle A^2, dA \rangle \\ &= \langle dA, dA \rangle + \langle dA, A^2 \rangle + \langle A^2, dA \rangle \\ &= \langle F, F \rangle - \langle A^2, A^2 \rangle = \langle F, F \rangle, \end{aligned} \tag{4.14}$$

since (see the Appendix) $\langle A^2, A^2 \rangle = 0$ for minimal deformations.

In verifying that $\langle F, F \rangle = 0$ is closed and exact, we have relied on the condition of minimality. It is not clear whether or not the calculation can be successfully carried out for more general systems. We know it is not possible to do for all systems. This is because we are able to find a counter example. The latter corresponds to take $U_q(2)$ as the gauge group which is known [35,22] not to admit nontrivial minimal deformations. We have checked, using the explicit formulae given in [22], that $d\langle F, F \rangle \neq 0$ for $U_q(2)$ (unless $r = q = 1$, in which case there is no deformation at all) and this implies that no CS term can exist in this case. This appears to be a further sign that minimality might not only be a sufficient, but in fact, a necessary condition for a deformed CS term to exist, although we have not been able to prove it.

5. Canonical formalism

In this section we present the canonical formalism for our deformed CS action. We compute the deformed Poisson brackets among the components of the connection and then show that the zero-curvature constraints are the generators of infinitesimal gauge transformations, in complete analogy with the undeformed case. Finally, we prove that the algebra of constraints is closed with respect to the deformed Poisson bracket.

Our procedure will be to obtain the symplectic structure starting from the canonical one form $\theta_{\mathcal{L}} = \sum \delta QP$ of classical mechanics. Now since here the Q 's and P 's denote noncommuting coordinates and momenta, care must be taken in their ordering. (An alternative procedure would be to carry out a constraint analysis on the noncommuting configuration space, as, for example was done in [36].)

We consider the q -CS action (4.1) on a manifold with the topology of a solid cylinder $M = \Sigma \times \mathbf{R}$, where Σ is some two-manifold that we think of as space while \mathbf{R} accounts for time. The action S_{CS} , being a three form, is invariant under the diffeomorphisms of M , hence it does not allow a natural choice of the time coordinate. Since a time function is necessary in the canonical approach, we arbitrarily choose a time function, called x^0 , and consider any constant x^0 slice (diffeomorphic to Σ) as our space, with coordinates

$\bar{x} \equiv (x^1, x^2)$. According with this separation of space and time coordinates, we split the connection A in its time and space parts

$$A = A_0 dx^0 + A_a dx^a \equiv A_0 dx^0 + \bar{A}, \quad (5.1)$$

$$a = 1, 2.$$

(In the rest of this section the first latin letters a, b, \dots will refer to space coordinates.) As it happens in the undeformed case, the q -CS action does not contain time derivatives of A_0 . We thus interpret it as a Lagrange multiplier for the constraint

$$\bar{F} \equiv \bar{d}\bar{A} + \bar{A}^2 = \frac{1}{2} F_{ab} dx^a dx^b \approx 0, \quad (5.2)$$

where $\bar{d} = dx^a \partial_a$. The phase space is then spanned by the space components of A , A_1 and A_2 and we read off the canonical one-form $\theta_{\mathcal{L}}$ directly from the part of the action which contains time derivatives of these fields:

$$\theta_{\mathcal{L}} = \int_{\Sigma} d^2 \bar{x} \delta A_a^i A_b^j g_{ij} \epsilon^{ab}, \quad \epsilon^{12} = -\epsilon^{21} = 1 \quad (5.3)$$

(the fields are evaluated at the same time x^0). From here we get the symplectic form

$$\omega = \delta \theta_{\mathcal{L}} = - \int_{\Sigma} d^2 \bar{x} \delta A_a^i \wedge \delta A_b^j g_{ij} \epsilon^{ab}, \quad (5.4)$$

where we apply the usual rules for exterior differentiation. Inverting the symplectic form we get the Poisson Brackets (PB) among the space components of the connection

$$\{A_a^i(\bar{x}), A_b^j(\bar{y})\} = \epsilon_{ab} g^{ij} \delta(\bar{x} - \bar{y}), \quad (5.5)$$

$$\epsilon_{12} = -\epsilon_{21} = 1,$$

g^{ij} being the inverse of g_{ij} . Because the deformed metric g_{ij} is, in general, non-symmetric, the PB's are, in general, not skewsymmetric.

Consider now two arbitrary function(al)s B and C of the fields $A_a(\bar{x})$. B and C are assumed to be some polynomials of the fields $A_a(\bar{x})$. Again since $A_a(\bar{x})$ are q -fields, attention must be paid to the ordering of the fields in the definition of $\{B, C\}$. We set

$$\{B, C\} = \int \int_{\Sigma \times \Sigma} d^2 \bar{w} d^2 \bar{u} \frac{\delta_R B}{\delta A_a^i(\bar{w})} \{A_a^i(\bar{w}), A_b^j(\bar{u})\} \times \frac{\delta_L C}{\delta A_b^j(\bar{u})}, \quad (5.6)$$

where R (L) denotes the right (left) derivative and it is computed after pulling A_a^i to the right (left).

We next investigate whether the constraints (5.2) are the generators of gauge transformations as in the undeformed case. We first smear the field strength \bar{F} with a Lie algebra valued function $\epsilon = \epsilon^i T_i$ so that

$$G(\epsilon) = \int_{\Sigma} \langle \epsilon, \bar{F} \rangle \approx 0. \quad (5.7)$$

We assume for ϵ the commutation relations (2.12). Then we look for the Poisson bracket of $G(\epsilon)$ with the components of the connection

$$\{G(\epsilon), A_c^j(\bar{y})\}. \quad (5.8)$$

We find

$$\begin{aligned} \{G(\epsilon), A_c^j(\bar{y})\} &= \left\{ \int_{\Sigma} \langle \epsilon, \bar{F} \rangle, A_c^j(\bar{y}) \right\} \\ &= \int_{\Sigma} d^2 \bar{x} \epsilon^k(\bar{x}) \{(\partial_a A_b^i(\bar{x}) \\ &\quad + \frac{1}{2} C_{rs}^i A_a^r(\bar{x}) A_b^s(\bar{x})) \epsilon^{ab} g_{ki}, A_c^j(\bar{y})\} \\ &= \partial_c \epsilon^j(\bar{y}) - C_{kr}^j \epsilon^k(\bar{y}) A_c^r(\bar{y}). \end{aligned} \quad (5.9)$$

We can see that this is the infinitesimal transformation of the gauge field $A_c^j(\bar{y})$ with gauge parameter $-\epsilon$. Hence we have found that $G(\epsilon)$ are the generators of the deformed gauge transformations:

$$\{G(\epsilon), \bar{A}(\bar{y})\} = -\delta_{\epsilon} \bar{A}(\bar{y}). \quad (5.10)$$

As a consequence of this result the algebra of constraints (5.7) closes, as can be seen by computing the PB for two constraints (the details of the computation can be found in the Appendix):

$$\{G(\epsilon_1), G(\epsilon_2)\} = G(\epsilon_1 \epsilon_2 - \epsilon_2 \epsilon_1). \quad (5.11)$$

This result too is completely analogous to the one found in the undeformed case. It is consistent with the closure of gauge transformations found in [22] for minimal deformations.

6. Conclusions

As we have already stressed, the CS action we propose for minimally deformed gauge theories has the advantage of enjoying all the topological and algebraic features of the undeformed case. In particular, once the existence of a new invariant scalar product has been taken into account, a deformed Hamiltonian formalism naturally arises, leading to a closed algebra for the appropriately smeared constraints, which generate the corresponding gauge transformations. We are then ready to explore the implications of the deformed CS action in the realm of 1+1 physics by simply applying our results to a manifold with boundary [20]. Furthermore, as the differential calculus for minimal $ISO_q(2, 1)$ is available together with the corresponding gauge theory, we are ready to explore the physics of 2 + 1 q -gravity [21].

Appendix A

Below we derive some equations, which hold in the case of minimal deformations. The first one is

$$\langle A^2, A^2 \rangle = 0. \tag{A.1}$$

The proof consists in showing that the left hand side is proportional to the q -Jacobi identity, Eq. (2.5).

$$\begin{aligned} \langle A^2, A^2 \rangle &= \frac{1}{4} A^i A^j A^k A^l \langle [T_i, T_j], [T_k, T_l] \rangle \\ &= \frac{1}{4} A^i A^j A^k A^l \langle [[T_i, T_j], T_k], T_l \rangle; \end{aligned} \tag{A.2}$$

upon permuting the middle A 's and using (3.13), it can be rewritten as

$$\langle A^2, A^2 \rangle = -\frac{1}{4} A^i A^j A^k A^l \Lambda_{jk}^{rs} \langle [[T_i, T_r], T_s], T_l \rangle; \tag{A.3}$$

permuting the last two A 's it can also be rewritten as

$$\langle A^2, A^2 \rangle = \frac{1}{4} A^i A^j A^k A^l \Lambda_{js}^{rm} \Lambda_{kl}^{sn} \langle [T_i, T_r], [T_m, T_n] \rangle. \tag{A.4}$$

This expression can be further manipulated using (2.6), (2.15), (3.16), in the given order. We have

$$\begin{aligned} \Lambda_{js}^{rm} \Lambda_{kl}^{sn} \langle [T_i, T_r], [T_m, T_n] \rangle &= \Lambda_{sl}^{rp} C_{jk}^s \langle [T_i, T_r], T_p \rangle \\ &= -\langle T_i, [[T_j, T_k], T_l] \rangle = -\langle [T_i, [T_j, T_k]], T_l \rangle. \end{aligned} \tag{A.5}$$

Putting together the three different ways that we have found to write $\langle A^2, A^2 \rangle$, that is Eqs. (A.2), (A.3), (A.5), we finally have

$$\begin{aligned} \langle A^2, A^2 \rangle &= \frac{1}{12} A^i A^j A^k A^l \langle \{ [[T_i, T_j], T_k] \\ &\quad - \Lambda_{jk}^{rs} [[T_i, T_r], T_s] - [T_i, [T_j, T_k]] \}, T_l \rangle \\ &= 0 \end{aligned} \tag{A.6}$$

the expression between the curly brackets being zero because of q -Jacobi identity.

We now turn to the computation of the Poisson algebra of the constraints. First, we derive the identity

$$C_{ij}^l \Lambda_{lk}^{mn} \langle T_m, T_n \rangle = \langle T_i, [T_j, T_k] \rangle. \tag{A.7}$$

It can be proven using equations (2.6), (3.16), (3.15), (2.15), (3.16) in the given order:

$$\begin{aligned} \Lambda_{lk}^{mn} C_{ij}^l g_{mn} &= \Lambda_{ip}^{mw} \Lambda_{jk}^{pq} C_{vq}^n g_{mn} = \Lambda_{ip}^{mw} \Lambda_{jk}^{pq} C_{mw}^n g_{nq} \\ &= -\Lambda_{jk}^{pq} C_{ip}^m g_{mq} = C_{jk}^m g_{im}, \end{aligned} \tag{A.8}$$

which is Eq. (A.7). Going back to the PB of two constraints, and using Eq. (5.10), we have

$$\begin{aligned} \{G(\epsilon_1), G(\epsilon_2)\} &= \int_{\Sigma} \{G(\epsilon_1), \langle \epsilon_2(\bar{y}), \bar{F}(\bar{y}) \rangle\} \\ &= \int_{\Sigma} \Lambda_{kl}^{ij} \{G(\epsilon_1), \bar{F}^k(\bar{y})\} \epsilon_2^l g_{ij} \\ &= - \int_{\Sigma} \Lambda_{kl}^{ij} C_{rs}^k g_{ij} \epsilon_1^r \bar{F}^s \epsilon_2^l. \end{aligned} \tag{A.9}$$

We now use Eq. (A.7) to write the above expression as:

$$\begin{aligned} - \int_{\Sigma} C_{sl}^k g_{rk} \epsilon_1^r \bar{F}^s \epsilon_2^l &= - \int_{\Sigma} C_{sl}^k \Lambda_{pq}^{sl} g_{rk} \epsilon_1^r \epsilon_2^p \bar{F}^q \\ &= \int_{\Sigma} C_{pq}^k g_{rk} \epsilon_1^r \epsilon_2^p \bar{F}^q = \int_{\Sigma} C_{rp}^k g_{kq} \epsilon_1^r \epsilon_2^p \bar{F}^q \\ &= G(\epsilon_1 \epsilon_2 - \epsilon_2 \epsilon_1), \end{aligned} \tag{A.10}$$

where we have made use of Eqs. (3.15) and (3.16).

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