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SO(10) group theory for the unified model building

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The complete tables of Clebsch–Gordan (CG) coefficients for a wide class of SO(10) SUSY grand unified theories (GUTs) are given. Explicit expressions of states of all corresponding multiplets under standard model gauge group $G_{321} = \mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$, necessary for evaluation of the CG coefficients are presented. The SUSY SO(10) GUT model considered here includes most of the Higgs irreducible representations usually used in the literature, **10**, **45**, **54**, **120**, **126**, $\overline{\mathbf{126}}$, and **210**. Mass matrices of all G_{321} multiplets are found for the most general superpotential. These results are indispensable for the precision calculations of the gauge couplings unification and proton decay, etc. © *2005 American Institute of Physics.* [DOI: 10.1063/1.1847709]

I. INTRODUCTION

A particularly attractive idea for the physics beyond the standard model (SM) is the possible appearance of grand unified theories (GUTs).[1] The idea of GUTs bears several profound features. Perhaps the most obvious one is that GUTs have the potential to unify the diverse set of particle representations and parameters found in the SM into a single, comprehensive, and hopefully predictive framework. For example, through the GUT symmetry one might hope to explain the quantum numbers of the fermion spectrum, or even the origins of fermion mass. Moreover, by unifying all U(1) generators within a non-Abelian theory, GUTs would also provide an explanation for the quantization of electric charge. By combining GUTs with supersymmetry (SUSY), we hope to unify the attractive features of GUTs simultaneously with those of SUSY into a single theory, SUSY GUTs.[2] The apparent gauge couplings unification of the minimal supersymmetric standard model (MSSM) is strong circumstantial evidence in favor of the emergence of a SUSY GUT near $M_G \simeq 2 \times 10^{16}$ GeV.[3]

While there are *a priori* many choices for such possible groups, the list can be narrowed down by requiring groups of rank $\geqslant 4$ that have complex representations. The smallest groups satisfying these requirements are SU(5), SU(6), SO(10), and E_6. Amongst these choices, SO(10) is particu-

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larly attractive,[4] because SO(10) is the smallest simple Lie group for which a single anomaly-free irreducible representation (irrep) (namely the spinor **16** representation) can accommodate the entire SM fermion content of each generation.

Once we fix SO(10) as the gauge group, we have also many choices of the Higgs fields though they are limited by the gauge symmetry. The Higgs fields play an essential role in the spontaneous symmetry breaking of the SO(10) gauge group and as a source of the observed fermion masses. The SO(10) gauge group must be broken down to the standard model gauge group $G_{321} = SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge group, and each SO(10) irrep must be decomposed into G_{321} multiplets. In this paper, we make an explicit construction of the states of these G_{321} multiplets. Using these states, we calculate the CG coefficients appearing in the mass matrices for the states belonging to G_{321} irreps and corresponding mass matrices in a wide class of the SO(10) models. The purpose of the present paper is to give detailed structures of the SO(10) GUTs based on a general model as far as has been possible, and to serve a wide range of unified model builders.

The paper is organized as follows. In Sec. II, we give a class of SUSY SO(10) GUTs and give an explicit form of the most general superpotential. In such a superpotential, we postulate a renormalizability in order to keep the predictability.[5–7] However, the result developed here is also applicable to the nonrenormalizable models.[8–14] The symmetry breakings are considered in Sec. III. Section IV is devoted to present explicit forms of the states in the G_{321} multiplets for all SO(10) irreps. This is the central part of the present paper. Using these tables, we give in Sec. V the mass matrices with the CG coefficients for a class of SUSY SO(10) GUTs, together with suitable tests and consistency checks for them. In Sec. VI, we consider the quark and lepton mass matrices in general SUSY SO(10) models. Section VII is devoted to conclusion. We list the decompositions of each SO(10) irreps under G_{321} subgroup in Appendix A. In Appendix B, we present the complete list of the CG coefficients for the G_{321} multiplets for all SO(10) irreps.

II. A CLASS OF SUSY SO(10) GUTs

In this section, we consider a class of renormalizable SUSY SO(10) models. They include three families of matter fields Ψ_i $(i=1,2,3)$ transforming as 16 dimensional fundamental spinor representation, **16**, gauge fields contained in the adjoint representation, and set of SO(10) multiplets of Higgs fields, enabling most general Yukawa couplings. The most general Yukawa couplings follow from decomposition of $\mathbf{16}\times\mathbf{16}=\mathbf{10}+\mathbf{120}+\mathbf{126}$, i.e., they include the Higgs fields in $H=\mathbf{10}$, $D=\mathbf{120}$, $\overline{\Delta}=\overline{\mathbf{126}}$ irreps, respectively. Furthermore, to consider as general case of the symmetry breaking of SO(10) to the standard model gauge group G_{321} as possible, we add several Higgs fields containing G_{321} singlets. They are $A=\mathbf{45}$, $\Delta=\mathbf{126}$, $\Phi=\mathbf{210}$, and $E=\mathbf{54}$ irreps. Of course, that is not the most general case. However, this set of Higgs fields is quite rich and gives rise to several realistic SUSY SO(10) models. Our aim is to give a systematic method for treatment of models with complicated Higgs sectors. This is a generalization of the method proposed in Refs. 15–17. We shall assume that the SUSY is preserved so that we consider the breaking of SUSY SO(10) to the MSSM.

Then the Yukawa couplings are

$$W_Y = Y_{10}^{ij}\Psi_i H \Psi_j + Y_{120}^{ij}\Psi_i D \Psi_j + Y_{126}^{ij}\Psi_i \overline{\Delta}\Psi_j, \tag{1}$$

where $i,j=1,2,3$ denote the generation indices. Note that H is a fundamental SO(10) irrep, and A, D, Φ, and $\Delta+\overline{\Delta}$ are antisymmetric tensors of rank 2, 3, 4, and 5, respectively. E is a symmetric traceless tensor of rank 2.

The most general Higgs superpotential is given by

$$W=\frac{1}{2}m_1\Phi^2+m_2\overline{\Delta}\Delta+\frac{1}{2}m_3H^2+\frac{1}{2}m_4A^2+\frac{1}{2}m_5E^2+\frac{1}{2}m_6D^2+\lambda_1\Phi^3+\lambda_2\Phi\overline{\Delta}\Delta+(\lambda_3\Delta+\lambda_4\overline{\Delta})H\Phi$$
$$+\lambda_5A^2\Phi-i\lambda_6A\overline{\Delta}\Delta+\frac{\lambda_7}{120}\varepsilon A\Phi^2+E(\lambda_8E^2+\lambda_9A^2+\lambda_{10}\Phi^2+\lambda_{11}\Delta^2+\lambda_{12}\overline{\Delta}^2+\lambda_{13}H^2)$$
$$+D^2(\lambda_{14}E+\lambda_{15}\Phi)+D\{\lambda_{16}HA+\lambda_{17}H\Phi+(\lambda_{18}\Delta+\lambda_{19}\overline{\Delta})A+(\lambda_{20}\Delta+\lambda_{21}\overline{\Delta})\Phi\},\tag{2}$$

where SO(10) invariants are defined in the fundamental SO(10) basis $1',2',\ldots,9',0'$ and in the Y diagonal basis $1,2,\ldots,9,0$ (which we will define in the next section) as follows:

$$\Phi^2\equiv\Phi_{a'b'c'd'}\Phi_{a'b'c'd'}=\Phi_{abcd}\Phi_{\bar a\bar b\bar c\bar d},$$

$$\overline{\Delta}\Delta\equiv\overline{\Delta}_{a'b'c'd'e'}\Delta_{a'b'c'd'e'}=\overline{\Delta}_{abcde}\Delta_{\bar a\bar b\bar c\bar d\bar e},$$

$$H^2\equiv H_{a'}H_{a'}=H_aH_{\bar a},$$

$$A^2\equiv A_{a'b'}A_{a'b'}=A_{ab}A_{\bar a\bar b},$$

$$E^2\equiv E_{a'b'}E_{a'b'}=E_{ab}E_{\bar a\bar b},$$

$$D^2\equiv D_{a'b'c'}D_{a'b'c'}=D_{abc}D_{\bar a\bar b\bar c},$$

$$\Phi^3\equiv\Phi_{a'b'c'd'}\Phi_{a'b'e'f'}\Phi_{c'd'e'f'}=\Phi_{\bar a\bar b\bar c\bar d}\Phi_{abef}\Phi_{cd\bar e\bar f},$$

$$\Phi\overline{\Delta}\Delta\equiv\Phi_{a'b'c'd'}\overline{\Delta}_{a'b'e'f'g'}\Delta_{c'd'e'f'g'}=\Phi_{\bar a\bar b\bar c\bar d}\overline{\Delta}_{abefg}\Delta_{cd\bar e\bar f\bar g},$$

$$\Delta H\Phi\equiv\Delta_{a'b'c'd'e'}H_{a'}\Phi_{b'c'd'e'}=\Delta_{\bar a\bar b\bar c\bar d\bar e}H_a\Phi_{bcde},$$

$$\overline{\Delta}H\Phi\equiv\overline{\Delta}_{a'b'c'd'e'}H_{a'}\Phi_{b'c'd'e'}=\overline{\Delta}_{\bar a\bar b\bar c\bar d\bar e}H_a\Phi_{bcde},$$

$$A^2\Phi\equiv A_{a'b'}A_{c'd'}\Phi_{a'b'c'd'}=A_{\bar a\bar b}A_{cd}\Phi_{ab\bar c\bar d},$$

$$-iA\overline{\Delta}\Delta\equiv-iA_{a'b'}\overline{\Delta}_{a'c'd'e'f'}\Delta_{b'c'd'e'f'}=-iA_{\bar a\bar b}\overline{\Delta}_{acdef}\Delta_{bc\bar d\bar e\bar f},$$

$$\tfrac{1}{120}\varepsilon A\Phi^2\equiv\tfrac{1}{120}\varepsilon_{a'_1a'_2a'_3a'_4a'_5a'_6a'_7a'_8a'_9a'_0}A_{a'_1a'_2}\Phi_{a'_3a'_4a'_5a'_6}\Phi_{a'_7a'_8a'_9a'_0}$$
$$=\tfrac{1}{120}\varepsilon_{\bar a_1\bar a_2\bar a_3\bar a_4\bar a_5\bar a_6\bar a_7\bar a_8\bar a_9\bar a_0}A_{a_1a_2}\Phi_{a_3a_4a_5a_6}\Phi_{a_7a_8a_9a_0},$$

$$E^3\equiv E_{a'b'}E_{a'c'}E_{b'c'}=E_{\bar a\bar b}E_{ac}E_{b\bar c},\tag{3}$$

$$EA^2\equiv E_{a'b'}A_{a'c'}A_{b'c'}=E_{\bar a\bar b}A_{ac}A_{b\bar c},$$

$$E\Phi^2\equiv E_{a'b'}\Phi_{a'c'd'e'}\Phi_{b'c'd'e'}=E_{\bar a\bar b}\Phi_{acde}\Phi_{b\bar c\bar d\bar e},$$

$$E\Delta^2\equiv E_{a'b'}\Delta_{a'c'd'e'f'}\Delta_{b'c'd'e'f'}=E_{\bar a\bar b}\Delta_{acdef}\Delta_{b\bar c\bar d\bar e\bar f},$$

$$E\overline{\Delta}^2 \equiv E_{a'b'}\overline{\Delta}_{a'c'd'e'f'}\overline{\Delta}_{b'c'd'e'f'} = E_{\overline{a}\overline{b}}\overline{\Delta}_{acdef}\overline{\Delta}_{b\overline{c}\overline{d}\overline{e}\overline{f}},$$

$$EH^2 \equiv E_{a'b'}H_{a'}H_{b'} = E_{\overline{a}\overline{b}}H_aH_b,$$

$$ED^2 \equiv E_{a'b'}D_{a'c'd'}D_{b'c'd'} = E_{\overline{a}\overline{b}}D_{acd}D_{b\overline{c}\overline{d}},$$

$$D^2\Phi \equiv D_{a'b'c'}D_{a'd'e'}\Phi_{b'c'd'e'} = D_{\overline{a}\overline{b}\overline{c}}D_{ade}\Phi_{bc\overline{d}\overline{e}},$$

$$DHA \equiv D_{a'b'c'}H_{a'}A_{b'c'} = D_{\overline{a}\overline{b}\overline{c}}H_aA_{bc},$$

$$DH\Phi \equiv D_{a'b'c'}H_{d'}\Phi_{a'b'c'd'} = D_{\overline{a}\overline{b}\overline{c}}H_d\Phi_{abc\overline{d}},$$

$$D\Delta A \equiv D_{a'b'c'}\Delta_{a'b'c'd'e'}A_{d'e'} = D_{\overline{a}\overline{b}\overline{c}}\Delta_{abcde}A_{\overline{d}\overline{e}},$$

$$D\overline{\Delta}A \equiv D_{a'b'c'}\overline{\Delta}_{a'b'c'd'e'}A_{d'e'} = D_{\overline{a}\overline{b}\overline{c}}\overline{\Delta}_{abcde}A_{\overline{d}\overline{e}},$$

$$D\Delta\Phi \equiv D_{a'b'c'}\Delta_{a'b'd'e'f'}\Phi_{c'd'e'f'} = D_{\overline{a}\overline{b}\overline{c}}\Delta_{abdef}\Phi_{c\overline{d}\overline{e}\overline{f}},$$

$$D\overline{\Delta}\Phi \equiv D_{a'b'c'}\overline{\Delta}_{a'b'd'e'f'}\Phi_{c'd'e'f'} = D_{\overline{a}\overline{b}\overline{c}}\overline{\Delta}_{abdef}\Phi_{c\overline{d}\overline{e}\overline{f}}.$$

Here $a',b',c',\ldots$ $(a,b,c,\ldots)$ run over all the SO(10) vector (Y diagonal) indices and ε is a totally antisymmetric SO(10) invariant tensor with

$$\varepsilon_{1'2'3'4'5'6'7'8'9'0'} = i\varepsilon_{1234567890} = 1. \tag{4}$$

III. SYMMETRY BREAKING

Here we first introduce Y diagonal basis (see also Ref. 18): $1=1'+2'i$, $2=1'-2'i$, $3=3'+4'i$, $4=3'-4'i$, $5=5'+6'i$, $6=5'-6'i$, $7=7'+8'i$, $8=7'-8'i$, $9=9'+0'i$, $0=9'-0'i$, up to the normalization factor $1/\sqrt{2}$. It is more convenient since (1,3,5,7,9) transforms as **5**-plet and (2,4,6,8,0) transforms as $\overline{\mathbf{5}}$-plet under $SU(5)\times U(1)_X$ [for that reason Y diagonal basis could also be called SU(5) basis]. Consequently, (1,3) and (2,4) are $SU(2)_L$ doublets with definite hypercharges $Y=\frac{1}{2}$ and $Y=-\frac{1}{2}$, respectively. Similarly, (5,7,9) and (6,8,0) transform under $SU(3)_C$ as **3** and $\overline{\mathbf{3}}$ with definite hypercharges $Y=-\frac{1}{3}$ and $Y=\frac{1}{3}$, respectively. Note that under the complex conjugation (c.c.), $\overline{1}=2$, $\overline{3}=4$, $\overline{5}=6$, $\overline{7}=8$, $\overline{9}=0$, and vice versa. The SO(10) invariants are built in such a way that an index a is contracted (summed) with the corresponding c.c. index $\overline{a}$, for example, $T_{\cdots a\cdots}T_{\cdots \overline{a}\cdots}$.

The basis in $A=\mathbf{45}$, $D=\mathbf{120}$, $\Phi=\mathbf{210}$, and $\Delta+\overline{\Delta}=\mathbf{126}+\overline{\mathbf{126}}$ dimensional spaces are defined by totally antisymmetric (unit) tensors $(a'b')$, $(a'b'c')$, $(a'b'c'd')$, and $(a'b'c'd'e')$, respectively, and similarly in a, b, c, d, e indices in Y diagonal basis. The states of the Δ and $\overline{\Delta}$ have additional properties,

$$i\varepsilon_{\overline{a}_1\overline{a}_2\overline{a}_3\overline{a}_4\overline{a}_5\overline{a}_6\overline{a}_7\overline{a}_8\overline{a}_9\overline{a}_{10}}\overline{\Delta}_{a_6a_7a_8a_9a_{10}} = \overline{\Delta}_{\overline{a}_1\overline{a}_2\overline{a}_3\overline{a}_4\overline{a}_5},$$
$$i\varepsilon_{\overline{a}_1\overline{a}_2\overline{a}_3\overline{a}_4\overline{a}_5\overline{a}_6\overline{a}_7\overline{a}_8\overline{a}_9\overline{a}_{10}}\Delta_{a_6a_7a_8a_9a_{10}} = -\Delta_{\overline{a}_1\overline{a}_2\overline{a}_3\overline{a}_4\overline{a}_5}, \tag{5}$$

that allow one to project out the Δ and $\overline{\Delta}$ states, respectively, from the 256 antisymmetric states $(abcde)$. The explicit expressions for antisymmetric tensors are, for example,

$$(ab) = ab - ba,$$

$$(abc) = abc + cab + bca - bac - acb - cba, \tag{6}$$

etc. Important relations are

$$\begin{aligned}
(12) &= -i(1'2'),\\
(34) &= -i(3'4'),\\
(56) &= -i(5'6'),\\
(78) &= -i(7'8'),\\
(90) &= -i(9'0').
\end{aligned} \tag{7}$$

Symmetric $E=\mathbf{54}$ dimensional space is spanned by traceless symmetric states $\{a'b'\}\equiv a'b' + b'a'(a',b'=1',2',\ldots,9',0')$ and $\Sigma_{a'}c_{a'}\{a'a'\}$ with $\Sigma_{a'}c_{a'}\equiv 0$. Also, important relations are

$$\begin{aligned}
\{12\} &= 1'1' + 2'2',\\
\{34\} &= 3'3' + 4'4',\\
\{56\} &= 5'5' + 6'6',\\
\{78\} &= 7'7' + 8'8',\\
\{90\} &= 9'9' + 0'0'.
\end{aligned} \tag{8}$$

Now, the Higgs fields A, E, Δ, $\overline{\Delta}$, and Φ contain eight directions of singlets under the G_{321} subgroup (see Appendix A). The corresponding vacuum expectation values (VEVs) are defined by

$$\langle A\rangle = \sum_{i=1}^{2} A_i \hat{A}_i, \tag{9}$$

$$\langle E\rangle = E\hat{E}, \tag{10}$$

$$\langle \Delta\rangle = v_R \hat{v_R}, \tag{11}$$

$$\langle \overline{\Delta}\rangle = \overline{v_R}\widehat{\overline{v_R}}, \tag{12}$$

$$\langle \Phi\rangle = \sum_{i=1}^{3} \Phi_i \hat{\Phi}_i, \tag{13}$$

where unit directions $\hat{A}_i$ $(i=1,2)$, $\hat{E}$, $\hat{v_R}$, $\widehat{\overline{v_R}}$ and $\hat{\Phi}_i$ $(i=1,2,3)$ in the Y diagonal basis are

$$\hat{A}_1 = \hat{A}_{(1,1,3)}^{(1,1,0)} = \frac{i}{2}(12+34), \tag{14}$$

$$\hat{A}_2 = \hat{A}^{(1,1,0)}_{(15,1,1)} = \frac{i}{\sqrt{6}}(56 + 78 + 90), \tag{15}$$

$$\hat{E} = \hat{E}^{(1,1,0)}_{(1,1,1)} = \frac{1}{\sqrt{60}}\{\mathit{3} \times [12 + 34] - \mathit{2} \times [56 + 78 + 90]\}, \tag{16}$$

$$\widehat{v_R} = \hat{\Delta}^{(1,1,0)}_{(10,1,3)} = \frac{1}{\sqrt{120}}(24\,680), \tag{17}$$

$$\widehat{\overline{v}_R} = \hat{\overline{\Delta}}^{(1,1,0)}_{(10,1,3)} = \frac{1}{\sqrt{120}}(13\,579), \tag{18}$$

$$\hat{\Phi}_1 = \hat{\Phi}^{(1,1,0)}_{(1,1,1)} = -\frac{1}{\sqrt{24}}(1234), \tag{19}$$

$$\hat{\Phi}_2 = \hat{\Phi}^{(1,1,0)}_{(15,1,1)} = -\frac{1}{\sqrt{72}}(5678 + 5690 + 7890), \tag{20}$$

$$\hat{\Phi}_3 = \hat{\Phi}^{(1,1,0)}_{(15,1,3)} = -\frac{1}{12}([12 + 34][56 + 78 + 90]). \tag{21}$$

Here and hereafter, the upper and the lower indices indicate the $SU(3)_C \times SU(2)_L \times U(1)_Y$, $SU(4) \times SU(2)_L \times SU(2)_R$ quantum numbers, respectively in the case of double indices. A word about notation: the square brackets are used for grouping of indices. This grouping of indices is used to emphasize the $SU(2)_L$ and $SU(3)_C$ structures within the state vectors. The square brackets satisfy usual distributive law with respect to summation of indices and tensor product of indices, e.g.,

$$\begin{aligned}([12 + 34][56 + 78 + 90]) &= (1256 + 1278 + 1290 + 3456 + 3478 + 3490) \\ &= (1256) + (1278) + (1290) + (3456) + (3478) + (3490),\end{aligned}$$

$$\begin{aligned}([1,3][5[78 + 90],7[56 + 90],9[56 + 78]]) &= (1578 + 1590, 1756 + 1790, 1956 + 1978, 3578 \\ &\quad + 3590, 3756 + 3790, 3956 + 3978) \\ &= (1578, 1756, 1956, 3578, 3756, 3956) \\ &\quad + (1590, 1790, 1978, 3590, 3790, 3978).\end{aligned} \tag{22}$$

Further, the numerical factors which could be misinterpreted as additional SO(10) indices are written in italics.

The unit directions appearing in VEVs satisfy the following orthonormality relations:

$$\hat{A}_i \cdot \hat{A}_j = \delta_{ij} \quad (i,j = 1,2),$$

$$\hat{E}^2 = 1,$$

$$\widehat{v_R} \cdot \widehat{v_R} = \widehat{\overline{v}_R} \cdot \widehat{\overline{v}_R} = 0, \tag{23}$$

$$\widehat{v_R}\cdot\widehat{\overline{v_R}}=1,$$

$$\widehat{\Phi_i}\cdot\widehat{\Phi_j}=\delta_{ij}\quad (i,j=1,2,3).$$

Due to the D-flatness condition the absolute values of the VEVs, v_R and $\overline{v_R}$ are equal,

$$|v_R|=|\overline{v_R}|. \tag{24}$$

The superpotential of Eq. (2) calculated at the VEVs in Eqs. (14)–(21) is

$$\langle W\rangle=\frac{1}{2}m_1\langle\Phi\rangle^2+m_2\langle\overline{\Delta}\rangle\langle\Delta\rangle+\frac{1}{2}m_4\langle A\rangle^2+\frac{1}{2}m_5\langle E\rangle^2+\lambda_1\langle\Phi\rangle^3+\lambda_2\langle\Phi\rangle\langle\overline{\Delta}\rangle\langle\Delta\rangle+\lambda_5\langle A\rangle^2\langle\Phi\rangle$$
$$-i\lambda_6\langle A\rangle\langle\overline{\Delta}\rangle\langle\Delta\rangle+\frac{\lambda_7}{120}\varepsilon\langle A\rangle\langle\Phi\rangle^2+\langle E\rangle[\lambda_8\langle E\rangle^2+\lambda_9\langle A\rangle^2+\lambda_{10}\langle\Phi\rangle^2]. \tag{25}$$

Inserting the VEVs from Eqs. (14)–(21), one obtains

$$\langle W\rangle=\frac{1}{2}m_1[\Phi_1^2+\Phi_2^2+\Phi_3^2]+m_2v_R\overline{v_R}+\frac{1}{2}m_4(A_1^2+A_2^2)+\frac{1}{2}m_5E^2$$
$$+\lambda_1\left[\Phi_2^3\frac{1}{9\sqrt{2}}+3\Phi_1\Phi_3^2\frac{1}{6\sqrt{6}}+3\Phi_2\Phi_3^2\frac{1}{9\sqrt{2}}\right]+\lambda_2\left[\Phi_1\frac{1}{10\sqrt{6}}+\Phi_2\frac{1}{10\sqrt{2}}+\Phi_3\frac{1}{10}\right]v_R\overline{v_R}$$
$$+\lambda_5\left[A_1^2\Phi_1\frac{1}{\sqrt{6}}+A_2^2\Phi_2\frac{\sqrt{2}}{3}+A_1A_2\Phi_3\frac{2}{\sqrt{6}}\right]+\lambda_6\left[A_1\left(-\frac{1}{5}\right)+A_2\left(-\frac{3}{5\sqrt{6}}\right)\right]v_R\overline{v_R}$$
$$+\lambda_7\left[2A_2\Phi_1\Phi_2\frac{\sqrt{2}}{5}+A_2\Phi_3^2\frac{2\sqrt{2}}{5\sqrt{3}}+2A_1\Phi_2\Phi_3\frac{\sqrt{2}}{5}\right]+\lambda_8E^3\frac{1}{2\sqrt{15}}+\lambda_9E\left[A_1^2\frac{\sqrt{3}}{2\sqrt{5}}+A_2^2\left(-\frac{1}{\sqrt{15}}\right)\right]$$
$$+\lambda_{10}E\left[\Phi_1^2\frac{\sqrt{3}}{2\sqrt{5}}+\Phi_2^2\left(-\frac{1}{\sqrt{15}}\right)+\Phi_3^2\frac{1}{4\sqrt{15}}\right]. \tag{26}$$

The VEVs are determined by the following equation:

$$\left\{\frac{\partial}{\partial\Phi_1},\frac{\partial}{\partial\Phi_2},\frac{\partial}{\partial\Phi_3},\frac{\partial}{\partial v_R},\frac{\partial}{\partial\overline{v_R}},\frac{\partial}{\partial A_1},\frac{\partial}{\partial A_2},\frac{\partial}{\partial E}\right\}\langle W\rangle=0. \tag{27}$$

From Eq. (27), we obtain seven equations for Φ_1, Φ_2, Φ_3, A_1, A_1, E, and $v_R\overline{v_R}$. They are the following:

$$0=m_1\Phi_1+\frac{\lambda_1\Phi_3^2}{2\sqrt{6}}+\frac{\lambda_2v_R\overline{v_R}}{10\sqrt{6}}+\frac{\lambda_5A_1^2}{\sqrt{6}}+\frac{2\sqrt{2}\lambda_7A_2\Phi_2}{5}+\frac{\sqrt{3}\lambda_{10}\Phi_1E}{\sqrt{5}},$$

$$0=m_1\Phi_2+\frac{\lambda_1\Phi_2^2}{3\sqrt{2}}+\frac{\lambda_1\Phi_3^2}{3\sqrt{2}}+\frac{\lambda_2v_R\overline{v_R}}{10\sqrt{2}}+\frac{\sqrt{2}}{3}\lambda_5A_2^2+\frac{2\sqrt{2}\lambda_7\Phi_1A_2}{5}+\frac{2\sqrt{2}\lambda_7A_1\Phi_3}{5}-\frac{2\lambda_{10}\Phi_2E}{\sqrt{15}},$$

$$0=m_1\Phi_3+\frac{\lambda_1\Phi_1\Phi_3}{\sqrt{6}}+\frac{\sqrt{2}\lambda_1\Phi_2\Phi_3}{3}+\frac{\lambda_2v_R\overline{v_R}}{10}+\frac{\sqrt{2}\lambda_5A_1A_2}{\sqrt{3}}+\frac{2\sqrt{2}\lambda_7A_1\Phi_2}{5}+\frac{4\sqrt{2}\lambda_7A_2\Phi_3}{5\sqrt{3}}+\frac{\lambda_{10}\Phi_3E}{2\sqrt{15}},$$

$$0=v_R\overline{v_R}\left[m_2+\frac{\lambda_2\Phi_1}{10\sqrt{6}}+\frac{\lambda_2\Phi_2}{10\sqrt{2}}+\frac{\lambda_2\Phi_3}{10}-\frac{\lambda_6A_1}{5}-\frac{\sqrt{3}\lambda_6A_2}{5\sqrt{2}}\right], \tag{28}$$

$$0 = m_4A_1 + \frac{\sqrt{2}\lambda_5A_1\Phi_1}{\sqrt{3}} + \frac{\sqrt{2}\lambda_5A_2\Phi_3}{\sqrt{3}} - \frac{\lambda_6 v_R\overline{v_R}}{5} + \frac{2\sqrt{2}\lambda_7\Phi_2\Phi_3}{5} + \frac{\sqrt{3}\lambda_9A_1E}{\sqrt{5}},$$

$$0 = m_4A_2 + \frac{\sqrt{2}\lambda_5A_1\Phi_3}{\sqrt{3}} + \frac{2\sqrt{2}\lambda_5A_2\Phi_2}{3} - \frac{\sqrt{3}\lambda_6 v_R\overline{v_R}}{5\sqrt{2}} + \frac{2\sqrt{2}\lambda_7\Phi_3^2}{5\sqrt{3}} + \frac{2\sqrt{2}\lambda_7\Phi_1\Phi_2}{5} - \frac{2\lambda_9A_2E}{\sqrt{15}},$$

$$0 = m_5E + \frac{\sqrt{3}\lambda_8E^2}{2\sqrt{5}} + \frac{\sqrt{3}\lambda_9A_1^2}{2\sqrt{5}} - \frac{\lambda_9A_2^2}{\sqrt{15}} + \frac{\sqrt{3}\lambda_{10}\Phi_1^2}{2\sqrt{5}} - \frac{\lambda_{10}\Phi_2^2}{\sqrt{15}} + \frac{\lambda_{10}\Phi_3^2}{4\sqrt{15}}.$$

If we assume $v_R\overline{v_R}\neq 0$, we obtain five quadratic equations and one linear equation for Φ_i, A_i, and E. For that case there are 32 solutions. Two of them correspond to SU(5) symmetry and remaining 30 solutions to G_{321} standard gauge group symmetry solutions. If we set $v_R=0$, we find six quadratic equations for Φ_1, Φ_2, Φ_3, A_1, A_2, and E with 64 solutions with symmetry groups having rank 5. They are isomorphic to $G_{3211}\equiv \mathrm{SU}(3)_C\times \mathrm{SU}(2)_L\times \mathrm{U}(1)_R\times \mathrm{U}(1)_{B-L}$. However, there are solutions with higher symmetries. They are G_{3221}, G_{421}, G_{422}, and G_{51} (for the reader's convenience, we list the decompositions of each representation in Appendix A). For general coupling constants $\lambda_1,\ldots,\lambda_{21}$, $m_1,\ldots,m_8$, the solutions with higher symmetries are specified by the following relations. Solutions with higher symmetries are characterized by the following:

(1) $\mathrm{SU}(5)\times \mathrm{U}(1)_X$ and $(\mathrm{SU}(5)\times \mathrm{U}(1))^{\mathrm{flipped}}$ symmetry solutions,

$$E = v_R = 0,$$
$$\Phi_1 = \frac{\varepsilon}{\sqrt{6}}\Phi_3, \quad \Phi_2 = \frac{\varepsilon}{\sqrt{2}}\Phi_3, \quad A_1 = \frac{2\varepsilon}{\sqrt{6}}A_2, \tag{29}$$

where $\varepsilon=1$ and $\varepsilon=-1$ correspond to the $\mathrm{SU}(5)\times \mathrm{U}(1)_X$ symmetric vacua and $(\mathrm{SU}(5)\times \mathrm{U}(1))^{\mathrm{flipped}}$ symmetric vacua, respectively.

(2) SU(5) symmetry solutions,

$$E = 0,$$
$$\Phi_1 = \frac{1}{\sqrt{6}}\Phi_3, \quad \Phi_2 = \frac{1}{\sqrt{2}}\Phi_3, \quad A_1 = \frac{2}{\sqrt{6}}A_2, \quad v_R \neq 0. \tag{30}$$

(3) $G_{422}\equiv \mathrm{SU}(4)\times \mathrm{SU}(2)_L\times \mathrm{SU}(2)_R$ symmetry solutions,

$$\Phi_2 = \Phi_3 = A_1 = A_2 = v_R = 0,$$
$$\Phi_1 \neq 0, \quad E \neq 0. \tag{31}$$

(4) $G_{3221}\equiv \mathrm{SU}(3)_C\times \mathrm{SU}(2)_L\times \mathrm{SU}(2)_R\times \mathrm{U}(1)_{B-L}$ symmetry solutions,

$$\Phi_3 = A_1 = v_R = 0,$$
$$\Phi_1 \neq 0, \quad \Phi_2 \neq 0, \quad A_2 \neq 0, \quad E \neq 0. \tag{32}$$

(5) $G_{421}\equiv \mathrm{SU}(4)\times \mathrm{SU}(2)_L\times \mathrm{U}(1)$ symmetry solutions,

$$\Phi_2 = \Phi_3 = A_2 = v_R = 0,$$
$$\Phi_1 \neq 0, \quad A_1 \neq 0, \quad E \neq 0. \tag{33}$$

(6) $G_{3211}\equiv \mathrm{SU}(3)_C\times \mathrm{SU}(2)_L\times \mathrm{U}(1)_R\times \mathrm{U}(1)_{B-L}$ symmetry solutions,

$$v_R = 0,$$
$$\Phi_i \neq 0 \quad (i=1,2,3), \quad A_i \neq (i=1,2), \quad E \neq 0. \tag{34}$$

The higher symmetry solutions given in Eqs. (29)–(34) lead to the crucial consistency checks for all results in this paper.

IV. THE STATES AND CLEBSCH–GORDAN COEFFICIENTS

A. The states in the G_{321} multiplets

In order to obtain and study the mass matrices, it is convenient to decompose the Higgs representations under the G_{321} gauge group. The explicit decompositions of **10**, **45**, **54**, **120**, **126**, $\overline{\mathbf{126}}$, and **210** representations in Y diagonal basis are presented according to the G_{321} multiplets with the same quantum numbers which generally mix among themselves. The eight singlets $(\mathbf{1},\mathbf{1},0)$ are already given in Eqs. (14)–(21). There are $45-12=33$ would-be NG modes. They are in the following multiplets: $(\mathbf{1},\mathbf{1},0)$, $[(\mathbf{3},\mathbf{2},-\frac{5}{6})+\text{c.c.}]$, $[(\mathbf{1},\mathbf{1},1)+\text{c.c.}]$, $[(\mathbf{3},\mathbf{1},\frac{2}{3})+\text{c.c.}]$, and $[(\mathbf{3},\mathbf{2},\frac{1}{6})+\text{c.c.}]$. The corresponding orthonormal states are listed in Table 1 (Ref. 19) in Y-diagonal basis. The physically important modes, so-called, Higgs doublets $[(\mathbf{1},\mathbf{2},\frac{1}{2})+\text{c.c.}]$ and color triplets $[(\mathbf{3},\mathbf{1},-\frac{1}{3})+\text{c.c.}]$ states are listed in Tables 2 (Ref. 19) and 3 (Ref. 19), respectively. The remaining G_{321} multiplets are listed in Tables 4 (Ref. 19) and 5 (Ref. 19). There are altogether 691 states accommodated in G_{321} multiplets [see Table 6 (Ref. 19)]. There are 26 mass matrices, five containing NG modes, one containing doublets, one containing color triplets, 19 containing the other modes. There are five multiplets (33 states) with zero mass and 69 G_{321} multiplets with masses different from zero (containing $691-33=658$ states). Hence the mass spectrum contains 70 different mass eigenvalues. For the SU (5) solutions, there are only 21 different masses and G_{321} multiplets are grouped into multiplets transforming under the SU (5) group. For the G_{422} solutions, there are 27 different mass eigenvalues and G_{321} multiplets are grouped into multiplets transforming under the G_{422} group (see Appendix A). The higher symmetries serve as a strong consistency check of our mass matrices and CG coefficients.

We point out that the main basic blocks in all 691 states are $SU(2)_L$ irreps **1**, **2**, and **3**, and SU (3) irreps **1**, **3**, $\overline{\mathbf{3}}$, **6**, $\overline{\mathbf{6}}$, and **8**:

$$SU(2)_L,$$
$$\mathbf{1} \quad [12+34],\ (13),(24),\ (1234),$$
$$\mathbf{2} \quad [1,3],\ [-3,1],\ [2,4],[-4,2],$$
$$\mathbf{3} \quad \left[14,32,\frac{12-34}{\sqrt{2}}\right],\ \left\{\frac{11}{2},-\frac{33}{2},-\frac{13}{\sqrt{2}}\right\},\ \left\{\frac{22}{2},-\frac{44}{2},-\frac{24}{\sqrt{2}}\right\} \tag{35}$$
$$SU(3)_C,$$
$$\mathbf{1} \quad [56+78+90],\ (579),\ (680),\ (5678+5690+7890),$$
$$\mathbf{3} \quad [5,7,9],\ (80,06,68),\ (5680,7806,9068),$$
$$\overline{\mathbf{3}} \quad [6,8,0],\ (79,95,57),\ (5697,7859,9075),$$

$$\mathbf{6} \quad \left(580,670,689,\frac{6[90-78]}{\sqrt{2}},\frac{8[56-90]}{\sqrt{2}},\frac{0[78-56]}{\sqrt{2}}\right),$$

$$\left\{\frac{55}{2},\frac{77}{2},\frac{99}{2},\frac{79}{\sqrt{2}},\frac{95}{\sqrt{2}},\frac{57}{\sqrt{2}}\right\}, \tag{36}$$

$$\overline{\mathbf{6}} \quad \left(679,589,570,\frac{5[09-78]}{\sqrt{2}},\frac{7[65-09]}{\sqrt{2}},\frac{9[87-65]}{\sqrt{2}}\right),$$

$$\left\{\frac{66}{2},\frac{88}{2},\frac{00}{2},\frac{\{80\}}{\sqrt{2}},\frac{\{06\}}{\sqrt{2}},\frac{\{68\}}{\sqrt{2}}\right\},$$

$$\mathbf{8} \quad \left[58,50,70,76,96,98,\frac{56-78}{\sqrt{2}},\frac{56+78-2\times 90}{\sqrt{6}}\right],$$

$$\left(5890,5078,7056,7690,9678,9856,\frac{5690-7890}{\sqrt{2}},\frac{2\times 5678-5690-7890}{\sqrt{6}}\right).$$

All states can be constructed combining and antisymmetrizing or symmetrizing the basic blocks. The basic blocks (35) and (36) which appear only in antisymmetric tensors are embraced by parentheses, the basic blocks which appear only in the symmetric tensors are embraced by curly brackets ($\{aa\}/2=aa$), while the basic blocks that appear both in symmetric and antisymmetric tensors are embraced by square brackets.

B. $\mathcal{H}$ operators and Clebsch–Gordan coefficients

Let us denote by R the sum of all representations **10**, **45**, **54**, **120**, **126**, $\overline{\mathbf{126}}$, and **210**,

$$R=\sum_I R^I, \quad \dim R = 691. \tag{37}$$

There are 21 cubic invariants [see Eq. (3)],

$$\mathcal{I}(R^I,R^J,R^{\overline{K}}) \equiv \mathcal{I}^{IJ\overline{K}}, \quad \overline{R^K} \equiv R^{\overline{K}}, \tag{38}$$

where $R^I \times R^J = \Sigma_K R^K$.

Let us denote $\mathcal{H}$-operators (see Refs. 15 and 16),

$$\mathcal{H}_K(R^I,R^J)=\mathcal{H}_K(R^J,R^I) \sim R^K \tag{39}$$

transforming as R^K and

$$\mathcal{H}_K(R^I,R^J)=\frac{1}{N}\frac{\partial \mathcal{I}^{IJ\overline{K}}}{\partial R^{\overline{K}}}. \tag{40}$$

The normalization factor N is chosen so that

$$\mathcal{H}_K(R^I,R^J)R^{\overline{K}}=\mathcal{I}^{IJ\overline{K}}. \tag{41}$$

For example, in the Y-diagonal basis

$$[\mathcal{H}_\Phi(\Phi_1,\Phi_2)]_{abcd} = \tfrac{1}{6}[(\Phi_1)_{abef}(\Phi_2)_{cd\overline{ef}} - (\Phi_1)_{acef}(\Phi_2)_{bd\overline{ef}} + (\Phi_1)_{adef}(\Phi_2)_{bc\overline{ef}} + (\Phi_1)_{cdef}(\Phi_2)_{ab\overline{ef}} - (\Phi_1)_{bdef}(\Phi_2)_{ac\overline{ef}} + (\Phi_1)_{bcef}(\Phi_2)_{ad\overline{ef}}]. \quad (42)$$

For invariants of the type $\mathcal{I}^{III}$, there is only one $\mathcal{H}$ operator, $\mathcal{H}_I$, for invariants of the type $\mathcal{I}^{II\overline{K}}$ there are two $\mathcal{H}$ operators, $\mathcal{H}_{\overline{I}}$ and $\mathcal{H}_K$, and for invariants of the type $\mathcal{I}^{IJK}$ there are three $\mathcal{H}$ operators, $\mathcal{H}_{\overline{I}}$, $\mathcal{H}_{\overline{J}}$, and $\mathcal{H}_K$. The $\mathcal{H}$ operators are symmetric in R^I and R^J, $\mathcal{H}(R^I,R^J)=\mathcal{H}(R^J,R^I)$. More generally,

$$\mathcal{I}^{IJ\overline{K}} = \mathcal{I}^{JI\overline{K}} = \mathcal{I}^{I\overline{K}J} = \mathcal{I}^{J\overline{K}I} = \mathcal{I}^{\overline{K}IJ} = \mathcal{I}^{\overline{K}JI}. \quad (43)$$

We are especially interested in CG coefficients when at least one of the states transforms as a singlet under the G_{321} group. That requires the decomposition of each SO(10) irrep R^I into G_{321} irreps,

$$R^I = \sum_i R_i^I, \quad (44)$$

where indices "i" just enumerate the G_{321} irreps with fixed Y contained in a specific R^I irrep. There are eight singlets, denoted here shortly by S_i^I, defined in Eqs. (14)–(21). For this choice of G_{321}-singlet states, the action of $\mathcal{H}$ operators are reduced to the invariant subspaces of states with fixed G_{321} quantum numbers, which are listed in Tables 1–5 (Ref. 19),

$$\mathcal{H}_K(\hat{S}_i^I,R_j^J) = \sum_k C_{ijk}^{IJK} R_k^K, \quad (45)$$

where R_j^J and R_k^K transform as identical G_{321} irreps (have the same G_{321} quantum numbers). For example, for the evaluation of the first column in the $(\mathbf{1},\mathbf{3},1)$ mass matrix [see Eq. (74)], the following $\mathcal{H}$ operators must be evaluated:

$$\mathcal{H}_E(\hat{E},\hat{E}_{(1,3,3)}^{(1,3,1)}) = \frac{\sqrt{3}}{2\sqrt{5}}\hat{E}_{(1,3,3)}^{(1,3,1)},$$
$$\mathcal{H}_{\overline{\Delta}}(\widehat{v_R},\hat{E}_{(1,3,3)}^{(1,3,1)}) = \frac{1}{5}\hat{\overline{\Delta}}_{(\overline{10},3,1)}^{(1,3,1)}. \quad (46)$$

Note that

$$C_{ijk}^{IJK} = \mathcal{I}(\hat{S}_i^I,\hat{R}_j^J,\hat{R}_k^{\overline{K}}) \equiv \hat{S}_i^I\hat{R}_j^J\hat{R}_k^{\overline{K}} \quad (47)$$

[the second part of Eq. (47) defines shortand notation] where

$$\hat{R}_i^{\overline{I}} \equiv \overline{\hat{R}_i^I} \quad (48)$$

is the complex conjugated irrep of $\hat{R}_i^I$. The CG coefficients are listed in 25 tables in Appendix B. Note that the CG coefficients depend only on the indices $i,j,\ldots$, that represent G_{321} multiplets and not on the specific states within the G_{321} multiplets. That can be used as a consistency check of the states belonging to the specific G_{321} multiplets.

We choose all SO(10) invariants in Eq. (3) to be real, except for invariants containing the Δ or the $\overline{\Delta}$ field separately (for example, $\Delta H\Phi$ or $E\overline{\Delta}^2$). In this last case, the sum of the two invariants, one containing Δ field and the other having $\overline{\Delta}$ field in place of Δ, are real. Generally, CG coefficients are complex (in our case they are either real or pure imaginary). Starting with cubic invariants in Eq. (3), our choice of phases of states in Tables 1–5 (Ref. 19) is such that it leads to the minimal number of imaginary terms in the mass matrices.

Symmetry relations imply

$$C_{ijk}^{IJK} = C_{jik}^{JIK}. \tag{49}$$

CG coefficients also satisfy Hermiticity relations,

$$\overline{C_{ijk}^{IJK}} = C_{\overline{i}kj}^{\overline{I}KJ}. \tag{50}$$

Here the $\overline{i}$ represents the label assigned to the irrep complex conjugated to the irrep designated by i.

Furthermore, following relations are valid:

$$\sum_j C_{ij\overline{j}}^{IJ\overline{J}} \dim R_j^J = 0, \quad \sum_j \dim R_j^J = \dim R^J, \tag{51}$$

$$\sum_i \sum_j C_{ijk}^{IJK} C_{\overline{i}\overline{j}l}^{\overline{I}\overline{J}K} \dim R_i^I \dim R_j^J = C(I,J,K)\delta_{k\overline{l}},$$

$$\sum_i \dim R_i^I = \dim R^I, \quad \sum_j \dim R_j^J = \dim R^J, \tag{52}$$

where $C(I,J,K)$ are constants depending on irreps R^I, R^J, and R^K.

V. MASS MATRICES

A. Mass matrices

For the sum of all representations R [see Eq. (37)], we define fluctuations of the Higgs field around the VEVs as

$$R = \langle R \rangle + \sum_{I,i} r_i^I. \tag{53}$$

Then the matrix element of the mass matrix corresponding to G_{321} multiplets i,j (R_i^I and R_j^J transform identically under G_{321}) is

$$\mathcal{M}_{ij}^{IJ} = \left(\frac{\partial^2 W}{\partial r_i^I \partial r_{\overline{j}}^{\overline{J}}} \right)_{R=\langle R \rangle}, \tag{54}$$

$$\mathcal{M}_{ij}^{IJ} = m_I \delta_{IJ} \delta_{ij} + \sum_{K,k} \lambda^{IJK} S_i^I C_{ijk}^{IJK}, \tag{55}$$

where $\lambda^{III}=6\lambda_p$, $\lambda^{IIK}=2\lambda_p$, and $\lambda^{IJK}=\lambda_p$ for invariants containing three identical representations, two identical representations (different from the third one), and three different representations, respectively. Specifically, $\{\lambda^{IJK}\}=\{6\lambda_1, 6\lambda_8, 2\lambda_5, 2\lambda_7, 2\lambda_9, 2\lambda_{10}, 2\lambda_{11}, 2\lambda_{12}, 2\lambda_{13}, 2\lambda_{14}, 2\lambda_{15}, \lambda_2, \lambda_3, \lambda_4, \lambda_6, \lambda_{16}, \lambda_{17}, \lambda_{18}, \lambda_{19}, \lambda_{20}, \lambda_{21}\}$.

According to Eq. (51) the trace of the total mass matrix over all $\dim R$ states in R is

$$\mathrm{Tr}\, \mathcal{M} = \sum_I m_I \dim R^I. \tag{56}$$

The mass matrices are generally non-Hermitian. So the squares of physical masses are equal to the eigenvalues of matrices $\mathcal{M}^\dagger \mathcal{M}$ and $\mathcal{M}\mathcal{M}^\dagger$. (One can obtain Hermitian matrices $\mathcal{M}^\dagger \mathcal{M}$, i.e., $\mathcal{M}\mathcal{M}^\dagger$ with the same spectra.) For a real superpotential, that is for $\lambda_3=\lambda_4$, $\lambda_{11}=\lambda_{12}$, $\lambda_{18}=\lambda_{19}$, and $\lambda_{20}=\lambda_{21}$ and all coupling constants and VEVs real, the matrices are Hermitian due to the Hermiticity relation (50).

Now we are ready to present the explicit forms of the mass matrices calculated from the superpotential of Eq. (2). Every matrix is designated with the corresponding G_{321} multiplet and appears $\dim R_i^I$ times in the total mass matrix $\mathcal{M}$. A mass matrix associated with a G_{321} multiplet and the mass matrix associated with the corresponding complex conjugated G_{321} multiplet are equal up to transposition, and therefore for multiplets with $Y \neq 0$ we list only one of the two mass matrices. Of course, when enumerating the total degrees of freedom, one must include all mass eigenvalues (691 in total). The basis designating the columns ($\mathbf{c}$:) of the mass matrices listed below is given in the same way as shown in Tables 1, 2, 3, 4, and 5 (Ref. 19), while the rows are designated by the corresponding complex conjugated G_{321} multiplets ($\mathbf{r}$:),

$(\mathbf{1},\mathbf{1},0)$

$\mathbf{c}$: $\hat{A}^{(1,1,0)}_{(1,1,3)}, \hat{A}^{(1,1,0)}_{(15,1,1)}, \hat{E}^{(1,1,0)}_{(1,1,1)}, \hat{\Delta}^{(1,1,0)}_{(\overline{10},1,3)}, \hat{\overline{\Delta}}^{(1,1,0)}_{(10,1,3)}, \hat{\Phi}^{(1,1,0)}_{(1,1,1)}, \hat{\Phi}^{(1,1,0)}_{(15,1,1)}, \hat{\Phi}^{(1,1,0)}_{(15,1,3)}$

$\mathbf{r}$: $\hat{A}^{(1,1,0)}_{(1,1,3)}, \hat{A}^{(1,1,0)}_{(15,1,1)}, \hat{E}^{(1,1,0)}_{(1,1,1)}, \hat{\overline{\Delta}}^{(1,1,0)}_{(10,1,3)}, \hat{\Delta}^{(1,1,0)}_{(\overline{10},1,3)}, \hat{\Phi}^{(1,1,0)}_{(1,1,1)}, \hat{\Phi}^{(1,1,0)}_{(15,1,1)}, \hat{\Phi}^{(1,1,0)}_{(15,1,3)}$

$$\begin{pmatrix}
m_{11}^{(1,1,0)} & \sqrt{\frac{2}{3}}\lambda_5\Phi_3 & \sqrt{\frac{3}{5}}\lambda_9 A_1 & -\frac{\lambda_6\overline{v_R}}{5} & -\frac{\lambda_6 v_R}{5} & \sqrt{\frac{2}{3}}\lambda_5 A_1 & \frac{2\sqrt{2}\lambda_7\Phi_3}{5} & m_{81}^{(1,1,0)} \\
\sqrt{\frac{2}{3}}\lambda_5\Phi_3 & m_{22}^{(1,1,0)} & -\frac{2\lambda_9 A_2}{\sqrt{15}} & -\frac{1}{5}\sqrt{\frac{3}{2}}\lambda_6\overline{v_R} & -\frac{1}{5}\sqrt{\frac{3}{2}}\lambda_6 v_R & \frac{2\sqrt{2}\lambda_7\Phi_2}{5} & m_{72}^{(1,1,0)} & m_{82}^{(1,1,0)} \\
\sqrt{\frac{3}{5}}\lambda_9 A_1 & -\frac{2\lambda_9 A_2}{\sqrt{15}} & m_{33}^{(1,1,0)} & 0 & 0 & \sqrt{\frac{3}{5}}\lambda_{10}\Phi_1 & -\frac{2\lambda_{10}\Phi_2}{\sqrt{15}} & \frac{\lambda_{10}\Phi_3}{2\sqrt{15}} \\
-\frac{\lambda_6 v_R}{5} & -\frac{1}{5}\sqrt{\frac{3}{2}}\lambda_6 v_R & 0 & m_{44}^{(1,1,0)} & 0 & \frac{\lambda_2 v_R}{10\sqrt{6}} & \frac{\lambda_2 v_R}{10\sqrt{2}} & \frac{\lambda_2 v_R}{10} \\
-\frac{\lambda_6\overline{v_R}}{5} & -\frac{1}{5}\sqrt{\frac{3}{2}}\lambda_6\overline{v_R} & 0 & 0 & m_{44}^{(1,1,0)} & \frac{\lambda_2\overline{v_R}}{10\sqrt{6}} & \frac{\lambda_2\overline{v_R}}{10\sqrt{2}} & \frac{\lambda_2\overline{v_R}}{10} \\
\sqrt{\frac{2}{3}}\lambda_5 A_1 & \frac{2\sqrt{2}\lambda_7\Phi_2}{5} & \sqrt{\frac{3}{5}}\lambda_{10}\Phi_1 & \frac{\lambda_2\overline{v_R}}{10\sqrt{6}} & \frac{\lambda_2 v_R}{10\sqrt{6}} & m_{66}^{(1,1,0)} & \frac{2\sqrt{2}\lambda_7 A_2}{5} & \frac{\lambda_1\Phi_3}{\sqrt{6}} \\
\frac{2\sqrt{2}\lambda_7\Phi_3}{5} & m_{72}^{(1,1,0)} & -\frac{2\lambda_{10}\Phi_2}{\sqrt{15}} & \frac{\lambda_2\overline{v_R}}{10\sqrt{2}} & \frac{\lambda_2 v_R}{10\sqrt{2}} & \frac{2\sqrt{2}\lambda_7 A_2}{5} & m_{77}^{(1,1,0)} & m_{87}^{(1,1,0)} \\
m_{81}^{(1,1,0)} & m_{82}^{(1,1,0)} & \frac{\lambda_{10}\Phi_3}{2\sqrt{15}} & \frac{\lambda_2\overline{v_R}}{10} & \frac{\lambda_2 v_R}{10} & \frac{\lambda_1\Phi_3}{\sqrt{6}} & m_{87}^{(1,1,0)} & m_{88}^{(1,1,0)}
\end{pmatrix}, \tag{57}$$

where

$$m_{11}^{(1,1,0)} \equiv m_4 + \sqrt{\tfrac{2}{3}}\lambda_5\Phi_1 + \sqrt{\tfrac{3}{5}}\lambda_9 E,$$

$$m_{22}^{(1,1,0)} \equiv m_4 + \frac{2\sqrt{2}\lambda_5\Phi_2}{3} - \frac{2\lambda_9 E}{\sqrt{15}},$$

$$m_{33}^{(1,1,0)} \equiv m_5 + \sqrt{\tfrac{3}{5}}\lambda_8 E,$$

$$m_{44}^{(1,1,0)} \equiv m_2 + \frac{\lambda_2\Phi_1}{10\sqrt{6}} + \frac{\lambda_2\Phi_2}{10\sqrt{2}} + \frac{\lambda_2\Phi_3}{10} - \frac{\lambda_6 A_1}{5} - \frac{\sqrt{3}\lambda_6 A_2}{5\sqrt{2}}$$

$$m_{66}^{(1,1,0)} \equiv m_1 + \sqrt{\tfrac{3}{5}}\lambda_{10} E,$$

$$m_{72}^{(1,1,0)} \equiv \frac{2\sqrt{2}\lambda_5 A_2}{3} + \frac{2\sqrt{2}\lambda_7 \Phi_1}{5}, \tag{58}$$

$$m_{77}^{(1,1,0)} \equiv m_1 + \frac{\sqrt{2}\lambda_1 \Phi_2}{3} - \frac{2\lambda_{10} E}{\sqrt{15}},$$

$$m_{81}^{(1,1,0)} \equiv \sqrt{\frac{2}{3}}\lambda_5 A_2 + \frac{2\sqrt{2}\lambda_7 \Phi_2}{5},$$

$$m_{82}^{(1,1,0)} \equiv \sqrt{\frac{2}{3}}\lambda_5 A_1 + \frac{4}{5}\sqrt{\frac{2}{3}}\lambda_7 \Phi_3,$$

$$m_{87}^{(1,1,0)} \equiv \frac{\sqrt{2}\lambda_1 \Phi_3}{3} + \frac{2\sqrt{2}\lambda_7 A_1}{5},$$

$$m_{88}^{(1,1,0)} \equiv m_1 + \frac{\lambda_1 \Phi_1}{\sqrt{6}} + \frac{\sqrt{2}\lambda_1 \Phi_2}{3} + \frac{4}{5}\sqrt{\frac{2}{3}}\lambda_7 A_2 + \frac{\lambda_{10} E}{2\sqrt{15}}.$$

$[(\mathbf{1},\mathbf{1},1)+\text{c.c.}]$

$\mathbf{c}$: $\hat{A}^{(1,1,1)}_{(1,1,3)}, \hat{D}^{(1,1,1)}_{(\overline{10},1,1)}, \hat{\Delta}^{(1,1,1)}_{(\overline{10},1,3)}$

$\mathbf{r}$: $\hat{A}^{(1,1,-1)}_{(1,1,3)}, \hat{D}^{(1,1,-1)}_{(10,1,1)}, \hat{\Delta}^{(1,1,-1)}_{(10,1,3)}$

$$\begin{pmatrix} m_4 + \frac{\sqrt{2}\lambda_5\Phi_1}{\sqrt{3}} + \frac{\sqrt{3}\lambda_9 E}{\sqrt{5}} & -\frac{i\lambda_{19}\overline{v_R}}{\sqrt{10}} & -\frac{\lambda_6\overline{v_R}}{5} & -\sqrt{\frac{2}{3}}\lambda_5 A_2 - \frac{2\sqrt{2}\lambda_7\Phi_2}{5} \\ \frac{i\lambda_{18}v_R}{\sqrt{10}} & m_6 - \frac{2\lambda_{14}E}{\sqrt{15}} + \frac{\sqrt{2}\lambda_{15}\Phi_2}{3} & -\frac{i\lambda_{18}A_1}{\sqrt{10}} - \frac{\lambda_{20}\Phi_3}{2\sqrt{10}} & -\frac{\lambda_{20}v_R}{\sqrt{10}} \\ -\frac{\lambda_6 v_R}{5} & \frac{i\lambda_{19}A_1}{\sqrt{10}} - \frac{\lambda_{21}\Phi_3}{2\sqrt{10}} & m_{33}^{(1,1,1)} & -\frac{\lambda_2 v_R}{10} \\ -\sqrt{\frac{2}{3}}\lambda_5 A_2 - \frac{2\sqrt{2}\lambda_7\Phi_2}{5} & -\frac{\lambda_{21}\overline{v_R}}{\sqrt{10}} & -\frac{\lambda_2\overline{v_R}}{10} & m_{44}^{(1,1,1)} \end{pmatrix}, \tag{59}$$

where

$$m_{33}^{(1,1,1)} \equiv m_2 + \frac{\lambda_2 \Phi_1}{10\sqrt{6}} + \frac{\lambda_2 \Phi_2}{10\sqrt{2}} - \frac{1}{5}\sqrt{\frac{3}{2}}\lambda_6 A_2,$$

$$m_{44}^{(1,1,1)} \equiv m_1 + \frac{\lambda_1 \Phi_1}{\sqrt{6}} + \frac{\sqrt{2}\lambda_1 \Phi_2}{3} + \frac{4}{5}\sqrt{\frac{2}{3}}\lambda_7 A_2 + \frac{\lambda_{10} E}{2\sqrt{15}}. \tag{60}$$

$[(\mathbf{3},\mathbf{1},\frac{2}{3})+\text{c.c.}]$

$\mathbf{c}$: $\hat{A}^{(3,1,2/3)}_{(\underline{15},1,1)}, \hat{D}^{(3,1,2/3)}_{(6,\underline{1},3)}, \overline{\hat{\Delta}}^{(3,1,2/3)}_{(10,\underline{1},3)}, \hat{\Phi}^{(3,1,2/3)}_{(15,1,\underline{1})}, \hat{\Phi}^{(3,1,2/3)}_{(15,1,3)}$

$\mathbf{r}$: $\hat{A}^{(3,1,-2/3)}_{(15,1,1)}, \hat{D}^{(3,1,-2/3)}_{(6,1,3)}, \hat{\Delta}^{(3,1,-2/3)}_{(\overline{10},1,3)}, \hat{\Phi}^{(3,1,-2/3)}_{(15,1,1)}, \overline{\hat{\Phi}}^{(3,1,-2/3)}_{(15,1,3)}$

$$\begin{pmatrix} m_{11}^{(3,1,2/3)} & -\frac{i\lambda_{18}v_R}{\sqrt{10}} & -\frac{\lambda_6 v_R}{5} & -\frac{\sqrt{2}}{3}\lambda_5 A_2 - \frac{2\sqrt{2}\lambda_7\Phi_1}{5} & -\sqrt{\frac{2}{3}}\lambda_5 A_1 - \frac{2}{5}\sqrt{\frac{2}{3}}\lambda_7\Phi_3 \\ \frac{i\lambda_{19}\overline{v_R}}{\sqrt{10}} & m_{22}^{(3,1,2/3)} & m_{23}^{(3,1,2/3)} & \frac{\lambda_{21}\overline{v_R}}{2\sqrt{30}} & \frac{\lambda_{21}\overline{v_R}}{2\sqrt{15}} \\ -\frac{\lambda_6\overline{v_R}}{5} & m_{32}^{(3,1,2/3)} & m_{33}^{(3,1,2/3)} & -\frac{\lambda_2\overline{v_R}}{10\sqrt{3}} & -\frac{\lambda_2\overline{v_R}}{5\sqrt{6}} \\ -\frac{\sqrt{2}}{3}\lambda_5 A_2 - \frac{2\sqrt{2}\lambda_7\Phi_1}{5} & \frac{\lambda_{20}v_R}{2\sqrt{30}} & -\frac{\lambda_2 v_R}{10\sqrt{3}} & m_{44}^{(3,1,2/3)} & m_{45}^{(3,1,2/3)} \\ -\sqrt{\frac{2}{3}}\lambda_5 A_1 - \frac{2}{5}\sqrt{\frac{2}{3}}\lambda_7\Phi_3 & \frac{\lambda_{20}v_R}{2\sqrt{15}} & -\frac{\lambda_2 v_R}{5\sqrt{6}} & m_{45}^{(3,1,2/3)} & m_{55}^{(3,1,2/3)} \end{pmatrix}, \tag{61}$$

where

$$m_{11}^{(3,1,2/3)} \equiv m_4 + \frac{\sqrt{2}\lambda_5\Phi_2}{3} - \frac{2\lambda_9 E}{\sqrt{15}},$$

$$m_{22}^{(3,1,2/3)} \equiv m_6 + \frac{4\lambda_{14}E}{3\sqrt{15}} + \frac{1}{3}\sqrt{\frac{2}{3}}\lambda_{15}\Phi_1 + \frac{2\lambda_{15}\Phi_3}{9},$$

$$m_{23}^{(3,1,2/3)} \equiv -\frac{i\lambda_{19}A_2}{\sqrt{15}} + \frac{\lambda_{21}\Phi_2}{6\sqrt{5}} + \frac{\lambda_{21}\Phi_3}{3\sqrt{10}},$$

$$m_{32}^{(3,1,2/3)} \equiv \frac{i\lambda_{18}A_2}{\sqrt{15}} + \frac{\lambda_{20}\Phi_2}{6\sqrt{5}} + \frac{\lambda_{20}\Phi_3}{3\sqrt{10}}, \tag{62}$$

$$m_{33}^{(3,1,2/3)} \equiv m_2 + \frac{\lambda_2\Phi_1}{10\sqrt{6}} + \frac{\lambda_2\Phi_2}{30\sqrt{2}} + \frac{\lambda_2\Phi_3}{30} - \frac{\lambda_6 A_1}{5} - \frac{\lambda_6 A_2}{5\sqrt{6}},$$

$$m_{44}^{(3,1,2/3)} \equiv m_1 + \frac{\lambda_1\Phi_2}{3\sqrt{2}} - \frac{2\lambda_{10}E}{\sqrt{15}},$$

$$m_{45}^{(3,1,2/3)} \equiv \frac{\lambda_1\Phi_3}{3\sqrt{2}} + \frac{2\sqrt{2}\lambda_7 A_1}{5},$$

$$m_{55}^{(3,1,2/3)} \equiv m_1 + \frac{\lambda_1\Phi_1}{\sqrt{6}} + \frac{\lambda_1\Phi_2}{3\sqrt{2}} + \frac{2}{5}\sqrt{\frac{2}{3}}\lambda_7 A_2 + \frac{\lambda_{10}E}{2\sqrt{15}}.$$

$[(\mathbf{3},\mathbf{2},-\frac{5}{6})+\mathrm{c.c.}]$

c: $\hat{A}_{(\underline{6},2,2)}^{(3,2,-5/6)}, \hat{E}_{(\underline{6},2,2)}^{(3,2,-5/6)}, \hat{\Phi}_{(\underline{6},2,2)}^{(3,2,-5/6)} \hat{\Phi}_{(\underline{10},2,2)}^{(3,2,-5/6)}$

r: $\hat{A}_{(6,2,2)}^{(3,2,5/6)}, \hat{E}_{(6,2,2)}^{(3,2,5/6)}, \hat{\Phi}_{(6,2,2)}^{(3,2,5/6)} \hat{\Phi}_{(\overline{10},2,2)}^{(3,2,5/6)}$

$$\begin{pmatrix} m_4-\frac{\lambda_5\Phi_3}{3}+\frac{\lambda_9E}{2\sqrt{15}} & -\frac{\lambda_9A_1}{2}-\frac{\lambda_9A_2}{\sqrt{6}} & -\frac{\lambda_5A_1}{\sqrt{3}}+\frac{2}{5}\sqrt{\frac{2}{3}}\lambda_7\Phi_2 & -\frac{2\lambda_5A_2}{3}+\frac{2}{5}\sqrt{\frac{2}{3}}\lambda_7\Phi_3 \\ -\frac{\lambda_9A_1}{2}-\frac{\lambda_9A_2}{\sqrt{6}} & m_5+\frac{1}{2}\sqrt{\frac{3}{5}}\lambda_8E & \frac{\lambda_{10}\Phi_1}{2\sqrt{2}}+\frac{\lambda_{10}\Phi_3}{4\sqrt{3}} & \frac{\lambda_{10}\Phi_2}{2\sqrt{3}}+\frac{\lambda_{10}\Phi_3}{2\sqrt{6}} \\ -\frac{\lambda_5A_1}{\sqrt{3}}+\frac{2}{5}\sqrt{\frac{2}{3}}\lambda_7\Phi_2 & \frac{\lambda_{10}\Phi_1}{2\sqrt{2}}+\frac{\lambda_{10}\Phi_3}{4\sqrt{3}} & m_1-\frac{\lambda_1\Phi_3}{6}+\frac{7\lambda_{10}E}{4\sqrt{15}} & \frac{\lambda_1\Phi_3}{3\sqrt{2}}-\frac{4\lambda_7A_2}{5\sqrt{3}} \\ -\frac{2\lambda_5A_2}{3}+\frac{2}{5}\sqrt{\frac{2}{3}}\lambda_7\Phi_3 & \frac{\lambda_{10}\Phi_2}{2\sqrt{3}}+\frac{\lambda_{10}\Phi_3}{2\sqrt{6}} & \frac{\lambda_1\Phi_3}{3\sqrt{2}}-\frac{4\lambda_7A_2}{5\sqrt{3}} & m_{44}^{(3,2,-5/6)} \end{pmatrix}, \tag{63}$$

where

$$m_{44}^{(3,2,-5/6)} \equiv m_1+\frac{\lambda_1\Phi_2}{3\sqrt{2}}-\frac{\lambda_1\Phi_3}{6}-\frac{2\lambda_7A_1}{5}-\frac{1}{4}\sqrt{\frac{3}{5}}\lambda_{10}E. \tag{64}$$

$[(\mathbf{3},\mathbf{2},\frac{1}{6})+\mathrm{c.c.}]$

$\mathbf{c}$: $\hat{A}^{(3,2,1/6)}_{(6,2,2)}, \hat{E}^{(3,2,1/6)}_{(6,2,2)}, \hat{D}^{(3,2,1/6)}_{(15,2,2)}, \hat{\Delta}^{(3,2,1/6)}_{(15,2,2)}, \hat{\overline{\Delta}}^{(3,2,1/6)}_{(15,2,2)}, \hat{\Phi}^{(3,2,1/6)}_{(6,2,2)}, \hat{\Phi}^{(3,2,1/6)}_{(10,2,2)}$

$\mathbf{r}$: $\hat{A}^{(\overline{3},2,-1/6)}_{(6,2,2)}, \hat{E}^{(\overline{3},2,-1/6)}_{(6,2,2)}, \hat{D}^{(\overline{3},2,-1/6)}_{(15,2,2)}, \hat{\overline{\Delta}}^{(\overline{3},2,-1/6)}_{(15,2,2)}, \hat{\Delta}^{(\overline{3},2,-1/6)}_{(15,2,2)}, \hat{\Phi}^{(\overline{3},2,-1/6)}_{(6,2,2)}, \hat{\Phi}^{(\overline{3},2,-1/6)}_{(\overline{10},2,2)}$

$$\begin{pmatrix} m_{11}^{(3,2,1/6)} & \frac{\lambda_9A_1}{2}-\frac{\lambda_9A_2}{\sqrt{6}} & -\frac{i\lambda_{19}\overline{v_R}}{\sqrt{10}} & -\frac{\lambda_6\overline{v_R}}{5} & 0 & m_{61}^{(3,2,1/6)} & m_{71}^{(3,2,1/6)} \\ \frac{\lambda_9A_1}{2}-\frac{\lambda_9A_2}{\sqrt{6}} & m_5+\frac{1}{2}\sqrt{\frac{3}{5}}\lambda_8E & 0 & 0 & -\frac{2\lambda_{12}v_R}{5} & \frac{\lambda_{10}\Phi_3}{4\sqrt{3}}-\frac{\lambda_{10}\Phi_1}{2\sqrt{2}} & \frac{\lambda_{10}\Phi_2}{2\sqrt{3}}-\frac{\lambda_{10}\Phi_3}{2\sqrt{6}} \\ \frac{i\lambda_{18}v_R}{\sqrt{10}} & 0 & m_{33}^{(3,2,1/6)} & m_{34}^{(3,2,1/6)} & m_{35}^{(3,2,1/6)} & -\frac{\lambda_{20}v_R}{2\sqrt{30}} & -\frac{\lambda_{20}v_R}{2\sqrt{15}} \\ -\frac{\lambda_6v_R}{5} & 0 & m_{43}^{(3,2,1/6)} & m_{44}^{(3,2,1/6)} & \frac{\lambda_{12}E}{\sqrt{15}} & -\frac{\lambda_2v_R}{10\sqrt{3}} & -\frac{\lambda_2v_R}{5\sqrt{6}} \\ 0 & -\frac{2\lambda_{11}v_R}{5} & m_{53}^{(3,2,1/6)} & \frac{\lambda_{11}E}{\sqrt{15}} & m_{55}^{(3,2,1/6)} & 0 & 0 \\ m_{61}^{(3,2,1/6)} & \frac{\lambda_{10}\Phi_3}{4\sqrt{3}}-\frac{\lambda_{10}\Phi_1}{2\sqrt{2}} & -\frac{\lambda_{21}\overline{v_R}}{2\sqrt{30}} & -\frac{\lambda_2\overline{v_R}}{10\sqrt{3}} & 0 & m_{66}^{(3,2,1/6)} & \frac{\lambda_1\Phi_3}{3\sqrt{2}}+\frac{4\lambda_7A_2}{5\sqrt{3}} \\ m_{71}^{(3,2,1/6)} & \frac{\lambda_{10}\Phi_2}{2\sqrt{3}}-\frac{\lambda_{10}\Phi_3}{2\sqrt{6}} & -\frac{\lambda_{21}\overline{v_R}}{2\sqrt{15}} & -\frac{\lambda_2\overline{v_R}}{5\sqrt{6}} & 0 & \frac{\lambda_1\Phi_3}{3\sqrt{2}}+\frac{4\lambda_7A_2}{5\sqrt{3}} & m_{77}^{(3,2,1/6)} \end{pmatrix}, \tag{65}$$

where

$$m_{11}^{(3,2,1/6)} \equiv m_4+\frac{\lambda_5\Phi_3}{3}+\frac{\lambda_9E}{2\sqrt{15}},$$

$$m_{33}^{(3,2,1/6)} \equiv m_6-\frac{\lambda_{14}E}{3\sqrt{15}}+\frac{\sqrt{2}\lambda_{15}\Phi_2}{9}+\frac{2\lambda_{15}\Phi_3}{9},$$

$$m_{34}^{(3,2,1/6)} \equiv -\frac{i\lambda_{18}A_1}{2\sqrt{10}}-\frac{i\lambda_{18}A_2}{2\sqrt{15}}-\frac{\lambda_{20}\Phi_1}{4\sqrt{15}}-\frac{\lambda_{20}\Phi_2}{6\sqrt{5}}-\frac{\lambda_{20}\Phi_3}{4\sqrt{10}},$$

$$
\begin{aligned}
m_{35}^{(3,2,1/6)} &\equiv -\frac{i\lambda_{19}A_1}{2\sqrt{10}}+\frac{i\lambda_{19}A_2}{2\sqrt{15}}-\frac{\lambda_{21}\Phi_1}{4\sqrt{15}}+\frac{\lambda_{21}\Phi_2}{6\sqrt{5}}-\frac{\lambda_{21}\Phi_3}{12\sqrt{10}},\\
m_{43}^{(3,2,1/6)} &\equiv \frac{i\lambda_{19}A_1}{2\sqrt{10}}+\frac{i\lambda_{19}A_2}{2\sqrt{15}}-\frac{\lambda_{21}\Phi_1}{4\sqrt{15}}-\frac{\lambda_{21}\Phi_2}{6\sqrt{5}}-\frac{\lambda_{21}\Phi_3}{4\sqrt{10}},\\
m_{44}^{(3,2,1/6)} &\equiv m_2+\frac{\lambda_2\Phi_2}{30\sqrt{2}}+\frac{\lambda_2\Phi_3}{20}-\frac{\lambda_6A_1}{10}-\frac{1}{5}\sqrt{\frac{2}{3}}\lambda_6A_2,\\
m_{53}^{(3,2,1/6)} &\equiv \frac{i\lambda_{18}A_1}{2\sqrt{10}}-\frac{i\lambda_{18}A_2}{2\sqrt{15}}-\frac{\lambda_{20}\Phi_1}{4\sqrt{15}}+\frac{\lambda_{20}\Phi_2}{6\sqrt{5}}-\frac{\lambda_{20}\Phi_3}{12\sqrt{10}},\\
m_{55}^{(3,2,1/6)} &\equiv m_2+\frac{\lambda_2\Phi_2}{30\sqrt{2}}+\frac{\lambda_2\Phi_3}{60}+\frac{\lambda_6A_1}{10}+\frac{1}{5}\sqrt{\frac{2}{3}}\lambda_6A_2,\\
m_{61}^{(3,2,1/6)} &\equiv -\frac{\lambda_5A_1}{\sqrt{3}}-\frac{2}{5}\sqrt{\frac{2}{3}}\lambda_7\Phi_2,\\
m_{66}^{(3,2,1/6)} &\equiv m_1+\frac{\lambda_1\Phi_3}{6}+\frac{7\lambda_{10}E}{4\sqrt{15}},\\
m_{71}^{(3,2,1/6)} &\equiv -\frac{2\lambda_5A_2}{3}-\frac{2}{5}\sqrt{\frac{2}{3}}\lambda_7\Phi_3,\\
m_{77}^{(3,2,1/6)} &\equiv m_1+\frac{\lambda_1\Phi_2}{3\sqrt{2}}+\frac{\lambda_1\Phi_3}{6}+\frac{2\lambda_7A_1}{5}-\frac{1}{4}\sqrt{\frac{3}{5}}\lambda_{10}E.
\end{aligned}
\tag{66}
$$

$[(\mathbf{1},\mathbf{2},\tfrac{1}{2})+\text{c.c.}]$

$\mathbf{c}$: $\hat{H}_{(1,2,2)}^{(1,2,1/2)},\hat{D}_{(1,2,2)}^{(1,2,1/2)},\hat{D}_{(15,2,2)}^{(1,2,1/2)},\hat{\Delta}_{(15,2,2)}^{(1,2,1/2)},\hat{\overline{\Delta}}_{(15,2,2)}^{(1,2,1/2)},\hat{\Phi}_{(6,2,2)}^{(1,2,1/2)}$

$\mathbf{r}$: $\hat{H}_{(1,2,2)}^{(1,2,-1/2)},\hat{D}_{(1,2,2)}^{(1,2,-1/2)},\hat{D}_{(15,2,2)}^{(1,2,-1/2)},\hat{\overline{\Delta}}_{(15,2,2)}^{(1,2,-1/2)},\hat{\Delta}_{(15,2,2)}^{\left(1,2,-\frac{1}{2}\right)},\hat{\Phi}_{(6,2,2)}^{(1,2,-1/2)}$

$$
\begin{pmatrix}
m_3+\sqrt{\frac{3}{5}}\lambda_{13}E & -\frac{i\lambda_{16}A_1}{\sqrt{6}}-\frac{\lambda_{17}\Phi_1}{2} & -\frac{i\lambda_{16}A_2}{\sqrt{3}}-\frac{\lambda_{17}\Phi_3}{2\sqrt{2}} & \frac{\lambda_3\Phi_2}{\sqrt{10}}-\frac{\lambda_3\Phi_3}{2\sqrt{5}} & -\frac{\lambda_4\Phi_2}{\sqrt{10}}-\frac{\lambda_4\Phi_3}{2\sqrt{5}} & -\frac{\lambda_4\overline{v_R}}{\sqrt{5}}\\
\frac{i\lambda_{16}A_1}{\sqrt{6}}-\frac{\lambda_{17}\Phi_1}{2} & m_6+\sqrt{\frac{3}{5}}\lambda_{14}E & \frac{\lambda_{15}\Phi_3}{3\sqrt{3}} & -\frac{i\lambda_{18}A_2}{2\sqrt{5}}+\frac{\lambda_{20}\Phi_3}{4\sqrt{30}} & -\frac{i\lambda_{19}A_2}{2\sqrt{5}}+\frac{\lambda_{21}\Phi_3}{4\sqrt{30}} & -\frac{\lambda_{21}\overline{v_R}}{2\sqrt{30}}\\
\frac{i\lambda_{16}A_2}{\sqrt{3}}-\frac{\lambda_{17}\Phi_3}{2\sqrt{2}} & \frac{\lambda_{15}\Phi_3}{3\sqrt{3}} & m_{33}^{(1,2,1/2)} & m_{34}^{(1,2,1/2)} & m_{35}^{(1,2,1/2)} & -\frac{\lambda_{21}\overline{v_R}}{2\sqrt{10}}\\
\frac{\lambda_4\Phi_2}{\sqrt{10}}-\frac{\lambda_4\Phi_3}{2\sqrt{5}} & \frac{i\lambda_{19}A_2}{2\sqrt{5}}+\frac{\lambda_{21}\Phi_3}{4\sqrt{30}} & m_{43}^{(1,2,1/2)} & m_{44}^{(1,2,1/2)} & \frac{\lambda_{12}E}{\sqrt{15}} & 0\\
-\frac{\lambda_3\Phi_2}{\sqrt{10}}-\frac{\lambda_3\Phi_3}{2\sqrt{5}} & \frac{i\lambda_{18}A_2}{2\sqrt{5}}+\frac{\lambda_{20}\Phi_3}{4\sqrt{30}} & m_{53}^{(1,2,1/2)} & \frac{\lambda_{11}E}{\sqrt{15}} & m_{55}^{(1,2,1/2)} & \frac{\lambda_2\overline{v_R}}{10}\\
-\frac{\lambda_3v_R}{\sqrt{5}} & -\frac{\lambda_{20}v_R}{2\sqrt{30}} & -\frac{\lambda_{20}v_R}{2\sqrt{10}} & 0 & \frac{\lambda_2v_R}{10} & m_{66}^{(1,2,1/2)}
\end{pmatrix},
\tag{67}
$$

where

$$
\begin{aligned}
m_{33}^{(1,2,1/2)} &\equiv m_6 - \frac{\lambda_{14}E}{3\sqrt{15}} + \frac{2\sqrt{2}\lambda_{15}\Phi_2}{9},\\
m_{34}^{(1,2,1/2)} &\equiv -\frac{i\lambda_{18}A_1}{2\sqrt{10}} + \frac{i\lambda_{18}A_2}{\sqrt{15}} + \frac{\lambda_{20}\Phi_1}{4\sqrt{15}} - \frac{\lambda_{20}\Phi_3}{6\sqrt{10}},\\
m_{35}^{(1,2,1/2)} &\equiv -\frac{i\lambda_{19}A_1}{2\sqrt{10}} - \frac{i\lambda_{19}A_2}{\sqrt{15}} + \frac{\lambda_{21}\Phi_1}{4\sqrt{15}} + \frac{\lambda_{21}\Phi_3}{6\sqrt{10}},\\
m_{43}^{(1,2,1/2)} &\equiv \frac{i\lambda_{19}A_1}{2\sqrt{10}} - \frac{i\lambda_{19}A_2}{\sqrt{15}} + \frac{\lambda_{21}\Phi_1}{4\sqrt{15}} - \frac{\lambda_{21}\Phi_3}{6\sqrt{10}},\\
m_{44}^{(1,2,1/2)} &\equiv m_2 + \frac{\lambda_2\Phi_2}{15\sqrt{2}} - \frac{\lambda_2\Phi_3}{30} + \frac{\lambda_6A_1}{10},\\
m_{53}^{(1,2,1/2)} &\equiv \frac{i\lambda_{18}A_1}{2\sqrt{10}} + \frac{i\lambda_{18}A_2}{\sqrt{15}} + \frac{\lambda_{20}\Phi_1}{4\sqrt{15}} + \frac{\lambda_{20}\Phi_3}{6\sqrt{10}},\\
m_{55}^{(1,2,1/2)} &\equiv m_2 + \frac{\lambda_2\Phi_2}{15\sqrt{2}} + \frac{\lambda_2\Phi_3}{30} - \frac{\lambda_6A_1}{10},\\
m_{66}^{(1,2,1/2)} &\equiv m_1 + \frac{\lambda_1\Phi_2}{\sqrt{2}} + \frac{\lambda_1\Phi_3}{2} + \frac{2\lambda_7A_1}{5} - \frac{1}{4}\sqrt{\frac{3}{5}}\lambda_{10}E.
\end{aligned}
\tag{68}
$$

$[(\mathbf{3},\mathbf{1},-\frac{1}{3})+\text{c.c.}]$

c: $\hat{H}^{(3,1,-1/3)}_{(6,1,1)}$, $\hat{D}^{(3,1,-1/3)}_{(6,1,3)}$, $\hat{D}^{(3,1,-1/3)}_{(10,1,1)}$, $\hat{\Delta}^{(3,1,-1/3)}_{(6,1,1)}$, $\hat{\overline{\Delta}}^{(3,1,-1/3)}_{(6,1,1)}$, $\hat{\overline{\Delta}}^{(3,1,-1/3)}_{(10,1,3)}$, $\hat{\Phi}^{(3,1,-1/3)}_{(15,1,3)}$

r: $\hat{H}^{(\overline{3},1,1/3)}_{(6,1,1)}$, $\hat{D}^{(\overline{3},1,1/3)}_{(6,1,3)}$, $\hat{D}^{(\overline{3},1,1/3)}_{(\overline{10},1,1)}$, $\hat{\overline{\Delta}}^{(\overline{3},1,1/3)}_{(6,1,1)}$, $\hat{\Delta}^{(\overline{3},1,1/3)}_{(6,1,1)}$, $\hat{\Delta}^{(\overline{3},1,1/3)}_{(\overline{10},1,3)}$, $\hat{\Phi}^{(\overline{3},1,1/3)}_{(15,1,3)}$

$$
\begin{pmatrix}
m_3 - \frac{2\lambda_{13}E}{\sqrt{15}} & m_{12}^{(3,1,-1/3)} & m_{13}^{(3,1,-1/3)} & m_{14}^{(3,1,-1/3)} & m_{15}^{(3,1,-1/3)} & -\sqrt{\frac{2}{15}}\lambda_4\Phi_3 & \frac{\lambda_4\overline{v_R}}{\sqrt{5}}\\
m_{21}^{(3,1,-1/3)} & m_{22}^{(3,1,-1/3)} & \frac{2\lambda_{15}\Phi_3}{9} & m_{24}^{(3,1,-1/3)} & m_{25}^{(3,1,-1/3)} & m_{26}^{(3,1,-1/3)} & \frac{\lambda_4\overline{v_R}}{2\sqrt{15}}\\
m_{31}^{(3,1,-1/3)} & \frac{2\lambda_{15}\Phi_3}{9} & m_{33}^{(3,1,-1/3)} & m_{34}^{(3,1,-1/3)} & m_{35}^{(3,1,-1/3)} & m_{36}^{(3,1,-1/3)} & \frac{\lambda_{21}\overline{v_R}}{2\sqrt{15}}\\
m_{41}^{(3,1,-1/3)} & m_{42}^{(3,1,-1/3)} & m_{43}^{(3,1,-1/3)} & m_2 + \frac{\lambda_6A_2}{5\sqrt{6}} & \frac{2\lambda_{12}E}{\sqrt{15}} & 0 & 0\\
m_{51}^{(3,1,-1/3)} & m_{52}^{(3,1,-1/3)} & m_{53}^{(3,1,-1/3)} & \frac{2\lambda_{11}E}{\sqrt{15}} & m_2 - \frac{\lambda_6A_2}{5\sqrt{6}} & \frac{\lambda_2\Phi_3}{15\sqrt{2}} & -\frac{\lambda_2\overline{v_R}}{10\sqrt{3}}\\
-\sqrt{\frac{2}{15}}\lambda_3\Phi_3 & m_{62}^{(3,1,-1/3)} & m_{63}^{(3,1,-1/3)} & 0 & \frac{\lambda_2\Phi_3}{15\sqrt{2}} & m_{66}^{(3,1,-1/3)} & -\frac{\lambda_2\overline{v_R}}{5\sqrt{6}}\\
\frac{\lambda_3v_R}{\sqrt{5}} & \frac{\lambda_{20}v_R}{2\sqrt{15}} & \frac{\lambda_{20}v_R}{2\sqrt{15}} & 0 & -\frac{\lambda_2v_R}{10\sqrt{3}} & -\frac{\lambda_2v_R}{5\sqrt{6}} & m_{77}^{(3,1,-1/3)}
\end{pmatrix},
\tag{69}
$$

where

$$
\begin{aligned}
m_{12}^{(3,1,-1/3)} &\equiv -\frac{i\lambda_{16}A_1}{\sqrt{3}} - \frac{\lambda_{17}\Phi_3}{2\sqrt{3}},\\
m_{13}^{(3,1,-1/3)} &\equiv -\frac{i\sqrt{2}\lambda_{16}A_2}{3} - \frac{\lambda_{17}\Phi_2}{\sqrt{6}},\\
m_{14}^{(3,1,-1/3)} &\equiv \frac{\lambda_3\Phi_2}{\sqrt{30}} - \frac{\lambda_3\Phi_1}{\sqrt{10}},\\
m_{15}^{(3,1,-1/3)} &\equiv -\frac{\lambda_4\Phi_1}{\sqrt{10}} - \frac{\lambda_4\Phi_2}{\sqrt{30}},\\
m_{21}^{(3,1,-1/3)} &\equiv \frac{i\lambda_{16}A_1}{\sqrt{3}} - \frac{\lambda_{17}\Phi_3}{2\sqrt{3}},\\
m_{22}^{(3,1,-1/3)} &\equiv m_6 + \frac{4\lambda_{14}E}{3\sqrt{15}} + \frac{1}{3}\sqrt{\frac{2}{3}}\lambda_{15}\Phi_1,\\
m_{24}^{(3,1,-1/3)} &\equiv -\frac{i\lambda_{18}A_1}{2\sqrt{5}} + \frac{\lambda_{20}\Phi_3}{12\sqrt{5}},\\
m_{25}^{(3,1,-1/3)} &\equiv -\frac{i\lambda_{19}A_1}{2\sqrt{5}} + \frac{\lambda_{21}\Phi_3}{12\sqrt{5}},\\
m_{26}^{(3,1,-1/3)} &\equiv -\frac{i\lambda_{19}A_2}{\sqrt{15}} + \frac{\lambda_{21}\Phi_2}{6\sqrt{5}},\\
m_{31}^{(3,1,-1/3)} &\equiv \frac{i\sqrt{2}\lambda_{16}A_2}{3} - \frac{\lambda_{17}\Phi_2}{\sqrt{6}},\\
m_{33}^{(3,1,-1/3)} &\equiv m_6 - \frac{2\lambda_{14}E}{\sqrt{15}} + \frac{\sqrt{2}\lambda_{15}\Phi_2}{9}\\
m_{34}^{(3,1,-1/3)} &\equiv \frac{i\lambda_{18}A_2}{\sqrt{30}} - \frac{\lambda_{20}\Phi_2}{6\sqrt{10}},\\
m_{35}^{(3,1,-1/3)} &\equiv -\frac{i\lambda_{19}A_2}{\sqrt{30}} + \frac{\lambda_{21}\Phi_2}{6\sqrt{10}},\\
m_{36}^{(3,1,-1/3)} &\equiv -\frac{i\lambda_{19}A_1}{\sqrt{10}} + \frac{\lambda_{21}\Phi_3}{6\sqrt{10}},\\
m_{41}^{(3,1,-1/3)} &\equiv \frac{\lambda_4\Phi_2}{\sqrt{30}} - \frac{\lambda_4\Phi_1}{\sqrt{10}},
\end{aligned}
\tag{70}
$$

$$m_{42}^{(3,1,-1/3)} \equiv \frac{i\lambda_{19}A_1}{2\sqrt{5}} + \frac{\lambda_{21}\Phi_3}{12\sqrt{5}},$$

$$m_{43}^{(3,1,-1/3)} \equiv -\frac{i\lambda_{19}A_2}{\sqrt{30}} - \frac{\lambda_{21}\Phi_2}{6\sqrt{10}},$$

$$m_{51}^{(3,1,-1/3)} \equiv -\frac{\lambda_3\Phi_1}{\sqrt{10}} - \frac{\lambda_3\Phi_2}{\sqrt{30}},$$

$$m_{52}^{(3,1,-1/3)} \equiv \frac{i\lambda_{18}A_1}{2\sqrt{5}} + \frac{\lambda_{20}\Phi_3}{12\sqrt{5}},$$

$$m_{53}^{(3,1,-1/3)} \equiv \frac{i\lambda_{18}A_2}{\sqrt{30}} + \frac{\lambda_{20}\Phi_2}{6\sqrt{10}},$$

$$m_{62}^{(3,1,-1/3)} \equiv \frac{i\lambda_{18}A_2}{\sqrt{15}} + \frac{\lambda_{20}\Phi_2}{6\sqrt{5}},$$

$$m_{63}^{(3,1,-1/3)} \equiv \frac{i\lambda_{18}A_1}{\sqrt{10}} + \frac{\lambda_{20}\Phi_3}{6\sqrt{10}},$$

$$m_{66}^{(3,1,-1/3)} \equiv m_2 + \frac{\lambda_2\Phi_1}{10\sqrt{6}} + \frac{\lambda_2\Phi_2}{30\sqrt{2}} - \frac{\lambda_6A_2}{5\sqrt{6}},$$

$$m_{77}^{(3,1,-1/3)} \equiv m_1 + \frac{\lambda_1\Phi_1}{\sqrt{6}} + \frac{\lambda_1\Phi_2}{3\sqrt{2}} + \frac{2\lambda_1\Phi_3}{3} + \frac{2}{5}\sqrt{\frac{2}{3}}\lambda_7A_2 + \frac{\lambda_{10}E}{2\sqrt{15}}.$$

$[(\mathbf{1},\mathbf{1},2)+\text{c.c.}]$
$\mathbf{c}$: $\hat{\Delta}_{(\overline{10},1,3)}^{(1,1,2)}$
$\mathbf{r}$: $\hat{\overline{\Delta}}_{(10,1,3)}^{(1,1,-2)}$

$$m_2 + \frac{\lambda_2\Phi_1}{10\sqrt{6}} + \frac{\lambda_2\Phi_2}{10\sqrt{2}} - \frac{\lambda_2\Phi_3}{10} + \frac{\lambda_6A_1}{5} - \frac{1}{5}\sqrt{\frac{3}{2}}\lambda_6A_2. \tag{71}$$

$[(\mathbf{1},\mathbf{2},\frac{3}{2})+\text{c.c.}]$
$\mathbf{c}$: $\hat{\Phi}_{(\overline{10},2,2)}^{(1,\overline{2},3/2)}$
$\mathbf{r}$: $\hat{\Phi}_{(10,2,2)}^{(1,2,-3/2)}$

$$m_1 + \frac{\lambda_1\Phi_2}{\sqrt{2}} - \frac{\lambda_1\Phi_3}{2} - \frac{2}{5}\lambda_7A_1 - \frac{1}{4}\sqrt{\frac{3}{5}}\lambda_{10}E. \tag{72}$$

$(\mathbf{1},\mathbf{3},0)$
$\mathbf{c}$: $\hat{A}_{(1,3,1)}^{(1,3,0)}$, $\hat{E}_{(1,3,3)}^{(1,3,0)}$, $\hat{\Phi}_{(15,3,1)}^{(1,3,0)}$
$\mathbf{r}$: $\hat{A}_{(1,3,1)}^{(1,3,0)}$, $\hat{E}_{(1,3,3)}^{(1,3,0)}$, $\hat{\Phi}_{(15,3,1)}^{(1,3,0)}$

$$\begin{pmatrix} m_4 - \sqrt{\frac{2}{3}}\lambda_5\Phi_1 + \sqrt{\frac{3}{5}}\lambda_9 E & \lambda_9 A_1 & -\sqrt{\frac{2}{3}}\lambda_5 A_2 + \frac{2}{5}\sqrt{2}\lambda_7\Phi_2 \\ \lambda_9 A_1 & m_5 + 3\sqrt{\frac{3}{5}}\lambda_8 E & -\frac{\lambda_{10}\Phi_3}{2} \\ -\sqrt{\frac{2}{3}}\lambda_5 A_2 + \frac{2}{5}\sqrt{2}\lambda_7\Phi_2 & -\frac{\lambda_{10}\Phi_3}{2} & m_1 - \frac{\lambda_1\Phi_1}{\sqrt{6}} + \frac{\sqrt{2}\lambda_1\Phi_2}{3} - \frac{4}{5}\sqrt{\frac{2}{3}}\lambda_7 A_2 + \frac{\lambda_{10}E}{2\sqrt{15}} \end{pmatrix}. \tag{73}$$

$[(\mathbf{1},\mathbf{3},1)+\text{c.c.}]$
c: $\hat{E}^{(1,3,1)}_{(1,3,3)}$, $\overline{\hat{\Delta}}^{(1,3,1)}_{(\overline{10},3,1)}$
r: $\hat{E}^{(1,3,-1)}_{(1,3,3)}$, $\hat{\Delta}^{(1,3,-1)}_{(10,3,1)}$

$$\begin{pmatrix} m_5 + 3\sqrt{\frac{3}{5}}\lambda_8 E & \frac{2\lambda_{12}\overline{v_R}}{5} \\ \frac{2\lambda_{11}v_R}{5} & m_2 - \frac{\lambda_2\Phi_1}{10\sqrt{6}} + \frac{\lambda_2\Phi_2}{10\sqrt{2}} + \frac{1}{5}\sqrt{\frac{3}{2}}\lambda_6 A_2 \end{pmatrix}. \tag{74}$$

$[(\mathbf{3},\mathbf{1},-\frac{4}{3})+\text{c.c.}]$
c: $\hat{D}^{(3,1,-4/3)}_{(6,1,3)}$, $\overline{\hat{\Delta}}^{(3,1,-4/3)}_{(10,1,3)}$
r: $\overline{\hat{D}}^{(\overline{3},1,4/3)}_{(6,1,3)}$, $\hat{\Delta}^{(\overline{3},1,4/3)}_{(\overline{10},1,3)}$

$$\begin{pmatrix} m_6 + \frac{4\lambda_{14}E}{3\sqrt{15}} + \frac{1}{3}\sqrt{\frac{2}{3}}\lambda_{15}\Phi_1 - \frac{2\lambda_{15}\Phi_3}{9} & -\frac{i\lambda_{19}A_2}{\sqrt{15}} + \frac{\lambda_{21}\Phi_2}{6\sqrt{5}} - \frac{\lambda_{21}\Phi_3}{3\sqrt{10}} \\ \frac{i\lambda_{18}A_2}{\sqrt{15}} + \frac{\lambda_{20}\Phi_2}{6\sqrt{5}} - \frac{\lambda_{20}\Phi_3}{3\sqrt{10}} & m_2 + \frac{\lambda_2\Phi_1}{10\sqrt{6}} + \frac{\lambda_2\Phi_2}{30\sqrt{2}} - \frac{\lambda_2\Phi_3}{30} + \frac{\lambda_6 A_1}{5} - \frac{\lambda_6 A_2}{5\sqrt{6}} \end{pmatrix}. \tag{75}$$

$[(\mathbf{3},\mathbf{1},\frac{5}{3})+\text{c.c.}]$
c: $\hat{\Phi}^{(3,1,5/3)}_{(15,1,3)}$
r: $\hat{\Phi}^{(\overline{3},1,-5/3)}_{(15,1,3)}$

$$m_1 + \frac{\lambda_1\Phi_1}{\sqrt{6}} + \frac{\lambda_1\Phi_2}{3\sqrt{2}} - \frac{2\lambda_1\Phi_3}{3} + \frac{2}{5}\sqrt{\frac{2}{3}}\lambda_7 A_2 + \frac{\lambda_{10}E}{2\sqrt{15}}. \tag{76}$$

$[(\mathbf{3},\mathbf{2},\frac{7}{6})+\text{c.c.}]$
c: $\hat{D}^{(3,2,7/6)}_{(15,2,2)}$, $\hat{\Delta}^{(3,2,7/6)}_{(15,2,2)}$, $\overline{\hat{\Delta}}^{(3,2,7/6)}_{(15,2,2)}$
r: $\hat{D}^{(\overline{3},2,-7/6)}_{(15,2,2)}$, $\overline{\hat{\Delta}}^{(\overline{3},2,-7/6)}_{(15,2,2)}$, $\hat{\Delta}^{(\overline{3},2,7/6)}_{(15,2,2)}$

$$\begin{pmatrix} m_{11}^{(3,2,7/6)} & m_{12}^{(3,2,7/6)} & m_{13}^{(3,2,7/6)} \\ m_{21}^{(3,2,7/6)} & m_{22}^{(3,2,7/6)} & \frac{\lambda_{12}E}{\sqrt{15}} \\ m_{31}^{(3,2,7/6)} & \frac{\lambda_{11}E}{\sqrt{15}} & m_{33}^{(3,2,7/6)} \end{pmatrix}, \tag{77}$$

where

$$m_{11}^{(3,2,7/6)} \equiv m_6 - \frac{\lambda_{14}E}{3\sqrt{15}} + \frac{\sqrt{2}\lambda_{15}\Phi_2}{9} - \frac{2\lambda_{15}\Phi_3}{9},$$

$$m_{12}^{(3,2,7/6)} \equiv -\frac{i\lambda_{18}A_1}{2\sqrt{10}} + \frac{i\lambda_{18}A_2}{2\sqrt{15}} + \frac{\lambda_{20}\Phi_1}{4\sqrt{15}} + \frac{\lambda_{20}\Phi_2}{6\sqrt{5}} - \frac{\lambda_{20}\Phi_3}{4\sqrt{10}},$$

$$m_{13}^{(3,2,7/6)} \equiv -\frac{i\lambda_{19}A_1}{2\sqrt{10}} - \frac{i\lambda_{19}A_2}{2\sqrt{15}} + \frac{\lambda_{21}\Phi_1}{4\sqrt{15}} - \frac{\lambda_{21}\Phi_2}{6\sqrt{5}} - \frac{\lambda_{21}\Phi_3}{12\sqrt{10}},$$

$$m_{21}^{(3,2,7/6)} \equiv \frac{i\lambda_{19}A_1}{2\sqrt{10}} - \frac{i\lambda_{19}A_2}{2\sqrt{15}} + \frac{\lambda_{21}\Phi_1}{4\sqrt{15}} + \frac{\lambda_{21}\Phi_2}{6\sqrt{5}} - \frac{\lambda_{21}\Phi_3}{4\sqrt{10}}, \tag{78}$$

$$m_{22}^{(3,2,7/6)} \equiv m_2 + \frac{\lambda_2\Phi_2}{30\sqrt{2}} - \frac{\lambda_2\Phi_3}{20} + \frac{\lambda_6A_1}{10} - \frac{1}{5}\sqrt{\frac{2}{3}}\lambda_6A_2,$$

$$m_{31}^{(3,2,7/6)} \equiv \frac{i\lambda_{18}A_1}{2\sqrt{10}} + \frac{i\lambda_{18}A_2}{2\sqrt{15}} + \frac{\lambda_{20}\Phi_1}{4\sqrt{15}} - \frac{\lambda_{20}\Phi_2}{6\sqrt{5}} - \frac{\lambda_{20}\Phi_3}{12\sqrt{10}},$$

$$m_{33}^{(3,2,7/6)} \equiv m_2 + \frac{\lambda_2\Phi_2}{30\sqrt{2}} - \frac{\lambda_2\Phi_3}{60} - \frac{\lambda_6A_1}{10} + \frac{1}{5}\sqrt{\frac{2}{3}}\lambda_6A_2.$$

$[(\mathbf{3},\mathbf{3},-\frac{1}{3})+\mathrm{c.c.}]$

c: $\hat{D}^{(3,3,-1/3)}_{(\overline{6},3,1)}$, $\hat{\Delta}^{(3,3,-1/3)}_{(\overline{10},3,1)}$

r: $\hat{D}^{(\overline{3},3,1/3)}_{(6,3,1)}$, $\hat{\overline{\Delta}}^{(3,\overline{3},1/3)}_{(10,3,1)}$

$$\begin{pmatrix} m_6 + \frac{4\lambda_{14}E}{3\sqrt{15}} - \frac{1}{3}\sqrt{\frac{2}{3}}\lambda_{15}\Phi_1 & -\frac{i\lambda_{18}A_2}{\sqrt{15}} + \frac{\lambda_{20}\Phi_2}{6\sqrt{5}} \\ \frac{i\lambda_{19}A_2}{\sqrt{15}} + \frac{\lambda_{21}\Phi_2}{6\sqrt{5}} & m_2 - \frac{\lambda_2\Phi_1}{10\sqrt{6}} + \frac{\lambda_2\Phi_2}{30\sqrt{2}} + \frac{\lambda_6A_2}{5\sqrt{6}} \end{pmatrix}. \tag{79}$$

$[(\mathbf{3},\mathbf{3},\frac{2}{3})+\mathrm{c.c.}]$

c: $\hat{\Phi}^{(3,\overline{3},2/3)}_{(\overline{15},3,1)}$

r: $\hat{\Phi}^{(3,3,-2/3)}_{(15,3,1)}$

$$m_1 - \frac{\lambda_1\Phi_1}{\sqrt{6}} + \frac{\lambda_1\Phi_2}{3\sqrt{2}} - \frac{2}{5}\sqrt{\frac{2}{3}}\lambda_7A_2 + \frac{\lambda_{10}E}{2\sqrt{15}}. \tag{80}$$

$[(\mathbf{6},\mathbf{1},-\frac{2}{3})+\mathrm{c.c.}]$

c: $\hat{E}^{(6,1,-2/3)}_{(20',1,1)}$, $\hat{\overline{\Delta}}^{(6,1,-2/3)}_{(10,1,3)}$

r: $\hat{E}^{(6,1,2/3)}_{(20',1,1)}$, $\hat{\Delta}^{(6,1,2/3)}_{(\overline{10},1,3)}$

$$\begin{pmatrix} m_5 - 2\sqrt{\frac{3}{5}}\lambda_8E & \frac{2\lambda_{12}\overline{v_R}}{5} \\ \frac{2\lambda_{11}v_R}{5} & m_2 + \frac{\lambda_2\Phi_1}{10\sqrt{6}} - \frac{\lambda_2\Phi_2}{30\sqrt{2}} + \frac{\lambda_2\Phi_3}{30} + \frac{\lambda_6A_1}{5} + \frac{\lambda_6A_2}{5\sqrt{6}} \end{pmatrix}. \tag{81}$$

$[(\mathbf{6},\mathbf{1},\frac{1}{3})+\text{c.c.}]$

$\mathbf{c}$: $\hat{D}^{(6,1,1/3)}_{(10,1,1)}, \hat{\overline{\Delta}}^{(6,1,1/3)}_{(10,1,3)}$

$\mathbf{r}$: $\hat{D}^{(6,1,-1/3)}_{(\overline{10},1,1)}, \hat{D}^{(6,1,-1/3)}_{(\overline{10},1,3)}$

$$\begin{pmatrix} m_6-\dfrac{2\lambda_{14}E}{\sqrt{15}}-\dfrac{\sqrt{2}\lambda_{15}\Phi_2}{9} & -\dfrac{i\lambda_{19}A_1}{\sqrt{10}}-\dfrac{\lambda_{21}\Phi_3}{6\sqrt{10}} \\ \dfrac{i\lambda_{18}A_1}{\sqrt{10}}-\dfrac{\lambda_{20}\Phi_3}{6\sqrt{10}} & m_2+\dfrac{\lambda_2\Phi_1}{10\sqrt{6}}-\dfrac{\lambda_2\Phi_2}{30\sqrt{2}}+\dfrac{\lambda_6A_2}{5\sqrt{6}} \end{pmatrix}. \tag{82}$$

$[(\mathbf{6},\mathbf{1},\frac{4}{3})+\text{c.c.}]$

$\mathbf{c}$: $\hat{\overline{\Delta}}^{(6,1,4/3)}_{(10,1,3)}$

$\mathbf{r}$: $\hat{\Delta}^{(6,1,-4/3)}_{(\overline{10},1,3)}$

$$m_2+\frac{\lambda_2\Phi_1}{10\sqrt{6}}-\frac{\lambda_2\Phi_2}{30\sqrt{2}}-\frac{\lambda_2\Phi_3}{30}-\frac{\lambda_6A_1}{5}+\frac{\lambda_6A_2}{5\sqrt{6}}. \tag{83}$$

$[(\mathbf{6},\mathbf{2},-\frac{1}{6})+\text{c.c.}]$

$\mathbf{c}$: $\hat{\phi}^{(6,2,-1/6)}_{(10,2,2)}$

$\mathbf{r}$: $\hat{\Phi}^{(6,2,1/6)}_{(\overline{10},2,2)}$

$$m_1-\frac{\lambda_1\Phi_2}{3\sqrt{2}}+\frac{\lambda_1\Phi_3}{6}-\frac{2}{5}\lambda_7A_1-\frac{1}{4}\sqrt{\frac{3}{5}}\lambda_{10}E. \tag{84}$$

$[(\mathbf{6},\mathbf{2},\frac{5}{6})+\text{c.c.}]$

$\mathbf{c}$: $\hat{\Phi}^{(6,2,5/6)}_{(10,2,2)}$

$\mathbf{r}$: $\hat{\Phi}^{(6,2,-5/6)}_{(\overline{10},2,2)}$

$$m_1-\frac{\lambda_1\Phi_2}{3\sqrt{2}}-\frac{\lambda_1\Phi_3}{6}+\frac{2}{5}\lambda_7A_1-\frac{1}{4}\sqrt{\frac{3}{5}}\lambda_{10}E. \tag{85}$$

$[(\mathbf{6},\mathbf{3},1/3)+\text{c.c.}]$

$\mathbf{c}$: $\hat{\Delta}^{(6,3,1/3)}_{(10,3,1)}$

$\mathbf{r}$: $\hat{\overline{\Delta}}^{(6,3,-1/3)}_{(\overline{10},3,1)}$

$$m_2-\frac{\lambda_2\Phi_1}{10\sqrt{6}}-\frac{\lambda_2\Phi_2}{30\sqrt{2}}-\frac{\lambda_6A_2}{5\sqrt{6}}. \tag{86}$$

$(\mathbf{8},\mathbf{1},0)$

$\mathbf{c}$: $\hat{A}^{(8,1,0)}_{(15,1,1)}, \hat{E}^{(8,1,0)}_{(20',1,1)}, \hat{\Phi}^{(8,1,0)}_{(15,1,1)}, \hat{\Phi}^{(8,1,0)}_{(15,1,3)}$

$\mathbf{r}$: $\hat{A}^{(8,1,0)}_{(15,1,1)}, \hat{E}^{(8,1,0)}_{(20',1,1)}, \hat{\Phi}^{(8,1,0)}_{(15,1,1)}, \hat{\Phi}^{(8,1,0)}_{(15,1,3)}$

$$\begin{pmatrix} m_4-\frac{\sqrt{2}\lambda_5\Phi_2}{3}-\frac{2\lambda_9E}{\sqrt{15}} & \sqrt{\frac{2}{3}}\lambda_9A_2 & -\frac{\sqrt{2}\lambda_5A_2}{3}+\frac{2\sqrt{2}\lambda_7\Phi_1}{5} & -\sqrt{\frac{2}{3}}\lambda_5A_1+\frac{2}{5}\sqrt{\frac{2}{3}}\lambda_7\Phi_3 \\ \sqrt{\frac{2}{3}}\lambda_9A_2 & m_5-2\sqrt{\frac{3}{5}}\lambda_8E & -\frac{\lambda_{10}\Phi_2}{\sqrt{6}} & -\frac{\lambda_{10}\Phi_3}{\sqrt{6}} \\ -\frac{\sqrt{2}\lambda_5A_2}{3}+\frac{2}{5}\sqrt{2}\lambda_7\Phi_1 & -\frac{\lambda_{10}\Phi_2}{\sqrt{6}} & m_1-\frac{\lambda_1\Phi_2}{3\sqrt{2}}-\frac{2\lambda_{10}E}{\sqrt{15}} & \frac{\lambda_1\Phi_3}{3\sqrt{2}}-\frac{2\sqrt{2}\lambda_7A_1}{5} \\ -\sqrt{\frac{2}{3}}\lambda_5A_1+\frac{2}{5}\sqrt{\frac{2}{3}}\lambda_7\Phi_3 & -\frac{\lambda_{10}\Phi_3}{\sqrt{6}} & \frac{\lambda_1\Phi_3}{3\sqrt{2}}-\frac{2\sqrt{2}\lambda_7A_1}{5} & m_{44}^{(8,1,0)} \end{pmatrix}, \qquad (87)$$

where

$$m_{44}^{(8,1,0)} \equiv m_1+\frac{\lambda_1\Phi_1}{\sqrt{6}}-\frac{\lambda_1\Phi_2}{3\sqrt{2}}-\frac{2}{5}\sqrt{\frac{2}{3}}\lambda_7A_2+\frac{\lambda_{10}E}{2\sqrt{15}}. \qquad (88)$$

$[(\mathbf{8},\mathbf{1},1)+\text{c.c.}]$
c: $\hat{\Phi}^{(8,1,1)}_{(15,1,3)}$
r: $\hat{\Phi}^{(8,1,-1)}_{(15,1,3)}$

$$m_1+\frac{\lambda_1\Phi_1}{\sqrt{6}}-\frac{\lambda_1\Phi_2}{3\sqrt{2}}-\frac{2}{5}\sqrt{\frac{2}{3}}\lambda_7A_2+\frac{\lambda_{10}E}{2\sqrt{15}}. \qquad (89)$$

$[(\mathbf{8},\mathbf{2},-\frac{1}{2})+\text{c.c.}]$
c: $\hat{D}^{(8,2,1/2)}_{(15,2,2)},\hat{\Delta}^{(8,2,1/2)}_{(15,2,2)},\hat{\overline{\Delta}}^{(8,2,1/2)}_{(15,2,2)}$
r: $\hat{D}^{(8,2,-1/2)}_{(15,2,2)},\hat{\overline{\Delta}}^{(8,2,-1/2)}_{(15,2,2)},\hat{\Delta}^{(8,2,-1/2)}_{(15,2,2)}$

$$\begin{pmatrix} m_{11}^{(8,2,-1/2)} & m_{12}^{(8,2,-1/2)} & m_{13}^{(8,2,-1/2)} \\ m_{21}^{(8,2,-1/2)} & m_{22}^{(8,2,-1/2)} & \frac{\lambda_{12}E}{\sqrt{15}} \\ m_{31}^{(8,2,-1/2)} & \frac{\lambda_{11}E}{\sqrt{15}} & m_{33}^{(8,2,-1/2)} \end{pmatrix}, \qquad (90)$$

where

$$m_{11}^{(8,2,-1/2)} \equiv m_6-\frac{\lambda_{14}E}{3\sqrt{15}}-\frac{\sqrt{2}\lambda_{15}\Phi_2}{9},$$

$$m_{12}^{(8,2,-1/2)} \equiv -\frac{i\lambda_{18}A_1}{2\sqrt{10}}-\frac{i\lambda_{18}A_2}{2\sqrt{15}}+\frac{\lambda_{20}\Phi_1}{4\sqrt{15}}+\frac{\lambda_{20}\Phi_3}{12\sqrt{10}},$$

$$m_{13}^{(8,2,-1/2)} \equiv -\frac{i\lambda_{19}A_1}{2\sqrt{10}}+\frac{i\lambda_{19}A_2}{2\sqrt{15}}+\frac{\lambda_{21}\Phi_1}{4\sqrt{15}}-\frac{\lambda_{21}\Phi_3}{12\sqrt{10}},$$

$$m_{21}^{(8,2,-1/2)} \equiv \frac{i\lambda_{19}A_1}{2\sqrt{10}}+\frac{i\lambda_{19}A_2}{2\sqrt{15}}+\frac{\lambda_{21}\Phi_1}{4\sqrt{15}}+\frac{\lambda_{21}\Phi_3}{12\sqrt{10}}, \qquad (91)$$

$$m_{22}^{(8,2,-1/2)} \equiv m_2-\frac{\lambda_2\Phi_2}{30\sqrt{2}}+\frac{\lambda_2\Phi_3}{60}+\frac{\lambda_6A_1}{10},$$

$$m_{31}^{(8,2,-1/2)} \equiv \frac{i\lambda_{18}A_1}{2\sqrt{10}} - \frac{i\lambda_{18}A_2}{2\sqrt{15}} + \frac{\lambda_{20}\Phi_1}{4\sqrt{15}} - \frac{\lambda_{20}\Phi_3}{12\sqrt{10}},$$

$$m_{33}^{(8,2,-1/2)} \equiv m_2 - \frac{\lambda_2\Phi_2}{30\sqrt{2}} - \frac{\lambda_2\Phi_3}{60} - \frac{\lambda_6 A_1}{10}.$$

$(\mathbf{8},\mathbf{3},0)$
c: $\hat{\Phi}_{(15,3,1)}^{(8,3,0)}$
r: $\hat{\Phi}_{(15,3,1)}^{(8,3,0)}$

$$m_1 - \frac{\lambda_1\Phi_1}{\sqrt{6}} - \frac{\lambda_1\Phi_2}{3\sqrt{2}} + \frac{2}{5}\sqrt{\frac{2}{3}}\lambda_7 A_2 + \frac{\lambda_{10}E}{2\sqrt{15}}. \tag{92}$$

B. Tests and consistency checks of the total mass matrix

The following consistency checks of the total mass matrix have been performed (see also Ref. 17):

(1) The CG coefficients appearing in the total mass matrix have been found to satisfy the Hermiticity relation (50).

(2) The trace of the total mass matrix has been evaluated and it has been found that it satisfies Eq. (56).

(3) For the G_{321} symmetric vacuum, the number of the would-be NG modes have been found to be equal to the number of the broken generators, i.e., massive gauge bosons, $45-12=33$. Here, 45 represents the number of gauge bosons in the adjoint irrep of the SO(10) group, and 12 represents the number of the gauge bosons in the standard model (G_{321} group). The check has been performed numerically for several tens of randomly chosen sets of the parameters of the superpotential $\lambda_1,\ldots,\lambda_{21}$ and $m_1,\ldots,m_6$, constrained by VEV Eqs. (28).

(4) For the SU(5) symmetric vacuum, it has been found that the number of the different mass eigenvalues is 21. The number of independent SU(5) irreps contained in the total representation of the model $\underline{R}$ is 22 (see Appendix A), and the would-be NG bosons are in two multiplets $[(\mathbf{1})+(\mathbf{10}+\overline{\mathbf{10}})]$. Therefore, there are in total 21 mass eigenvalues: 20 different from zero and one equal to zero. All other states must be accommodated in the SU(5) irreps. Further, for the SU(5) symmetric vacuum, all eigenvalues of the doublet Higgs matrix $(\mathbf{1},\mathbf{2},\frac{1}{2})$ are contained in the spectrum of the triplet Higgs matrix $(\mathbf{3},\mathbf{1},-\frac{1}{3})$. The only mass eigenvalue of the triplet Higgs matrix spectrum not contained in the doublet mass spectrum belongs to the SU(5) multiplet **50**, leading to the following relation between determinants of these two matrices:

$$\det(M^{(3,1,-1/3)} - \lambda \times \mathbf{1}) = (m_{50}^{(3,1,-1/3)} - \lambda)\det(M^{(1,2,1/2)} - \lambda \times \mathbf{1}). \tag{93}$$

The above relation gives a very strong test for these two matrices. All above checks have been performed numerically, in the same way as explained in the previous item. Our mass matrices passed all checks. For the SU(5) symmetric vacuum, all VEVs, except E, are different from zero, and they are nontrivially correlated through the VEV Eqs. (28) and SU(5) symmetry conditions (30). Therefore, these tests serve as a strong check of the total mass matrix up to terms which depend on E.

(5) For the G_{422} symmetric vacuum, it has been found that the number of different mass eigenvalues is 27, what is just the number of the independent G_{422} irreps (see Appendix A). The number of NG states, contained in G_{422} irrep $(\mathbf{6},\mathbf{2},\mathbf{2})$ is $45-21=24$. These checks have been performed numerically as explained above. All other states have been found to be accommodated in G_{422} irreps. For the G_{422} symmetric vacuum, VEVs different from zero are Φ_1 and E, and therefore this serves as a check of the E-dependent parts of the total mass matrix.

(6) Similar tests are satisfied for other higher symmetries G_{51}, G_{421}, G_{3221}, and G_{3211}.

We stress that all mass matrices in Ref. 17, derived for minimal SO(10) model ($R=\mathbf{10}+\mathbf{126}+\overline{\mathbf{126}}+\mathbf{210}$), satisfy all of the above consistency checks, and are just a special case of the mass matrices in this paper. Further, all results including CG coefficients and mass matrices were obtained analytically including checks.

In the recent calculations in Refs. 20 and 21, that appeared after Ref. 17, the necessary condition (93) between doublets and triplets is not satisfied. The Hermiticity condition (50) and the total trace relation (56) are also not satisfied in these references. Further, none of the higher symmetry tests is satisfied. That is a consequence of different phase conventions in Ref. 17 and phase conventions in Refs. 20 and 21.

In this paper, all results for the CG coefficients and mass matrices have been also obtained analytically. Checks have been performed numerically and it has been found that the mass matrices satisfy ALL consistency checks.

VI. MASS MATRICES OF QUARKS AND LEPTONS

After the symmetry breaking down to the G_{321} subgroup, the electroweak symmetry breaking $SU(2)_L\times U(1)_Y\to U(1)_{em}$ can be achieved by the VEVs of doublets included in the fields H, D, Δ, $\overline{\Delta}$, and Φ. These fields are given by [see Table 2 (Ref. 19)] $\tilde{H}_{u/d}=H^{(1,2,\pm1/2)}_{(1,2,2)}$, $D_{u/d}=D^{(1,2,\pm1/2)}_{(1,2,2)}$, $\tilde{D}_{u/d}=D^{(1,2,\pm1/2)}_{(15,2,2)}$, $\overline{\Delta}_{u/d}=\overline{\Delta}^{(1,2,\pm1/2)}_{(15,2,2)}$, $\Delta_{u/d}=\Delta^{(1,2,\pm1/2)}_{(15,2,2)}$, $\Phi_{u/d}=\Phi^{(1,2,\pm1/2)}_{(\overline{10},2,2)}$. The Yukawa couplings of Eq. (1) including these doublets can be written as follows:

$$\begin{aligned}W_Y &= U^c_i(Y^{ij}_{10}\tilde{H}_u + Y^{ij}_{120}D_u + Y^{ij}_{120}\tilde{D}_u + Y^{ij}_{126}\overline{\Delta}_u)Q_j + D^c_i(Y^{ij}_{10}\tilde{H}_d + Y^{ij}_{120}D_d + Y^{ij}_{120}\tilde{D}_d + Y^{ij}_{126}\overline{\Delta}_d)Q_j \\ &\quad + N^c_i(Y^{ij}_{10}\tilde{H}_u + Y^{ij}_{120}D_u - 3Y^{ij}_{120}\tilde{D}_u - 3Y^{ij}_{126}\overline{\Delta}_u)L_j + E^c_i(Y^{ij}_{10}\tilde{H}_d + Y^{ij}_{120}D_d - 3Y^{ij}_{120}\tilde{D}_d - 3Y^{ij}_{126}\overline{\Delta}_d)L_j,\end{aligned} \tag{94}$$

where U^c, D^c, N^c, and E^c are the right-handed $SU(2)_L$ singlet quark and lepton superfields, Q and L are the left-handed $SU(2)_L$ doublet quark and lepton superfields, respectively. This is a generalization of the renormalizable minimal SO(10) model,[6,7] including **120**. Note that the successful gauge-couplings unification is realized with only the MSSM particle contents. This means that only one pair of Higgs doublets remains light and others should be heavy ($\geqslant M_G$). Here we accept the simple picture that the low-energy superpotential is described by only one pair of light Higgs doublets (H_u and H_d) in the MSSM. But, in general, these Higgs fields are admixtures of all Higgs doublets having the same quantum numbers in the original model such as

$$\begin{aligned}H_u &= \tilde{\alpha}^1_u\tilde{H}_u + \tilde{\alpha}^2_u D_u + \tilde{\alpha}^3_u\tilde{D}_u + \tilde{\alpha}^4_u\overline{\Delta}_u + \tilde{\alpha}^5_u\Delta_u + \tilde{\alpha}^6_u\Phi_u, \\ H_d &= \tilde{\alpha}^1_d\tilde{H}_d + \tilde{\alpha}^2_d D_d + \tilde{\alpha}^3_d\tilde{D}_d + \tilde{\alpha}^4_d\overline{\Delta}_d + \tilde{\alpha}^5_d\Delta_d + \tilde{\alpha}^6_d\Phi_d,\end{aligned} \tag{95}$$

where $\tilde{\alpha}^i_{u,d}$ $(i=1,2,\ldots,5,6)$ denote elements of the unitary matrix which rotate the flavor basis in the original model into the (SUSY) mass eigenstates. As mentioned above, the low-energy superpotential is described only by the light Higgs doublets H_u and H_d,

$$\begin{aligned}W_Y &= U^c_i(\alpha^1_u Y^{ij}_{10} + \alpha^2_u Y^{ij}_{120} + \alpha^3_u Y^{ij}_{120} + \alpha^4_u Y^{ij}_{126})H_u Q_j + D^c_i(\alpha^1_d Y^{ij}_{10} + \alpha^2_d Y^{ij}_{120} + \alpha^3_d Y^{ij}_{120} + \alpha^4_d Y^{ij}_{126})H_d Q_j \\ &\quad + N^c_i(\alpha^1_u Y^{ij}_{10} + \alpha^2_u Y^{ij}_{120} - 3\alpha^3_u Y^{ij}_{120} - 3\alpha^4_u Y^{ij}_{126})H_u L_j + E^c_i(\alpha^1_d Y^{ij}_{10} + \alpha^2_d Y^{ij}_{120} - 3\alpha^3_d Y^{ij}_{120} \\ &\quad - 3\alpha^4_d Y^{ij}_{126})H_d L_j,\end{aligned} \tag{96}$$

where the formulas of the inverse unitary transformation of Eq. (95),

$$\tilde{H}_u = \alpha^1_u H_u, \quad D_u = \alpha^2_u H_u,$$

$$\widetilde{D}_u = \alpha_u^3 H_u, \quad \overline{\Delta}_u = \alpha_u^4 H_u,$$
$$\widetilde{H}_d = \alpha_d^1 H_d, \quad D_d = \alpha_d^2 H_d, \tag{97}$$
$$\widetilde{D}_d = \alpha_d^3 H_d, \quad \overline{\Delta}_d = \alpha_d^4 H_d,$$

have been used.

Providing the Higgs VEVs, $H_u = v \sin\beta$ and $H_d = v\cos\beta$ with $v \simeq 174.1(\mathrm{GeV})$, the quark and lepton mass matrices can be read off as

$$M_u = c_{10} M_{10} + c_{120} M_{120} + \tilde{c}_{120} \widetilde{M}_{120} + c_{126} M_{126},$$
$$M_d = M_{10} + M_{120} + \widetilde{M}_{120} + M_{126},$$
$$M_D = c_{10} M_{10} + c_{120} M_{120} - 3\tilde{c}_{120} \widetilde{M}_{120} - 3 c_{126} M_{126}, \tag{98}$$
$$M_e = M_{10} + M_{120} - 3\widetilde{M}_{120} - 3M_{126},$$
$$M_R = c_R M_{126},$$

where M_u, M_d, M_D, M_e, and M_R denote the up-type quark, down-type quark, neutrino Dirac, charged-lepton, and right-handed neutrino Majorana mass matrices, respectively. Note that the mass matrices at the right-hand side of Eq. (98) are defined as $M_{10} = Y_{10} \alpha_d^1 v \cos\beta$, $M_{120} = Y_{120} \alpha_d^2 v \cos\beta$, $\widetilde{M}_{120} = Y_{120} \alpha_d^3 v \cos\beta$, and $M_{126} = Y_{126} \alpha_d^4 v \cos\beta$, respectively, and the coefficients are defined as $c_{10} = (\alpha_u^1/\alpha_d^1)\tan\beta$, $c_{120} = (\alpha_u^2/\alpha_d^2)\tan\beta$, $\tilde{c}_{120} = (\alpha_u^3/\alpha_d^3)\tan\beta$, $c_{126} = (\alpha_u^4/\alpha_d^4)\tan\beta$, and $c_R = v_R/(\alpha_d^4 v\cos\beta)$. These mass matrices are directly connected with low-energy observations and are crucial to model judgement.

VII. CONCLUSION

We have presented a simple method for the calculation of CG coefficients. We have constructed all states for all antisymmetric and symmetric SO(10) tensor irreps. We list all tables for the CG coefficients for the SO(10) irreps **10**, **45**, **54**, **120**, **126**, $\overline{\mathbf{126}}$, and **210**, for all possible cubic invariants. We have constructed all mass matrices for the corresponding Higgs-Higgsino sector is SUSY GUT SO(10) models. We have found a set of consistency checks for the CG coefficients and mass matrices which proved the correctness of all our results. The results obtained here are useful for a wide class of GUT models based on the SO (10) group.

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APPENDIX A: DECOMPOSITION OF REPRESENTATIONS UNDER G_{321}

Here we list the decompositions of **10, 16, 45, 54, 120**, $\overline{\mathbf{126}}$, and **210** representations under the chain of subgroups $G_{422} \supset G_{3122} \supset G_{3121} \supset G_{321}$ in Tables 7, 8, 9, 10, 11, 12, and 13 (Ref. 19), where U(1) groups are related to $\mathrm{SU}(4) \to \mathrm{SU}(3)_C \times \mathrm{U}(1)_{B-L}$, $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \to \mathrm{SU}(2)_L$

$\times \mathrm{U}(1)_R$. [We use the same notation as Slansky[22] but with the proper U(1) normalizations.] Note also that we may consider another chain of subgroups $\mathrm{SO}(10) \to \mathrm{SU}(5) \times \mathrm{U}(1)_X$ and $\mathrm{SU}(5) \to G_{321}$. The relations between the generators of $\mathrm{U}(1)_{X,Y}$ and $\mathrm{U}(1)_{B-L,R}$ are

$$\frac{1}{10}(-X + 4Y) = B - L, \qquad Y = B - L + T_R^3, \tag{A1}$$

where T_R^3 denotes the $\mathrm{U}(1)_R$ generator.

APPENDIX B: CG COEFFICIENTS

The CG coefficients for HH, AA, EE, DD, $\Delta\Delta$, $\overline{\Delta\Delta}$, $\overline{\Delta}\Delta$, $\Phi\Phi$, EA, $A\Phi$, $E\Delta$, $E\overline{\Delta}$, $E\Phi$, ΔH, $\overline{\Delta}H$, $H\Phi$, $\Phi\Delta$, $\Phi\overline{\Delta}$, $A\Delta$, $A\overline{\Delta}$, DH, DA, $D\Delta$, $D\overline{\Delta}$, and $D\Phi$ combinations are listed in Tables 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, and 39, respectively.[19]

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[19] See EPAPS Document No. E-JMAPAQ-46-030502 for the tables of G321 states, table of G321 multiplets, tables of decompositions of SO(10) representations, and tables of CG coefficients. A direct link to this documentation may be found in the online article's HTML reference section. The documentation may also be reached via the EPAPS homepage (http://www.aip.org/pubservs/epaps.html) or from ftp.aip.org in the directory/epaps/. See the EPAPS homepage for more information.
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