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Deposited 05/01/2019

Citation of published version:

Pinzul, A., Stern, A. (2008): Gauge Theory of the Star Product. *Nuclear Physics B*, 791(3).

DOI: <https://doi.org/10.1016/j.nuclphysb.2007.09.012>



Gauge theory of the star product

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Received 10 July 2007; accepted 17 September 2007

Available online 25 September 2007

Abstract

The choice of a star product realization for non-commutative field theory can be regarded as a gauge choice in the space of all equivalent star products. With the goal of having a gauge invariant treatment, we develop tools, such as integration measures and covariant derivatives on this space. The covariant derivative can be expressed in terms of connections in the usual way giving rise to new degrees of freedom for non-commutative theories.

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PACS: 11.10.Nx; 02.40.Gh

Keywords: Star product; Non-commutative gauge theory

1. Introduction

Deformation quantization [1] replaces the commutative algebra of functions on a Poisson manifold with a non-commutative algebra, where multiplication for the latter is given by some associative star product. Kontsevich [2] showed that this program can be carried out for any smooth Poisson manifold. An explicit construction for the star product (the Kontsevich star product) was given in [2] which was completely determined by the Poisson bi-vector α . The Kontsevich star product belongs to a very large equivalence class $\{\star\}_\alpha$ of star products. The different star products in $\{\star\}_\alpha$ are related by gauge transformations, where the gauge group \mathcal{G}_α is generated by all differential operators. \mathcal{G}_α also includes transformations, such as standard non-commutative $U(1)$ gauge transformations, which leave the star product invariant [3,4].

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Other ingredients, in addition to the non-commutative algebra, are needed in order to write down field theories on the non-commutative spaces. Among them are the trace and derivative. Concerning the former, theorems have been given which show the existence of the trace for the deformation of a symplectic manifold [5] and, more generally, for any regular Poisson manifold for which a Poisson trace exists [6,7]. It was shown, for example, that the usual integral with the commutative measure satisfies the necessary conditions for a trace when the topology of the manifold is \mathbb{R}^{2d} and one restricts to the Kontsevich star. In another example, corrections were computed in [8] to the commutative measure for a star product constructed from deformed coherent states [9]. Concerning the derivative, much is known for the case of constant non-commutativity, where the Kontsevich star reduces to the familiar Groenewold–Moyal star [10,11]. For that case, the standard partial derivative can be realized as an inner derivative on the algebra. This is not true for non-constant non-commutativity where the star product is position dependent. Derivatives have nevertheless been defined in the general case, after specializing to the Kontsevich star product [12].

As most previous treatments of non-commutative field theory have relied heavily on one particular star product in $\{\star\}_\alpha$, namely the Kontsevich star product, it is of interest to search for gauge invariant approaches, where here the gauge group is \mathcal{G}_α . This is addressed in the current article. One approach is to simply map the non-commutative field theory written with the Kontsevich star to a non-commutative field theory associated with an arbitrary star product in the equivalence class, resulting in no new physical degrees of freedom. Alternatively, one can introduce the notion of covariance with respect to \mathcal{G}_α , where functions $\{f, g, \dots\}$ and their star products transform in the same manner, and are hence covariant. In addition, one can define the trace of the functions to be gauge invariant. Like in Yang–Mills theories, one can then also introduce a covariant derivative, now associated with gauge transformations between different star products in $\{\star\}_\alpha$, where the covariant derivative of functions $\{f, g, \dots\}$ transforms in the same manner as the functions $\{f, g, \dots\}$. In such an approach, which is what we follow here, one thereby obtains new degrees of freedom associated with the connections.¹ As \mathcal{G}_α is an infinite dimensional extension of the non-commutative $U(1)$ gauge group, there are in principle an infinite number of such degrees of freedom, which contains the standard non-commutative $U(1)$ gauge degrees of freedom. It then becomes possible to consider an infinite dimensional extension of non-commutative $U(1)$ gauge theory, with the dynamics for the gauge fields and matter fields written on the entire equivalence class $\{\star\}_\alpha$.

The plan of the paper is as follows. In Section 2 we introduce the integration measure, covariant derivative, connection and curvature for the special case where $\{\star\}_\alpha$ contains the Groenewold–Moyal star product, while the generalization to an arbitrary equivalence class is given in Section 3. Star products in $\{\star\}_\alpha$ can be expanded in the non-commutative parameter, which we denote by \hbar , and each order can be expressed in terms of an infinite number of bi-differential operators. Furthermore, \mathcal{G}_α is generated by an infinite number of differential operators at each order in \hbar . For practical purposes, we examine a restricted gauge group in Section 4 which is generated by a finite number of differential operators at each order in \hbar . We can then write down explicit formulae for components of the connection, curvature and field equations. We summarize the results and indicate possible future developments in Section 5.

¹ Only after restricting the connection to a certain pure gauge, will the covariant derivative be gauge equivalent to the derivative [12] written for the Kontsevich star.

2. Gauging the Groenewold–Moyal star product

We first review well-known facts about the Groenewold–Moyal star product. Here the Poisson bi-vector on \mathbb{R}^{2d} coordinatized by x^μ , $\mu = 1, 2, \dots, 2d$, is

$$\theta^{\mu\nu} \bar{\partial}_\mu \bar{\partial}_\nu, \tag{2.1}$$

where $\theta^{\mu\nu} = -\theta^{\nu\mu}$ are constants on \mathbb{R}^{2d} . $\bar{\partial}_\mu$ and $\vec{\partial}_\mu$ are left and right derivatives $\partial_\mu = \frac{\partial}{\partial x^\mu}$, respectively. Constant non-commutativity results after deformation quantization. Denote by \mathcal{A}_θ the non-commutative algebra of functions f_0, g_0, \dots on \mathbb{R}^{2d} with multiplication given by the Groenewold–Moyal star product \star_θ [10,11]. \star_θ is the bi-differential operator

$$\star_\theta = \exp \left\{ \frac{i\hbar}{2} \theta^{\mu\nu} \bar{\partial}_\mu \bar{\partial}_\nu \right\}. \tag{2.2}$$

Then at lowest order in \hbar , the star commutator of functions reduces to their Poisson bracket

$$[f_0, g_0]_{\star_\theta} \equiv f_0 \star_\theta g_0 - g_0 \star_\theta f_0 = i\hbar \{f_0, g_0\} + \mathcal{O}(\hbar^3), \tag{2.3}$$

where $\{f_0, g_0\} = f_0 \theta^{\mu\nu} \bar{\partial}_\mu \bar{\partial}_\nu g_0$. The derivative ∂_μ satisfies the usual Leibniz rule when acting on the Groenewold–Moyal star product of two functions. Using the standard measure $d^{2d}x$ on \mathbb{R}^{2d} , the integral serves as a trace for \mathcal{A}_θ . Moreover, the integral of the Groenewold–Moyal star product of two functions can be replaced with the integral of the pointwise product of the two functions, provided these functions vanish sufficiently rapidly at infinity

$$\int d^{2d}x f_0 \star_\theta g_0 = \int d^{2d}x f_0 g_0, \tag{2.4}$$

from which the trace property easily follows,

$$\int d^{2d}x [f_0, g_0]_{\star_\theta} = 0. \tag{2.5}$$

The Groenewold–Moyal star product \star_θ is an element of the equivalence class of star products $\{\star\}_\theta$. The equivalence class is generated from the set of all invertible operators T of the form

$$T = \mathbb{1} + \sum_{k=1}^{\infty} \hbar^k T_k, \tag{2.6}$$

where T_k are arbitrary differential operators. Under the action of T , functions f_0, g_0, \dots are mapped to

$$f = T(f_0), \quad g = T(g_0), \quad \dots, \tag{2.7}$$

while \star_θ is mapped to another associative star product $\star \in \{\star\}_\theta$, such that [2]

$$f \star g = T(f_0 \star_\theta g_0). \tag{2.8}$$

The new star commutator has the same $\hbar \rightarrow 0$ limit as in (2.3),

$$[f, g]_\star \equiv f \star g - g \star f = i\hbar \{f, g\} + \mathcal{O}(\hbar^2). \tag{2.9}$$

As before, the integral serves as the trace. However, the measure associated with the star product \star is, in general, no longer $d^{2d}x$. Call the transformed measure $d\mu_\star$. Invariance of the trace

implies

$$\int d\mu_\star f = \int d^{2d}x f_0, \tag{2.10}$$

for functions f_0 vanishing sufficiently rapidly at infinity. From (2.6), $d\mu_\star$ can differ from the flat measure $d^{2d}x$ at order \hbar . (An explicit expression for $d\mu_\star$ for restricted gauge transformations is given in Section 4.3.) The analogue of (2.4) does not hold for an arbitrary \star in the equivalence class; i.e., $\int d\mu_\star f \star g \neq \int d\mu_\star fg$. However, the trace property easily follows from (2.8) and (2.10)

$$\int d\mu_\star [f, g]_\star = \int d\mu_\star T([f_0, g_0]_{\star_\theta}) = \int d^{2d}x [f_0, g_0]_{\star_\theta} = 0. \tag{2.11}$$

Subsequent transformations can be performed to map between any two star products in $\{\star\}_\theta$. Say that \star' is obtained from \star using invertible operator Λ , which we assume to have a form analogous to (2.6),

$$\Lambda = \mathbb{1} + \sum_{k=1}^{\infty} \hbar^k \Lambda_k, \tag{2.12}$$

where Λ_k are arbitrary differential operators. Then functions, as well as star products of functions, transform ‘covariantly’ if:

$$f \rightarrow f' = \Lambda(f), \quad f \star g \rightarrow f' \star' g' = \Lambda(f \star g). \tag{2.13}$$

In order that the trace be an invariant for $\{\star\}_\theta$, the measure should, in general, transform, $d\mu_\star \rightarrow d\mu_{\star'}$, such that

$$\int d\mu_\star f = \int d\mu_{\star'} f', \tag{2.14}$$

for all functions f that vanish sufficiently rapidly at infinity. $d\mu_\star$ and $d\mu_{\star'}$ correspond to the flat measure $d^{2d}x$ at zeroth order in \hbar , while from (2.12), they, in general, differ at order \hbar . Since the measure transforms nontrivially for general Λ , gauge transformations cannot be considered to be internal transformations beyond zeroth order in \hbar .

There exists a subset of transformations (2.12) which leaves the star product and the measure invariant, i.e., $\star' = \star$ and $d\mu_{\star'} = d\mu_\star$. This is the case for inner automorphisms [3,4] $\Lambda = \hat{\Lambda}_\lambda$, parametrized by functions λ on \mathbb{R}^{2d} , where

$$\hat{\Lambda}_\lambda(f) = \lambda \star f \star \lambda_\star^{-1}, \tag{2.15}$$

and $\lambda \star \lambda_\star^{-1} = 1$. It is not surprising that general gauge transformations given by (2.13) are not internal beyond zeroth order in \hbar , because the same is true for the subset of inner automorphisms, even though the measure is invariant under the latter. For the case of the Groenewold–Moyal star, the inner automorphisms are known to contain (global) translations [13]. Here we need that $[\theta_{\mu\nu}]$ has an inverse:

$$e_\star^{-i\theta_{\rho\sigma}^{-1}c^\rho x^\sigma} \star_\theta x^\mu \star_\theta e_\star^{i\theta_{\rho\sigma}^{-1}c^\rho x^\sigma} = x^\mu + \hbar c^\mu, \tag{2.16}$$

where $e_\star^f = 1 + f + \frac{1}{2}f \star f + \frac{1}{3!}f \star f \star f + \dots$. So for c^μ of zeroth order in \hbar , one gets translations of order \hbar .

We denote the derivative associated with any $\star \in \{\star\}_\theta$ by $D[A]_\mu$, and require that it is covariant under the gauge transformations (2.13),

$$D[A]_\mu f \rightarrow D[A']_\mu f' = \Lambda(D[A]_\mu f), \quad (2.17)$$

or

$$D[A']_\mu \Lambda = \Lambda D[A]_\mu. \quad (2.18)$$

Since Λ is a differential operator of arbitrary order, so in general should be $D[A]_\mu$. As usual, let us write the covariant derivative in terms of potentials A_μ , which we expand according to

$$A_\mu = \sum_{k=1}^{\infty} \hbar^k A_{k,\mu}, \quad (2.19)$$

where $A_{k,\mu}$ are differential operators.² If we require $D[A]_\mu$ to reduce to the standard derivative in the absence of the potentials, then we can write the usual expression

$$D[A]_\mu = \partial_\mu + A_\mu, \quad (2.20)$$

and the potentials A_μ gauge transform as

$$A_\mu \rightarrow A'_\mu = \Lambda[\partial_\mu, \Lambda^{-1}] + \Lambda A_\mu \Lambda^{-1}. \quad (2.21)$$

Derivative-valued field strengths

$$F_{\mu\nu} = \sum_{k=1}^{\infty} \hbar^k F_{k,\mu\nu}, \quad (2.22)$$

where $F_{k,\mu\nu}$ are arbitrary differential operators, can also be introduced

$$F_{\mu\nu} = [D[A]_\mu, D[A]_\nu] = [\partial_\mu, A_\nu] - [\partial_\nu, A_\mu] + [A_\mu, A_\nu]. \quad (2.23)$$

They satisfy the Bianchi identity

$$[D[A]_\rho, F_{\mu\nu}] + [D[A]_\mu, F_{\nu\rho}] + [D[A]_\nu, F_{\rho\mu}] = 0, \quad (2.24)$$

and gauge transform according to

$$F_{\mu\nu} \rightarrow F'_{\mu\nu} = \Lambda F_{\mu\nu} \Lambda^{-1}. \quad (2.25)$$

For the special case where A_μ is the pure gauge $A_\mu = T[\partial_\mu, T^{-1}]$, $D[A]_\mu$ satisfies the usual Leibniz rule when acting on the star product \star of two functions, as we can use the inverse of (2.8) to map back to the Groenewold–Moyal star product. This, however, is not true for arbitrary connections A_μ .

$U(1)$ gauge theory on the non-commutative plane is contained in this system. Here we write $A_\mu = \hat{A}_\mu$ acting on functions f [which gauge transform as inner automorphisms (2.15)] according to $\hat{A}_\mu(f) = [a_\mu, f]_{\star_\theta}$, where a_μ are the non-commutative $U(1)$ potentials. So the

² Derivative-valued gauge fields have been considered previously in different contexts [3,4,15].

derivative-valued potentials acting on covariant functions can be written as³

$$\hat{A}_\mu = 2i a_\mu \sin \left\{ \frac{\hbar}{2} \theta^{\rho\sigma} \bar{\partial}_\rho \bar{\partial}_\sigma \right\}. \tag{2.26}$$

Upon restricting to the Groenewold–Moyal star \star_θ , (2.15) leads to the usual non-commutative $U(1)$ gauge transformations for a_μ ,

$$a_\mu \rightarrow a'_\mu = \lambda \star_\theta a_\mu \star_\theta \lambda_\star^{-1} - \partial_\mu \lambda \star_\theta \lambda_\star^{-1}. \tag{2.27}$$

From (2.23), the field strength operator $F_{\mu\nu} = \hat{F}_{\mu\nu}$ acting on a function ϕ is $\hat{F}_{\mu\nu}(\phi) = [f_{\mu\nu}, \phi]_{\star_\theta}$, where $f_{\mu\nu}$ is the non-commutative $U(1)$ field strength tensor $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu + [a_\mu, a_\nu]_{\star_\theta}$. Thus

$$\hat{F}_{\mu\nu} = 2i f_{\mu\nu} \sin \left\{ \frac{\hbar}{2} \theta^{\rho\sigma} \bar{\partial}_\rho \bar{\partial}_\sigma \right\}, \tag{2.28}$$

and upon restricting Λ in (2.25) to (2.15), $f_{\mu\nu}$ gauge transform as inner automorphisms, $f_{\mu\nu} \rightarrow f'_{\mu\nu} = \lambda \star_\theta f_{\mu\nu} \star_\theta \lambda_\star^{-1}$.

Field theory actions can now be written down which are invariants for the equivalence class $\{\star\}_\theta$. If we assume the field ϕ on \mathbb{R}^{2d} transforms covariantly with respect to the above gauge transformations, $\phi \rightarrow \phi' = \Lambda(\phi)$, then an invariant action is

$$S_{\phi,A} = \frac{1}{2} \int d\mu_\star \eta^{\mu\nu} D[A]_\mu \phi \star D[A]_\nu \phi, \tag{2.29}$$

where $\eta^{\mu\nu}$ is the flat metric. It is equivalent to the commutative action for a free massless scalar field after restricting A_μ to the pure gauge $A_\mu = T[\partial_\mu, T^{-1}]$ and making the field redefinition from $\phi = T(\phi_0)$ to ϕ_0 . This is since then $D[A]_\mu \phi = T \partial_\mu \phi_0$, and we can re-express the action in terms of the Groenewold–Moyal star product and use (2.4).

More exciting is the possibility of writing down a kinetic term for A_μ . This would require a trace over the operator-valued fields. A possible candidate is the Wodzicki residue [14]. Alternatively, one can adopt the usual Yang–Mills form for the field equations:

$$[D[A]_\mu, F^{\mu\nu}] = J^\nu. \tag{2.30}$$

The right-hand side represents a matter current source which can be expanded

$$J_\mu = \sum_{k=1}^{\infty} \hbar^k J_{k,\mu}, \tag{2.31}$$

where $J_{k,\mu}$ are differential operators and J_μ gauge transforms as the field strengths $F_{\mu\nu}$,

$$J^\mu \rightarrow J'^\mu = \Lambda J^\mu \Lambda^{-1}. \tag{2.32}$$

Moreover, it must be covariantly conserved,

$$[D[A]_\mu, J^\mu] = 0. \tag{2.33}$$

³ Alternatively, one can define the action of \hat{A} on fields ϕ_{fund} in the fundamental representation. Such fields are not covariant, in that they do not gauge transform according to (2.13) with $\Lambda = \hat{\Lambda}_\lambda$, but rather with the left action $\phi_{\text{fund}} \rightarrow \phi'_{\text{fund}} = \lambda \star \phi_{\text{fund}}$. On such fields one has

$$\hat{A}_\mu = a_\mu \exp \left\{ \frac{i\hbar}{2} \theta^{\rho\sigma} \bar{\partial}_\rho \bar{\partial}_\sigma \right\}.$$

Since J^μ takes values in an infinite dimensional vector space, (2.33) then corresponds to infinitely many conservation laws.

3. Generalization to the Kontsevich star product

Now we go to the case of a general Poisson bi-vector

$$\vec{\partial}_\mu \alpha^{\mu\nu} \vec{\partial}_\nu, \tag{3.1}$$

where $\alpha^{\mu\nu} = -\alpha^{\nu\mu}$, $\mu, \nu = 1, 2, \dots, 2d$, are functions on an open subset M^{2d} of \mathbb{R}^{2d} . Corresponding star products can be given in terms of series expansions in the non-commutativity parameter \hbar , where the terms in the expansions are bi-differential operators $B_n, n = 1, 2, 3, \dots$,

$$f \star g = fg + \sum_{n=1}^{\infty} \hbar^n B_n(f, g). \tag{3.2}$$

In the Kontsevich construction of the star product [2], which we denote using \star_α , B_1 is proportional to the Poisson bi-vector field. Acting between functions f_0 and g_0 , \star_α is, up to second order in \hbar , given by⁴

$$\begin{aligned} f_0 \star_\alpha g_0 &= f_0 g_0 + \frac{i\hbar}{2} \alpha^{\mu\nu} \partial_\mu f_0 \partial_\nu g_0 - \frac{\hbar^2}{8} \alpha^{\mu\nu} \alpha^{\rho\sigma} \partial_{\mu,\rho} f_0 \partial_{\nu,\sigma} g_0 \\ &\quad - \frac{\hbar^2}{12} \alpha^{\mu\nu} \partial_\nu \alpha^{\rho\sigma} (\partial_{\mu,\rho} f_0 \partial_\sigma g_0 - \partial_\rho f_0 \partial_{\mu,\sigma} g_0) + \mathcal{O}(\hbar^3), \end{aligned} \tag{3.3}$$

where $\partial_{\mu,\nu,\dots,\rho} = \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} \dots \frac{\partial}{\partial x^\rho}$. The Poisson bracket is again recovered at lowest order in \hbar from the star commutator

$$[f_0, g_0]_{\star_\alpha} \equiv f_0 \star_\alpha g_0 - g_0 \star_\alpha f_0 = i\hbar \{f_0, g_0\} + \mathcal{O}(\hbar^3), \tag{3.4}$$

where $\{f_0, g_0\} = f_0 \vec{\partial}_\mu \alpha^{\mu\nu} \vec{\partial}_\nu g_0$.

The integral with measure $d\mu_0 = d^{2d}x \Omega_0(x)$ can serve as a trace for a star product associated with the Poisson bi-vector (3.1), provided that $\Omega_0(x)$ satisfies [5–7]

$$\partial_\mu (\Omega_0 \alpha^{\mu\nu}) = 0. \tag{3.5}$$

From this relation the cyclicity property easily follows at first order in \hbar ,

$$\int d\mu_0 [f_0, g_0]_{\star_\alpha} = \mathcal{O}(\hbar^2), \tag{3.6}$$

provided functions f_0 and g_0 vanish sufficiently rapidly at infinity. For the special case of symplectic manifolds, Ω_0 is proportional to $|\det \alpha|^{-1/2}$. More generally, it is known [6] that there exists a star product \star_0 , that is gauge equivalent to \star_α , for which the cyclicity property is guaranteed to all orders in \hbar using measure $d\mu_0$. Call T_α the map from \star_α to \star_0 , and define a measure $d\mu_\alpha = d^{2d}x \Omega(x)$ associated with star product \star_α such that

$$\int d\mu_\alpha f_0 = \int d\mu_0 T_\alpha(f_0), \tag{3.7}$$

⁴ Third order terms were computed in [16].

for all f_0 that vanish sufficiently rapidly at infinity. Thus

$$\int d\mu_\alpha [f_0, g_0]_{\star_\alpha} = 0, \tag{3.8}$$

and so $\int d\mu_\alpha$ serves as a trace for the star product \star_α .

Derivations δ_X^α can be defined for the star product \star_α satisfying the standard Leibniz rule

$$\delta_X^\alpha (f_0 \star_\alpha g_0) = \delta_X^\alpha f_0 \star_\alpha g_0 + f_0 \star_\alpha \delta_X^\alpha g_0, \tag{3.9}$$

provided the Lie derivative \mathcal{L}_X of α vanishes,

$$\mathcal{L}_X \alpha^{\mu\nu} = X^\rho \partial_\rho \alpha^{\mu\nu} - \alpha^{\mu\rho} \partial_\rho X^\nu + \alpha^{\nu\rho} \partial_\rho X^\mu = 0, \tag{3.10}$$

for some vectors $X = X^\mu \partial_\mu$. This is the same condition that is needed for X to be a derivation of the Poisson bracket; i.e., $X\{f_0, g_0\} = \{Xf_0, g_0\} + \{f_0, Xg_0\}$, and it also corresponds to a vanishing Schouten–Nijenhuis bracket of X with the Poisson bivector. An expansion for δ_X^α was given in [12]. Up to second order in \hbar , δ_X^α was found to be

$$\delta_X^\alpha = X^\mu \partial_\mu + \frac{\hbar^2}{12} \alpha^{\mu\nu} \partial_\nu \alpha^{\rho\sigma} \partial_{\mu,\sigma} X^\lambda \partial_{\rho,\lambda} - \frac{\hbar^2}{24} \alpha^{\mu\nu} \alpha^{\rho\sigma} \partial_{\mu,\rho} X^\lambda \partial_{\nu,\sigma,\lambda} + \mathcal{O}(\hbar^3). \tag{3.11}$$

The commutator of any two such derivatives δ_X^α and δ_Y^α is nonvanishing, with the zeroth order being the Lie bracket, $[\delta_X^\alpha, \delta_Y^\alpha] = \mathcal{L}_X Y + \mathcal{O}(\hbar^2)$.

As with the Groenewold–Moyal star product, \star_α belongs to an equivalence class of star products which we denote as $\{\star\}_\alpha$. The equivalence class is once again generated from the set of all invertible differential operators T , mapping functions f_0, g_0, \dots to f, g, \dots in (2.7), and the star product \star_α to \star , whose general form is given by (3.2), with

$$f \star g = T(f_0 \star_\alpha g_0). \tag{3.12}$$

The measure $d\mu_\star$ associated with the star product \star is related to $d\mu_\alpha$ by

$$\int d\mu_\star f = \int d\mu_\alpha f_0, \tag{3.13}$$

for functions f_0 vanishing sufficiently rapidly at infinity. Subsequent transformations can again be performed to map between any two star products in the equivalence class $\{\star\}_\alpha$ given by (2.13). The corresponding measures are related by (2.14). For covariant derivatives we again need (2.17). Now say that the covariant derivative $D[A]_X$ reduces to δ_X^α in the absence of the potentials, as is the case for

$$D[A]_X = \delta_X^\alpha + A_X. \tag{3.14}$$

The derivative-valued potentials A_X gauge transform as

$$A_X \rightarrow A'_X = \Lambda [\delta_X^\alpha, \Lambda^{-1}] + \Lambda A_X \Lambda^{-1}. \tag{3.15}$$

Given independent derivatives δ_X^α and δ_Y^α , one can define field strengths

$$F_{XY} = [D[A]_X, D[A]_Y] = [\delta_X^\alpha, \delta_Y^\alpha] + [\delta_X^\alpha, A_Y] - [\delta_Y^\alpha, A_X] + [A_X, A_Y], \tag{3.16}$$

which gauge transform as in (2.25). For the special case where A_X is the pure gauge $A_X = T[\delta_X^\alpha, T^{-1}]$, $D[A]_X$ satisfies the usual Leibniz rule when acting on the star product \star of two functions. Gauge invariant actions analogous to (2.29) can be written down after introducing a metric over the space of vector fields $\{X, Y, \dots\}$.

4. \hbar expansion

The most general T_k and Λ_k in (2.6) and (2.12), respectively, contain an infinite number of derivatives, and map to star products using (2.8), which then also contain an infinite number of derivatives at each order in \hbar beyond the zeroth order. Here for simplicity we shall restrict to operators T_k and Λ_k which have a finite number of derivatives. More specifically, terms of order n in \hbar in the equivalence map will be, at most, of order $2n$ in derivatives. As a result of this the star product \star , connections A_μ and curvature $F_{\mu\nu}$ can be written in terms of a finite number of derivatives at each order in \hbar .

4.1. Gauge group

We parametrize the set of all differential operators $\{T_k = T_k^{(s)}\}$ with an infinite number of symmetric tensors $s = (s^{\mu_1}, s^{\mu_1\mu_2}, s^{\mu_1\mu_2\mu_3}, \dots)$ which are functions on \mathbb{R}^{2d} and are polynomials in \hbar starting with order zero. The resulting expression for $T^{(s)} = \mathbb{1} + \sum_{k=1}^\infty \hbar^k T_k^{(s)}$ should be consistent with closure

$$T^{(s')}T^{(s)} = T^{(s'')}. \tag{4.1}$$

A possible solution is

$$T_k^{(s)} = \tau_{2k-1}^{(s)} + \tau_{2k}^{(s)}, \quad \tau_n^{(s)} = \frac{1}{n} s^{\mu_1\mu_2\cdots\mu_n} \partial_{\mu_1, \mu_2, \dots, \mu_n}. \tag{4.2}$$

The identity corresponds to $s = 0, T^{(0)} = \mathbb{1}$. From (4.1) one gets

$$\begin{aligned} s''^\mu &= s^\mu + s'^\mu + \hbar T_1^{(s')} s^\mu + \mathcal{O}(\hbar^2), \\ s''^{\mu\nu} &= s^{\mu\nu} + s'^{\mu\nu} + \hbar (T_1^{(s')} s^{\mu\nu} + s'^{(\mu} s^{\nu)}) + s'^{\lambda(\mu} \partial_\lambda s^{\nu)}) + \mathcal{O}(\hbar^2), \\ s''^{\mu\nu\lambda} &= s^{\mu\nu\lambda} + s'^{\mu\nu\lambda} + \frac{1}{4} (s'^{(\mu} s^{\nu\lambda)} + s^{(\mu} s'^{\nu\lambda)} + s'^{\eta(\mu} \partial_\eta s^{\nu\lambda)}) + \mathcal{O}(\hbar), \\ s''^{\mu\nu\lambda\eta} &= s^{\mu\nu\lambda\eta} + s'^{\mu\nu\lambda\eta} + \frac{1}{24} s'^{(\mu\nu} s^{\lambda\eta)} + \mathcal{O}(\hbar), \\ \dots, \end{aligned} \tag{4.3}$$

where $y^{(\mu\nu\cdots\rho)} = y^{\mu\nu\cdots\rho} +$ all symmetric combinations. Denoting the inverse of $T^{(s)}$ by $T^{(s_{\text{inv}})} = T^{(s)}^{-1}$, we get that

$$\begin{aligned} s_{\text{inv}}^\mu &= -s^\mu + \hbar T_1^{(s)} s^\mu + \mathcal{O}(\hbar^2), \\ s_{\text{inv}}^{\mu\nu} &= -s^{\mu\nu} + \hbar (T_1^{(s)} s^{\mu\nu} + s^{(\mu} s^{\nu)}) + s^{\lambda(\mu} \partial_\lambda s^{\nu)}) + \mathcal{O}(\hbar^2), \\ s_{\text{inv}}^{\mu\nu\lambda} &= -s^{\mu\nu\lambda} + \frac{1}{2} s^{(\mu} s^{\nu\lambda)} + \frac{1}{4} s^{\eta(\mu} \partial_\eta s^{\nu\lambda)} + \mathcal{O}(\hbar), \\ s_{\text{inv}}^{\mu\nu\lambda\eta} &= -s^{\mu\nu\lambda\eta} + \frac{1}{24} s^{(\mu\nu} s^{\lambda\eta)} + \mathcal{O}(\hbar), \\ \dots \end{aligned} \tag{4.4}$$

4.2. Star product

Using the operator $T^{(s)}$ in the equivalence relation (3.12), the Kontsevich star product \star_α is mapped to the star product given by (3.2), with the first two bi-differential operators B_1 and B_2 given by

$$\begin{aligned}
 B_1(f, g) &= b^{\mu\nu} \partial_\mu f \partial_\nu g, \\
 B_2(f, g) &= b_1^{\mu\nu} \partial_\mu f \partial_\nu g \\
 &\quad + \left(b^{\mu\nu\rho} + \frac{1}{6}(2b^{\mu\sigma} + b^{\sigma\mu})\partial_\sigma b^{[\nu\rho]} - \frac{1}{2}b^{\rho\sigma} \partial_\sigma b^{\mu\nu} \right) \partial_{\mu,\nu} f \partial_\rho g \\
 &\quad + \left(b^{\mu\nu\rho} - \frac{1}{6}(b^{\mu\sigma} + 2b^{\sigma\mu})\partial_\sigma b^{[\nu\rho]} - \frac{1}{2}b^{\rho\sigma} \partial_\sigma b^{\mu\nu} \right) \partial_\rho f \partial_{\mu,\nu} g \\
 &\quad + b^{\mu\nu\rho\sigma} (\partial_{\mu,\nu,\rho} f \partial_\sigma g + \partial_\sigma f \partial_{\mu,\nu,\rho} g) \\
 &\quad + \frac{1}{2}(3b^{\mu\nu\rho\sigma} + b^{\mu\rho} b^{\nu\sigma}) \partial_{\mu,\nu} f \partial_{\rho,\sigma} g,
 \end{aligned} \tag{4.5}$$

where the tensors $b^{\mu\nu\dots\rho}$ are expressed in terms of $\alpha^{\mu\nu}$ and $s^{\mu\nu\dots\rho}$ according to

$$\begin{aligned}
 b^\mu &= s^\mu, \\
 b^{\mu\nu} &= \frac{i}{2}\alpha^{\mu\nu} + s^{\mu\nu}, \\
 b_1^{\mu\nu} &= \frac{i}{2}T_1^{(s)}\alpha^{\mu\nu} - s^\mu s^\nu - \left(\frac{i}{2}\alpha^{\mu\sigma} + s^{\mu\sigma}\right)\partial_\sigma s^\nu - \left(\frac{i}{2}\alpha^{\sigma\nu} + s^{\sigma\nu}\right)\partial_\sigma s^\mu, \\
 b^{\mu\nu\rho} &= s^{\mu\nu\rho} - \frac{1}{4}s^{(\mu\nu} s^{\rho)}, \\
 b^{\mu\nu\rho\sigma} &= s^{\mu\nu\rho\sigma} - \frac{1}{48}s^{(\mu\nu} s^{\rho\sigma)},
 \end{aligned} \tag{4.6}$$

and $b^{[\mu\nu]} = b^{\mu\nu} - b^{\nu\mu}$. We introduced the vector field b^μ for the sake of completeness. Although it does not appear directly in the star product, it does appear in the measure. (4.5) reduces to (3.3) when $s = 0$, corresponding to the Kontsevich gauge. The star commutator is now

$$\begin{aligned}
 [f, g]_{\star_B} &= \hbar(b^{[\mu\nu]} + \hbar b_1^{[\mu\nu]})\partial_\mu f \partial_\nu g + \frac{\hbar^2}{2}(b^{\mu\rho} b^{\nu\sigma} - b^{\rho\mu} b^{\sigma\nu})\partial_{\mu,\nu} f \partial_{\rho,\sigma} g \\
 &\quad + \frac{\hbar^2}{2}(b^{(\mu\sigma})\partial_\sigma b^{[\nu\rho]} + b^{[\rho\sigma]}\partial_\sigma b^{\mu\nu})(\partial_{\mu,\nu} f \partial_\rho g - \partial_\rho f \partial_{\mu,\nu} g) + \mathcal{O}(\hbar^3).
 \end{aligned} \tag{4.7}$$

As in (3.4), the leading term is $i\hbar\{f, g\}$.

A subsequent gauge transformation can be performed using Λ in (2.12) to map the star product \star to \star' . Now write $\Lambda_k = T_k^{(\lambda)}$, with $T_k^{(\lambda)}$ given by (4.2) and λ denoting symmetric tensors $\lambda = (\lambda^{\mu_1}, \lambda^{\mu_1\mu_2}, \lambda^{\mu_1\mu_2\mu_3}, \dots)$. \star' is again of the form (3.2), with $b^{\mu\nu\dots\rho}$ in (4.5) replaced by $b'^{\mu\nu\dots\rho}$ defined by

$$\begin{aligned}
 b^\mu &\rightarrow b'^\mu = b^\mu + \lambda^\mu, \\
 b^{\mu\nu} &\rightarrow b'^{\mu\nu} = b^{\mu\nu} + \lambda^{\mu\nu}, \\
 b_1^{\mu\nu} &\rightarrow b_1'^{\mu\nu} = b_1^{\mu\nu} + T_1^{(\lambda)} b^{\mu\nu} - \lambda^\mu \lambda^\nu - b^{\mu\rho} \partial_\rho \lambda^\nu - b^{\rho\nu} \partial_\rho \lambda^\mu - \lambda^{\rho(\mu} \partial_\rho \lambda^{\nu)},
 \end{aligned}$$

$$\begin{aligned}
 b^{\mu\nu\rho} &\rightarrow b'^{\mu\nu\rho} = b^{\mu\nu\rho} + \lambda^{\mu\nu\rho} + \frac{1}{4}(\lambda^{\sigma(\mu}\partial_\sigma b^{\nu\rho)} - \lambda^{(\mu\nu}\lambda^{\rho)}), \\
 b^{\mu\nu\rho\sigma} &\rightarrow b'^{\mu\nu\rho\sigma} = b^{\mu\nu\rho\sigma} + \lambda^{\mu\nu\rho\sigma} - \frac{1}{48}\lambda^{(\mu\nu}\lambda^{\rho\sigma)}.
 \end{aligned}
 \tag{4.8}$$

A gauge invariant antisymmetric tensor can be constructed from the fields $b^{\mu\nu\dots\rho}$. It is simply the zeroth order term in an \hbar expansion expression for $\alpha^{\mu\nu}$ re-expressed in terms of $b^{\mu\nu\dots\rho}$. We remark that the subset of inner automorphisms (2.15) are contained in the group generated by (4.2). For example, in that case $T_1 = 2b^{\mu\nu}\lambda_\star^{-1}\partial_\mu\lambda\partial_\nu$.

4.3. Measure

The measure $d\mu_\star$ associated with the above star product should satisfy (3.13) (for functions f_0 vanishing sufficiently rapidly at infinity) and reduce to $d\mu_0 = d^{2d}x \Omega_0(x)$ in the commutative limit. If we expand $d\mu_\star$ in \hbar ,

$$d\mu_\star = d^{2d}x (\Omega_0 + \hbar\Omega_1 + \mathcal{O}(\hbar^2)). \tag{4.9}$$

Ω_0 is gauge invariant. We can substitute into (2.14) to obtain the gauge transformations of the higher order corrections to the measure. For example, if under the action of Λ , Ω_1 goes to Ω'_1 then

$$\int d^{2d}x (\Omega'_1 - \Omega_1)f = - \int d^{2d}x \Omega_0 T_1^{(\lambda)} f. \tag{4.10}$$

Upon integrating by parts and assuming functions f vanish sufficiently rapidly at infinity one gets

$$\Omega'_1 = \Omega_1 + \partial_\mu(\Omega_0\lambda^\mu) - \frac{1}{2}\partial_{\mu,\nu}(\Omega_0\lambda^{\mu\nu}). \tag{4.11}$$

4.4. Connection and curvature

For simplicity, here and in the following section, we work in the of equivalence class $\{\star\}_\theta$ containing the Groenewold–Moyal star product. Now write the differential operator-valued potentials $A_{k,\mu}$ in (2.19) according to $A_{k,\mu} = T_k^{(a_\mu)}$ given in (4.2), where here a_μ denote the tensors $a_\mu = (a_\mu^{\mu_1}, a_\mu^{\mu_1\mu_2}, a_\mu^{\mu_1\mu_2\mu_3}, \dots)$. From (2.21), using (4.3) and (4.4), we then deduce the following gauge transformations for $a_\mu^{\rho\sigma\dots\eta}$:

$$\begin{aligned}
 a_\mu^\rho &\rightarrow a'_\mu{}^\rho = (a_\mu - \partial_\mu\lambda)^\rho + \hbar(T_1^{(\lambda)}a_\mu^\rho - T_1^{(a_\mu - \partial_\mu\lambda)}\lambda^\rho) + \mathcal{O}(\hbar^2), \\
 a_\mu^{\rho\sigma} &\rightarrow a'_\mu{}^{\rho\sigma} = (a_\mu - \partial_\mu\lambda)^{\rho\sigma} + \hbar\left(T_1^{(\lambda)}a_\mu^{\rho\sigma} - T_1^{(a_\mu - \partial_\mu\lambda)}\lambda^{\rho\sigma} \right. \\
 &\quad \left. + \frac{1}{2}\partial_\mu(\lambda^{(\rho}\lambda^{\sigma)}) - (a_\mu - \partial_\mu\lambda)^{\xi(\rho}\partial_\xi\lambda^{\sigma)} + \lambda^{\xi(\rho}\partial_\xi a_\mu^{\sigma)}\right) + \mathcal{O}(\hbar^2), \\
 a_\mu^{\rho\sigma\eta} &\rightarrow a'_\mu{}^{\rho\sigma\eta} = (a_\mu - \partial_\mu\lambda)^{\rho\sigma\eta} + \frac{1}{4}(\lambda^{\xi(\rho}\partial_\xi(a_\mu - \partial_\mu\lambda)^{\sigma\eta)}) - (a_\mu - \partial_\mu\lambda)^{(\rho}\lambda^{\sigma\eta)} \\
 &\quad + \partial_\mu(\lambda^{\xi(\rho}\partial_\xi\lambda^{\sigma\eta)}) + \mathcal{O}(\hbar), \\
 a_\mu^{\rho\sigma\eta\xi} &\rightarrow a'_\mu{}^{\rho\sigma\eta\xi} = (a_\mu - \partial_\mu\lambda)^{\rho\sigma\eta\xi} + \frac{1}{48}\partial_\mu(\lambda^{(\rho\sigma}\lambda^{\eta\xi)}) + \mathcal{O}(\hbar).
 \end{aligned}
 \tag{4.12}$$

Using (2.23) we can construct the field strengths $F_{k,\mu\nu} = T_k^{(f_{\mu\nu})}$ where here $f_{\mu\nu}$ denotes the tensors $f_{\mu\nu}^{\mu_1}, f_{\mu\nu}^{\mu_1\mu_2}, f_{\mu\nu}^{\mu_1\mu_2\mu_3}, \dots$,

$$\begin{aligned} f_{\mu\nu}^\rho &= \partial_\mu a_\nu^\rho + \hbar T_1^{(a_\mu)} a_\nu^\rho - (\mu \rightleftharpoons \nu) + \mathcal{O}(\hbar^2), \\ f_{\mu\nu}^{\rho\sigma} &= \partial_\mu a_\nu^{\rho\sigma} + \hbar (T_1^{(a_\mu)} a_\nu^{\rho\sigma} + a_\mu^{\xi(\rho} \partial_\xi a_\nu^{\sigma)}) - (\mu \rightleftharpoons \nu) + \mathcal{O}(\hbar^2), \\ f_{\mu\nu}^{\rho\sigma\eta} &= \partial_\mu a_\nu^{\rho\sigma\eta} + \frac{1}{4} a_\mu^{\xi(\rho} \partial_\xi a_\nu^{\sigma\eta)} - (\mu \rightleftharpoons \nu) + \mathcal{O}(\hbar), \\ f_{\mu\nu}^{\rho\sigma\eta\xi} &= \partial_\mu a_\nu^{\rho\sigma\eta\xi} - (\mu \rightleftharpoons \nu) + \mathcal{O}(\hbar). \end{aligned} \tag{4.13}$$

They gauge transform according to

$$\begin{aligned} f_{\mu\nu}^\rho &\rightarrow f_{\mu\nu}'^\rho = f_{\mu\nu}^\rho + \hbar (T_1^{(\lambda)} f_{\mu\nu}^\rho - T_1^{(f_{\mu\nu})} \lambda^\rho) + \mathcal{O}(\hbar^2), \\ f_{\mu\nu}^{\rho\sigma} &\rightarrow f_{\mu\nu}'^{\rho\sigma} = f_{\mu\nu}^{\rho\sigma} + \hbar (T_1^{(\lambda)} f_{\mu\nu}^{\rho\sigma} - T_1^{(f_{\mu\nu})} \lambda^{\rho\sigma} + \lambda^{\xi(\rho} \partial_\xi f_{\mu\nu}^{\sigma)} - f_{\mu\nu}^{\xi(\rho} \partial_\xi \lambda^{\sigma)}) + \mathcal{O}(\hbar^2), \\ f_{\mu\nu}^{\rho\sigma\eta} &\rightarrow f_{\mu\nu}'^{\rho\sigma\eta} = f_{\mu\nu}^{\rho\sigma\eta} + \frac{1}{4} (\lambda^{\xi(\rho} \partial_\xi f_{\mu\nu}^{\sigma\eta)} - f_{\mu\nu}^{\xi(\rho} \partial_\xi \lambda^{\sigma\eta)}) + \mathcal{O}(\hbar), \\ f_{\mu\nu}^{\rho\sigma\eta\xi} &\rightarrow f_{\mu\nu}'^{\rho\sigma\eta\xi} = f_{\mu\nu}^{\rho\sigma\eta\xi} + \mathcal{O}(\hbar). \end{aligned} \tag{4.14}$$

4.5. Field equations

We can now substitute the above expansion for the field strength tensor into the sourceless Yang–Mills type equation (2.30)

$$\begin{aligned} \partial^\mu f_{\mu\nu}^\rho + \hbar T_1^{(a^\mu)} f_{\mu\nu}^\rho + \mathcal{O}(\hbar^2) &= 0, \\ \partial^\mu f_{\mu\nu}^{\rho\sigma} + \hbar (T_1^{(a^\mu)} f_{\mu\nu}^{\rho\sigma} + [a^\mu]^{\xi(\rho} \partial_\xi f_{\mu\nu}^{\sigma)}) + \mathcal{O}(\hbar^2) &= 0, \\ \partial^\mu f_{\mu\nu}^{\rho\sigma\eta} + \frac{1}{4} [a^\mu]^{\xi(\rho} \partial_\xi f_{\mu\nu}^{\sigma\eta)} + \mathcal{O}(\hbar) &= 0, \\ \partial^\mu f_{\mu\nu}^{\rho\sigma\eta\xi} + \mathcal{O}(\hbar) &= 0, \end{aligned} \tag{4.15}$$

where a^μ is obtained from a_μ assuming a flat metric.

From the action (2.29) we can also easily obtain the field equation for the scalar field ϕ in a background A_μ . For simplicity, choose \star to be the Groenewold–Moyal star. Variations of ϕ lead to an exactly conserved current

$$\partial_\mu k^\mu = 0, \tag{4.16}$$

where

$$k^\mu = D[A]^\mu \phi + \hbar a_\rho^\mu D[A]^\rho \phi - \frac{\hbar}{2} \partial_\rho (a_\sigma^{\mu\rho} D[A]^\sigma \phi) + \mathcal{O}(\hbar^2). \tag{4.17}$$

One recovers the commutative result $\partial_\mu \partial^\mu \phi = 0$ at zeroth order in \hbar . On the other hand, in order to couple ϕ to the above gauge theory one should search currents J^μ which are covariantly conserved, i.e., satisfy (2.33), and gauge transform as (2.32). Let us assume they exists and can be expanded as in (2.31), with $J_{k,\mu}$ given by $J_{k,\mu} = T_k^{(j_\mu)}$. j_μ denote the tensors $j_\mu = (j_\mu^{\mu_1}, j_\mu^{\mu_1\mu_2}, j_\mu^{\mu_1\mu_2\mu_3}, \dots)$, which now enter in the right-hand sides of (4.15). From (2.33), $T_1^{(j_\mu)}$ is exactly conserved at zeroth order in \hbar . Candidates for first two currents (up to an overall

factor) are

$$\begin{aligned} j_{\mu}^{\rho} &= \partial_{\mu}\phi\partial^{\rho}\phi - \phi\partial_{\mu}\partial^{\rho}\phi + \mathcal{O}(\hbar), \\ j_{\mu}^{\rho\sigma} &= \partial_{\mu}\phi\partial^{\rho,\sigma}\phi - \phi\partial_{\mu}\partial^{\rho,\sigma}\phi + \mathcal{O}(\hbar). \end{aligned} \quad (4.18)$$

The next order expressions for these currents will contain the potentials a^{μ} and they are expected to be nonlocal.

5. Conclusion

In the previous sections we developed tools for writing gauge theories on the space $\{\star\}_{\alpha}$ of equivalent star products associated with any given Poisson bi-vector α . The gauge theories can be regarded as an extension of non-commutative $U(1)$ gauge theory. Since general gauge transformations induce $\mathcal{O}(\hbar)$ corrections in the integration measure, they cannot be regarded as purely internal transformations.

Although it is not difficult to write down matter field actions, as for example in (2.29), a final ingredient is needed in order to introduce kinetic terms for the infinitely many gauge fields in $F_{\mu\nu}$, namely the trace Tr over differential operators. In addition to satisfying the usual trace property, $\text{Tr} F_{\mu\nu} F^{\mu\nu}$ should reduce to the usual action for non-commutative $U(1)$ gauge fields upon restricting $F_{\mu\nu}$ to (2.28). Other familiar non-commutative field theories may be contained in the full action. In order to make contact with physical theories, mechanisms, such as the Higgs mechanism, should be applied to give (large) masses to all but a finite number of the gauge fields. This may then involve introducing additional derivative-valued fields.

Finally, a more ambitious project would be to write down field theories on the space of *all* equivalence classes $\{\star\}_{\alpha}$ of star products. This means making the Poisson bi-vector α dynamical, and then as a result all of the bi-differential operators B_n in the star product (3.2) dynamical. Variations of these operators must then include diffeomorphisms on the underlying manifold, at all orders in \hbar , setting the possible framework for a quasi-classical approximation to quantum gravity [17].

Acknowledgement

The work of A.P. has been supported by FAPESP grant number 06/56056-0.

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