

Bicovariant Calculus in Quantum Theory and a Generalization
of the Gauss Law

G. Bimonte – Universita di Napoli, Italy

G. Marmo – Universita di Napoli, Italy

A. Stern – University of Alabama

Deposited 05/02/2019

Citation of published version:

Bimonte, G., Marmo, G., Stern, A. (2000): Bicovariant Calculus in Quantum Theory and a Generalization of the Gauss Law. *Physics Letters B*, 478(1-3).

DOI: [https://doi.org/10.1016/S0370-2693\(00\)00245-8](https://doi.org/10.1016/S0370-2693(00)00245-8)



ELSEVIER

6 April 2000

PHYSICS LETTERS B

Physics Letters B 478 (2000) 358–364

Bicovariant calculus in quantum theory and a generalization of the Gauss law

G. Bimonte ^{a,b}, G. Marmo ^{a,b}, A. Stern ^c^a *Dipartimento di Scienze Fisiche, Università di Napoli, Mostra d'Oltremare, Pad.20, I-80125, Napoli, Italy*^b *INFN, Sezione di Napoli, Napoli, Italy*^c *Department of Physics, University of Alabama, Tuscaloosa, AL 35487, USA*

Received 22 December 1999; accepted 11 February 2000

Editor: M. Cvetič

Abstract

We construct a deformation of the quantum algebra $\text{Fun}(T^*G)$ associated with Lie group G to the case where G is replaced by a quantum group G_q which has a bicovariant calculus. The deformation easily allows for the inclusion of the current algebra of left and right invariant one forms. We use it to examine a possible generalization of the Gauss law commutation relations for gauge theories based on G_q . © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Quantum groups; Bicovariant calculus; Cotangent bundles

1. Introduction

For many descriptions of dynamical systems where continuous symmetries are present, one requires more than the existence of a Lie group G and its corresponding Lie algebra g . One often relies on the existence of a bicovariant calculus on G . This is especially true when constructing currents and current algebras. A bicovariant calculus is also useful for gauge theories. Loosely speaking, the bicovariant calculus allows for compatible left and right actions on a bimodule constructed on the group. The bimodule is spanned by the left or right invariant one forms. While there is a natural construction of a bicovariant calculus for any Lie group G , the same cannot be said for quantum groups. The notion of a bicovariant calculus for quantum groups was introduced by Woronowicz [1]. (Also see Refs. [2–7].) Only for a class of quantum groups $\{G_q\}$ does a

structure analogous to the one for Lie groups exist. There have subsequently been several attempts to apply this structure to dynamical systems [8,9]. These works generalize the usual classical Lagrangians for gauge theories, including Chern–Simons theory and gravity, to the case where the ‘gauge group’ is G_q . However, such systems have the novel feature that the ‘classical’ field degrees of freedom end up being noncommuting operators. Their physical interpretation is, to say the least, obscure. The situation is somewhat reminiscent of the ‘classical’ Lagrangian description for fermions expressed in terms of Grassmann fields. That description derives a meaning due to the existence of a canonical quantization scheme which turns it into the usual quantum mechanical description of fermions. An analogous canonical quantization scheme for ‘classical’ Lagrangian descriptions of quantum group gauge symmetries is not obvious. This being the case, there is currently no

much use in having such Lagrangians, aside from being a curiosity.

Under these circumstances it is reasonable to skip the step of canonical quantization entirely, and start instead with the quantum theory. That is, we can try to incorporate the structure of a bicovariant calculus for a quantum group G_q in quantum theory. For ordinary Lie groups G , a natural setting for a bicovariant calculus is provided by the cotangent bundle T^*G . Left and right actions on T^*G are obtained via the pullback from G of right invariant or left invariant one-forms, and they are generated using the corresponding canonical momenta coordinatizing the fibre, via the trivialization $T^*G = G \otimes g^*$. (g^* is the co-Lie algebra associated with G .) In the quantum theory, one defines the algebra of functions $\text{Fun}(T^*G)$ on T^*G . This is the algebra of operators for the rigid rotor, and it is generated by elements of G and their conjugate momenta. The algebra is easily enlarged to include the current algebra of the left and right invariant one forms.

Our task is to find (i) a deformation of $\text{Fun}(T^*G)$ to an algebra \mathcal{B} where G is replaced by G_q , and then (ii) enlarge that algebra to include the left and right invariant one forms. For the latter, the existence of an exterior derivative is assumed with the usual Leibniz rule. (For an alternative to the usual Leibniz rule, see Ref. [10].) We shall carry out (i) and (ii) for quantum groups G_q having a bicovariant calculus. With regard to (i), deformations of $\text{Fun}(T^*G)$ known as Heisenberg doubles exist for a large class of quantum groups, which do not necessarily possess a bicovariant calculus [11]. We instead find it convenient to write down a deformation algebra \mathcal{B} exclusively for quantum groups which possess a bicovariant calculus, as it is easily extended to the bimodule. A bicovariant calculus is characterized by braiding matrices, structure constants, as well as the R matrix for G_q , and we express \mathcal{B} directly in terms of these constants. The algebra is shown to be consistent with associativity, and although it is not in general a Hopf algebra, it contains three Hopf subalgebras: the quantum group G_q , as well as two algebras \mathcal{A}_L and \mathcal{A}_R which are deformations of the Lie algebra associated with G . \mathcal{A}_L and \mathcal{A}_R contain the analogues of generators of the left and right actions of Lie groups, which we denote by $\{\ell_i\}$ and $\{r_i\}$, respectively. These algebras commute with each other, and they may or

may not be isomorphic. (We find them to be isomorphic for simple examples, like $G_q = U_q(2)$.) Their center elements are shown to coincide. All of the above properties of \mathcal{B} are shared by the Heisenberg double [11]¹, and most likely any \mathcal{B} is equivalent to some quantum double. As stated earlier, using \mathcal{B} we can easily carry out step (ii), i.e. extend the algebra to include the left and right invariant one forms. In this regard we get that $\{\ell_i\}$ commutes with the left invariant one form, and $\{r_i\}$ commutes with the right invariant one form, as in the undeformed case. The remaining commutation relations can be expressed in terms of deformed commutators.

The work presented here sets the stage for a variety of generalizations of quantum systems, such as Wess–Zumino–Witten models and gauge theories, to ones based on quantum groups. With regard to the latter, we propose a simple generalization of the Gauss law commutation relations. We find that the resulting algebra is associative only for a certain restricted class of deformations, known as ‘minimal’ [8]. For such deformations we express the Gauss law in terms of the analogue of potentials and electric fields and specify their algebra. We plan to explore its consequences in future works.

We first review the algebra $\text{Fun}(T^*G)$ of functions on the cotangent bundle T^*G associated with Lie group G . It is generated by group elements, along with elements of the Lie algebra. For the former one can write matrices $[T_b^a]$ in the fundamental representation.² The latter, which we denote by r_i (or ℓ_i) are the generators of right (or left) transformations of G on T^*G and project onto corresponding actions on the group G . (The indices a, b, \dots label the fundamental representation, while i, j, \dots are Lie algebra indices.) In terms of the right generators, the algebra is given by

$$[T_b^a, T_d^c] = 0 \quad (1)$$

$$[r_i, r_j] = c_{ji}^k r_k \quad (2)$$

$$[T_b^a, r_i] = T_c^a [\lambda_i]_b^c \quad (3)$$

where c_{ij}^k are structure constants and $\{[\lambda_i]_c^a\}$ spans the fundamental representation of the Lie algebra,

¹ We thank S. Frolov for bringing this to our attention.

² We restrict here our attention to subgroups of the general linear group.

i.e. $[\lambda_i, \lambda_j] = c_{ij}^k \lambda_k$. Eqs. (1-3) define the quantum algebra for a rigid rotor (where here we set $\hbar = 1$). The left generators ℓ_i can be expressed in terms of the right generators upon introducing adjoint matrices M_j^i satisfying

$$\lambda_j T = T \lambda_i M_j^i \quad (4)$$

Their commutation relations are

$$[M_j^i, M_\ell^k] = 0 \quad (5)$$

$$[M_j^i, r_k] = c_{\ell k}^i M_j^\ell \quad (6)$$

Defining $\ell_i = r_j M_i^j$, gives

$$[\ell_i, r_j] = 0 \quad (7)$$

$$[\ell_i, \ell_j] = c_{ij}^k \ell_k \quad (8)$$

$$[T_b^a, \ell_i] = [\lambda_i]_c^a T_b^c \quad (9)$$

where we used

$$c_{ij}^k M_k^\ell = M_i^k M_j^n c_{kn}^\ell \quad (10)$$

Upon introducing the left and right invariant one forms:

$$\omega_L = T^{-1} dT, \quad \omega_R = dTT^{-1} \quad (11)$$

it is easy to check that ω_L commutes with ℓ_i , and ω_R commutes with r_i . The nonvanishing commutation relations of the generators ℓ_i and r_i with $\omega_L = \omega_L^i \lambda_i$ and $\omega_R = \omega_R^i \lambda_i$ are

$$[\omega_L^i, r_j] = c_{kj}^i \omega_L^k \quad (12)$$

$$[\ell_j, \omega_R^i] = c_{kj}^i \omega_R^k \quad (12)$$

The components ω_L^i and ω_R^i are related by

$$\omega_L^i = \omega_R^j M_j^i \quad (13)$$

We now show that there is a completely analogous structure for quantum groups G_q which admit a bicovariant calculus in the sense of Woronowicz [1]. Quantum groups can be thought of as deformations of the (commutative) Hopf algebra $\text{Fun}(G)$ of functions on a Lie group G . The deformation of $\text{Fun}(G)$ to a noncommutative Hopf algebra G_q depends on at least one deformation parameter q , and reduces to $\text{Fun}(G)$ in the limit $q \rightarrow 1$. The Hopf algebra is

generated by quantum group matrices for which we again use the notation T . Their coproduct is the usual one: $\Delta(T_b^a) = T_c^a \otimes T_b^c$, and they satisfy commutation relations:

$$R_{ef}^{ab} T_c^e T_d^f - T_f^b T_e^a R_{cd}^{ef} = 0 \quad (14)$$

R obeys the Yang–Baxter equation, and this is consistent with having an associative algebra for the matrix elements T_b^a . We want to enlarge this algebra to an associative algebra \mathcal{B} which is a deformation of $\text{Fun}(T^*G)$. For this we reintroduce elements r_i , which now are the analogues of the right generators on the group. For their commutation relations we take:

$$r_i r_j - \Lambda_{ji}^{mn} r_n r_m = c_{ji}^k r_k \quad (15)$$

$$T_b^a r_i - r_j T_c^a [f_i^j]_b^c = T_c^a [\chi_i]_b^c \quad (16)$$

which, along with (14), generalize (1-3). Eq. (15) defines a deformed Lie algebra. Λ_{ij}^{km} and c_{ij}^k are c-numbers, which are respectively, the braiding matrices and structure constants. For $q \rightarrow 1$, $\Lambda_{ij}^{km} \rightarrow \delta_i^k \delta_j^m$. For $q \neq 1$, the structure constants are not in general antisymmetric, and do not obey the usual Jacobi identities. In systems where there is a bicovariant calculus, one has the following identities for Λ_{ij}^{km} and c_{ij}^k [6]:

$$\Lambda_{kl}^{ij} \Lambda_{sp}^{lm} \Lambda_{qu}^{ks} = \Lambda_{kl}^{jm} \Lambda_{qs}^{ik} \Lambda_{up}^{sl} \quad (17)$$

$$c_{mi}^r c_{rj}^n - \Lambda_{ij}^{kl} c_{mk}^r c_{rl}^n = c_{ij}^k c_{mk}^n \quad (18)$$

$$\Lambda_{mk}^{ir} \Lambda_{nl}^{ks} c_{rs}^j = \Lambda_{kl}^{ij} c_{mn}^k \quad (19)$$

$$\Lambda_{ri}^{jq} \Lambda_{kl}^{si} c_{ps}^r + \Lambda_{pi}^{jq} c_{kl}^i = c_{is}^j \Lambda_{rl}^{sq} \Lambda_{pk}^{ir} + c_{rl}^q \Lambda_{pk}^{jr} \quad (20)$$

The first condition is the Yang–Baxter equation, the second is the analogue of the Jacobi identity, while the last two equations are trivial in the limit $q \rightarrow 1$. From the above conditions the r_i 's (alone) generate a non cocommutative Hopf algebra \mathcal{A}_R . f_i^j and χ_i in Eq. (16) are matrices with c-number coefficients.³ f_i^j goes to δ_i^j times the identity matrix, while χ_i goes to λ_i , in the limit $q \rightarrow 1$, so that Eq. (16) reduces to Eq. (3). The matrices f_i^j and χ_i appear in

³ This differs from the notation in Ref. [6] where f_i^j and χ_i represent functions on G_q .

the construction of a bicovariant calculus. They are required to satisfy [6]

$$\Lambda_{ij}^{nm} f_k^i f_\ell^j = f_i^n f_j^m \Lambda_{k\ell}^{ij} \quad (21)$$

$$\chi_i \chi_j - \Lambda_{ij}^{km} \chi_k \chi_m = c_{ij}^k \chi_k \quad (22)$$

$$\Lambda_{k\ell}^{ij} f_i^n \chi_j = \chi_k f_\ell^n \quad (23)$$

$$c_{mn}^i f_j^m f_k^n + f_j^i \chi_k = \Lambda_{jk}^{mn} \chi_m f_n^i + c_{jk}^\ell f_\ell^i \quad (24)$$

The second equation states that $\{[\chi_i]_c^a\}$ ⁴ defines a matrix representation (the fundamental representation) of a deformed Lie algebra. (From (15) this is not, in general, the algebra generated by the r_i 's. Rather, as we shall show shortly, it is the algebra generated by the analogue of the left generators ℓ_i 's.) Eqs. (21-24) can be viewed as defining an operator algebra for χ_i and f_j^i . In addition to the assumption of the existence of a fundamental matrix representation for χ_i and f_j^i , eqs. (17-20) show that that there is another representation (the adjoint representation) for this algebra. It is obtained by setting $\chi_i = {}^{adj}\chi_i$ and $f_j^i = {}^{adj}f_j^i$, where

$$[{}^{adj}\chi_i]_k^j = c_{ki}^j \quad [{}^{adj}f_j^i]_\ell^k = \Lambda_{jk}^{i\ell} \quad (25)$$

With the help of the bicovariant calculus, i.e. the conditions (17-20), (21-24) and the Yang–Baxter equation for R , one can show that the algebra generated by T_b^a and r_i is consistent with an associative algebra. For example, if we consider the trinomial $Tr_i r_j$ (for simplicity, we suppress the matrix indices on T), by repeated usage of (15) and (16), T can be commuted all the way to the right in two different ways:

$$\begin{aligned} Tr_i r_j &= (r_k T f_i^k + T \chi_i) r_j \\ &= r_k (r_\ell T f_j^\ell + T \chi_j) f_i^k + (r_\ell T f_j^\ell + T \chi_j) \chi_i \\ &= (\Lambda_{\ell k}^{mn} r_n r_m + c_{\ell k}^m r_m) T f_j^\ell f_i^k + r_k T \chi_j f_i^k \\ &\quad + (r_k T f_j^k + T \chi_j) \chi_i \end{aligned} \quad (26)$$

$$\begin{aligned} Tr_i r_j &= T (\Lambda_{ji}^{\ell m} r_m r_\ell + c_{ji}^m r_m) \\ &= \Lambda_{ji}^{\ell m} (r_k T f_m^k + T \chi_m) r_\ell \\ &\quad + c_{ji}^m (r_k T f_m^k + T \chi_m) \\ &= \Lambda_{ji}^{\ell m} r_k (r_n T f_\ell^n + T \chi_\ell) f_m^k \\ &\quad + \Lambda_{ji}^{\ell m} (r_n T f_\ell^n + T \chi_\ell) \chi_m \\ &\quad + c_{ji}^m (r_n T f_\ell^n + T \chi_\ell) \end{aligned} \quad (27)$$

Equating the right hand sides of (26) and (27) leads to no new conditions. Rather, (26) and (27) are identical upon using the bicovariance conditions (21-24).

Another ingredient for a bicovariant calculus is the analogue of the adjoint matrix elements M_j^i which now have values in $\text{Fun}_q(G)$. Its coproduct is $\Delta(M_i^j) = M_i^k \otimes M_k^j$. The analogue of relation (4), i.e.

$$\chi_j T = T \chi_i M_j^i \quad (28)$$

can be adapted, only now the ordering is important since M_i^j are not c-numbers, and furthermore one has [6]

$$M_i^j f_k^i T = T f_i^j M_k^i \quad (29)$$

These relations can be utilized in obtaining the commutation properties for M_i^j :

$$\Lambda_{k\ell}^{ij} M_i^m M_j^n - M_k^i M_\ell^j \Lambda_{ij}^{mn} = 0 \quad (30)$$

$$M_i^j r_k - \Lambda_{nk}^{mj} r_m M_i^n = c_{\ell k}^j M_i^\ell \quad (31)$$

generalizing (5) and (6). They are the commutation relations (14) and (16) written in the adjoint representation (25). Now define the analogue of the left generators on the group $\ell_i = r_j M_i^j$. From (28) and (29), we get

$$T \ell_i - \ell_j f_i^j T = \chi_i T \quad (32)$$

generalizing (9). It is easy to check that ℓ_i commute with r_j as in the undeformed case:

$$\begin{aligned} \ell_i r_k &= r_j M_i^j r_k = r_j (\Lambda_{nk}^{mj} r_m M_i^n + c_{nk}^j M_i^n) \\ &= (\Lambda_{nk}^{mj} r_j r_m + c_{nk}^j r_j) M_i^n = r_k r_n M_i^n = r_k \ell_i \end{aligned} \quad (33)$$

⁴ When writing the elements of a matrix, it is understood that the left most index labels the rows, while the right most one labels the columns.

where we used (15) and (31). The generalization of commutation relations (8) can then be obtained

$$\begin{aligned} \ell_i^j \ell_k^j &= \ell_i^j r_j M_k^j = r_j \ell_i^j M_k^j = r_j r_\ell M_i^{\ell} M_k^j \\ &= (\Lambda_{\ell_j^m}^{m n} r_n r_m + c_{\ell_j^m}^n) M_i^{\ell} M_k^j \\ &= \Lambda_{ik}^{\ell j} r_n r_m M_\ell^m M_j^n + c_{\ell_j^m}^n M_i^{\ell} M_k^m \\ &= \Lambda_{ik}^{\ell j} \ell_\ell \ell_j + c_{ik}^j \ell_j \end{aligned} \quad (34)$$

where we again used (10) which remains valid in the deformed case. We also note that (10) corresponds to (28) written in the adjoint representation. From (34), the left generators satisfy the same commutation relations as the χ_i 's in (22). The χ_i 's therefore provide a matrix representation for the ℓ_i 's. It is the fundamental representation and thus irreducible.

The ℓ_i 's (alone) generate a non cocommutative Hopf algebra \mathcal{A}_L which is, in general, distinct from the Hopf algebra \mathcal{A}_R generated by the r_i 's. However, their centers, if they exist, coincide. For example, if one of the generators of \mathcal{A}_L , say ℓ_0 , belongs to the center of \mathcal{A}_L , then it also belongs to the center of \mathcal{A}_R . To see this, we note by Schur's lemma that the fundamental matrix representation χ_0 of ℓ_0 is a multiple of the identity. Then by (28)

$$T\chi_0 = \chi_0 T = T\chi_j M_0^j \quad (35)$$

Assuming linear independence of the χ_i 's, M_0^j is just δ_0^j times the unit I of the algebra G_q (which in fact we assume to be the same as the unit of the full algebra \mathcal{B}). This then implies that $\ell_0 = r_j M_0^j = r_0$, and since all ℓ_i 's commute with elements of \mathcal{A}_R , r_0 must be in its center. Similarly, we can show that if C_L is a quadratic Casimir for \mathcal{A}_L , it is also a quadratic Casimir for \mathcal{A}_R . In this regard, let us write

$$C_L = h^{ij} \ell_i^j \ell_j \quad (36)$$

h^{ij} being c -numbers, and unlike in the undeformed case, h^{ij} is not, in general, symmetric. Then by Schur's lemma $h^{ij} \chi_i \chi_j$ is a multiple of the identity, and

$$Th^{ij} \chi_i \chi_j = h^{ij} \chi_i \chi_j T = h^{ij} T \chi_\ell \chi_k M_i^{\ell} M_j^k \quad (37)$$

This is solved by

$$h^{\ell k} I = h^{ij} M_i^{\ell} M_j^k \quad (38)$$

We can use this to write C_L as a binomial in r_i as follows:

$$\begin{aligned} C_L &= h^{ij} \ell_i^j r_k M_j^k = h^{ij} r_k \ell_i^j M_j^k = h^{ij} r_k r_\ell M_i^{\ell} M_j^k \\ &= h^{\ell k} r_k r_\ell \end{aligned} \quad (39)$$

Since (36) commutes with r_i , $C_R = h^{\ell k} r_k r_\ell$ is a Casimir for \mathcal{A}_R . It is a natural choice for the Hamiltonian of a q -rigid rotor. The above procedure can be repeated for any higher order Casimirs.

Finally, let us enlarge our algebra \mathcal{B} to include a bimodule Γ . One then assumes the existence of an exterior derivative d which maps G_q to Γ , and following [1] satisfies the usual Leibniz rule. We next introduce the left and right invariant one forms. We define them as in (11). They obey the Maurer–Cartan equations:

$$\begin{aligned} d\omega_L + \omega_L \wedge \omega_L &= 0 \\ d\omega_R - \omega_R \wedge \omega_R &= 0 \end{aligned} \quad (40)$$

If we postulate that dT has the same commutation relations with ℓ_i and r_i as does T , it then follows that ω_L commutes with ℓ_i , and ω_R commutes with r_i , as in the undeformed case. For the latter, multiply $dTr_i - r_j dTf_i^j = dT\chi_i$ on the right by T^{-1} and $r_i T^{-1} - T^{-1} r_j T f_i^j T^{-1} = \chi_i T^{-1}$ on the left by dT , and then take the difference to find

$$[\omega_R, r_j] T f_i^j T^{-1} = 0 \quad (41)$$

The result that ω_R and r_j commute follows using the existence of an antipode $\kappa'(f_j^k)$ [6], where $\kappa'(f_j^k) f_i^j = f_j^k \kappa'(f_i^j) = \epsilon' \delta_i^k$. ϵ' is the co-unit, which for us is just the unit matrix in the defining representation. The nonvanishing commutation relations of the generators ℓ_i and r_i with the left and right invariant one forms are

$$\begin{aligned} [\omega_L, r_j \kappa'(f_i^j)] &= [\omega_L, \chi_j \kappa'(f_i^j)], \\ [\omega_R, \ell_j f_i^j] &= -[\omega_R, \chi_i] \end{aligned} \quad (42)$$

Upon writing $\omega_L = \omega_L^i \chi_i$ and $\omega_R = \omega_R^i \chi_i$, they can be rewritten

$$\begin{aligned} [\omega_L^i, r_j]_q &\equiv \omega_L^i r_j - \Lambda_{\ell_j^k}^{ki} r_k \omega_L^i = c_{kj}^i \omega_L^k, \\ [\ell_j, \omega_R^i]_q &\equiv \ell_j \omega_R^i - \Lambda_{\ell_j^k}^{ki} \omega_R^i \ell_k = c_{kj}^i \omega_R^k \end{aligned} \quad (43)$$

generalizing (12). Once again, the components ω_L^i and ω_R^i are related by (13). To derive the second equation in (43) one can substitute (13) into $[\omega_L^i, \mathcal{L}_j] = 0$.

This system suggests a natural generalization to gauge theories based on quantum groups. This was pursued in Refs. [8,9] from the Lagrangian perspective.⁵ Here we view the theory from the Hamiltonian perspective. For this one can now drop the zero curvature conditions (40), and let ω_L (or ω_R) denote arbitrary connection one forms. From (43), r_i (or \mathcal{L}_i) generate global transformations. Following Castellani [8], if we introduce ‘infinitesimal’ parameters ϵ^i , which are not c-numbers, infinitesimal variations of the one forms ω_L^i can be expressed by

$$\delta\omega_L^i = -[\omega_L^i, r_j]_q \epsilon^j = -c_{kj}^i \omega_L^k \epsilon^j \quad (44)$$

A natural question concerns the generators of local or gauge transformations, i.e. the analogue of the Gauss law constraints. From [8], local quantum group transformations are of the form

$$\delta\omega_L^i = -d\epsilon^i - c_{kj}^i \omega_L^k \epsilon^j, \quad (45)$$

where the ‘infinitesimal’ parameters ϵ^i are now functions on space-time, which is considered here as a commuting manifold. The Gauss law generators \mathcal{G}_i and space components A_μ^i of ω_L^i are also functions on space-time. The equal time commutator of Gauss law generators with A_μ^i should now include a central term

$$\begin{aligned} & [A_\mu^i(x), \mathcal{G}_j(y)]_q \\ & \equiv A_\mu^i(x) \mathcal{G}_j(y) - \Lambda_{\mathcal{L}_j}^{ki} \mathcal{G}_k(y) A_\mu^{\mathcal{L}}(x) \\ & = c_{kj}^i A_\mu^k(x) \delta(x-y) + \delta_j^i \frac{\partial}{\partial x^\mu} \delta(x-y) \end{aligned} \quad (46)$$

The Gauss law constraints should form a closed algebra if they are to be first class. A natural choice is a q-Lie algebra at every point in space:

$$\begin{aligned} & \mathcal{G}_i(x) \mathcal{G}_j(y) - \Lambda_{\mathcal{L}_i}^{mn} \mathcal{G}_n(y) \mathcal{G}_m(x) \\ & = c_{ji}^k \mathcal{G}_k(x) \delta(x-y) \end{aligned} \quad (47)$$

However, the algebra generated by A_μ^i and \mathcal{G}_j is, in general, nonassociative. In fact, the algebra for the

Gauss law generators alone is, in general, nonassociative. On the other hand, it can be checked that associativity is recovered for the so-called ‘minimal’ deformations [8,9], where

$$\Lambda_{k\mathcal{L}}^{ij} \Lambda_{mn}^{k\mathcal{L}} = \delta_m^i \delta_n^j \quad (48)$$

One can wonder if the Gauss law can be expressed in terms of some analogue of the electric fields π_i^μ , similar to what is done in the undeformed case. This should at least be possible in the minimal case. We thus write:

$$\mathcal{G}_i(x) = -\frac{\partial}{\partial x^\mu} \pi_i^\mu(x) + c_{ji}^k \pi_k^\mu(x) A_\mu^j(x) \quad (49)$$

To recover the derivative term in (46) one then needs

$$\begin{aligned} & [A_\mu^i(x), \pi_j^\nu(y)]_q \\ & \equiv A_\mu^i(x) \pi_j^\nu(y) - \Lambda_{\mathcal{L}_j}^{ki} \pi_k^\nu(y) A_\mu^{\mathcal{L}}(x) \\ & = \delta_j^i \delta_\mu^\nu \delta(x-y) \end{aligned} \quad (50)$$

giving a deformation of the canonical commutation relations. For the remaining term in (46) one needs

$$c_{\mathcal{L}_j}^m \Lambda_{km}^{ni} A_\mu^k(x) A_\nu^{\mathcal{L}}(y) = c_{mk}^n \Lambda_{\mathcal{L}_j}^{ki} A_\nu^m(y) A_\mu^{\mathcal{L}}(x) \quad (51)$$

This is indeed valid for the minimal case if one chooses the following commutation relations for the gauge potentials:

$$A_\mu^k(x) A_\nu^{\mathcal{L}}(y) = \Lambda_{m_j}^{k\mathcal{L}} A_\nu^m(y) A_\mu^j(x) \quad (52)$$

which agree with those found in Ref. [8]. Concerning the commutation relations between the electric fields, this can be obtained (for minimal deformations) by again appealing to associativity. The algebra of the operators A_μ^i and π_j^ν is associative for

$$\pi_i^\mu(x) \pi_j^\nu(y) = \Lambda_{ji}^{k\mathcal{L}} \pi_k^\nu(y) \pi_i^\mu(x) \quad (53)$$

From the above discussion, it may appear that gauge theories based on quantum groups are possible only for ‘minimal’ deformations. This is consistent with the result found in Ref. [8] that the set of gauge transformations (45) closes only for such deformations. For us, minimality was needed to make the algebra generated by $\mathcal{G}_i(x)$, satisfying (47), be asso-

⁵ Gauge theories based on quantum groups were also studied on the lattice [12,13]. The continuum limit for these theories is nontrivial. On the other hand, in Refs. [8,9] and what follows one works directly on the continuum.

ciative. On the other hand, (47) may be too simple a generalization of the global symmetry algebra defined by (15). A less restrictive choice would have space-dependent braiding matrices, R -matrices and structure constants. More precisely, $\mathcal{E}_i(x)$ can be regarded as ‘right’ generators having both a continuous and discrete index, whose corresponding braiding matrices \tilde{A} and structure constants \tilde{c} have the simple form

$$\begin{aligned}\tilde{A}_{(jy)(ix)}^{(mz)(nw)} &= \Lambda_{ji}^{mn} \delta(x-z) \delta(y-w), \\ \tilde{c}_{(jy)(ix)}^{(kz)} &= c_{ji}^k \delta(x-z) \delta(x-y)\end{aligned}\quad (54)$$

This choice is consistent with the bicovariance conditions if (48) is satisfied. Whether or not one can have a more general expression for \tilde{A} and \tilde{c} would be of interest. Moreover, it is perhaps also too restrictive to require that space-time defines a normal manifold, when this is not the case for the target manifold. Possibly, a more general class of gauge theories based on quantum groups can be developed if one is able to drop this requirement as well.

Acknowledgements

A.S. thanks S. Frolov for discussions and the members of Dipartimento di Scienze Fisiche, Università di Napoli, for their hospitality and support. This work was partially supported by PRIN 97

‘‘SIN.TE.SI’’ A.S. was supported in part by the U.S. Department of Energy under contract number DE-FG02-96ER40967.

References

- [1] S.L. Woronowicz, *Comm. Math. Phys.* 111 (1987) 613; 122 (1989) 125.
- [2] D. Bernard, *Prog. Theor. Phys. Supp.* 102 (1990) 49.
- [3] B. Jurčo, *Lett. Math. Phys.* 22 (1991) 177.
- [4] U. Carow-Watamura, M. Schlieker, S. Watamura, W. Weich, *Comm. Math. Phys.* 142 (1991) 905.
- [5] P. Schupp, P. Watts, B. Zumino, *Comm. Math. Phys.* 157 (1993) 305.
- [6] P. Aschieri, L. Castellani, *Int. J. Mod. Phys. A* 8 (1993) 1667.
- [7] K. Schmudgen, A. Schuler, *Comm. Math. Phys.* 167 (1995) 635; 170 (1995) 315.
- [8] L. Castellani, *Mod. Phys. Lett. A* 9 (1994) 2835; *Phys. Lett. B* 292 (1992) 93; *B* 327 (1994) 22.
- [9] G. Bimonte, R. Musto, A. Stern, P. Vitale, *Phys. Lett. B* 406 (1997) 205; *Nucl. Phys. B* 525 (1998) 483; *Int. J. Mod. Phys. A* 13 (1998) 4023; *Phys. Lett. B* 441 (1998) 69.
- [10] L.D. Faddeev, P.N. Pyatov, ‘The Differential Calculus on Quantum Linear Groups’, hep-th/9402070.
- [11] M.A. Semenov-Tian-Shansky, *Theor. Math. Phys.* 93 (1992) 1292.
- [12] S.A. Frolov, *Mod. Phys. Lett. A* 10 (1995) 2619; *A* 10 (1995) 2885; *Physical Phase Space of Lattice Yang–Mills Theory and the Moduli Space of Flat Connections on a Riemann Surface*, hep-th/9511018.
- [13] G. Bimonte, A. Stern, P. Vitale, *Phys. Rev. D* 54 (1996) 1054.