VOLATILITY ANALYSIS FOR HIGH FREQUENCY FINANCIAL DATA

by

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ABSTRACT

Measuring and modeling financial volatility are key steps for derivative pricing and risk management. In financial markets, there are two kinds of data: low-frequency financial data and high-frequency financial data. Most research has been done based on low-frequency data. In this dissertation we focus on high-frequency data. In theory, the sum of squares of log returns sampled at high frequency estimates their variance. For log price data following a diffusion process without noise, the realized volatility $\sum_{j=0}^{n-1}\left(X_{t_{j+1}} - X_{t_j}\right)^2$ converges to its quadratic variation. When log price data contain market microstructure noise, the realized volatility explodes as the sampling interval converges to 0.

In this dissertation, we generalize the fundamental Ito isometry and analyze the speed with which stochastic processes approach to their quadratic variations. We determine the difference between realized volatility and quadratic variation under mean square constraints for Brownian motion and general case. We improve the estimation for quadratic variation. The estimators found by us converge to quadratic variation at a higher rate, which is $O(n^{-2})$.

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CONTENTS

CHAPTER 1

INTRODUCTION

High-frequency financial data, which are intra-day financial data, possess important features absent in data measured at lower frequencies, which is related to daily or longer time horizons, and analysis of these data poses interesting challenges to econometric modeling and statistical analysis. First, the number of observations in high-frequency data sets can be overwhelming. The average daily number of quotes in the USD/EUR spot market can easily exceed 20,000, and the average daily number of observations of an actively traded NYSE stock can be even higher. Second, transaction-by-transaction data on trades and quotes are, by nature, irregularly spaced time series with random daily numbers of observations.

Modeling and measuring financial volatility are key steps for derivative pricing, portfolio allocation and risk management. Parametric models like stochastic volatility models, or implied volatilities from option pricing models such as the Black-Scholes model(Reference [13], page 188) have been well studied and established when dealing with low frequency data.

How can we model and measure high frequency data? In theory, the sum of squares of log returns sampled at high frequency estimates their variance(Andersen, Bollerslev, Diebold and Labys, (2003); Barndor-Nielsen and Shephard,(2002)). For log price data following a diffusion process without noise, the realized volatility converges to its quadratic variation. When log price data contain market microstructure noise, the realized volatility explodes as the sampling interval converges to 0.

One intuitive question is how fast does the sum of squares of log returns converge to the variance? Is it possible to obtain a better estimator for quadratic variation?

It is the aim of our work to obtain a better approximation to the quadratic variation. In this dissertation, we determine the difference between realized volatility and quadratic variation under mean square. We improved the estimation for quadratic variation. In this dissertation, we also discuss quadratic variation for high dimension problems. The dissertation is organized in this way: Chapter 2 deals with preliminaries about probability space, random variables and stochastic processes, Chapter 3 is about models for high frequency data, while Chapter 4 is concerned with higher order approximations for quadratic variation.

CHAPTER 2

PRELIMINARIES

In econometrics, the research object is asset price. The mathematical model for an asset price requires that it be positive and satisfies a Geometric Brownian motion, which is a special stochastic process. The definition of Geometric Brownian motion is provided in Definition 2.2.5. Before we provide it, we need some concepts from general probability theory.

The following provides a detailed discussion of some important concepts of the theory of stochastic processes along with a discussion of several of their financial interpretations.

2.1. Probability Spaces and Measurable Functions

DEFINITION 2.1.1. *If Ω is a given set, then a σ-*algebra *$\mathcal{F}$ on Ω is a family of subsets of Ω satisfying the following properties*

- $\phi \in \mathcal{F}$
- $F \in \mathcal{F} \Rightarrow F^c \in \mathcal{F}$, *where $F^c = \Omega \setminus F$ is the complement of F in Ω*
- $A_1, A_2, \cdots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

The *pair* $(\Omega, \mathcal{F})$ is called a *measurable space.* A probability measure $\mathbf{P}$ on a *measurable space* $(\Omega, \mathcal{F})$ is a *function* $\mathbf{P} : \mathcal{F} \to [0, 1]$ such that

(1) $\mathbf{P}(\phi) = 0, \ \mathbf{P}(\Omega) = 1$

(2) If $A_1, A_2, \cdots \in \mathcal{F}$ and $\{A_i\}_{i=1}^{\infty}$ is disjoint, then

$$\mathbf{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbf{P}(A_i)$$

The *triple* $(\Omega, \mathcal{F}, \mathbf{P})$ is called a *probability space.*

DEFINITION 2.1.2. *If* $(\Omega, \mathcal{F}, \mathbf{P})$ *is a given probability space, then a function* $Y : \Omega \to R^n$ *is called* $\mathcal{F}$*-measurable if*

$$Y^{-1}(U) = \{\omega \in \Omega; Y(\omega) \in U\} \in \mathcal{F}$$

for all open sets $U \in R^n$.

If $X : \Omega \to R^n$ is any function, then the σ*-algebra* $\mathcal{H}_X$ generated by X is the smallest σ*-algebra* on Ω containing all sets

$$X^{-1}(U); U \subset R^n \text{ open.}$$

2.2. Random Variables and Stochastic Processes

In the following, we let $(\Omega, \mathcal{F}, \mathbf{P})$ denote a given probability space. A *random variable* X is an $\mathcal{F}$-measurable function $X : \Omega \to \mathbf{R}^n$. Every random variable X induces a probability measure μ_X on $\mathbf{R}^n$, defined by

$$\mu_X(B) = \mathbf{P}(X^{-1}(B)), \; B \in Borel \;\; \sigma - algebra \; \mathcal{B},$$

μ_X is called the *distribution* of X.

If $\int_\Omega |X(\omega)| \, d\mathbf{P}(\omega) < \infty$, then the number

$$E(X) = \int_\Omega X(\omega) d\mathbf{P}(\omega) = \int_{\mathbf{R}^n} x \cdot d\mu_X(x)$$

is called the *expectation* of X(w.r.t $\mathbf{P}$).

More generally, if $f : \mathbf{R}^n \to \mathbf{R}$ is Borel measurable and $\int_\Omega |f(X(\omega))| \, d\mathbf{P}(\omega) < \infty$, then we have

$$E[f(X)] := \int_\Omega f(X(\omega)) d\mathbf{P}(\omega) = \int_{\mathbf{R}^n} f(x) d\mu_X(x)$$

The mathematical model for *independence* is the following:

Definition 2.2.1. *Two subsets* $A, B \in \mathcal{F}$ *are called* independent *if*

$$\mathbf{P}(A \cap B) = \mathbf{P}(A) \cdot \mathbf{P}(B).$$

A collection $\mathcal{A} = \{\mathcal{H}_i : i \in I\}$ *of families* $\mathcal{H}_i$ *of measurable sets is* independent *if*

$$\mathbf{P}(H_{i_1} \cap \ldots \cap H_{i_k}) = \mathbf{P}(H_{i_1}) \ldots \mathbf{P}(H_{i_k})$$

for all choices of $H_{i_1} \in \mathcal{H}_{i_1}, \ldots, H_{i_k} \in \mathcal{H}_{i_k}$ *with different indices* $i_1, \ldots, i_k$.

A collection of random variables $\{X_i; i \in I\}$ *is* independent *if the collection of generated* σ*-algebras* $\mathcal{H}_{X_i}$ *is independent.*

If two random variables $X, Y : \Omega \to R$ are independent, then

$$E[XY] = E[X] \cdot E[Y]$$

provided that $E[|X|] < \infty$ and $E[|Y|] < \infty$.

Furthermore, if two random variables $X, Y : \Omega \to R$ are independent, and the functions g and h are continuous, then $g(X)$, $h(Y)$ are also independent.

After the definition of random variable, we can define *stochastic process.*

Definition 2.2.2. *A* stochastic process *is a parameterized collection of random variables*

$$\{X_t\}_{t \in T}$$

defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ *and assuming values in* $\mathbf{R}^n$.

Note that for each $t \in T$ fixed, we have a random variable

$$\omega \to X_t(\omega);\ \omega \in \Omega$$

On the other hand, fixing $\omega \in \Omega$, we can consider the function

$$t \to X_t(\omega);\; t \in T$$

which is called a *path* of X_t.

Intuitively, we can think of t as "time" and each ω as an individual "particle"or "experiment". With this picture $X_t(\omega)$ would represent the position (or result) at time t of the particle ω. Sometimes it's convenient to write $X(t, \omega)$ instead of $X_t(\omega)$. Thus we may also regard the process as a function of two variables

$$(t, \omega) \to X(t, \omega)$$

from $T \times \Omega$ into $\mathbf{R}^n$. This is often a natural point of view in stochastic analysis.

In financial markets, the price of a security is usually assumed to follow a stochastic process. A stochastic process as considered in probability theory is the counterpart of a deterministic process. Instead of dealing only with one possible outcome of how the process might evolve under time (as it is the case, for example, for solutions of an ordinary differential equation), in a random process there is some indeterminacy in its future evolution described by probability distributions. This means that even if the initial condition (or starting point) is known, there are multiple possibilities to the process to follow, but some paths are more probable than others.

DEFINITION 2.2.3. *A stochastic process X_t is called* stationary *if X_t has same distribution as X_{t+h} for any $h > 0$.*

DEFINITION 2.2.4. *A stochastic process $\{B_t\}_{t \geq 0}$ is said to be a standard Brownian motion process if:*

(1) $B_0 = 0$;

(2) $\{B_t\}_{t \geq 0}$ *has stationary independent increments;*

(3) *for every $t > 0$, B_t is normally distributed with mean 0 and variance t.*

The standard Brownian Motion is the most fundamental continuous time stochastic process. R. Brown in 1826 observed the irregular motion of pollen particles moving in water. He and others noted that the path of a given particle is very irregular, having a tangent at no point, and the motions of two distinct particles appear to be independent. In 1900 L. Bachelier attempted to describe fluctuations in stock prices mathematically from the viewpoint of stochastic processes.

DEFINITION 2.2.5. G*eometric Brownian motion: let* μ *and* $\sigma > 0$ *be constants and define the Geometric Brownian motion:*

$$S_t = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t}.$$

If S_t represents the stock price, then μ is the mean value of the return of the stock and σ^2 is the variance of the return of the stock's value. For low frequency data, variance σ is assumed to be constant, however, for high frequency data, we will see in Chapter 4 that this assumption is inaccurate.

DEFINITION 2.2.6. *Let* $\{\mathcal{N}_t\}_{t \geq 0}$ *be an increasing family of* σ*-algebras of subsets of* Ω*. A process* $g\,(t, \omega) : [0, \infty) \times \Omega \to R^n$ *is called* $\mathcal{N}_t$*-adapted if for each* $t \geq 0$ *the function* $\omega \to g\,(t, \omega)$ *is* $\mathcal{N}_t$*-measurable.*

In particular, for Brownian motion, $\mathcal{N}_t$ is the smallest σ-algebra containing all sets of the form

$$\{\omega;\ B_s(\omega) \in U\},$$

where $s \leq t$ and $U \subset R^n$ is a Borel set and we commonly use the notation $\mathcal{F}_t$ instead of $\mathcal{N}_t$ in this case.

DEFINITION 2.2.7. *Let* $\mathcal{V} = \mathcal{V}(S, T)$ *be the class of functions*

$$f\,(t, \omega) : [0, \infty] \times \Omega \to R$$

such that

(1) $(t,\omega) \to f(t,\omega)$ *is* $\mathcal{B} \times \mathcal{F}$*-measurable, where* $\mathcal{B}$ *denotes the Borel* σ*-algebra on* $[0,\infty)$*;*

(2) $f(t,\omega)$ *is* $\mathcal{F}_t$*-adapted;*

(3) $E[\int_S^T f^2(t,\omega)\,dt] < \infty$.

Lemma 2.2.1. (The Ito isometry) (page 26, [4]). *For all* $f \in \mathcal{V}(S,T)$,

$$E\left[\left(\int_S^T f(t,\omega)\,dB_t\right)^2\right] = E\left[\int_S^T f^2(t,\omega)\,dt\right].$$

Definition 2.2.8. *An Ito process or stochastic integral is a stochastic process on* $(\Omega, \mathcal{F}, \mathbf{P})$ *which can be represented in the form*

$$X(t) = X(0) + \int_0^t U(s)\,ds + \int_0^t V(s)\,dB_s, \tag{2.2.1}$$

where $U(t)$ *and* $V(t)$ *are* $\mathcal{F}_t$*-adapted stochastic process. As a shorthand notation, it is usually written as*

$$dX(t) = U(t)\,dt + V(t)\,dB_t.$$

2.3. Relation between Convergence in Probability and in Mean Square

Definition 2.3.1. *A sequence* X_n *of random variables is said to converge to* X *in probability if*

$$\lim_{n\to\infty} p(|X_n - X| \geq \varepsilon) = 0, \ \ \forall\varepsilon > 0.$$

Convergence in probability is often denoted by adding the letter $\mathbf{P}$ over an arrow indicating convergence: $X_n \xrightarrow{\mathbf{P}} X$, which is also denoted by $p\lim_{m\to\infty} X_n = X$, or simply $p\lim X_n = X$. Convergence in probability is the notion of convergence used in the weak law of large numbers. It's well know that convergence in probability implies convergence in distribution.

DEFINITION 2.3.2. *A sequence X_n of random variables is said to converge to X in mean square if*

$$\lim_{n\to\infty} E\left(|X_n - X|^2\right) = 0.$$

By Markov's Inequality (page 68, [11]) it follows that convergence in mean square is stronger than convergence in probability.

Markov's Inequality: Let $u(X)$ be a nonnegative function of the random variable X. If $E[u(X)]$ exists, then for every positive constant c,

$$p\left(u\left(X\right) \geq c\right) \leq \frac{Eu(X)}{c}.$$

CHAPTER 3

MODELS FOR HIGH FREQUENCY DATA

3.1. Non-parametric Models

High frequency market data are non-stationary([1]). Classical economics assumes that financial markets are stationary, and that short-term price movements follow a discrete random walk. The complex structure of statistical properties discovered with high frequency data is the result of the non-stationary nature of financial markets. Market participants trade with different time horizons, some take positions for only minutes, others for hours, days, weeks or months. Depending on their trading horizon, they react differently to the same news events. Clearly, volatility models at the daily level cannot readily accommodate high-frequency data, and parametric models specified directly for intra-daily data generally fail to capture inter-daily volatility movements (Andersen, Bollerslev, Diebold and Labys, 2003).

Therefore, we use a nonparametric approach for high-frequency volatility analysis. One popular nonparametric method is realized volatility, which converges to quadratic variation in probability if the limit exists.

Let S_t denote the price process of a security, and suppose its log price

$$X_t = logS_t,$$

follows an Ito process:

$$dX_t = \mu_t dt + \sigma_t dB_t,$$

where B_t is a standard Brownian motion, μ_t is the drift coefficient and σ_t^2 is the instantaneous variance of the return process X_t, then

$$\sum_{j=0}^{n-1} \left(X_{t_{j+1}} - X_{t_j}\right)^2$$

is called the *realized volatility* of the stochastic process X_t, where $\{ 0 = t_0 \leq t_1 \leq ... \leq t_n = t \}$ is any partition of the interval $[\, 0,\, t\,]$.

3.2. Quadratic Variation and Application

Suppose that X_t is a real-valued stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with time index t ranging over the non-negative real numbers. Its *quadratic variation* is the process, written as $[X, X]_t$, defined by the following

$$[X, X]_t = p \lim_{\|\Pi\| \to 0} \sum_{j=1}^{n} \left(X_{t_j} - X_{t_{j-1}}\right)^2, \tag{3.2.1}$$

where $\prod = \{ 0 = t_0 \leq t_1 ... \leq t_n = t \}$ is any partition of the interval $[\, 0,\, t\,]$ and the norm of the partition $\prod$ is the length of the longest of these subintervals, that is $\max\{|t_j - t_{j-1}| : j = 1, ..., n\}$. By the definition, we can see quadratic variation is the p-limit of the realized volatility.

Quadratic variation is used in the analysis of stochastic processes. Quadratic variation is just one kind of variation of a stochastic process.

More generally, the *quadratic covariation* of two processes X_t and Y_t is defined by the following

$$[X, Y]_t = p \lim_{\|\Pi\| \to 0} \sum_{j=1}^{n} \left(X_{t_j} - X_{t_{j-1}}\right) \left(Y_{t_j} - Y_{t_{j-1}}\right). \tag{3.2.2}$$

.

For the Ito process defined in (2.2.1), we are interested in finding its quadratic variation since quadratic variation can be used as a measure of volatility. Quadratic covariation can be used as a measure of co-volatility.

Quadratic variation has important applications. For the Brownian motion B_t, the quadratic variation is

$$[B, B]_t = t.$$

This result is in accordance with the definition of Brownian motion. By the definition of Brownian motion, we know t is the variance of B_t.

For the general Ito process, it has the following quadratic variation

LEMMA 3.2.1. (page 131, [17]) *If $V(t)$ is an $\mathcal{F}_t$-adapted stochastic process, then the quadratic variation of the Ito integral $I(t) = \int_0^t V(s)\, dB_s$ is*

$$[I, I]_t = \int_0^t V^2(s)\, ds.$$

PROPOSITION 3.2.1. (page 144, [17]) *Let B_t $t \geq 0$ be the Brownian motion, $U(t)$, $V(t)$ are $\mathcal{F}_t$-adapted stochastic processes, then the quadratic variation of the Ito process* (2.2.1) *is*

$$[X, X]_t = \int_0^t V^2(s)\, ds.$$

This proposition is needed later. For convenience, we provide a proof.

PROOF. We denote $I(t) = \int_0^t V(s)\, dB_s$ and $R(t) = \int_0^t U(s)\, ds$. Then both processes are continuous with respect to their upper limit of integration t. To determine the quadratic variation of X_t on $[0,\ t]$, we choose a partition $\prod = \{ 0 = t_0 \leq t_1 ... \leq t_n = t \}$ on $[0, t]$. The corresponding realized volatility is

$$\sum_{j=0}^{n-1} \left(X_{t_{j+1}} - X_{t_j}\right)^2 = \sum_{j=0}^{n-1} \left(I_{t_{j+1}} - I_{t_j}\right)^2 + \sum_{j=0}^{n-1} \left(R_{t_{j+1}} - R_{t_j}\right)^2$$
$$+2\sum_{j=0}^{n-1} \left(I_{t_{j+1}} - I_{t_j}\right)\left(R_{t_{j+1}} - R_{t_j}\right).$$

As $\| \prod \| \to 0$, $\sum\limits_{j=0}^{n-1} \left(I_{t_{j+1}} - I_{t_j}\right)^2 \to \int_0^t V^2(s)\, ds$ (by Lemma 3.2.1).

Since

$$\begin{aligned} &\sum_{j=0}^{n-1} \left(R_{t_{j+1}} - R_{t_j}\right)^2 \\ &\le \max_{0 \le j \le n-1} \mid R_{t_{j+1}} - R_{t_j} \mid \cdot \sum_{j=0}^{n-1} \mid R_{t_{j+1}} - R_{t_j} \mid \\ &= \max_{0 \le j \le n-1} \mid R_{t_{j+1}} - R_{t_j} \mid \cdot \sum_{j=0}^{n-1} \mid \int_{t_j}^{t_{j+1}} U(s)\, ds \mid \\ &\le \max_{0 \le j \le n-1} \mid R_{t_{j+1}} - R_{t_j} \mid \cdot \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \mid U(s) \mid ds \\ &= \max_{0 \le j \le n-1} \mid R_{t_{j+1}} - R_{t_j} \mid \cdot \int_0^t \mid U(s) \mid ds, \end{aligned}$$

so when $\| \prod \| \to 0$, $\max\limits_{0 \le j \le n-1} \mid R_{t_{j+1}} - R_{t_j} \mid \to 0$, because $R(s)$ is continuous. Hence, $\sum\limits_{j=0}^{n-1} \left(R_{t_{j+1}} - R_{t_j}\right)^2 \to 0$.

Furthermore, by the Cauchy-Schwarz inequality,

$$\begin{aligned} &\left(\sum_{j=0}^{n-1} \left(I_{t_{j+1}} - I_{t_j}\right)\left(R_{t_{j+1}} - R_{t_j}\right)\right)^2 \\ &\le \sum_{j=0}^{n-1} \left(I_{t_{j+1}} - I_{t_j}\right)^2 \sum_{j=0}^{n-1} \left(R_{t_{j+1}} - R_{t_j}\right)^2 \end{aligned}$$

and by previous proof, $\sum\limits_{j=0}^{n-1} \left(R_{t_{j+1}} - R_{t_j}\right)^2$ converges to 0, while $\sum\limits_{j=0}^{n-1} \left(I_{t_{j+1}} - I_{t_j}\right)^2$ converges to $\int_0^t V^2(s)\, ds$.

Hence

$$\left(\sum_{j=0}^{n-1}(I_{t_{j+1}} - I_{t_j})(R_{t_{j+1}} - R_{t_j})\right)^2$$

converges to 0 as $\| \prod \| \to 0$, and we conclude $[X, X]_t = \int_0^t V^2(s)\, ds$.

□

Remarks:

- $\int_0^t V^2(s)\, ds$ is also called the integrated volatility of the stochastic process.
- We can get $[B, B]_t = \int_0^t ds = t$ directly by this proposition.

For high dimension problems, assume we have p stocks, denoted as $X_1(t)$, $X_2(t)$, ... , $X_p(t)$, and that each of them satisfies a diffusion process:

$$dX_i(t) = \mu_{it}dt + \sigma_{it}dB_t, i = 1, 2, ..., p. \tag{3.2.3}$$

Writting in this way means that $X_i(t)(1 \le i \le p)$ are independent. We will consider the dependent case later. First of all, we are interested to find the quadratic variation for a group of stocks. Consider

$$X_t = \omega_1 X_1(t) + \omega_2 X_2(t) + ... + \omega_p X_p(t),$$

where $\omega_1 + ... + \omega_p = 1$, then the realized volatility is

$$\begin{aligned}
& \sum_{j=0}^{n-1} \left(X_{t_{j+1}} - X_{t_j}\right)^2 \\
= & \sum_{j=0}^{n-1} \left(\omega_1 X_1(t_{j+1}) + ... + \omega_p X_p(t_{j+1}) - (\omega_1 X_1(t_j) + ... + \omega_p X_p(t_j))\right)^2 \\
= & \sum_{j=0}^{n-1} \left(\sum_{k=1}^{p} \omega_k \left(X_k(t_{j+1}) - X_k(t_j)\right)\right)^2 \\
= & \sum_{j=0}^{n-1}\sum_{k=1}^{p} \omega_k^2 \left(X_k(t_{j+1}) - X_k(t_j)\right)^2 \\
& + \sum_{j=0}^{n-1}\sum_{m \ne n}^{p} \omega_m \left(X_m(t_{j+1}) - X_m(t_j)\right) \omega_n \left(X_n(t_{j+1}) - X_n(t_j)\right) \\
= & \sum_{k=1}^{p} \omega_k^2 \sum_{j=0}^{n-1} \left(X_k(t_{j+1}) - X_k(t_j)\right)^2 \\
& + \sum_{m \ne n}^{p}\sum_{j=0}^{n-1} [\omega_m \omega_n \left(X_m(t_{j+1}) - X_m(t_j)\right)\left(X_n(t_{j+1}) - X_n(t_j)\right)].
\end{aligned}$$

If both sides converge, then we can get the following proposition by taking the limit on both sides.

Proposition 3.2.2. *Suppose p stocks satisfy a diffusion process:*

$$dX_i(t) = \mu_{it}dt + \sigma_{it}dB_t, \ i = 1, 2, ..., p.$$

and

$$X_t = \omega_1 X_1(t) + \omega_2 X_2(t) + ... + \omega_p X_p(t),$$

where $\omega_1 + ... + \omega_p = 1$, *then the quadratic variation is*

$$[X, X]_t = \sum_{k=1}^{p} \omega_k^2 [X_k(t), X_k(t)]_t + \sum_{m \neq n}^{p} \omega_m \omega_n [X_m(t), X_n(t)]_t.$$

Note that $\omega_1 + ... + \omega_p$ does not necessarily equal to 1. Theoretically speaking, investors can change their portfolios.

3.3. Realized Co-volatility

By equation (3.2.2), the quadratic covariation of two process X_t and Y_t is

$$[X, Y]_t = p \lim_{\|\Pi\| \to 0} \sum_{k=1}^{n} \left(X_{t_k} - X_{t_{k-1}}\right)\left(Y_{t_k} - Y_{t_{k-1}}\right).$$

People may ask what is the quadratic covariation of p stocks ? The following proposition answers the question.

Proposition 3.3.1. *For a stochastic process* $X_i(t)$, $i = 1, ..., p$, *denote the vector* $\overrightarrow{X_t} = (X_1(t), X_2(t), ..., X_p(t))'$ *satisfying* $d\overrightarrow{X_t} = \overrightarrow{\mu_t}dt + \Sigma_t d\overrightarrow{B_t}$, *where* $\overrightarrow{B_t}$ *is a p-vector standard Brownian motion,* $\overrightarrow{\mu_t}$ *is a drift taking values in* R^p, *and* $\Sigma_t = (\sigma_{ij}(t))$ *is a* $p \times p$ *matrix, then*

$$[\overrightarrow{X}, \overrightarrow{X}]_t = \int_0^t \Sigma_s \Sigma_s^T ds.$$

Proof. The proof of this proposition is very similar to the proof of Proposition 3.2.1. Since $\overrightarrow{X_t}$ satisfies

$$d\overrightarrow{X_t} = \overrightarrow{\mu_t}dt + \Sigma_t d\overrightarrow{B_t},$$

then for stock i,

$$dX_i(t) = \mu_i(t)dt + \sum_{j=1}^{p} \sigma_{ij}(t)dB_j(t).$$

Similar to Proposition 3.2.1, we introduce $I_i(t) = \sum_{j=1}^{p} \int_0^t \sigma_{ij}(s)\, dB_s$ and $R_i(t) = \int_0^t \mu_i(s)ds$, then both processes are continuous with respect to their upper limits of integration t. To determine the quadratic variation of $\overrightarrow{X_t}$ on $[\ 0,\ t\ ]$, we choose a partition $\prod = \{0 = t_0 \le t_1 \le ... \le t_n = t\}$ on $[\ 0,\ t\ ]$, hence the corresponding realized volatility is

$$\begin{aligned}
& \sum_{k=0}^{n-1} \left(\overrightarrow{X}_{t_{k+1}} - \overrightarrow{X}_{t_k}\right)\left(\overrightarrow{X}_{t_{k+1}} - \overrightarrow{X}_{t_k}\right)^T \\
= \; & \sum_{k=0}^{n-1} \left(I_{t_{k+1}} - I_{t_k}\right)\left(I_{t_{k+1}} - I_{t_k}\right)^T + \sum_{k=0}^{n-1} \left(R_{t_{k+1}} - R_{t_k}\right)\left(R_{t_{k+1}} - R_{t_k}\right)^T \\
& + \sum_{k=0}^{n-1} \left(I_{t_{k+1}} - I_{t_k}\right)\left(R_{t_{k+1}} - R_{t_k}\right)^T + \sum_{k=0}^{n-1} \left(R_{t_{k+1}} - R_{t_k}\right)\left(I_{t_{k+1}} - I_{t_k}\right)^T,
\end{aligned}$$

where I, R are p dimension vectors.

Since

$$\begin{aligned}
& \sum_{k=0}^{n-1} \left(I_{t_{k+1}} - I_{t_k}\right)\left(I_{t_{k+1}} - I_{t_k}\right)^T \\
= \; & \sum_{k=0}^{n-1} \left(\left(\sum_{j=1}^{p} \int_{t_{k-1}}^{t_k} \sigma_{mj}(s)\, dB_s\right)\left(\sum_{j=1}^{p} \int_{t_{k-1}}^{t_k} \sigma_{nj}(s)\, dB_s\right)\right)_{p\times p} \\
= \; & \sum_{k=0}^{n-1} \left(\sum_{j=1}^{p} \int_{t_{k-1}}^{t_k} \sigma_{mj}(s)\, \sigma_{nj}(s)\, ds\right)_{p\times p} \\
= \; & \sum_{j=1}^{p} \left(\sum_{k=0}^{n-1} \int_{t_{k-1}}^{t_k} \sigma_{mj}(s)\, \sigma_{nj}(s)\, ds\right)_{p\times p} \\
= \; & \sum_{j=1}^{p} \left(\int_0^t \sigma_{mj}(s)\, \sigma_{nj}(s)\, ds\right)_{p\times p},
\end{aligned}$$

then as $\| \prod \| \to 0$, $\sum\limits_{k=0}^{n-1} \left(I_{t_{k+1}} - I_{t_k}\right) \left(I_{t_{k+1}} - I_{t_k}\right)^T \to \int_0^t \Sigma_s \Sigma_s^T ds$.

$$\text{Indeed,} \qquad \sum_{k=0}^{n-1} \left(R_{t_{k+1}} - R_{t_k}\right) \left(R_{t_{k+1}} - R_{t_k}\right)^T$$

$$= \left(\sum_{k=0}^{n-1} \left(R_i\left(t_{k+1}\right) - R_i\left(t_k\right)\right) \left(R_j\left(t_{k+1}\right) - R_j\left(t_k\right)\right) \right)_{p \times p},$$

as $\| \prod \| \to 0$, $\sum\limits_{k=0}^{n-1} \left(R_i\left(t_{k+1}\right) - R_i\left(t_k\right)\right) \left(R_j\left(t_{k+1}\right) - R_j\left(t_k\right)\right) \to 0$.

$$\text{Also,} \qquad \sum_{k=0}^{n-1} \left(I_{t_{k+1}} - I_{t_k}\right) \left(R_{t_{k+1}} - R_{t_k}\right)^T$$

$$= \left(\sum_{k=0}^{n-1} \left(I_i\left(t_{k+1}\right) - I_i\left(t_k\right)\right) \left(R_j\left(t_{k+1}\right) - R_j\left(t_k\right)\right) \right)_{p \times p},$$

similar to Proposition 3.2.1, as $\| \prod \| \to 0$,

$$\sum_{k=0}^{n-1} \left(I_i\left(t_{k+1}\right) - I_i\left(t_k\right)\right) \left(R_j\left(t_{k+1}\right) - R_j\left(t_k\right)\right) \to 0.$$

$$\text{And finally,} \qquad \sum_{k=0}^{n-1} \left(R_{t_{k+1}} - R_{t_k}\right) \left(I_{t_{k+1}} - I_{t_k}\right)^T$$

$$= \left(\sum_{k=0}^{n-1} \left(R_i\left(t_{k+1}\right) - R_i\left(t_k\right)\right) \left(I_j\left(t_{k+1}\right) - I_j\left(t_k\right)\right) \right)_{p \times p},$$

as $\| \prod \| \to 0$, $\sum\limits_{k=0}^{n-1} \left(R_i\left(t_{k+1}\right) - R_i\left(t_k\right)\right) \left(I_j\left(t_{k+1}\right) - I_j\left(t_k\right)\right) \to 0$.

Adding these four terms together and taking limits on both sides, we obtain the quadratic covariation

$$[\overrightarrow{X}, \overrightarrow{X}]_t = \int_0^t \Sigma_s \Sigma_s^T ds$$

for a group of assets. □

Remark: One shortcut to prove this proposition is to denote

$$Y_t = \lambda_1 X_1(t) + \lambda_2 X_2(t) + ... + \lambda_p X_p(t) = \begin{pmatrix} \lambda_1 & ... & \lambda_m \end{pmatrix} \begin{pmatrix} X_1(t) \\ ... \\ X_p(t) \end{pmatrix} = \lambda^T \overrightarrow{X_t},$$

then by Proposition 3.2.1,

$$[Y, Y]_t = \int_0^t \lambda^T \Sigma_s \Sigma_s^T \lambda ds = \lambda^T \int_0^t \Sigma_s \Sigma_s^T ds \lambda,$$

and

$$\lambda^T \overrightarrow{X_t} \overrightarrow{X_t}^T \lambda \to \lambda^T \int_0^t \Sigma_s \Sigma_s^T ds \lambda,$$

hence

$$[\overrightarrow{X}, \overrightarrow{X}]_t = \int_0^t \Sigma_s \Sigma_s^T ds.$$

When λ is constant, we obtain the desired conclusion. What if λ changes with time t, i.e, λ is function of time t, denoted by $\lambda(t)$? Then

$$Y_t = \lambda_1(t) X_1(t) + \lambda_2(t) X_2(t) + ... + \lambda_p(t) X_p(t).$$

PROPOSITION 3.3.2. *For a stochastic process $X_i(t)$, $i = 1, ..., p$, denote the vector $\overrightarrow{X_t} = (X_1(t), X_2(t), ..., X_p(t))'$ satisfying $d\overrightarrow{X_t} = \overrightarrow{\mu_t} dt + \Sigma_t d\overrightarrow{B_t}$, where $\overrightarrow{B_t}$ is a p-vector standard Brownian motion, $\overrightarrow{\mu_t}$ is a drift taking values in R^p, and Σ_t is a $p \times p$ matrix, while*

$$Y_t = \lambda_1(t) X_1(t) + \lambda_2(t) X_2(t) + ... + \lambda_p(t) X_p(t).$$

then

$$[Y, Y]_t = \int_0^t \lambda(s)^T \Sigma_s \Sigma_s^T \lambda(s) ds.$$

PROOF. Since

$$Y_t = \lambda_1(t) X_1(t) + \lambda_2(t) X_2(t) + ... + \lambda_p(t) X_p(t)$$

$$
\begin{aligned}
&= \begin{pmatrix} \lambda_1(t) & ... & \lambda_p(t) \end{pmatrix} \begin{pmatrix} X_1(t) \\ ... \\ X_p(t) \end{pmatrix} \\
&= \lambda(t)^T \overrightarrow{X_t},
\end{aligned}
$$

then by Proposition 3.2.1,

$$[Y, Y]_t = \int_0^t \lambda(s)^T \Sigma_s \Sigma_s^T \lambda(s)\, ds.$$

Since in this case, $\lambda(t)$ is function of time t, it can not be moved to the outside of integral. □

It can been easily seen that Proposition 3.3.1 is a special situation of Proposition 3.3.2 .

What if λ not only changes with time t, but also is a random variable of event ω? Then denote

$$Y_t = \lambda_1(t, \omega) X_1(t) + \lambda_2(t, \omega) X_2(t) + ... + \lambda_p(t, \omega) X_p(t).$$

PROPOSITION 3.3.3. *For a stochastic process* $X_i(t)$, $i = 1, ..., p$, *denote vector* $\overrightarrow{X_t} = (X_1(t), X_2(t), ..., X_p(t))'$ *satisfying* $d\overrightarrow{X_t} = \overrightarrow{\mu_t} dt + \Sigma_t d\overrightarrow{B_t}$, *where* $\overrightarrow{B_t}$ *is a p-vector standard Brownian motion,* $\overrightarrow{\mu_t}$ *is a drift taking values in* R^p, *and* Σ_t *is a* $p \times p$ *matrix, while*

$$Y_t = \lambda_1(t, \omega) X_1(t) + \lambda_2(t, \omega) X_2(t) + ... + \lambda_p(t, \omega) X_p(t),$$

then

$$[Y, Y]_t = \int_0^t \lambda(s, \omega)^T \Sigma_s \Sigma_s^T \lambda(s, \omega)\, ds.$$

PROOF. Similar to Proposition 3.3.2 □

CHAPTER 4

HIGHER ORDER APPROXIMATION FOR QUADRATIC VARIATION

4.1. The Difference between Quadratic Variation and Realized Volatility

As an example, we consider the Brownian motion B_t. By Proposition 3.2.1, we have already seen that

$$[B, B]_t = p\lim_{n\to\infty}\sum_{j=0}^{n-1}\left(B_{t_{j+1}} - B_{t_j}\right)^2 = t,$$

where $\{\,0 = t_0 \leq t_1 ... \leq t_n = t\,\}$ is a partition on $[\,0,\ t\,]$. This t is the quadratic variation of Brownian motion. In addition to convergence in probability, we can even prove that the realized volatility $\sum_{j=0}^{n-1}\left(B_{t_{j+1}} - B_{t_j}\right)^2$ converges to t in mean square (convergence in mean square is stronger than convergence in probability). One intuitive question is how fast this realized volatility approaches to the quadratic volatility. To answer this question, we investigate $E\left(\sum_{j=0}^{n-1}\left(B_{t_{j+1}} - B_{t_j}\right)^2 - t\right)^2$ first.

Lemma 4.1.1. (page 15, [4]) *For Brownian motion, we have*

$$E[B_t^{2k}] = \frac{2k!}{2^k \cdot k!}t^k, \ \ k \in \mathbb{N}.$$

Using Lemma 4.1.1, we can calculate

$$\begin{aligned}
& E\left(\sum_{j=0}^{n-1}\left(B_{t_{j+1}} - B_{t_j}\right)^2 - t\right)^2 \\
= \ \ & E\left(\left(\sum_{j=0}^{n-1}\left(B_{t_{j+1}} - B_{t_j}\right)^2\right)^2 + t^2 - 2t\sum_{j=0}^{n-1}\left(B_{t_{j+1}} - B_{t_j}\right)^2\right)
\end{aligned}$$

$$
\begin{aligned}
&= E\left(\sum_{j=0}^{n-1}\left(B_{t_{j+1}}-B_{t_j}\right)^4\right)+\sum_{i\neq j}^{n-1}E\left(\left(B_{t_{i+1}}-B_{t_j}\right)^2\left(B_{t_{j+1}}-B_{t_j}\right)^2\right) \\
&\quad +t^2-2t\sum_{j=0}^{n-1}E\left(B_{t_{j+1}}-B_{t_j}\right)^2 \\
&= \sum_{j=0}^{n-1}3\left(t_{j+1}-t_j\right)^2+\sum_{i\neq j}^{n-1}\left(t_{i+1}-t_i\right)\left(t_{j+1}-t_j\right)+t^2-2t^2 \\
&= \sum_{j=0}^{n-1}2\left(t_{j+1}-t_j\right)^2+\sum_{j=0}^{n-1}\left(t_{j+1}-t_j\right)^2+\sum_{i\neq j}^{n-1}\left(t_{i+1}-t_i\right)\left(t_{j+1}-t_j\right)+t^2-2t^2 \\
&= \sum_{j=0}^{n-1}2\left(t_{j+1}-t_j\right)^2 \\
&\leq 2t\max_{1\leq j\leq n-1}|t_{j+1}-t_j|\to 0,\ as\max_{1\leq j\leq n-1}|t_{j+1}-t_j|\to 0.
\end{aligned}
$$

If $t_{j+1}-t_j=\frac{t}{n}$ for all $0\leq j\leq n-1$, then

$$
\begin{aligned}
E\left(\sum_{j=0}^{n-1}\left(B_{t_{j+1}}-B_{t_j}\right)^2-t\right)^2 &= \sum_{j=0}^{n-1}2\left(t_{j+1}-t_j\right)^2 \\
&= \sum_{j=0}^{n-1}2\frac{t^2}{n^2} \\
&= \frac{2}{n}t^2,
\end{aligned}
$$

thus

$$
\sum_{j=0}^{n-1}\left(B_{t_{j+1}}-B_{t_j}\right)^2-t\approx\sqrt{\frac{2}{n}}t. \tag{4.1.1}
$$

This approximation is trivially true because when n is large, $\sqrt{\frac{2}{n}}t$ is close to 0. Therefore, we say that for Brownian motion B_t over the interval $[\,0,\,t\,]$, the difference between its quadratic variation and realized volatility can be estimated by $\sqrt{\frac{2}{n}}t$.

Can this estimation be improved? This is the central question of this chapter. In this dissertation, the largest contribution is that we improved the approximation by combining it with the realized bipower variation or the fourth order power variation, which will be discussed later.

The following discussion concerns the difference analysis between realized volatility and quadratic variation for the Ito process. To find the difference between realized volatility and quadratic variation, we have to generalize Ito's isometry since Ito's isometry only refers to expectation of square.

We start by generalizing Ito's isometry as follows.

PROPOSITION 4.1.1. *If* $f\,(t,\omega) \in \mathcal{V}(S,T)$*, then for n=1,2,...*

$$E\left[\left(\int_S^T f\,(s,\omega)\,dB_s(\omega)\right)^{2n}\right] = (2n-1)!!\left[\left(E\int_S^T f^2\,(s,\omega)\,ds\right)^{n}\right].$$

PROOF. We only need to consider the case of elementary functions, which is

$$f\,(t,\omega) = \sum_j e_j(\omega)\cdot\chi_{[t_j,t_{j+1})}(t),$$

where $\{0 \le t_1 \le t_2 \le \cdots \le t_n = t\}$ is a partition of $[0, t]$, $e_j(\omega)$ is the value on $[t_j, t_{j+1})$, and $\chi_{[t_j,t_{j+1})}(t)$ is an an indicator function or a characteristic function defined on $[t_j, t_{j+1})$ as

$$\chi_{[t_j,t_{j+1})}(t) = \begin{cases} 1 & \text{if } t \in [t_j, t_{j+1}) \\ 0 & \text{otherwise} \end{cases}.$$

Denote $\Delta B_j = B_{t_{j+1}} - B_{t_j}$. Then

$$\int_S^T f\,(s,\omega)\,dB_s(\omega) = \int_S^T \sum_j e_j(\omega)\chi_{[t_j,t_{j+1})}dB_s(\omega) = \sum_j e_j(\omega)\Delta B_j(\omega).$$

Direct computation yields

$$E\left(\int_S^T f\,(s,\omega)\,dB_s(\omega)\right)^{2n}$$

$$
\begin{aligned}
&= E\left(\sum_j e_j(\omega)\Delta B_j(\omega)\right)^{2n} \\
&= E\sum_{j_1+j_2+\cdots+j_s=2n}\frac{(2n)!}{(j_1)!(j_2)!\cdots(j_s)!}\left(e_1\Delta B_1\right)^{j_1}\left(e_2\Delta B_2\right)^{j_2}\cdots\left(e_s\Delta B_s\right)^{j_s} \\
&= \sum_{j_1+j_2+\cdots+j_s=2n}\frac{(2n)!}{(j_1)!(j_2)!\cdots(j_s)!}E\left(e_1\Delta B_1\right)^{j_1}E\left(e_2\Delta B_2\right)^{j_2}\cdots E\left(e_s\Delta B_s\right)^{j_s}
\end{aligned}
$$

Since $E\left(\Delta B_{j_s}\right)^{j_s}=0$ (page 56, [5]), if the $j_s's$ are odd, therefore, when the $j_s's$ are even, there exist $i_s's$ such that $j_s=2i_s$. Then it follows that

$$
\begin{aligned}
&= \sum_{i_1+i_2+\cdots+i_s=n}\frac{(2n)!}{(2i_1)!(2i_2)!\cdots(2i_s)!}E\left(e_1\Delta B_1\right)^{2i_1}E\left(e_2\Delta B_2\right)^{2i_2}\cdots E\left(e_s\Delta B_s\right)^{2i_s} \\
&\quad \text{(by Lemma 4.1.1)} \\
&= \sum_{i_1+i_2+\cdots+i_s=n}\frac{(2n)!}{(2i_1)!(2i_2)!\cdots(2i_s)!}E\left(e_1^{2i_1}\right)\left(\Delta t_1\right)^{i_1}\frac{(2i_1)!}{2^{i_1}i_1!}E\left(e_2^{2i_2}\right)\left(\Delta t_2\right)^{i_2}\frac{(2i_2)!}{2^{i_2}i_2!} \\
&\quad \cdots E\left(e_s^{2i_s}\right)\left(\Delta t_s\right)^{i_s}\frac{(2i_s)!}{2^{i_s}i_s!} \\
&= \sum_{i_1+i_2+\cdots+i_s=n}\frac{(2n)!}{2^n n!}\frac{n!}{(i_1)!(i_2)!\cdots(i_s)!}\left(Ee_1^2\Delta t_1\right)^{i_1}\left(Ee_2^2\Delta t_2\right)^{i_2}\cdots\left(Ee_s^2\Delta t_s\right)^{i_s} \\
&= (2n-1)!![(Ee_1^2\Delta t_1+\cdots+Ee_s^2\Delta t_s)^n] \\
&= (2n-1)!!\left[E\int_S^T f^2(s,\omega)\,ds\right]^n.
\end{aligned}
$$

The proof is complete. □

Proposition 4.1.1 is one of our main contributions to this dissertation. It plays an important role for the following research. Most calculations are based on this proposition. By Proposition 4.1.1, we can readily obtain the following property which tells us that the moment generation function is suitable for dealing with stochastic processes.

Proposition 4.1.2. *Suppose* $\sigma(t,\omega) \in \mathcal{V}(S,T)$, *then*

$$E\left[\exp\left(t\int_S^T \sigma(s,\omega)\,dB_s\right)\right] = \exp\left(\frac{t^2}{2}\left(E\int_S^T \sigma^2(s,\omega)\,ds\right)\right).$$

Proof. The function $\exp\left(t\int_S^T \sigma(s,\omega)\,dB_s\right)$ can be represented by the following MacLaurin's series

$$\exp\left(t\int_S^T \sigma(s,\omega)\,dB_s\right) = \sum_{n=0}^{\infty}\frac{t^n}{n!}\left(\int_S^T \sigma(s,\omega)\,dB_s\right)^n.$$

Taking expectations on both sides, we get

$$E\left[\exp\left(t\int_S^T \sigma(s,\omega)\,dB_s\right)\right] = \sum_{n=0}^{\infty}\frac{t^n}{n!}E\left(\int_S^T \sigma(s,\omega)\,dB_s\right)^n.$$

Since terms with odd exponent are 0, then by Proposition 4.1.1, we have

$$\begin{aligned}
E\left[\exp\left(t\int_S^T \sigma(s,\omega)\,dB_s\right)\right] &= \sum_{n=0}^{\infty}\frac{t^{2n}}{(2n)!}E\left(\int_S^T \sigma(s,\omega)\,dB_s\right)^{2n} \\
&= \sum_{n=0}^{\infty}\frac{t^{2n}}{(2n)!}(2n-1)!!\left(E\int_S^T \sigma^2(s,\omega)\,ds\right)^n \\
&= \sum_{n=0}^{\infty}\frac{t^{2n}}{(2n)!}\frac{(2n)!}{2^n n!}\left(E\int_S^T \sigma^2(s,\omega)\,ds\right)^n \\
&= \exp\left(\frac{t^2}{2}\left(E\int_S^T \sigma^2(s,\omega)\,ds\right)\right).
\end{aligned}$$

The proof is complete. □

When σ is constant, this formula becomes

$$E\left[\exp\left(t\sigma B_x\right)\right] = \exp\left(\frac{t^2\sigma^2 x}{2}\right),$$

which is the moment generation function of the stochastic process σB_x.

From Proposition 4.1.1, when $n = 1$, we can obtain Ito's isometry. When $n = 2$ or 3, and $\sigma(t, \omega) \in \mathcal{V}(S, T)$, then we get the following two important identities.

$$E\left[\left(\int_S^T \sigma(s,\omega)\, dB_s\right)^4\right] = 3\left[E\int_S^T \sigma^2(s,\omega)\, dt\right]^2. \tag{4.1.2}$$

$$E\left[\left(\int_S^T \sigma(s,\omega)\, dB_s\right)^6\right] = 15\left[E\int_S^T \sigma^2(s,\omega)\, ds\right]^3. \tag{4.1.3}$$

As mentioned before, Proposition 4.1.1 is fundamental. E.g., we can use it to find the difference between the realized volatility and the quadratic variation of the Ito process.

By Proposition 3.2.1, we know the quadratic variation to Ito process

$$X_t = \int_0^t \sigma(s,\omega)\, dB_s$$

is

$$[X, X]_t = \int_0^t \sigma^2(s,\omega)\, ds.$$

Suppose $\{0 = t_0 \le t_1 \le ... \le t_n = t\}$ is a partition of the interval $[0,\ t]$, then the difference between the realized volatility and the quadratic variation for the Ito process is

$$\begin{aligned}
& E\left(\sum_{j=0}^{n-1}\left(\int_{t_j}^{t_{j+1}} \sigma(s,\omega)\, dB_s\right)^2 - E\int_0^t \sigma^2(s,\omega)\, ds\right)^2 \\
= \; & E\left(\sum_{j=0}^{n-1}\left(\int_{t_j}^{t_{j+1}} \sigma(s,\omega)\, dB_s\right)^2\right)^2 + \left(E\int_0^t \sigma^2(s,\omega)\, ds\right)^2 \\
& -2E\int_0^t \sigma^2(s,\omega)\, ds \sum_{j=0}^{n-1} E\left(\int_{t_j}^{t_{j+1}} \sigma(s,\omega)\, dB_s\right)^2 \\
= \; & E\sum_{j=0}^{n-1}\left(\int_{t_j}^{t_{j+1}} \sigma(s,\omega)\, dB_s\right)^4 + E\sum_{i\neq j}\left(\int_{t_i}^{t_{i+1}} \sigma(s,\omega)\, dB_s\right)^2\left(\int_{t_j}^{t_{j+1}} \sigma(s,\omega)\, dB_s\right)^2
\end{aligned}$$

$$+\left(E\int_0^t \sigma^2(s,\omega)\,ds\right)^2 - 2E\int_0^t \sigma^2(s,\omega)\,ds\sum_{j=0}^{n-1}E\left(\int_{t_j}^{t_{j+1}}\sigma(s,\omega)\,dB_s\right)^2$$

$$= \sum_{j=0}^{n-1}E\left(\int_{t_j}^{t_{j+1}}\sigma(s,\omega)\,dB_s\right)^4 + \sum_{i\neq j}^{n-1}E\left(\int_{t_i}^{t_{i+1}}\sigma(s,\omega)\,dB_s\right)^2\left(\int_{t_j}^{t_{j+1}}\sigma(s,\omega)\,dB_s\right)^2$$

$$+\left(E\int_0^t \sigma^2(s,\omega)\,ds\right)^2 - 2E\int_0^t \sigma^2(s,\omega)\,ds\sum_{j=0}^{n-1}E\left(\int_{t_j}^{t_{j+1}}\sigma(s,\omega)\,dB_s\right)^2$$

By Equation (4.1.2), it equals

$$\sum_{j=0}^{n-1}3\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)^2 + \sum_{i\neq j}^{n-1}\left(E\int_{t_i}^{t_{i+1}}\sigma^2(s,\omega)\,ds\right)\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)$$

$$+\left(E\int_0^t \sigma^2(s,\omega)\,ds\right)^2 - 2E\int_0^t \sigma^2(s,\omega)\,dsE\int_0^t \sigma^2(s,\omega)\,ds$$

$$= \sum_{j=0}^{n-1}3\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\right)^2 + \sum_{i\neq j}^{n-1}\left(E\int_{t_i}^{t_{i+1}}\sigma^2(s,\omega)\,ds\right)\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)$$

$$-\left(E\int_0^t \sigma^2(s,\omega)\,ds\right)^2$$

$$= \sum_{j=0}^{n-1}2\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)^2 + \left(\sum_{j=0}^{n-1}E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)^2 - \left(E\int_0^t \sigma^2(s,\omega)\,ds\right)^2$$

$$= 2\sum_{j=0}^{n-1}\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)^2.$$

In conclusion, we have the following result.

Proposition 4.1.3. *Suppose* $X_t = \int_0^t \sigma(s,\omega)\,dB_s$, *and* $\sigma(t,\omega) \in \mathcal{V}(0,t)$, *then*

$$E\left(\sum_{j=0}^{n-1}\left(X_{t_{j+1}} - X_{t_j}\right)^2 - E\int_0^t \sigma^2(s,\omega)\,ds\right)^2 = 2\sum_{j=0}^{n-1}\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)^2.$$

By Proposition 4.1.3, we can say

$$\sum_{j=0}^{n-1}\left(X_{t_{j+1}} - X_{t_j}\right)^2 - E\int_0^t \sigma^2(s,\omega)\,ds \approx \sqrt{2\sum_{j=0}^{n-1}\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)^2}.$$

This approximation is also trivially true because when n is large and $\sigma(t,\omega)$ is bounded,

$$\sqrt{2\sum_{j=0}^{n-1}\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)^2}$$

is close to $C\sqrt{\frac{2}{n}}t$, where C is a positive constant. Therefore, we say that for the Ito process I_t over the interval $[0,\, t]$, the difference between its quadratic variation and the realized volatility can be estimated by

$$\sqrt{2\sum_{j=0}^{n-1}\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)^2}.$$

In particular, if $\sigma(t,\omega)=1$, and $t_{j+1}-t_j=\frac{t}{n}$, we obtain

$$\sum_{j=0}^{n-1}\left(B_{t_{j+1}}-B_{t_j}\right)^2-t\approx\sqrt{\frac{2}{n}}t.$$

As one kind of extension to Proposition 4.1.3, it is also true for the m-th order power variation.

Proposition 4.1.4. *Suppose* $X_t=\int_0^t\sigma(s,\omega)\,dB_s$, $t_{j+1}-t_j=\frac{t}{n}$ *for all* j, *and* $\sigma(t,\omega)\in\mathcal{V}(0,t)$ *is bounded, then*

$$E\left[\frac{1}{(2m-1)!!}\left(\sum_{j=0}^{n-1}\left(X_{t_{j+1}}-X_{t_j}\right)^{2m}\right)-\sum_{j=0}^{n-1}\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)^m\right]^2$$
$$=\left[\left(\frac{(4m-1)!!}{((2m-1)!!)^2}-1\right)\sum_{j=0}^{n-1}\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)^{2m}\right].$$

Proof. To prove the proposition, we have to calculate

$$E\left[\frac{1}{(2m-1)!!}\left(\sum_{j=0}^{n-1}\left(X_{t_{j+1}}-X_{t_j}\right)^{2m}\right)-\sum_{j=0}^{n-1}\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)^m\right]^2$$
$$=E\left[\frac{1}{(2m-1)!!}\sum_{j=0}^{n-1}\left(X_{t_{j+1}}-X_{t_j}\right)^{2m}\right]^2+\left[\sum_{j=0}^{n-1}\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)^m\right]^2$$

$$-2\frac{1}{(2m-1)!!}E\left[\sum_{j=0}^{n-1}\left(X_{t_{j+1}}-X_{t_j}\right)^{2m}\right]\sum_{j=0}^{n-1}\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)^m$$

By Proposition 4.1.1, it equals

$$\begin{aligned}
&E\left[\left(\frac{1}{(2m-1)!!}\right)^2\left(\sum_{j=0}^{n-1}\left(X_{t_{j+1}}-X_{t_j}\right)^{2m}\right)^2\right]+\left(\sum_{j=0}^{n-1}\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)^m\right)^2\\
&-\frac{2}{(2m-1)!!}\cdot(2m-1)!!\sum_{j=0}^{n-1}\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)^m\sum_{j=0}^{n-1}\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)^m\\
=\;&E\left[\left(\frac{1}{(2m-1)!!}\right)^2\left(\sum_{j=0}^{n-1}\left(X_{t_{j+1}}-X_{t_j}\right)^{2m}\right)^2\right]-\left[\sum_{j=0}^{n-1}\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)^m\right]^2\\
=\;&E\left(\frac{1}{(2m-1)!!}\right)^2\left(\sum_{j=0}^{n-1}\left(X_{t_{j+1}}-X_{t_j}\right)^{4m}+\sum_{j\neq i}(X_{t_{j+1}}-X_{t_j})^{2m}\left(X_{t_{i+1}}-X_{t_i}\right)^{2m}\right)\\
&-\left[\sum_{j=0}^{n-1}\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)^m\right]^2
\end{aligned}$$

Using Proposition 4.1.1 again, it equals

$$\begin{aligned}
&\left[\left(\frac{1}{(2m-1)!!}\right)^2(4m-1)!!\sum_{j=0}^{n-1}\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)^{2m}\right]\\
&+\left[\sum_{j\neq i}\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)^m\left(E\int_{t_i}^{t_{i+1}}\sigma^2(s,\omega)\,ds\right)^m\right]\\
&-\left[\sum_{j=0}^{n-1}\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)^m\right]^2\\
=\;&\left[\frac{(4m-1)!!}{((2m-1)!!)^2}\sum_{j=0}^{n-1}\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)^{2m}\right]\\
&+\left[\sum_{j\neq i}\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)^m\left(E\int_{t_i}^{t_{i+1}}\sigma^2(s,\omega)\,ds\right)^m\right]
\end{aligned}$$

$$
\begin{aligned}
&\quad - \left[\sum_{j=0}^{n-1} \left(E \int_{t_j}^{t_{j+1}} \sigma^2 (s, \omega) \, ds \right)^m \right]^2 \\
&= \left[\left(\frac{(4m-1)!!}{((2m-1)!!)^2} - 1 \right) \sum_{j=0}^{n-1} \left(E \int_{t_j}^{t_{j+1}} \sigma^2 (s, \omega) \, ds \right)^{2m} \right] \\
&\quad + \sum_{j=0}^{n-1} \left(E \int_{t_j}^{t_{j+1}} \sigma^2 (s, \omega) \, ds \right)^{2m} + \sum_{j \neq i} \left(E \int_{t_j}^{t_{j+1}} \sigma^2 (s, \omega) \, ds \right)^m \\
&\quad \times \left(E \int_{t_i}^{t_{i+1}} \sigma^2 (s, \omega) \, ds \right)^m - \left[\sum_{j=0}^{n-1} \left(E \int_{t_j}^{t_{j+1}} \sigma^2 (s, \omega) \, ds \right)^m \right]^2 \\
&= \left[\left(\frac{(4m-1)!!}{((2m-1)!!)^2} - 1 \right) \sum_{j=0}^{n-1} \left(E \int_{t_j}^{t_{j+1}} \sigma^2 (s, \omega) \, ds \right)^{2m} \right].
\end{aligned}
$$

Hence, if $\sigma^2(t, \omega)$ is bounded, then

$$
\sum_{j=0}^{n-1} \left(E \int_{t_j}^{t_{j+1}} \sigma^2 (s, \omega) \, ds \right)^{2m}
$$

approaches 0. This shows that

$$
\frac{1}{(2m-1)!!} \left(\sum_{j=0}^{n-1} \left(X_{t_{j+1}} - X_{t_j} \right)^{2m} \right)
$$

converges to

$$
\sum_{j=0}^{n-1} \left(E \int_{t_j}^{t_{j+1}} \sigma^2 (s, \omega) \, ds \right)^m
$$

in mean square, absolutely, and it converges in probability. □

More generally, suppose we introduce a weight function $\rho(t)$ for the Ito process. Similar to Proposition 4.1.4, we obtain the following proposition which includes a weight function.

Proposition 4.1.5. *Suppose* $X_t = \int_0^t \sigma(s,\omega)\,dB_s$, $t_{j+1} - t_j = \frac{t}{n}$ *for all* j, $\rho(t)$ *is a weight function, and* $\sigma(t,\omega) \in \mathcal{V}(0,t)$ *is bounded, then*

$$E\left[\frac{1}{(2m-1)!!}\left(\sum_{j=0}^{n-1}\rho(t_j)\left(X_{t_{j+1}} - X_{t_j}\right)^{2m}\right) - \sum_{j=0}^{n-1}\left(E\int_{t_j}^{t_{j+1}}\rho(s)\sigma^2(s,\omega)\,ds\right)^m\right]^2$$

$$= \left[\left(\frac{(4m-1)!!}{((2m-1)!!)^2} - 1\right)\sum_{j=0}^{n-1}\rho^2(t_j)\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)^{2m}\right].$$

Proof. Since the proof is very similar to the proof of Proposition 4.1.4, we will skip some calculations. To prove convergence in probability, we can prove convergence in mean square first. To prove convergence in mean square, we have to calculate

$$E\left[\frac{1}{(2m-1)!!}\left(\sum_{j=0}^{n-1}\rho(t_j)\left(X_{t_{j+1}} - X_{t_j}\right)^{2m}\right) - \sum_{j=0}^{n-1}\left(E\int_{t_j}^{t_{j+1}}\rho(s)\sigma^2(s,\omega)\,ds\right)^m\right]^2$$

$$= E\left[\left(\frac{1}{(2m-1)!!}\right)^2\left(\sum_{j=0}^{n-1}\rho(t_j)\left(X_{t_{j+1}} - X_{t_j}\right)^{2m}\right)^2\right]$$

$$- \left[\sum_{j=0}^{n-1}\left(E\int_{t_j}^{t_{j+1}}\rho(s)\sigma^2(s,\omega)\,ds\right)^m\right]^2$$

Using Proposition 4.1.1, it equals

$$\left[\left(\frac{1}{(2m-1)!!}\right)^2(4m-1)!!\sum_{j=0}^{n-1}\rho^2(t_j)\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)^{2m}\right]$$

$$+ \left[\sum_{j\neq i}\rho(t_j)\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)^m\rho(t_i)\left(E\int_{t_i}^{t_{i+1}}\sigma^2(s,\omega)\,ds\right)^m\right]$$

$$- \left[\sum_{j=0}^{n-1}\left(E\int_{t_j}^{t_{j+1}}\rho(s)\sigma^2(s,\omega)\,ds\right)^m\right]^2$$

$$= \left[\left(\frac{(4m-1)!!}{((2m-1)!!)^2} - 1\right)\sum_{j=0}^{n-1}\rho^2(t_j)\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)^{2m}\right]$$

$$+\sum_{j=0}^{n-1}\rho^2(t_j)\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)^{2m}+\sum_{j\neq i}\rho(t_j)\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)^{m}$$

$$\times\rho(t_i)\left(E\int_{t_i}^{t_{i+1}}\sigma^2(s,\omega)\,ds\right)^{m}-\left[\sum_{j=0}^{n-1}\left(E\int_{t_j}^{t_{j+1}}\rho(s)\sigma^2(s,\omega)\,ds\right)^{m}\right]^2$$

$$=\quad\left[\left(\frac{(4m-1)!!}{((2m-1)!!)^2}-1\right)\sum_{j=0}^{n-1}\rho^2(t_j)\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)^{2m}\right].$$

Hence, if $\sigma^2(t,\omega)$ is bounded, then

$$\sum_{j=0}^{n-1}\rho^2(t_j)\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)^{2m}$$

approaches 0. This shows that

$$\frac{1}{(2m-1)!!}\left(\sum_{j=0}^{n-1}\rho^2(t_j)\left(X_{t_{j+1}}-X_{t_j}\right)^{2m}\right)$$

converges to

$$\sum_{j=0}^{n-1}\left(E\int_{t_j}^{t_{j+1}}\rho(s)\sigma^2(s,\omega)\,ds\right)^{m}$$

in mean square, absolutely, and it converges in probability. □

In particular, if the weight function $\rho(t)\equiv 1$, then it becomes Proposition 4.1.4.

Both Proposition 4.1.3 and Proposition 4.1.5 show the difference between realized volatility and quadratic variation. Meanwhile, both of them show that the realized volatility converges to quadratic variation in mean square, and of course, converges to the quadratic variation in probability. From these two propositions, we also see that when n is not so large, the difference can be quite large.

As one kind of application, we apply Proposition 4.1.3 to observed data in the financial market. As we know, observed data is very noisy, and therefore we assume the observed data Y_t satisfies

$$Y_t = X_t + \varepsilon_t,$$

where X_t is actual price and ε_t is a noise term. The realized volatility to Y_t is

$$
\begin{aligned}
& \sum_{j=0}^{n-1} \left(Y_{t_{j+1}} - Y_{t_j}\right)^2 \\
= & \sum_{j=0}^{n-1} \left(X_{t_{j+1}} - X_{t_j}\right)^2 + 2\sum_{j=0}^{n-1} \left(X_{t_{j+1}} - X_{t_j}\right)\left(\epsilon_{t_{j+1}} - \epsilon_{t_j}\right) + \sum_{j=0}^{n-1} \left(\epsilon_{t_{j+1}} - \epsilon_{t_j}\right)^2 .
\end{aligned}
$$

If we assume the noise item satisfies a standard normal $\mathcal{N}(0,1)$, and is independent of X_t, then we see that

$$
\begin{aligned}
E\left(\sum_{j=0}^{n-1} \left(\epsilon_{t_{j+1}} - \epsilon_{t_j}\right)^2\right) &= n, \\
E\left(\sum_{j=0}^{n-1} \left(X_{t_{j+1}} - X_{t_j}\right)\left(\epsilon_{t_{j+1}} - \epsilon_{t_j}\right)\right) &= 0, \\
E\left(\sum_{j=0}^{n-1} \left(X_{t_{j+1}} - X_{t_j}\right)^2\right) &= E\int_0^t \sigma^2(s,\omega)\, ds,
\end{aligned}
$$

thus

$$
E\sum_{j=0}^{n-1} \left(Y_{t_{j+1}} - Y_{t_j}\right)^2 = E\int_0^t \sigma^2(s,\omega)\, ds + n,
$$

when n approaches ∞. This realized volatility does not converge to the quadratic variation.

The next several sections will focus improvements to the approximation.

4.2. An Application of the Second-order Difference to the Ito Process

From numerical analysis, we know that

$$
f''(x) = \frac{1}{h^2}[f(x-h) - 2f(x) + f(x+h)] + O(h^2). \tag{4.2.1}
$$

If we rewrite 4.2.1, then we obtain

$$
\frac{1}{h^2}[f(x-h) - 2f(x) + f(x+h)] - f''(x) = O(h^2).
$$

We see that the left side converges at the rate h^2. Starting from this idea, we are curious about stochastic processes. Does a similar convergence rate exist for the Ito process?

PROPOSITION 4.2.1. *Suppose* $X_t = \int_0^t \sigma(s,\omega)\, dB_s$, $\delta = \frac{t}{n}$, $t_j = \frac{jt}{n}, j = 0, 1, 2, ..., n$, *and* $\sigma(t,\omega) \in \mathcal{V}(0,t)$, *then*

$$E\left(\sum_{j=1}^{n-1}\left(X_{t_{j+1}} + X_{t_{j-1}} - 2X_{t_j}\right)^2 - \left(E\int_{t_j}^{t_{j+1}} \sigma^2(s,\omega)\, ds + E\int_{t_{j-1}}^{t_j} \sigma^2(s,\omega)\, ds\right)^2\right)^2$$

$$= \sum_{j=1}^{n-1} 2\left(E\int_{t_j}^{t_{j+1}} \sigma^2(s,\omega)\, ds + E\int_{t_{j-1}}^{t_j} \sigma^2(s,\omega)\, ds\right)^2$$

PROOF. If the stochastic process $X_t = \int_0^t \sigma(s,\omega)\, dB_s$, then $X_{t_{j+1}} + X_{t_{j-1}} - 2X_{t_j}$ can be written as $\int_{t_j}^{t_{j+1}} \sigma(s,\omega)\, dB_s - \int_{t_{j-1}}^{t_j} \sigma(s,\omega)\, dB_s$. So

$$\begin{aligned} & E\left(\sum_{j=1}^{n-1}\left(X_{t_{j+1}} + X_{t_{j-1}} - 2X_{t_j}\right)^2 - \left(E\int_{t_j}^{t_{j+1}} \sigma^2(s,\omega)\, ds + E\int_{t_{j-1}}^{t_j} \sigma^2(s,\omega)\, ds\right)^2\right)^2 \\ = \quad & Ea^2 + b^2 - 2Eab \end{aligned}$$

where

$$\begin{aligned} a &= \sum_{j=1}^{n-1}\left(\int_{t_j}^{t_{j+1}} \sigma(s,\omega)\, dB_s, - \int_{t_{j-1}}^{t_j} \sigma(s,\omega)\, dB_s\right)^2, \\ b &= \sum_{j=1}^{n-1}\left(E\int_{t_j}^{t_{j+1}} \sigma^2(s,\omega)\, ds + E\int_{t_{j-1}}^{t_j} \sigma^2(s,\omega)\, ds\right) \approx 2E\int_0^t \sigma^2(s,\omega)\, ds, \end{aligned}$$

and b is a function of t. Therefore, we just have to find Ea^2 and Eab.

$$\text{Now } a^2 = \left(\sum_{j=1}^{n-1}\left(\int_{t_j}^{t_{j+1}} \sigma(s,\omega)\, dB_s - \int_{t_{j-1}}^{t_j} \sigma(s,\omega)\, dB_s\right)^2\right)^2$$

$$
\begin{aligned}
&= \sum_{j=1}^{n-1}\left(\int_{t_j}^{t_{j+1}} \sigma(s,\omega)\,dB_s - \int_{t_{j-1}}^{t_j} \sigma(s,\omega)\,dB_s\right)^4 \\
&\quad + \sum_{j_1\neq j_2}^{n-1}\left(\int_{t_{j_1}}^{t_{j_1+1}} \sigma(s,\omega)\,dB_s - \int_{t_{j_1-1}}^{t_{j_1}} \sigma(s,\omega)\,dB_s\right)^2 \\
&\quad \times \left(\int_{t_{j_2}}^{t_{j_2+1}} \sigma(s,\omega)\,dB_s - \int_{t_{j_2-1}}^{t_{j_2}} \sigma(s,\omega)\,dB_s\right)^2 \\
&= \sum_{j=1}^{n-1} X + \sum_{j_1\neq j_2} YZ,
\end{aligned}
$$

where X denotes

$$
\left(\int_{t_j}^{t_{j+1}} \sigma(s,\omega)\,dB_s - \int_{t_{j-1}}^{t_j} \sigma(s,\omega)\,dB_s\right)^4,
$$

Y denotes

$$
\left(\int_{t_{j_1}}^{t_{j_1+1}} \sigma(s,\omega)\,dB_s - \int_{t_{j_1-1}}^{t_{j_1}} \sigma(s,\omega)\,dB_s\right)^2,
$$

and Z denotes

$$
\left(\int_{t_{j_2}}^{t_{j_2+1}} \sigma(s,\omega)\,dB_s - \int_{t_{j_2-1}}^{t_{j_2}} \sigma(s,\omega)\,dB_s\right)^2.
$$

It's easy to find $E(YZ)$, that is

$$
\begin{aligned}
E(YZ) &= E\left(\int_{t_{j_1}}^{t_{j_1+1}} \sigma(s,\omega)\,dB_s - \int_{t_{j_1-1}}^{t_{j_1}} \sigma(s,\omega)\,dB_s\right)^2 \\
&\quad \times \left(\int_{t_{j_2}}^{t_{j_2+1}} \sigma(s,\omega)\,dB_s - \int_{t_{j_2-1}}^{t_{j_2}} \sigma(s,\omega)\,dB_s\right)^2 \\
&= \left(E\int_{t_{j_1}}^{t_{j_1+1}} \sigma^2(s,\omega)\,ds + E\int_{t_{j_1-1}}^{t_{j_1}} \sigma^2(s,\omega)\,ds\right) \\
&\quad \times \left(E\int_{t_{j_2}}^{t_{j_2+1}} \sigma^2(s,\omega)\,ds + E\int_{t_{j_2-1}}^{t_{j_2}} \sigma^2(s,\omega)\,ds\right).
\end{aligned}
$$

Meanwhile, using Equation (4.1.2), we get

$$
\begin{aligned}
Ea^2 &= \sum_{j=1}^{n-1} EX + \sum_{j_1 \neq j_2} EYZ \\
&= \sum_{j=1}^{n-1} 3\left(E\int_{t_j}^{t_{j+1}} \sigma^2(s,\omega)\,ds\right)^2 + 6E\int_{t_j}^{t_{j+1}} \sigma^2(s,\omega)\,dsE\int_{t_{j-1}}^{t_j} \sigma^2(s,\omega)\,ds \\
&\quad +3\left(E\int_{t_{j-1}}^{t_j} \sigma^2(s,\omega)\,ds\right)^2 + \sum_{j_1 \neq j_2}^{n-1} \left(E\int_{t_{j_1}}^{t_{j_1+1}} \sigma^2(s,\omega)\,ds + E\int_{t_{j_1-1}}^{t_{j_1}} \sigma^2(s,\omega)\,ds\right) \\
&\quad \times \left(E\int_{t_{j_2}}^{t_{j_2+1}} \sigma^2(s,\omega)\,ds + E\int_{t_{j_2-1}}^{t_{j_2}} \sigma^2(s,\omega)\,ds\right) \\
&= \sum_{j=1}^{n-1} 3\left(E\int_{t_j}^{t_{j+1}} \sigma^2(s,\omega)\,ds + E\int_{t_{j-1}}^{t_j} \sigma^2(s,\omega)\,ds\right)^2 \\
&\quad + \sum_{j_1 \neq j_2}^{n-1} \left(E\int_{t_{j_1}}^{t_{j_1+1}} \sigma^2(s,\omega)\,ds + E\int_{t_{j_1-1}}^{t_{j_1}} \sigma^2(s,\omega)\,ds\right) \\
&\quad \times \left(E\int_{t_{j_2}}^{t_{j_2+1}} \sigma^2(s,\omega)\,ds + E\int_{t_{j_2-1}}^{t_{j_2}} \sigma^2(s,\omega)\,ds\right) \\
&= \sum_{j=1}^{n-1} 2\left(E\int_{t_j}^{t_{j+1}} \sigma^2(s,\omega)\,ds + E\int_{t_{j-1}}^{t_j} \sigma^2(s,\omega)\,ds\right)^2 \\
&\quad + \left(\sum_{j=0}^{n-1} \left(E\int_{t_j}^{t_{j+1}} \sigma^2(s,\omega)\,ds + E\int_{t_{j-1}}^{t_j} \sigma^2(s,\omega)\,ds\right)\right)^2 \\
&= \sum_{j=1}^{n-1} 2\left(E\int_{t_j}^{t_{j+1}} \sigma^2(s,\omega)\,ds + E\int_{t_{j-1}}^{t_j} \sigma^2(s,\omega)\,ds\right)^2 + b^2
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
Eab &= bE\left(\sum_{j=1}^{n-1} \left(\int_{t_j}^{t_{j+1}} \sigma(s,\omega)\,dB_s\right)^2 + \left(\int_{t_{j-1}}^{t_j} \sigma(s,\omega)\,dB_s\right)^2\right) \\
&\quad -2\left(E\int_{t_j}^{t_{j+1}} \sigma(s,\omega)\,dB_s\right)\left(E\int_{t_{j-1}}^{t_j} \sigma(s,\omega)\,dB_s\right)
\end{aligned}
$$

$$
\begin{aligned}
&= b\left(\sum_{j=1}^{n-1} E\int_{t_j}^{t_{j+1}} \sigma^2(s,\omega)\,ds + E\int_{t_{j-1}}^{t_j} \sigma^2(s,\omega)\,ds\right) \\
&= b^2.
\end{aligned}
$$

In conclusion , we find that the difference between second order difference and the corresponding quadratic variation is

$$
\sum_{j=1}^{n-1} 2\left(E\int_{t_j}^{t_{j+1}} \sigma^2(s,\omega)\,ds + E\int_{t_{j-1}}^{t_j} \sigma^2(s,\omega)\,ds\right)^2 .
$$

□

This proposition tells us that a second difference stochastic process can't improve the rate of convergence. The rate of $\sum_{j=1}^{n-1} 2\left(E\int_{t_j}^{t_{j+1}} \sigma^2(s,\omega)\,ds + E\int_{t_{j-1}}^{t_j} \sigma^2(s,\omega)\,ds\right)^2$ is still n^{-1}.

4.3. The Bipower Variation Process and Fourth Order Estimators

In this section, we focus on bipower variation. Indeed we find that if we combine it with power variation, then we can improve the speed that the realized volatility approaches to quadratic variation. Realized bipower variation is an extension of realized volatility. Realized bipower variation is a cross term estimator. It is defined by:

$$
\sum_{j=1}^{n-1} \left(X_{t_j} - X_{t_{j-1}}\right)^2 \left(X_{t_{j+1}} - X_{t_j}\right)^2 .
$$

By reference [15], we know

$$
\left(\frac{n}{t}\right)\left(\sum_{j=1}^{n-1} \left(X_{t_{j+1}} - X_{t_j}\right)^2 \left(X_{t_j} - X_{t_{j-1}}\right)^2\right) \xrightarrow{\mathbf{P}} E\int_0^t \sigma^4(s,\omega)\,ds. \tag{4.3.1}
$$

By Proposition 4.1.3, we have the following result:

$$
E\left(\sum_{j=0}^{n-1} \left(X_{t_{j+1}} - X_{t_j}\right)^2 - E\int_0^t \sigma^2(s,\omega)\,ds\right)^2 \;=\; 2\sum_{j=0}^{n-1}\left(E\int_{t_j}^{t_{j+1}} \sigma^2(s,\omega)\,ds\right)^2
$$

$$\approx \quad 2\delta E \int_0^t \sigma^4(s,\omega)\,ds,$$

in other words, $2\delta E \int_0^t \sigma^4(s,\omega)\,ds$ can be approximated by

$$\left(\sum_{j=0}^{n-1} \left(X_{t_{j+1}} - X_{t_j} \right)^2 - E \int_0^t \sigma^2(s,\omega)\,ds \right)^2 .$$

One intuitive idea is to combine these two approximations. In (4.3.1), we replace $2\delta E \int_0^t \sigma^4(s,\omega)\,ds$ by

$$\left(\sum_{j=0}^{n-1} \left(X_{t_{j+1}} - X_{t_j} \right)^2 - E \int_0^t \sigma^2(s,\omega)\,ds \right)^2 .$$

To find a better approximation for the quadratic variation, we begin by investigating $E\left(2\sum_{j=1}^{n-1} \left(X_{t_{j+1}} - X_{t_j} \right)^2 \left(X_{t_j} - X_{t_{j-1}} \right)^2 - \left(\sum_{j=0}^{n-1} \left(X_{t_{j+1}} - X_{t_j} \right)^2 - E \int_0^t \sigma^2(s,\omega)\,ds \right)^2 \right)^2 .$

For simplicity, we look at $X_t = B_t$ first.

$$\begin{aligned}
& E\left(2\sum_{i=1}^{n-1} \left(B_{t_{i+1}} - B_{t_i} \right)^2 \left(B_{t_i} - B_{t_{i-1}} \right)^2 - \left(\sum_{i=0}^{n-1} \left(B_{t_{i+1}} - B_{t_i} \right)^2 - t \right)^2 \right)^2 \\
= \quad & E\left(2\sum_{i=1}^{n-1} (\Delta B_i)^2 (\Delta B_{i-1})^2 - \left(\sum_{j=0}^{n-1} (\Delta B_j)^2 - t \right)^2 \right)^2 \\
= \quad & 4E\left(\sum_{i=1}^{n-1} (\Delta B_i)^2 (\Delta B_{i-1})^2 \right)^2 - 4E\sum_{i=1}^{n-1} (\Delta B_i)^2 (\Delta B_{i-1})^2 \left(\sum_{j=0}^{n-1} (\Delta B_j)^2 - t \right)^2 \\
& + E\left(\sum_{j=0}^{n-1} (\Delta B_j)^2 - t \right)^4 \\
= \quad & 4E\sum_{i,j=1}^{n-1} (\Delta B_i)^2 (\Delta B_{i-1})^2 (\Delta B_j)^2 (\Delta B_{j-1})^2 \\
& - 4E\sum_{i=1}^{n-1} (\Delta B_i)^2 (\Delta B_{i-1})^2 \left[\left(\sum_{j=0}^{n-1} (\Delta B_j)^2 \right)^2 - 2t\sum_{j=0}^{n-1} (\Delta B_j)^2 + t^2 \right]
\end{aligned}$$

$$+E\left(\sum_{j=0}^{n-1}(\Delta B_j)^2\right)^4 - 4E\left(\sum_{j=0}^{n-1}(\Delta B_j)^2\right)^3 t + 6E\left(\sum_{j=0}^{n-1}(\Delta B_j)^2\right)^2 t^2$$

$$-4E\left(\sum_{j=0}^{n-1}(\Delta B_j)^2\right) t^3 + t^4$$

$$= \quad 4\sum_{i=0}^{n-1} E\Delta B_i^4 \Delta B_{i-1}^4 + 8\sum_{i=1}^{n-2} E\Delta B_{i+1}^2 \Delta B_i^4 \Delta B_{i-1}^2 + 4\sum_{|i-j|\geq 2} E\Delta B_i^2 \Delta B_{i-1}^2 \Delta B_j^2 \Delta B_{j-1}^2$$

$$-4\sum_{i=1}^{n-2} E\Delta B_i^2 \Delta B_{i-1}^2 \sum_{j,k=0}^{n-2} E\Delta B_j^2 \Delta B_k^2 + 8t\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} E\Delta B_i^2 \Delta B_{j-1}^2 \Delta B_j^2$$

$$-4t^2\sum_{i=0}^{n-1} E\Delta B_i^2 \Delta B_{i-1}^2 + \sum_{i,j,k,l=0}^{n-1} E\Delta B_i^2 \Delta B_j^2 \Delta B_k^2 \Delta B_l^2 - 4t\sum_{i,j,k=0}^{n-1} E\Delta B_i^2 \Delta B_j^2 \Delta B_k^2$$

$$+6t^2\sum_{i,j}^{n-1} E\Delta B_i^2 \Delta B_j^2 - 4t^3\sum_{i}^{n-1} E\Delta B_i^2 + t^4$$

Since $E(\Delta B_i)^4 = 3(\Delta t)^2 = 3(\frac{t}{n})^2$, $E(\Delta B_i)^2 = \Delta t = \frac{t}{n}$, and ΔB_i, ΔB_j are independent if $i \neq j$, the term equals

$$4\cdot 9\sum_{i=1}^{n-1}\left(\frac{t}{n}\right)^4 + 8\sum_{i=1}^{n-2}\left(\frac{t}{n}\right)^4\cdot 3 + 4\sum_{|i-j|\geq 2}^{n-1}\left(\frac{t}{n}\right)^4$$

$$-4\left[2\sum_{i=1}^{n-1}15\left(\frac{t}{n}\right)^4 + \sum_{i=1}^{n-1}\sum_{j\neq i,i-1}3\left(\frac{t}{n}\right)^4 + 2\sum_{i=1}^{n-1}9\left(\frac{t}{n}\right)^4 + \sum_{i=1}^{n-1}\sum_{k,j\neq i,i-1}3\left(\frac{t}{n}\right)^4\right]$$

$$+8t\left[2\sum_{i=1}^{n-1}3\left(\frac{t}{n}\right)^3 + \sum_{i=1}^{n-1}\sum_{j\neq i,i-1}\left(\frac{t}{n}\right)^3\right] - 4t^2\sum_{i=1}^{n-1}\left(\frac{t}{n}\right)^2 + \sum_{i=0}^{n-1}105\left(\frac{t}{n}\right)^4$$

$$+\sum_{i=0}^{n-1}\sum_{j\neq i}4\cdot 15\left(\frac{t}{n}\right)^4 + \sum_{i=0}^{n-1}\sum_{j\neq i}9\left(\frac{t}{n}\right)^4 + \sum_{i=0}^{n-1}\sum_{j\neq i,k\neq i,j\neq k}3\left(\frac{t}{n}\right)^4$$

$$+\sum_{i\neq j\neq k\neq l}\left(\frac{t}{n}\right)^4 - 4t\left[\sum_{i=1}^{n-1}15\left(\frac{t}{n}\right)^3 + \sum_{i=0}^{n-1}\sum_{j\neq i}3\cdot 3\left(\frac{t}{n}\right)^3 + 2\sum_{i\neq j\neq k}\left(\frac{t}{n}\right)^3\right]$$

$$+6t^2\left[\sum_{i=1}^{n-1}3\left(\frac{t}{n}\right)^2 + \sum_{j\neq i}3\left(\frac{t}{n}\right)^2\right] - 4t^3\sum_{i=1}^{n-1}3\left(\frac{t}{n}\right) + t^4$$

Counting the coefficients of t^4, we get

$$-4+1+6-4+1=0,$$

so no t^4 appears in the expression. Counting coefficient of $\frac{t^4}{n}$, we get

$$-4+8-4+18-36+18=0,$$

so no $\frac{t^4}{n}$ appears in the expression. Counting coefficient of $\frac{t^4}{n^2}$, we get

$$4-12+48-4+60+9-60=45,$$

so there is $\frac{t^4}{n^2}$ term appears in the expression. Counting the coefficient of $\frac{t^4}{n^3}$, we get

$$36+24-120+18+105=45,$$

so there is $\frac{t^4}{n^3}$ which appears in the expression.

This means that

$$E\left(2\sum_{j=1}^{n-1}\left(B_{t_{j+1}}-B_{t_j}\right)^2\left(B_{t_j}-B_{t_{j-1}}\right)^2-\left(\sum_{j=0}^{n-1}\left(B_{t_{j+1}}-B_{t_j}\right)^2-t\right)^2\right)^2$$

converges to 0 at a rate n^{-2}.

If we set

$$\left(2\sum_{j=1}^{n-1}\left(B_{t_{j+1}}-B_{t_j}\right)^2\left(B_{t_j}-B_{t_{j-1}}\right)^2-\left(\sum_{j=0}^{n-1}\left(B_{t_{j+1}}-B_{t_j}\right)^2-t\right)^2\right)^2=0,$$

we get

$$\pm\sqrt{2\sum_{j=1}^{n-1}\left(B_{t_{j+1}}-B_{t_j}\right)^2\left(B_{t_j}-B_{t_{j-1}}\right)^2}-\sum_{j=0}^{n-1}\left(B_{t_{j+1}}-B_{t_j}\right)^2+t=0,$$

and we can say the estimator

$$\sum_{j=0}^{n-1}\left(B_{t_{j+1}}-B_{t_j}\right)^2 \pm \sqrt{2\sum_{j=1}^{n-1}\left(B_{t_{j+1}}-B_{t_j}\right)^2\left(B_{t_j}-B_{t_{j-1}}\right)^2} \to t$$

at a rate of n^{-1}. Comparing it to approximation 4.1.1, we conclude that we have improved the rate of convergence from $n^{-\frac{1}{2}}$ to n^{-1}.

PROPOSITION 4.3.1. *For Brownian motion B_t, $t_{j+1}-t_j=\frac{t}{n}$, for all j, then the estimator*

$$\sum_{j=0}^{n-1}\left(B_{t_{j+1}}-B_{t_j}\right)^2 \pm \sqrt{2\sum_{j=1}^{n-1}\left(B_{t_{j+1}}-B_{t_j}\right)^2\left(B_{t_j}-B_{t_{j-1}}\right)^2}$$

converges to t in mean square at a rate of n^{-1}.

This proposition is also one of the main contributions in this dissertation.

The following investigation is useful in the general case:

$$E\left(2\sum_{j=1}^{n-1}\left(X_{t_{j+1}}-X_{t_j}\right)^2\left(X_{t_j}-X_{t_{j-1}}\right)^2-\left(\sum_{j=0}^{n-1}\left(X_{t_{j+1}}-X_{t_j}\right)^2-E\int_0^t\sigma^2(s,\omega)\,ds\right)^2\right)^2.$$

We approximate $2\delta\int_0^t\sigma^4(s,\omega)\,ds$ by $\left(\left(X_{t_{j+1}}-X_{t_j}\right)^2-E\int_0^t\sigma^2(s,\omega)\,ds\right)^2$.

Denote $\Delta X_j=X_{t_{j+1}}-X_{t_j}$, for $j=0, 1, ..., n-1$, and $A=E\int_0^t\sigma^2(s,\omega)\,ds$, then we get

$$\begin{aligned}
& E\left(2\sum_{i=1}^{n-1}\left(X_{t_{i+1}}-X_{t_i}\right)^2\left(X_{t_i}-X_{t_{i-1}}\right)^2-\left(\sum_{i=0}^{n-1}\left(X_{t_{i+1}}-X_{t_i}\right)^2-A\right)^2\right)^2 \\
= \; & E\left(2\sum_{i=1}^{n-1}(\Delta X_i)^2(\Delta X_{i-1})^2-\left(\sum_{j=0}^{n-1}(\Delta X_j)^2-A\right)^2\right)^2 \\
= \; & 4E\left(\sum_{i=1}^{n-1}(\Delta X_i)^2(\Delta X_{i-1})^2\right)^2
\end{aligned}$$

$$-4E\sum_{i=1}^{n-1}(\Delta X_i)^2(\Delta X_{i-1})^2\left(\sum_{j=0}^{n-1}(\Delta X_j)^2-A\right)^2+E\left(\sum_{j=0}^{n-1}(\Delta X_j)^2-A\right)^4$$

$$= 4E\sum_{i,j=1}^{n-1}(\Delta X_i)^2(\Delta X_{i-1})^2(\Delta X_j)^2(\Delta X_{j-1})^2$$

$$-4E\sum_{i=1}^{n-1}(\Delta X_i)^2(\Delta X_{i-1})^2\left[\left(\sum_{j=0}^{n-1}(\Delta X_j)^2\right)^2-2A\sum_{j=0}^{n-1}(\Delta X_j)^2\right]$$

$$-4E\sum_{i=1}^{n-1}(\Delta X_i)^2(\Delta X_{i-1})^2A^2+E\left(\sum_{j=0}^{n-1}(\Delta X_j)^2\right)^4-4E\left(\sum_{j=0}^{n-1}(\Delta X_j)^2\right)^3A$$

$$+6E\left(\sum_{j=0}^{n-1}(\Delta X_j)^2\right)^2A^2-4E\left(\sum_{j=0}^{n-1}(\Delta X_j)^2\right)A^3+A^4$$

$$= 4E\sum_{i=0}^{n-1}\Delta X_i^4\Delta X_{i-1}^4+8E\sum_{i=1}^{n-2}\Delta X_{i+1}^2\Delta X_i^4\Delta X_{i-1}^2+4E\sum_{|i-j|\geq 2}\Delta X_i^2\Delta X_{i-1}^2\Delta X_j^2\Delta X_{j-1}^2$$

$$-4E\sum_{i=1}^{n-2}\Delta X_i^2\Delta X_{i-1}^2\sum_{j,k=0}^{n-2}\Delta X_j^2\Delta X_k^2+8AE\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\Delta X_i^2\Delta X_{j-1}^2\Delta X_j^2$$

$$-4A^2E\sum_{i=0}^{n-1}\Delta X_i^2\Delta X_{i-1}^2+E\sum_{i,j,k,l=0}^{n-1}\Delta X_i^2\Delta X_j^2\Delta X_k^2\Delta X_l^2$$

$$-4AE\sum_{i,j,k=0}^{n-1}\Delta X_i^2\Delta X_j^2\Delta X_k^2+6A^2E\sum_{i,j}^{n-1}\Delta X_i^2\Delta X_j^2-4A^3E\sum_{i}^{n-1}\Delta X_i^2+A^4$$

By Proposition 4.1.1 and the independence property, it equals

$$4\cdot 9\sum_{i=1}^{n-1}\left(E\int_{t_i}^{t_{i+1}}\sigma^2(s,\omega)\,ds\right)^2\left(E\int_{t_{i-1}}^{t_i}\sigma^2(s,\omega)\,ds\right)^2$$

$$+8\sum_{i=1}^{n-2}\left(E\int_{t_{i+1}}^{t_{i+2}}\sigma(s,\omega)^2\,ds\right)3\cdot\left(E\int_{t_i}^{t_{i+1}}\sigma^2(s,\omega)\,ds\right)^4\left(E\int_{t_{i-1}}^{t_i}\sigma^2(s,\omega)\,ds\right)$$

$$+4\sum_{|i-j|\geq 2}E\int_{t_i}^{t_{i+1}}\sigma^2(s,\omega)\,dsE\int_{t_{i-1}}^{t_i}\sigma^2(s,\omega)\,dsE\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,dsE\int_{t_{j-1}}^{t_j}\sigma^2(s,\omega)\,ds$$

$$-4[2\sum_{i=1}^{n-1}15\left(E\int_{t_i}^{t_{i+1}}\sigma^2(s,\omega)\,ds\right)^3E\int_{t_{i-1}}^{t_i}\sigma^2(s,\omega)\,ds$$

$$+\sum_{i=1}^{n-1}\sum_{j\neq i,i-1}3\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)^2 E\int_{t_i}^{t_{i+1}}\sigma(s,\omega)^2\,dsE\int_{t_{i-1}}^{t_i}\sigma^2(s,\omega)\,ds$$

$$+2\sum_{i=1}^{n-1}9\left(E\int_{t_i}^{t_{i+1}}\sigma^2(s,\omega)\,ds\right)^2\left(E\int_{t_{i-1}}^{t_i}\sigma^2(s,\omega)\,ds\right)^2$$

$$+\sum_{i=1}^{n-1}\sum_{k,j\neq i,i-1}\left(E\int_{t_i}^{t_{i+1}}\sigma^2(s,\omega)\,ds\right)\left(E\int_{t_{i-1}}^{t_i}\sigma^2(s,\omega)\,ds\right)$$

$$\times\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)\left(E\int_{t_k}^{t_{k+1}}\sigma^2(s,\omega)\,ds\right)]$$

$$+8A[2\sum_{i=1}^{n-1}3\left(E\int_{t_i}^{t_{i+1}}\sigma^2(s,\omega)\,ds\right)^2 E\int_{t_{i-1}}^{t_i}\sigma^2(s,\omega)\,ds$$

$$+\sum_{i=1}^{n-1}\sum_{j\neq i,i-1}\left(E\int_{t_i}^{t_{i+1}}\sigma^2(s,\omega)\,ds\right)\left(E\int_{t_i}^{t_{i+1}}\sigma^2(s,\omega)\,ds\right)\left(E\int_{t_{i-1}}^{t_i}\sigma^2(s,\omega)\,ds\right)]$$

$$-4A^2\sum_{i=1}^{n-1}\left(E\int_{t_i}^{t_{i+1}}\sigma^2(s,\omega)\,ds\right)\left(E\int_{t_{i-1}}^{t_i}\sigma^2(s,\omega)\,ds\right)$$

$$+\sum_{i=0}^{n-1}105\left(E\int_{t_i}^{t_{i+1}}\sigma^2(s,\omega)\,ds\right)^4+\sum_{i=0}^{n-1}\sum_{j\neq i}4\cdot 15\left(E\int_{t_i}^{t_{i+1}}\sigma^2(s,\omega)\,ds\right)^4$$

$$+\sum_{i=0}^{n-1}\sum_{j\neq i}9\left(E\int_{t_i}^{t_{i+1}}\sigma^2(s,\omega)\,ds\right)^2\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)^2$$

$$+\sum_{i=0}^{n-1}\sum_{j\neq i,k\neq i,j\neq k}3\left(E\int_{t_i}^{t_{i+1}}\sigma^2(s,\omega)\,ds\right)^4+\sum_{i\neq j\neq k\neq l}\left(E\int_{t_i}^{t_{i+1}}\sigma^2(s,\omega)\,ds\right)$$

$$\times\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)\left(E\int_{t_k}^{t_{k+1}}\sigma^2(s,\omega)\,ds\right)\left(E\int_{t_l}^{t_{l+1}}\sigma^2(s,\omega)\,ds\right)$$

$$-4A\{\sum_{i=1}^{n-1}15\left(E\int_{t_i}^{t_{i+1}}\sigma^2(s,\omega)\,ds\right)^3$$

$$+\sum_{i=0}^{n-1}\sum_{j\neq i}3\cdot 3\left(E\int_{t_i}^{t_{i+1}}\sigma^2(s,\omega)\,ds\right)^2 E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds$$

$$+\sum_{i\neq j\neq k}\left(E\int_{t_i}^{t_{i+1}}\sigma^2(s,\omega)\,ds\right)\left(E\int_{t_j}^{t_{j+1}}\sigma^2(s,\omega)\,ds\right)\left(E\int_{t_k}^{t_{k+1}}\sigma^2(s,\omega)\,ds\right)\}$$

$$+6A^2\sum_{i=1}^{n-1}3\left(E\int_{t_i}^{t_{i+1}}\sigma^2\left(s,\omega\right)ds\right)^2$$

$$+6A^2\sum_{j\neq i}3\left(E\int_{t_i}^{t_{i+1}}\sigma^2\left(s,\omega\right)ds\right)\left(E\int_{t_j}^{t_{j+1}}\sigma^2\left(s,\omega\right)ds\right)$$

$$-4A^3\sum_{i=1}^{n-1}3\left(E\int_{t_i}^{t_{i+1}}\sigma^2\left(s,\omega\right)ds\right)+A^4$$

Similarly, if we count the coefficients of A^4, we get

$$-4+1+6-4+1=0.$$

No matter what $\sigma(t,\omega)$ is, no $\left(E\int_0^t\sigma^2\left(s,\omega\right)ds\right)^4$ appears in the expression. However, if $\sigma(t,\omega)$ is not constant, the coefficient of n^{-1} is not necessary 0. If $\sigma\left(t,\omega\right)$ is bounded, then we can still claim it converges at least in mean square, the result is still stronger than convergence in probability. This means that for the general case, the estimator:

$$E\left(2\sum_{j=1}^{n-1}\left(X_{t_{j+1}}-X_{t_j}\right)^2\left(X_{t_j}-X_{t_{j-1}}\right)^2-\left(\sum_{j=0}^{n-1}\left(X_{t_{j+1}}-X_{t_j}\right)^2-A\right)^2\right)^2$$

converges to to 0 in mean square at least at a rate of n^{-1}, not necessarily n^{-2}. It depends on $\sigma(t,\omega)$.

PROPOSITION 4.3.2. *For the Ito process* $X_t=\int_0^t\sigma\left(s,\omega\right)dB_s$*, when* $t_{j+1}-t_j=\frac{t}{n}$ *for all* j*, and* $\sigma\left(t,\omega\right)$ *is bounded, then*

$$E\left(2\sum_{j=1}^{n-1}\left(X_{t_{j+1}}-X_{t_j}\right)^2\left(X_{t_j}-X_{t_{j-1}}\right)^2-\left(\sum_{j=0}^{n-1}\left(X_{t_{j+1}}-X_{t_j}\right)^2-A\right)^2\right)^2$$

converges in mean square.

Set

$$\left(2\sum_{j=1}^{n-1}\left(X_{t_{j+1}}-X_{t_j}\right)^2\left(X_{t_j}-X_{t_{j-1}}\right)^2-\left(\sum_{j=0}^{n-1}\left(X_{t_{j+1}}-X_{t_j}\right)^2-A\right)^2\right)^2=0,$$

hence we obtain

$$\pm\sqrt{2\sum_{j=1}^{n-1}\left(X_{t_{j+1}}-X_{t_j}\right)^2\left(X_{t_j}-X_{t_{j-1}}\right)^2}-\sum_{j=0}^{n-1}\left(X_{t_{j+1}}-X_{t_j}\right)^2+A=0,$$

i.e., $\sum_{j=0}^{n-1}\left(X_{t_{j+1}}-X_{t_j}\right)^2\pm\sqrt{2\sum_{j=1}^{n-1}\left(X_{t_{j+1}}-X_{t_j}\right)^2\left(X_{t_j}-X_{t_{j-1}}\right)^2} \rightarrow \left(E\int_0^t\sigma^2(s,\omega)\,ds\right)$ at a rate of $n^{-\frac{1}{2}}$

For the bipower process, the cross terms were used. The question now is: can we just consider $\left(X_{t_{j+1}}-X_{t_j}\right)^4$ instead of the cross term

$$\left(X_{t_{j+1}}-X_{t_j}\right)^2\left(X_{t_j}-X_{t_{j-1}}\right)^2?$$

Using $\left(X_{t_{j+1}}-X_{t_j}\right)^4$ seems more reasonable and the computation process seems easier because no cross term appears in the expression. Proposition 4.1.1 can then be used directly during the calculation process.

For this case, in reference ([3]), the author proved

$$\frac{\sum_{j=0}^{n-1}\left(X_{t_{j+1}}-X_{t_j}\right)^2-E\int_0^t\sigma^2(s,\omega)\,ds}{\sqrt{\frac{2}{3}\sum_{j=0}^{n-1}\left(X_{t_{j+1}}-X_{t_j}\right)^4}}\rightarrow\mathcal{N}(0,1),$$

without improving the convergence rate. In our dissertation we have shown that the convergence rate is n^{-1}, which is better than $n^{-\frac{1}{2}}$.

For

$$E\left(\frac{2}{3}\sum_{j=0}^{n-1}\left(X_{t_{j+1}}-X_{t_j}\right)^4-\left(\sum_{j=0}^{n-1}\left(X_{t_{j+1}}-X_{t_j}\right)^2-E\int_0^t\sigma^2\left(s,\omega\right)ds\right)^2\right)^2,$$

if we denote $A=E\int_0^t\sigma^2\left(s,\omega\right)ds$, $\Delta X_j=X_{t_{j+1}}-X_{t_j}$, $C_j=E\int_{t_j}^{t_{j+1}}\sigma^2\left(s,\omega\right)ds$, then by Proposition 4.1.1, we have

$$E(\Delta X_j)^2=C_j,$$

$$E(\Delta X_j)^4=3(C_j)^2,$$

$$E(\Delta X_j)^6=15(C_j)^3,$$

$$E(\Delta X_j)^8=105(C_j)^4,$$

and a direct calculation yields

$$\begin{aligned}
& E\left(\frac{2}{3}\sum_{j=0}^{n-1}\left(X_{t_{j+1}}-X_{t_j}\right)^4-\left(\sum_{j=0}^{n-1}\left(X_{t_{j+1}}-X_{t_j}\right)^2-A\right)^2\right)^2 \\
= \; & E\left(\frac{2}{3}\sum_{j=0}^{n-1}(\Delta X_j)^4-\left(\sum_{j=0}^{n-1}(\Delta X_j)^2-A\right)^2\right)^2 \\
= \; & \frac{4}{9}E\left(\sum_{j=1}^{n-1}(\Delta X_j)^4\right)^2-2\cdot\frac{2}{3}E\sum_{j=0}^{n-1}(\Delta X_j)^4\left(\sum_{j=0}^{n-1}(\Delta X_j)^2-A\right)^2 \\
& +E\left(\sum_{j=0}^{n-1}(\Delta X_j)^2-A\right)^4 \\
= \; & \frac{4}{9}E\sum_{i}^{n-1}\sum_{j}^{n-1}(\Delta X_i)^4(\Delta X_j)^4 \\
& -\frac{4}{3}E\sum_{j=0}^{n-1}(\Delta X_j)^4\left[\left(\sum_{j=0}^{n-1}(\Delta X_j)^2\right)^2-2A\sum_{j=0}^{n-1}(\Delta X_j)^2+A^2\right]+E\left(\sum_{j=0}^{n-1}(\Delta X_j)^2\right)^4
\end{aligned}$$

$$
\begin{aligned}
& -4E\left(\sum_{j=0}^{n-1}(\Delta X_j)^2\right)^3 A + 6E\left(\sum_{j=0}^{n-1}(\Delta X_j)^2\right)^2 A^2 - 4E\left(\sum_{j=0}^{n-1}(\Delta X_j)^2\right) A^3 + A^4 \\
= \;& \frac{4}{9}E\left(\sum_{j=0}^{n-1}\Delta X_j^8 + \sum_{i\neq j}\Delta X_i^4 \Delta X_j^4\right) - \frac{4}{3}E\sum_{j=0}^{n-1}\Delta X_j^4 \sum_i\sum_k \Delta X_i^2 \Delta X_k^2 \\
& +\frac{8}{3}AE\sum_{j=0}^{n-1}\sum_{i=0}^{n-1}\Delta X_j^4 \Delta X_i^2 - \frac{4}{3}A^2 E\sum_{j=0}^{n-1}\Delta X_i^4 + E\sum_{i,j,k,l=0}^{n-1}\Delta X_i^2 \Delta X_j^2 \Delta X_k^2 \Delta X_l^2 \\
& -4AE\sum_{i,j,k=0}^{n-1}\Delta X_i^2 \Delta X_j^2 \Delta X_k^2 + 6A^2 E\sum_{i,j}^{n-1}\Delta X_i^2 \Delta X_j^2 - A^3 E\sum_{i}^{n-1}\Delta X_i^2 + A^4 \\
= \;& \frac{4}{9}E\left(\sum_{j=0}^{n-1}\Delta X_j^8 + \sum_{i\neq j}\Delta X_i^4 \Delta X_j^4\right) \\
& -\frac{4}{3}E\left(\sum_{j=0}^{n-1}\Delta X_j^4 \sum_i \Delta X_i^4 + \sum_{j=0}^{n-1}\sum_{i\neq k}\Delta X_j^4 \Delta X_i^2 \Delta X_k^2\right) \\
& +\frac{8}{3}AE\left(\sum_{j=0}^{n-1}\Delta X_j^6 + \sum_{j\neq i}\Delta X_j^4 \Delta X_i^2\right) - \frac{4}{3}A^2 E\sum_{j=0}^{n-1}\Delta X_i^4 \\
& +E\sum_{i=j=k=l}^{n-1}\Delta X_i^8 + 4E\sum_{i=j=k\neq l}^{n-1}\Delta X_i^6 \Delta X_l^2 + 3E\sum_{i=j\neq k=l}^{n-1}\Delta X_i^4 \Delta X_k^4 \\
& +6E\sum_{i=j\neq k\neq l}^{n-1}\Delta X_i^4 \Delta X_k^2 \Delta X_l^2 + E\sum_{i\neq j\neq k\neq l}^{n-1}\Delta X_i^2 \Delta X_j^2 \Delta X_k^2 \Delta X_l^2 \\
& -4AE\left(\sum_{i=j=k}^{n-1}\Delta X_i^6 + 3\sum_{i=j\neq k}^{n-1}\Delta X_i^4 \Delta X_l^2 + \sum_{i\neq j\neq k}^{n-1}\Delta X_i^2 \Delta X_j^2 \Delta X_k^2\right) \\
& +6A^2 E\left(\sum_{j}^{n-1}\Delta X_j^4 + \sum_{i\neq j}^{n-1}\Delta X_i^2 \Delta X_j^2\right) - 4A^3 E\sum_{i}^{n-1}\Delta X_i^2 + A^4 \\
= \;& \frac{4}{9}\left(\sum_{j=0}^{n-1}105\,(C_j)^4\right) + \frac{4}{9}\sum_{i\neq j}3\,(C_j)^2 \times 3\,(C_i)^2 - \frac{4}{3}\left(\sum_{j=0}^{n-1}105\,(C_j)^4\right) \\
& -\frac{4}{3}\left(\sum_{i\neq j}3\,(C_j)^2 \cdot 3\,(C_j)^2\right) - \frac{4}{3}\left(\sum_{j\neq i\neq k}^{n-1}3\,(C_j)^2\, C_i C_k\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{4}{3}\left(\sum_{i=j\neq k}^{n-1}15\left(C_j\right)^3C_k\right)-\frac{4}{3}\left(\sum_{j=k\neq i}^{n-1}15\left(C_i\right)\left(C_k\right)^3\right)+\frac{8}{3}A\sum_{j=0}^{n-1}15\left(C_j\right)^3\\
&+\frac{8}{3}A\left(\sum_{j\neq i}3\left(C_j\right)^2C_i\right)-\frac{4}{3}A^2\sum_{j=0}^{n-1}3\left(C_j\right)^2+\sum_{i=j=k=l}^{n-1}105\left(C_j\right)^4\\
&+4\sum_{i=j=k\neq l}^{n-1}15\left(C_i\right)^3\left(C_l\right)+3\sum_{i=j\neq k=l}^{n-1}3\left(C_i\right)^2 3\left(C_k\right)^2\\
&+6\sum_{i=j\neq k\neq l}^{n-1}3\left(C_i\right)^2\left(C_k\right)\left(C_l\right)+\sum_{i\neq j\neq k\neq l}^{n-1}\left(C_i\right)\left(C_j\right)\left(C_k\right)\left(C_l\right)\\
&-4A\sum_{i=j=k}^{n-1}15\left(C_j\right)^3-12A\sum_{i=j\neq k}^{n-1}3\left(C_i\right)^2\left(C_k\right)-4A\sum_{i\neq j\neq k}^{n-1}\left(C_i\right)\left(C_j\right)\left(C_k\right)\\
&+6A^2\sum_{j}^{n-1}3\left(C_j\right)^2+6A^2\left(\sum_{i\neq j}^{n-1}\left(C_i\right)\left(C_j\right)\right)-4A^3\sum_{i}^{n-1}\left(C_j\right)+A^4\\
=\;&\frac{32}{3}\sum_{j=0}^{n-1}\left(C_j\right)^4+16\sum_{i\neq j}\left(C_j\right)^2\left(C_i\right)^2+16\sum_{i\neq j}\left(C_j\right)^3C_i+8\sum_{i\neq j\neq k}\left(C_j\right)^2\left(C_i\right)\left(C_k\right)\\
&-16A\sum_{j=0}^{n-1}\left(C_j\right)^3-16A\sum_{i\neq j}\left(C_j\right)^2C_i+8A^2\sum_{j=0}^{n-1}\left(C_j\right)^2
\end{aligned}
$$

Thus, since $A=E\int_0^t\sigma^2\left(s,\omega\right)ds=\sum_{j=0}^{n-1}E\int_{t_j}^{t_{j+1}}\sigma^2\left(s,\omega\right)ds=\sum_{j=0}^{n-1}C_j$, we get a final result:

$$
\frac{8}{3}\sum_{j=0}^{n-1}\left(C_j\right)^4+8\sum_{i\neq j}\left(C_j\right)^2\left(C_i\right)^2=8\left(\sum_{j=0}^{n-1}\left(C_j\right)^2\right)^2-\frac{16}{3}\sum_{j=0}^{n-1}\left(C_j\right)^4.
$$

Obviously, the result indicates that there are only $O(n^{-2})$ and $O(n^{-3})$ terms. If $\sigma\left(t,\omega\right)$ is bounded, then we can say the estimator converges in mean square at a rate n^{-2}.

Proposition 4.3.3. *For the Ito process $X_t = \int_0^t \sigma(s,\omega)\, dB_s$, when $t_{j+1} - t_j = \frac{t}{n}$ for all j, and $\sigma(t,\omega)$ is bounded, then*

$$E\left(\frac{2}{3}\sum_{j=0}^{n-1}\left(X_{t_{j+1}} - X_{t_j}\right)^4 - \left(\sum_{j=0}^{n-1}\left(X_{t_{j+1}} - X_{t_j}\right)^2 - \left(E\int_0^t \sigma^2(s,\omega)\, ds\right)\right)^2\right)^2$$

converges in mean square at a rate n^{-2}.

If we set

$$\left(\frac{2}{3}\sum_{j=0}^{n-1}\left(X_{t_{j+1}} - X_{t_j}\right)^4 - \left(\sum_{j=0}^{n-1}\left(X_{t_{j+1}} - X_{t_j}\right)^2 - \left(\int_0^t \sigma^2(s,\omega)\, ds\right)\right)^2\right)^2 = 0,$$

then we obtain

$$\pm\sqrt{\frac{2}{3}\sum_{j=0}^{n-1}\left(X_{t_{j+1}} - X_{t_j}\right)^4} - \sum_{j=0}^{n-1}\left(X_{t_{j+1}} - X_{t_j}\right)^2 + \left(E\int_0^t \sigma^2(s,\omega)\, ds\right) = 0,$$

i.e

$$\sum_{j=0}^{n-1}\left(X_{t_{j+1}} - X_{t_j}\right)^2 \pm\sqrt{\frac{2}{3}\sum_{j=0}^{n-1}\left(X_{t_{j+1}} - X_{t_j}\right)^4} \to \left(E\int_0^t \sigma^2(s,\omega)\, ds\right)$$

at a rate of n^{-1}.

In particular, when X_t is the Brownian motion B_t, then

$$\begin{aligned} A &= E\int_0^t \sigma^2(s,\omega)\, ds = t, \\ C_j &= E\int_{t_j}^{t_{j+1}} \sigma^2(s,\omega)\, ds = \frac{t}{n}, \text{ for } j = 0, 1, ..., n-1, \end{aligned}$$

and
$$E\left(\frac{2}{3}\sum_{j=0}^{n-1}\left(B_{t_{j+1}} - B_{t_j}\right)^4 - \left(\sum_{j=0}^{n-1}\left(B_{t_{j+1}} - B_{t_j}\right)^2 - t\right)^2\right)^2$$
$$= 8\frac{t^4}{n^2} - \frac{16}{3}\frac{t^4}{n^3}.$$

This means that when n goes to infinity,

$$E\left(\frac{2}{3}\sum_{j=0}^{n-1}\left(B_{t_{j+1}}-B_{t_j}\right)^4-\left(\sum_{j=0}^{n-1}\left(B_{t_{j+1}}-B_{t_j}\right)^2-t\right)^2\right)^2$$

converges to 0 at a rate of n^{-2}.

If we set

$$\left(\frac{2}{3}\sum_{j=0}^{n-1}\left(B_{t_{j+1}}-B_{t_j}\right)^4-\left(\sum_{j=0}^{n-1}\left(B_{t_{j+1}}-B_{t_j}\right)^2-t\right)^2\right)^2=0,$$

it follows that

$$\pm\sqrt{\frac{2}{3}\sum_{j=0}^{n-1}\left(B_{t_{j+1}}-B_{t_j}\right)^4}-\sum_{j=0}^{n-1}\left(B_{t_{j+1}}-B_{t_j}\right)^2+t=0,$$

so we can say the estimator

$$\sum_{j=0}^{n-1}\left(B_{t_{j+1}}-B_{t_j}\right)^2\pm\sqrt{\frac{2}{3}\sum_{j=0}^{n-1}\left(B_{t_{j+1}}-B_{t_j}\right)^4}\rightarrow t$$

at a rate of n^{-1}. The $\pm$ can be explained by the reflection principle for Brownian motion.

This proposition is the main contribution in this dissertation. Now, we can say we improved the approximation for quadratic variation. The convergence rate is improved from $n^{-\frac{1}{2}}$ to n^{-1}.

4.4. Data Analysis

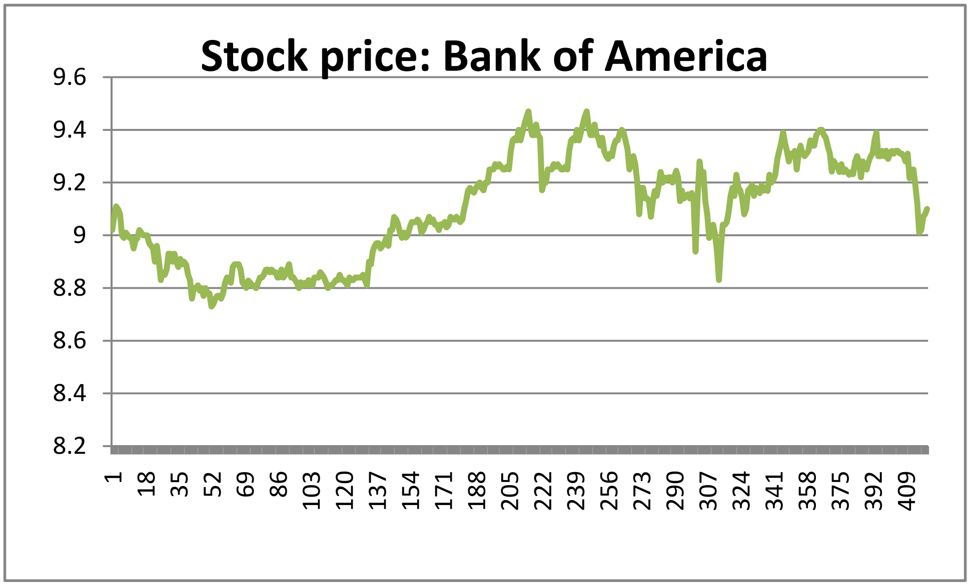
In this section, we take high frequency data from the financial market, and estimate the variance over a short period. The data were the price of Bank of America for one day(Fig 4.1). For simplicity, we take the data every minute.

By the approximation

$$\sum_{j=0}^{n-1} \left(X_{t_{j+1}} - X_{t_j}\right)^2 \pm \sqrt{\frac{2}{3}\sum_{j=0}^{n-1} \left(X_{t_{j+1}} - X_{t_j}\right)^4}$$

where X_t is the log stock price, i.e. $X_t = logS_t$, we can estimate the variance over a small period. We estimated the variance for every 15 minutes and 5 minutes cases. The result is shown in Fig 4.2 and Fig 4.3.

Stock price of BAC



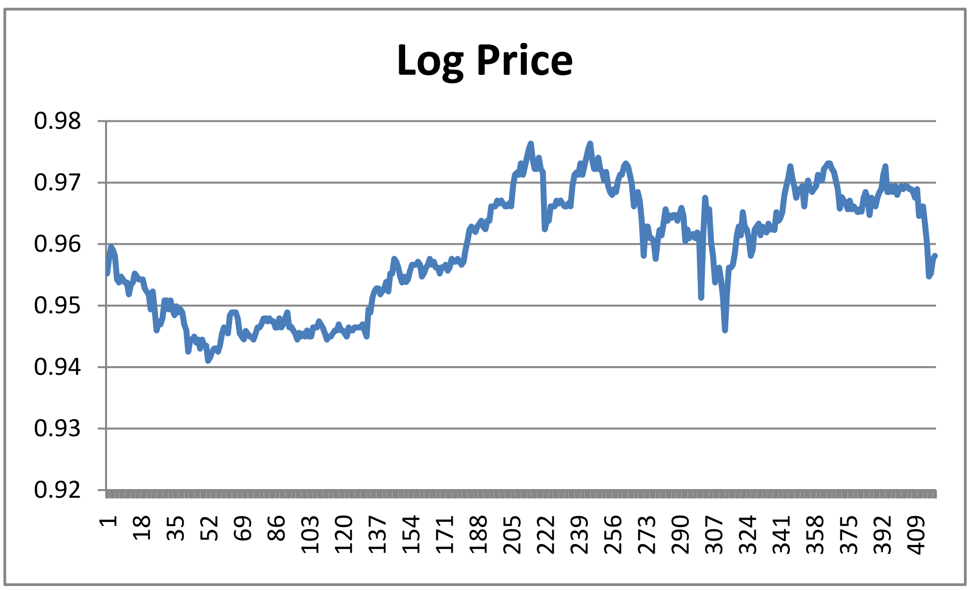
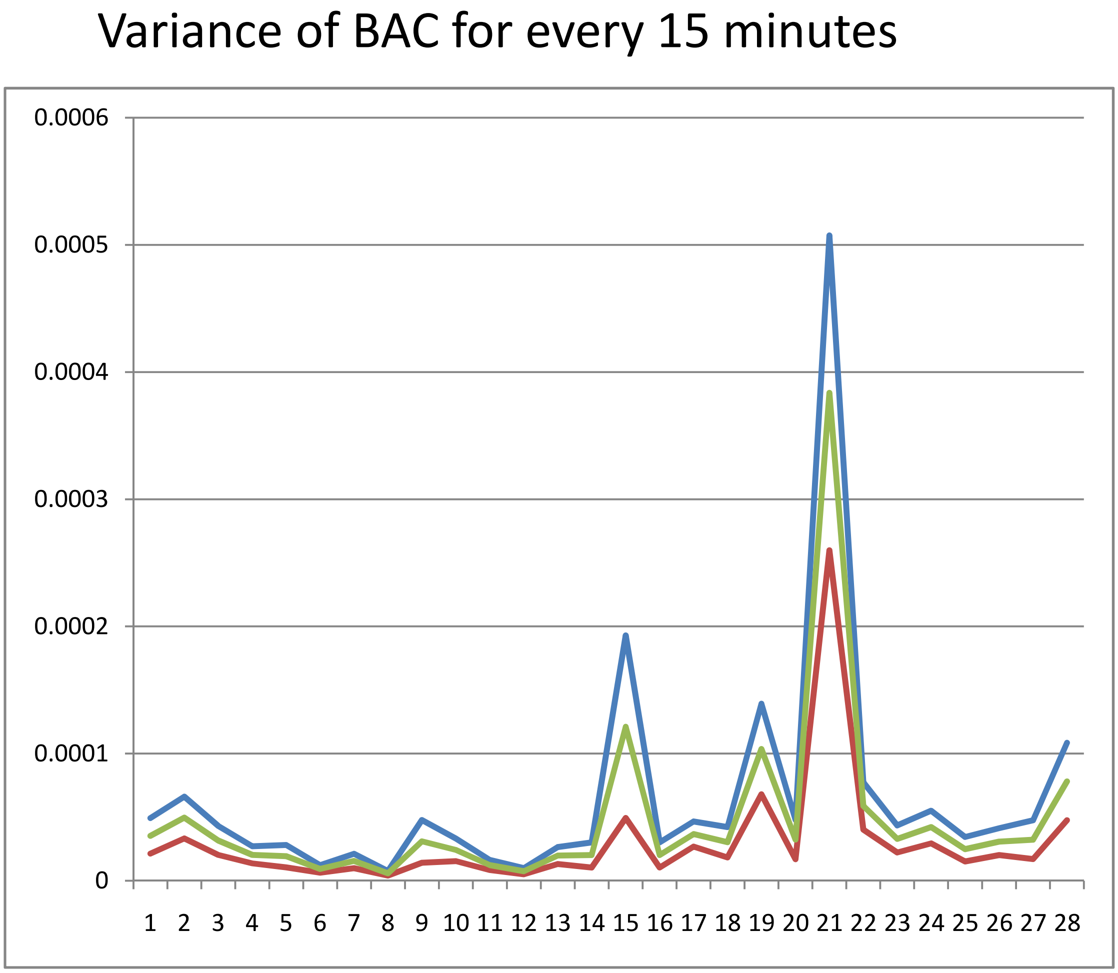


Figure 4.1

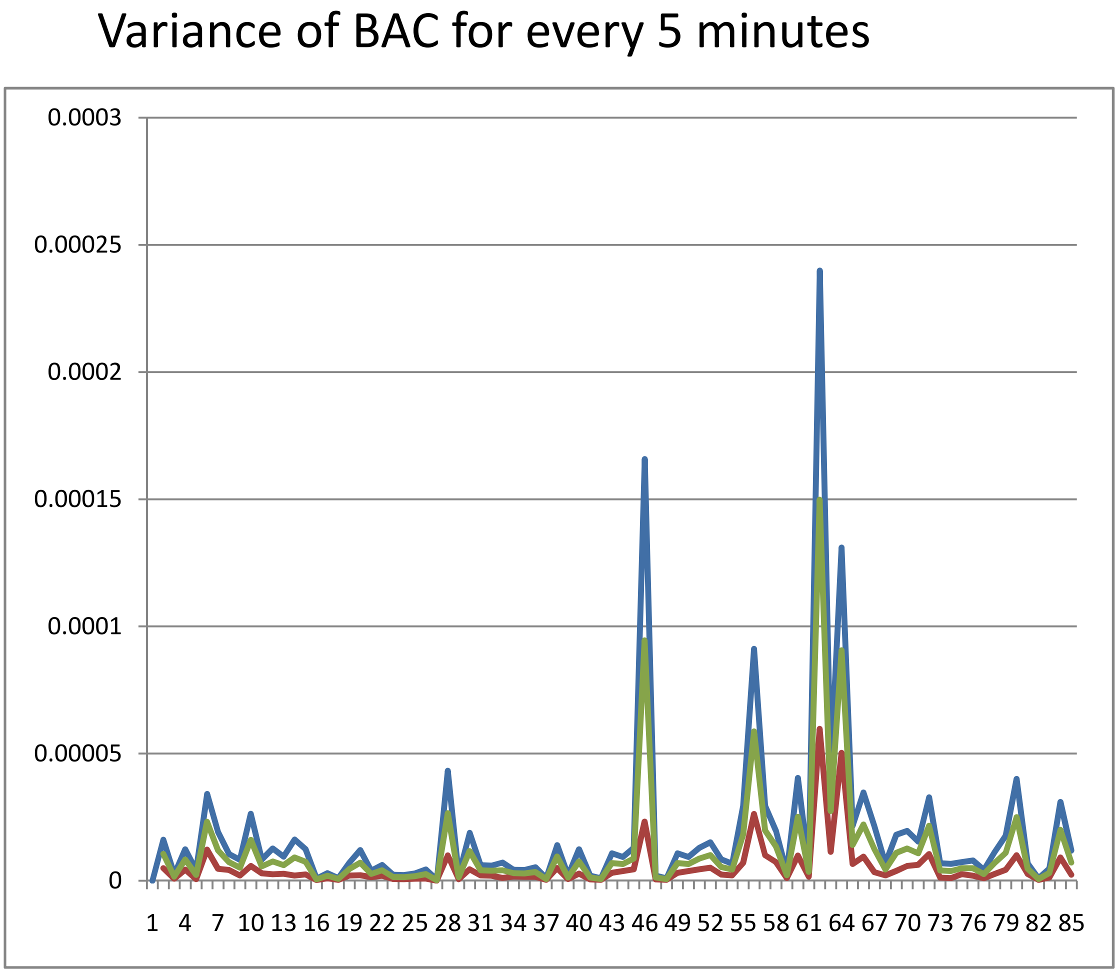


represents $\sum_{j=0}^{n-1}(X_{j+1}-X_j)^2+\sqrt{\frac{2}{3}\sum_{j=0}^{n-1}(X_{j+1}-X_j)^4}$ (blue)

represents $\sum_{j=0}^{n-1}(X_{j+1}-X_j)^2$ (green)

represents $\sum_{j=0}^{n-1}(X_{j+1}-X_j)^2-\sqrt{\frac{2}{3}\sum_{j=0}^{n-1}(X_{j+1}-X_j)^4}$ (red)

FIGURE 4.2



____ represents $\sum_{j=0}^{n-1}(X_{j+1}-X_j)^2+\sqrt{\frac{2}{3}\sum_{j=0}^{n-1}(X_{j+1}-X_j)^4}$

____ represents $\sum_{j=0}^{n-1}(X_{j+1}-X_j)^2$

____ represents $\sum_{j=0}^{n-1}(X_{j+1}-X_j)^2-\sqrt{\frac{2}{3}\sum_{j=0}^{n-1}(X_{j+1}-X_j)^4}$

FIGURE 4.3

From the graph, we can see that the traditional assumption about constant variance is not accurate.

CHAPTER 5

SUMMARY, CONCLUSION AND FUTURE RESEARCH

Summary and Conclusion

Modeling and measuring financial volatility are the key steps in derivative pricing, asset allocation and risk management. In this paper, we analyzed quadratic variation for the one dimension and high dimension problem.

In this dissertation, the results indicate that realized volatility for high frequency data not only converges to quadratic variation in probability, but also converges to quadratic volatility in mean square under some conditions. We obtained expressions for the difference between realized volatility and quadratic variation under mean square. More importantly, we improved the estimation for quadratic variation. The largest contribution is that we find a new approximation for quadratic variation. This new approximation converges in mean square to quadratic variation at a rate of n^{-2} instead of n^{-1}.

Future Research

The research that's been done up to the present has provided some insights about quadratic variation. This paper initiates a new research direction on volatility estimation in the field of high-frequency data. This paper still leaves some issues and raises several interesting questions for future investigation. We will make some recommendations about what to do in the future, but ultimately, we can't claim to have all the answers. Can we get an approximation which converges to quadratic variation at a higher rate, in other words, can we find an approximation which converges at a rate that is higher than n^{-2}?

Such a higher rate estimator might be

$$E\left[C\sum_{j=0}^{n-1}\left(X_{t_{j+1}}-X_{t_j}\right)^8-\left(\frac{2}{3}\sum_{j=0}^{n-1}\left(X_{t_{j+1}}-X_{t_j}\right)^4-\left(\sum_{j=0}^{n-1}\left(X_{t_{j+1}}-X_{t_j}\right)^2-A\right)^2\right)^2\right]^2,$$

where $A=E\int_0^t\sigma^2\left(s,\omega\right)ds$, C is some constant.

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APPENDIX

t	S_t	$X_t = logS_t$	$(X_{i+1} - X_i)^2$	$(X_{i+1} - X_i)^4$	Variance(5 mins)
0	9.02	0.955206538			
1	9.07	0.957607287	5.7636E-06	3.32191E-11	
2	9.11	0.959518377	3.65226E-06	1.3339E-11	
3	9.1	0.959041392	2.27514E-07	5.17628E-14	
4	9.08	0.958085849	9.13064E-07	8.33686E-13	1.61804E-05
5	9	0.954242509	1.47713E-05	2.1819E-10	
6	8.99	0.953759692	2.33113E-07	5.43416E-14	
7	9.01	0.954724791	9.31417E-07	8.67537E-13	
8	9	0.954242509	2.32595E-07	5.41007E-14	
9	8.99	0.953759692	2.33113E-07	5.43416E-14	2.45902E-06
10	8.99	0.953759692	0	0	
11	8.95	0.951823035	3.75064E-06	1.40673E-11	
12	8.9818	0.95336338	2.37266E-06	5.62953E-12	
13	8.99	0.953759692	1.57063E-07	2.46687E-14	
14	9.02	0.955206538	2.09336E-06	4.38217E-12	1.23824E-05
15	9.01	0.954724791	2.3208E-07	5.3861E-14	
16	9	0.954242509	2.32595E-07	5.41007E-14	
17	9	0.954242509	0	0	
18	9	0.954242509	0	0	
19	8.97	0.952792443	2.10269E-06	4.42132E-12	4.0626E-06
20	8.9599	0.952303163	2.39395E-07	5.73101E-14	
21	8.95	0.951823035	2.30522E-07	5.31405E-14	
22	8.9	0.949390007	5.91963E-06	3.5042E-11	
23	8.96	0.95230801	8.51474E-06	7.25008E-11	

t	S_t	$X_t = logS_t$	$(X_{i+1} - X_i)^2$	$(X_{i+1} - X_i)^4$	Variance
24	8.9	0.949390007	8.51474E-06	7.25008E-11	3.4137E-05
25	8.83	0.945960704	1.17601E-05	1.383E-10	
26	8.86	0.947433722	2.16978E-06	4.70796E-12	
27	8.85	0.946943271	2.40542E-07	5.78606E-14	
28	8.87	0.94792362	9.61084E-07	9.23683E-13	
29	8.93	0.950851459	8.57224E-06	7.34833E-11	1.92088E-05
30	8.93	0.950851459	0	0	
31	8.9	0.949390007	2.13584E-06	4.56182E-12	
32	8.93	0.950851459	2.13584E-06	4.56182E-12	
33	8.9	0.949390007	2.13584E-06	4.56182E-12	
34	8.88	0.948412966	9.54609E-07	9.11278E-13	1.04816E-05
35	8.91	0.949877704	2.14546E-06	4.60299E-12	
36	8.89	0.948901761	9.52465E-07	9.07189E-13	
37	8.9	0.949390007	2.38384E-07	5.68269E-14	
38	8.889	0.948852906	2.88477E-07	8.32189E-14	
39	8.85	0.946943271	3.64671E-06	1.32985E-11	8.21857E-06
40	8.83	0.945960704	9.65438E-07	9.32071E-13	
41	8.76	0.942504106	1.19481E-05	1.42756E-10	
42	8.8	0.944482672	3.91472E-06	1.53251E-11	
43	8.8	0.944482672	0	0	
44	8.81	0.944975908	2.43282E-07	5.91861E-14	2.63738E-05
45	8.79	0.943988875	9.74235E-07	9.49133E-13	
46	8.8	0.944482672	2.43836E-07	5.94558E-14	
47	8.7699	0.942994641	2.21424E-06	4.90284E-12	
48	8.8	0.944482672	2.21424E-06	4.90284E-12	
49	8.78	0.943494516	9.76453E-07	9.5346E-13	8.33435E-06

t	S_t	$X_t = logS_t$	$(X_{i+1} - X_i)^2$	$(X_{i+1} - X_i)^4$	Variance
50	8.78	0.943494516	0	0	
51	8.73	0.941014244	6.15175E-06	3.7844E-11	
52	8.74	0.941511433	2.47197E-07	6.11063E-14	
53	8.76	0.942504106	9.85401E-07	9.71015E-13	
54	8.77	0.942999593	2.45508E-07	6.0274E-14	1.27247E-05
55	8.7709	0.943044159	1.98614E-09	3.94475E-18	
56	8.76	0.942504106	2.91658E-07	8.50642E-14	
57	8.78	0.943494516	9.80911E-07	9.62187E-13	
58	8.82	0.945468585	3.89695E-06	1.51862E-11	
59	8.84	0.946452265	9.67626E-07	9.363E-13	9.52041E-06
60	8.83	0.945960704	2.41633E-07	5.83863E-14	
61	8.82	0.945468585	2.42181E-07	5.86514E-14	
62	8.88	0.948412966	8.66938E-06	7.51581E-11	
63	8.89	0.948901761	2.38921E-07	5.70831E-14	
64	8.89	0.948901761	0	0	1.62344E-05
65	8.8899	0.948896876	2.38655E-11	5.69561E-22	
66	8.87	0.94792362	9.47227E-07	8.97239E-13	
67	8.82	0.945468585	6.0272E-06	3.63271E-11	
68	8.81	0.944975908	2.4273E-07	5.8918E-14	
69	8.8001	0.944487607	2.38438E-07	5.68527E-14	1.24449E-05
70	8.8282	0.945872163	1.917E-06	3.67487E-12	
71	8.82	0.945468585	1.62875E-07	2.65284E-14	
72	8.81	0.944975908	2.4273E-07	5.8918E-14	
73	8.8085	0.944901959	5.46856E-09	2.99052E-17	
74	8.8	0.944482672	1.75801E-07	3.09061E-14	8.65422E-07
75	8.82	0.945468585	9.72024E-07	9.44831E-13	

t	S_t	$X_t = log S_t$	$(X_{i+1} - X_i)^2$	$(X_{i+1} - X_i)^4$	Variance
76	8.84	0.946452265	9.67626E-07	9.363E-13	
77	8.84	0.946452265	0	0	
78	8.8516	0.94702178	3.24347E-07	1.05201E-13	
79	8.869	0.947874655	7.27395E-07	5.29104E-13	3.04263E-06
80	8.87	0.94792362	2.39756E-09	5.74832E-18	
81	8.86	0.947433722	2.4E-07	5.76E-14	
82	8.87	0.94792362	2.4E-07	5.76E-14	
83	8.86	0.947433722	2.4E-07	5.76E-14	
84	8.8599	0.94742882	2.40274E-11	5.77314E-22	1.05944E-06
85	8.8395	0.9464277	1.00224E-06	1.00449E-12	
86	8.8401	0.946457178	8.68933E-10	7.55044E-19	
87	8.87	0.94792362	2.15045E-06	4.62444E-12	
88	8.84	0.946452265	2.16489E-06	4.68673E-12	
89	8.8505	0.946967806	2.65783E-07	7.06406E-14	7.0829E-06
90	8.872	0.948021533	1.11034E-06	1.23285E-12	
91	8.89	0.948901761	7.74801E-07	6.00317E-13	
92	8.84	0.946452265	6.00003E-06	3.60004E-11	
93	8.8418	0.946540687	7.81845E-09	6.11282E-17	
94	8.83	0.945960704	3.36381E-07	1.13152E-13	1.20663E-05
95	8.82	0.945468585	2.42181E-07	5.86514E-14	
96	8.8	0.944482672	9.72024E-07	9.44831E-13	
97	8.8218	0.945557208	1.15463E-06	1.33316E-12	
98	8.81	0.944975908	3.37909E-07	1.14182E-13	
99	8.82	0.945468585	2.4273E-07	5.8918E-14	3.98559E-06
100	8.81	0.944975908	2.4273E-07	5.8918E-14	

t	S_t	$X_t = logS_t$	$(X_{i+1} - X_i)^2$	$(X_{i+1} - X_i)^4$	Variance
101	8.83	0.945960704	9.69822E-07	9.40554E-13	
102	8.81	0.944975908	9.69822E-07	9.40554E-13	
103	8.81	0.944975908	0	0	
104	8.84	0.946452265	2.17963E-06	4.75078E-12	6.22195E-06
105	8.84	0.946452265	0	0	
106	8.84	0.946452265	0	0	
107	8.86	0.947433722	9.63258E-07	9.27865E-13	
108	8.85	0.946943271	2.40542E-07	5.78606E-14	
109	8.8384	0.946373653	3.24465E-07	1.05277E-13	2.3811E-06
110	8.82	0.945468585	8.19147E-07	6.71002E-13	
111	8.8	0.944482672	9.72024E-07	9.44831E-13	
112	8.8099	0.944970979	2.38443E-07	5.68553E-14	
113	8.81	0.944975908	2.43009E-11	5.90532E-22	
114	8.82	0.945468585	2.4273E-07	5.8918E-14	2.2941E-06
115	8.83	0.945960704	2.42181E-07	5.86514E-14	
116	8.83	0.945960704	0	0	
117	8.85	0.946943271	9.65438E-07	9.32071E-13	
118	8.8311	0.946014803	8.62053E-07	7.43135E-13	
119	8.83	0.945960704	2.9267E-09	8.56558E-18	2.88721E-06
120	8.82	0.945468585	2.42181E-07	5.86514E-14	
121	8.81	0.944975908	2.4273E-07	5.8918E-14	
122	8.84	0.946452265	2.17963E-06	4.75078E-12	
123	8.83	0.945960704	2.41633E-07	5.83863E-14	
124	8.83	0.945960704	0	0	4.46549E-06
125	8.84	0.946452265	2.41633E-07	5.83863E-14	

t	S_t	$X_t = logS_t$	$(X_{i+1} - X_i)^2$	$(X_{i+1} - X_i)^4$	Variance
126	8.84	0.946452265	0	0	
127	8.84	0.946452265	0	0	
128	8.84	0.946452265	0	0	
129	8.8501	0.946948178	2.4593E-07	6.04814E-14	4.4673E-07
130	8.83	0.945960704	9.75106E-07	9.50831E-13	
131	8.81	0.944975908	9.69822E-07	9.40554E-13	
132	8.9	0.949390007	1.94843E-05	3.79637E-10	
133	8.89	0.948901761	2.38384E-07	5.68269E-14	
134	8.9399	0.951332661	5.90927E-06	3.49195E-11	4.32461E-05
135	8.96	0.95230801	9.51305E-07	9.04982E-13	
136	8.97	0.952792443	2.34676E-07	5.50727E-14	
137	8.9701	0.952797285	2.34411E-11	5.49487E-22	
138	8.95	0.951823035	9.49162E-07	9.00908E-13	
139	8.96	0.95230801	2.352E-07	5.53191E-14	2.24016E-06
140	8.972	0.952889265	3.37858E-07	1.14148E-13	
141	8.9918	0.953846639	9.16564E-07	8.4009E-13	
142	8.96	0.95230801	2.36738E-06	5.60448E-12	
143	9.02	0.955206538	8.40146E-06	7.05846E-11	
144	9.02	0.955206538	0	0	1.88515E-05
145	9.0699	0.957602499	5.74063E-06	3.29548E-11	
146	9.06	0.957128198	2.24962E-07	5.06077E-14	
147	9.04	0.95616843	9.21153E-07	8.48523E-13	
148	9.01	0.954724791	2.08409E-06	4.34345E-12	
149	8.99	0.953759692	9.31417E-07	8.67537E-13	6.1799E-06
150	9.0101	0.954729611	9.40744E-07	8.84998E-13	

t	S_t	$X_t = logS_t$	$(X_{i+1} - X_i)^2$	$(X_{i+1} - X_i)^4$	Variance
151	8.9915	0.953832149	8.05439E-07	6.48732E-13	
152	9	0.954242509	1.68396E-07	2.83572E-14	
153	9.03	0.95568775	2.08872E-06	4.36276E-12	
154	9.05	0.956648579	9.23192E-07	8.52284E-13	5.96769E-06
155	9.0501	0.956653378	2.30286E-11	5.30314E-22	
156	9.05	0.956648579	2.30286E-11	5.30314E-22	
157	9.06	0.957128198	2.30034E-07	5.29156E-14	
158	9.05	0.956648579	2.30034E-07	5.29156E-14	
159	9.01	0.954724791	3.70096E-06	1.36971E-11	7.19453E-06
160	9.02	0.955206538	2.3208E-07	5.3861E-14	
161	9.04	0.95616843	9.25238E-07	8.56065E-13	
162	9.05	0.956648579	2.30543E-07	5.315E-14	
163	9.07	0.957607287	9.19121E-07	8.44783E-13	
164	9.05	0.956648579	9.19121E-07	8.44783E-13	4.31027E-06
165	9.06	0.957128198	2.30034E-07	5.29156E-14	
166	9.04	0.95616843	9.21153E-07	8.48523E-13	
167	9.04	0.95616843	0	0	
168	9.02	0.955206538	9.25238E-07	8.56065E-13	
169	9.0415	0.956240487	1.06905E-06	1.14287E-12	4.29323E-06
170	9.04	0.95616843	5.19209E-09	2.69578E-17	
171	9.05	0.956648579	2.30543E-07	5.315E-14	
172	9.03	0.95568775	9.23192E-07	8.52284E-13	
173	9.04	0.95616843	2.31053E-07	5.33857E-14	
174	9.07	0.957607287	2.07031E-06	4.28618E-12	5.32503E-06
175	9.06	0.957128198	2.29527E-07	5.26825E-14	

t	S_t	$X_t = logS_t$	$(X_{i+1} - X_i)^2$	$(X_{i+1} - X_i)^4$	Variance
176	9.06	0.957128198	0	0	
177	9.07	0.957607287	2.29527E-07	5.26825E-14	
178	9.06	0.957128198	2.29527E-07	5.26825E-14	
179	9.05	0.956648579	2.30034E-07	5.29156E-14	1.01393E-06
180	9.06	0.957128198	2.30034E-07	5.29156E-14	
181	9.1	0.959041392	3.66031E-06	1.33979E-11	
182	9.13	0.960470778	2.04314E-06	4.17443E-12	
183	9.17	0.962369336	3.60452E-06	1.29926E-11	
184	9.18	0.962842681	2.24056E-07	5.02011E-14	1.40498E-05
185	9.17	0.962369336	2.24056E-07	5.02011E-14	
186	9.1618	0.961980807	1.50955E-07	2.27873E-14	
187	9.18	0.962842681	7.42827E-07	5.51792E-13	
188	9.19	0.963315511	2.23568E-07	4.99828E-14	
189	9.2	0.963787827	2.23082E-07	4.97657E-14	2.01092E-06
190	9.1797	0.962828488	9.20331E-07	8.4701E-13	
191	9.1708	0.962407222	1.77465E-07	3.14939E-14	
192	9.2	0.963787827	1.90607E-06	3.6331E-12	
193	9.2	0.963787827	0	0	
194	9.25	0.966141733	5.54087E-06	3.07012E-11	1.24109E-05
195	9.25	0.966141733	0	0	
196	9.25	0.966141733	0	0	
197	9.27	0.967079734	8.79847E-07	7.7413E-13	
198	9.26	0.966610987	2.19724E-07	4.82787E-14	
199	9.2701	0.967084419	2.24138E-07	5.02379E-14	2.08644E-06
200	9.26	0.966610987	2.24138E-07	5.02379E-14	

t	S_t	$X_t = logS_t$	$(X_{i+1} - X_i)^2$	$(X_{i+1} - X_i)^4$	Variance
201	9.25	0.966141733	2.20199E-07	4.84877E-14	
202	9.25	0.966141733	0	0	
203	9.2601	0.966615677	2.24623E-07	5.04554E-14	
204	9.25	0.966141733	2.24623E-07	5.04554E-14	9.85038E-07
205	9.3211	0.969467167	1.10585E-05	1.22291E-10	
206	9.36	0.971275849	3.27133E-06	1.07016E-11	
207	9.368	0.971646882	1.37666E-07	1.89519E-14	
208	9.36	0.971275849	1.37666E-07	1.89519E-14	
209	9.4	0.973127854	3.42992E-06	1.17644E-11	1.08499E-05
210	9.36	0.971275849	3.42992E-06	1.17644E-11	
211	9.39	0.972665592	1.93139E-06	3.73026E-12	
212	9.42	0.974050903	1.91909E-06	3.68289E-12	
213	9.45	0.975431809	1.9069E-06	3.63627E-12	
214	9.47	0.976349979	8.43037E-07	7.10711E-13	9.40042E-06
215	9.41	0.973589623	7.61956E-06	5.80577E-11	
216	9.38	0.972202838	1.92317E-06	3.69859E-12	
217	9.3801	0.972207468	2.14367E-11	4.59533E-22	
218	9.42	0.974050903	3.39825E-06	1.15481E-11	
219	9.38	0.972202838	3.41534E-06	1.16646E-11	1.29724E-05
220	9.37	0.971739591	2.14598E-07	4.60524E-14	
221	9.1708	0.962407222	8.70931E-05	7.58521E-09	
222	9.2	0.963787827	1.90607E-06	3.6331E-12	
223	9.2	0.963787827	0	0	
224	9.25	0.966141733	5.54087E-06	3.07012E-11	0.000165812
225	9.25	0.966141733	0	0	

t	S_t	$X_t = logS_t$	$(X_{i+1} - X_i)^2$	$(X_{i+1} - X_i)^4$	Variance
226	9.25	0.966141733	0	0	
227	9.27	0.967079734	8.79847E-07	7.7413E-13	
228	9.26	0.966610987	2.19724E-07	4.82787E-14	
229	9.2701	0.967084419	2.24138E-07	5.02379E-14	2.08644E-06
230	9.26	0.966610987	2.24138E-07	5.02379E-14	
231	9.25	0.966141733	2.20199E-07	4.84877E-14	
232	9.25	0.966141733	0	0	
233	9.2601	0.966615677	2.24623E-07	5.04554E-14	
234	9.25	0.966141733	2.24623E-07	5.04554E-14	9.85038E-07
235	9.3211	0.969467167	1.10585E-05	1.22291E-10	
236	9.36	0.971275849	3.27133E-06	1.07016E-11	
237	9.368	0.971646882	1.37666E-07	1.89519E-14	
238	9.36	0.971275849	1.37666E-07	1.89519E-14	
239	9.4	0.973127854	3.42992E-06	1.17644E-11	1.08499E-05
240	9.36	0.971275849	3.42992E-06	1.17644E-11	
241	9.39	0.972665592	1.93139E-06	3.73026E-12	
242	9.42	0.974050903	1.91909E-06	3.68289E-12	
243	9.45	0.975431809	1.9069E-06	3.63627E-12	
244	9.47	0.976349979	8.43037E-07	7.10711E-13	9.40042E-06
245	9.41	0.973589623	7.61956E-06	5.80577E-11	
246	9.38	0.972202838	1.92317E-06	3.69859E-12	
247	9.3801	0.972207468	2.14367E-11	4.59533E-22	
248	9.42	0.974050903	3.39825E-06	1.15481E-11	
249	9.38	0.972202838	3.41534E-06	1.16646E-11	1.29724E-05
250	9.37	0.971739591	2.14598E-07	4.60524E-14	

t	S_t	$X_t = logS_t$	$(X_{i+1} - X_i)^2$	$(X_{i+1} - X_i)^4$	Variance
251	9.34	0.970346876	1.93965E-06	3.76226E-12	
252	9.37	0.971739591	1.93965E-06	3.76226E-12	
253	9.32	0.969415912	5.39948E-06	2.91544E-11	
254	9.3	0.968482949	8.70421E-07	7.57634E-13	1.5145E-05
255	9.29	0.968015714	2.18308E-07	4.76584E-14	
256	9.31	0.968949681	8.72294E-07	7.60897E-13	
257	9.3	0.968482949	2.17839E-07	4.74539E-14	
258	9.34	0.970346876	3.47423E-06	1.20702E-11	
259	9.36	0.971275849	8.6299E-07	7.44752E-13	8.44102E-06
260	9.36	0.971275849	0	0	
261	9.39	0.972665592	1.93139E-06	3.73026E-12	
262	9.4	0.973127854	2.13686E-07	4.56615E-14	
263	9.39	0.972665592	2.13686E-07	4.56615E-14	
264	9.359	0.971229447	2.06251E-06	4.25396E-12	6.74155E-06
265	9.3282	0.969797849	2.04947E-06	4.20034E-12	
266	9.25	0.966141733	1.33672E-05	1.78682E-10	
267	9.28	0.967547976	1.97752E-06	3.91059E-12	
268	9.3	0.968482949	8.74173E-07	7.64179E-13	
269	9.27	0.967079734	1.96901E-06	3.877E-12	2.93603E-05
270	9.2	0.963787827	1.08367E-05	1.17433E-10	
271	9.08	0.958085849	3.25126E-05	1.05707E-09	
272	9.18	0.962842681	2.26275E-05	5.12002E-10	
273	9.18	0.962842681	0	0	
274	9.1401	0.960950947	3.57866E-06	1.28068E-11	9.1193E-05
275	9.14	0.960946196	2.25773E-11	5.09733E-22	

t	S_t	$X_t = logS_t$	$(X_{i+1} - X_i)^2$	$(X_{i+1} - X_i)^4$	Variance
276	9.13	0.960470778	2.26022E-07	5.10862E-14	
277	9.07	0.957607287	8.19958E-06	6.72331E-11	
278	9.1292	0.960432722	7.98308E-06	6.37296E-11	
279	9.168	0.962274605	3.39253E-06	1.15093E-11	2.95488E-05
280	9.15	0.961421094	7.2848E-07	5.30684E-13	
281	9.19	0.963315511	3.58882E-06	1.28796E-11	
282	9.24	0.965671971	5.5529E-06	3.08347E-11	
283	9.2	0.963787827	3.55E-06	1.26025E-11	
284	9.219	0.964683815	8.02794E-07	6.44478E-13	1.96568E-05
285	9.2101	0.964264346	1.75955E-07	3.096E-14	
286	9.22	0.964730921	2.17693E-07	4.73901E-14	
287	9.22	0.964730921	0	0	
288	9.2	0.963787827	8.89426E-07	7.91078E-13	
289	9.222	0.964825118	1.07597E-06	1.15772E-12	3.33669E-06
290	9.245	0.965906915	1.17029E-06	1.36957E-12	
291	9.2201	0.964735631	1.37191E-06	1.88213E-12	
292	9.13	0.960470778	1.8189E-05	3.30839E-10	
293	9.1699	0.9623646	3.58656E-06	1.28634E-11	
294	9.14	0.960946196	2.01187E-06	4.04762E-12	4.04265E-05
295	9.15	0.961421094	2.25528E-07	5.08631E-14	
296	9.155	0.961658349	5.62897E-08	3.16853E-15	
297	9.14	0.960946196	5.07162E-07	2.57213E-13	
298	9.16	0.961895474	9.01129E-07	8.12033E-13	
299	9.13	0.960470778	2.02976E-06	4.11992E-12	5.35487E-06
300	8.9382	0.951250068	8.50215E-05	7.22865E-09	

t	S_t	$X_t = logS_t$	$(X_{i+1} - X_i)^2$	$(X_{i+1} - X_i)^4$	Variance
301	9.15	0.961421094	0.00010345	1.07019E-08	
302	9.28	0.967547976	3.75387E-05	1.40915E-09	
303	9.22	0.964730921	7.9358E-06	6.29769E-11	
304	9.24	0.965671971	8.85575E-07	7.84244E-13	0.000239902
305	9.13	0.960470778	2.70524E-05	7.31833E-10	
306	9.08	0.958085849	5.68789E-06	3.23521E-11	
307	8.99	0.953759692	1.87156E-05	3.50275E-10	
308	9.01	0.954724791	9.31417E-07	8.67537E-13	
309	9.04	0.95616843	2.08409E-06	4.34345E-12	4.34988E-05
310	9	0.954242509	3.70917E-06	1.3758E-11	
311	8.94	0.951337519	8.43897E-06	7.12162E-11	
312	8.83	0.945960704	2.89101E-05	8.35796E-10	
313	8.95	0.951823035	3.43669E-05	1.18109E-09	
314	9.0401	0.956173235	1.89242E-05	3.58127E-10	0.000131024
315	9.04	0.95616843	2.30795E-11	5.32665E-22	
316	9.05	0.956648579	2.30543E-07	5.315E-14	
317	9.09	0.958563883	3.66839E-06	1.34571E-11	
318	9.15	0.961421094	8.16365E-06	6.66452E-11	
319	9.18	0.962842681	2.02091E-06	4.08408E-12	2.15775E-05
320	9.15	0.961421094	2.02091E-06	4.08408E-12	
321	9.23	0.965201701	1.4293E-05	2.0429E-10	
322	9.18	0.962842681	5.56497E-06	3.09689E-11	
323	9.17	0.962369336	2.24056E-07	5.02011E-14	
324	9.14	0.960946196	2.02533E-06	4.10195E-12	3.47409E-05
325	9.0801	0.958090631	8.15425E-06	6.64917E-11	

t	S_t	$X_t = logS_t$	$(X_{i+1} - X_i)^2$	$(X_{i+1} - X_i)^4$	Variance
326	9.1	0.959041392	9.03946E-07	8.17119E-13	
327	9.17	0.962369336	1.10752E-05	1.2266E-10	
328	9.1801	0.962847412	2.28557E-07	5.22383E-14	
329	9.1901	0.963320237	2.23564E-07	4.99806E-14	2.1508E-05
330	9.15	0.961421094	3.60674E-06	1.30086E-11	
331	9.1799	0.96283795	2.00748E-06	4.02998E-12	
332	9.17	0.962369336	2.196E-07	4.8224E-14	
333	9.16	0.961895474	2.24545E-07	5.04205E-14	
334	9.19	0.963315511	2.01651E-06	4.0663E-12	6.8055E-06
335	9.17	0.962369336	8.95248E-07	8.0147E-13	
336	9.18	0.962842681	2.24056E-07	5.02011E-14	
337	9.1684	0.962293552	3.01542E-07	9.09278E-14	
338	9.23	0.965201701	8.45733E-06	7.15264E-11	
339	9.2	0.963787827	1.99904E-06	3.99616E-12	1.80843E-05
340	9.21	0.96425963	2.22598E-07	4.95498E-14	
341	9.23	0.965201701	8.87497E-07	7.87652E-13	
342	9.29	0.968015714	7.91867E-06	6.27053E-11	
343	9.32	0.969415912	1.96056E-06	3.84378E-12	
344	9.35	0.970811611	1.94797E-06	3.7946E-12	1.9601E-05
345	9.39	0.972665592	3.43725E-06	1.18147E-11	
346	9.35	0.970811611	3.43725E-06	1.18147E-11	
347	9.32	0.969415912	1.94797E-06	3.7946E-12	
348	9.28	0.967547976	3.48919E-06	1.21744E-11	
349	9.31	0.968949681	1.96478E-06	3.86035E-12	1.54322E-05
350	9.3	0.968482949	2.17839E-07	4.74539E-14	

t	S_t	$X_t = logS_t$	$(X_{i+1} - X_i)^2$	$(X_{i+1} - X_i)^4$	Variance
351	9.32	0.969415912	8.70421E-07	7.57634E-13	
352	9.25	0.966141733	1.07203E-05	1.14924E-10	
353	9.3115	0.969019648	8.28239E-06	6.8598E-11	
354	9.34	0.970346876	1.76154E-06	3.10301E-12	3.28114E-05
355	9.3097	0.968935686	1.99146E-06	3.9659E-12	
356	9.3	0.968482949	2.04971E-07	4.20133E-14	
357	9.3101	0.968954346	2.22215E-07	4.93797E-14	
358	9.32	0.969415912	2.13044E-07	4.53876E-14	
359	9.36	0.971275849	3.45936E-06	1.19672E-11	6.94025E-06
360	9.34	0.970346876	8.6299E-07	7.44752E-13	
361	9.3401	0.970351526	2.16207E-11	4.67456E-22	
362	9.38	0.972202838	3.42736E-06	1.17468E-11	
363	9.39	0.972665592	2.14141E-07	4.58564E-14	
364	9.4	0.973127854	2.13686E-07	4.56615E-14	6.66451E-06
35	9.4	0.973127854	0	0	
366	9.38	0.972202838	8.55653E-07	7.32142E-13	
367	9.3701	0.971744226	2.10325E-07	4.42368E-14	
368	9.34	0.970346876	1.95259E-06	3.81259E-12	
369	9.31	0.968949681	1.95215E-06	3.81091E-12	7.33713E-06
370	9.2418	0.965756566	1.0196E-05	1.03958E-10	
371	9.282	0.967641564	3.55322E-06	1.26254E-11	
372	9.27	0.967079734	3.15653E-07	9.96367E-14	
373	9.26	0.966610987	2.19724E-07	4.82787E-14	
374	9.24	0.965671971	8.8175E-07	7.77483E-13	7.97598E-06
375	9.27	0.967079734	1.9818E-06	3.92752E-12	

t	S_t	$X_t = logS_t$	$(X_{i+1} - X_i)^2$	$(X_{i+1} - X_i)^4$	Variance
376	9.24	0.965671971	1.9818E-06	3.92752E-12	
377	9.25	0.966141733	2.20676E-07	4.86978E-14	
378	9.24	0.965671971	2.20676E-07	4.86978E-14	
379	9.23	0.965201701	2.21154E-07	4.89091E-14	4.2923E-06
380	9.24	0.965671971	2.21154E-07	4.89091E-14	
381	9.2318	0.965286387	1.48675E-07	2.21043E-14	
382	9.28	0.967547976	5.11478E-06	2.6161E-11	
383	9.3	0.968482949	8.74173E-07	7.64179E-13	
384	9.28	0.967547976	8.74173E-07	7.64179E-13	1.131E-05
385	9.22	0.964730921	7.9358E-06	6.29769E-11	
386	9.28	0.967547976	7.9358E-06	6.29769E-11	
387	9.26	0.966610987	8.77949E-07	7.70795E-13	
388	9.25	0.966141733	2.20199E-07	4.84877E-14	
389	9.28	0.967547976	1.97752E-06	3.91059E-12	1.77299E-05
390	9.3	0.968482949	8.74173E-07	7.64179E-13	
391	9.31	0.968949681	2.17839E-07	4.74539E-14	
392	9.36	0.971275849	5.41106E-06	2.92795E-11	
393	9.39	0.972665592	1.93139E-06	3.73026E-12	
394	9.3	0.968482949	1.74945E-05	3.06058E-10	4.00906E-05
395	9.3	0.968482949	0	0	
396	9.32	0.969415912	8.70421E-07	7.57634E-13	
397	9.3	0.968482949	8.70421E-07	7.57634E-13	
398	9.321	0.969462508	9.59537E-07	9.20711E-13	
399	9.29	0.968015714	2.09321E-06	4.38154E-12	6.9255E-06
400	9.3101	0.968954346	8.8103E-07	7.76213E-13	

t	S_t	$X_t = logS_t$	$(X_{i+1} - X_i)^2$	$(X_{i+1} - X_i)^4$	Variance
401	9.32	0.969415912	2.13044E-07	4.53876E-14	
402	9.31	0.968949681	2.17372E-07	4.72505E-14	
403	9.321	0.969462508	2.62992E-07	6.91645E-14	
404	9.32	0.969415912	2.17115E-09	4.7139E-18	1.02402E-06
405	9.3102	0.968959011	2.08759E-07	4.35804E-14	
406	9.31	0.968949681	8.70402E-11	7.576E-21	
407	9.3	0.968482949	2.17839E-07	4.74539E-14	
408	9.28	0.967547976	8.74173E-07	7.64179E-13	
409	9.31	0.968949681	1.96478E-06	3.86035E-12	4.82171E-06
410	9.2165	0.964566027	1.92164E-05	3.69271E-10	
411	9.24	0.965671971	1.22311E-06	1.496E-12	
412	9.25	0.966141733	2.20676E-07	4.86978E-14	
413	9.19	0.963315511	7.98753E-06	6.38006E-11	
414	9.1212	0.960051979	1.06506E-05	1.13436E-10	3.09993E-05
415	9.01	0.954724791	2.83789E-05	8.05364E-10	
416	9.02	0.955206538	2.3208E-07	5.3861E-14	
417	9.07	0.957607287	5.7636E-06	3.32191E-11	
418	9.08	0.958085849	2.29021E-07	5.24507E-14	
419	9.1	0.959041392	9.13064E-07	8.33686E-13	1.19098E-05