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We construct a perturbative solution to classical noncommutative gauge theory on $\mathbb{R}^3$ minus the origin using the Groenewald-Moyal star product. The result describes a noncommutative point charge. Applying it to the quantum mechanics of the noncommutative hydrogen atom gives shifts in the $1S$ hyperfine splitting which are first order in the noncommutativity parameter.

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Introduction.—As position eigenstates do not occur in theories with space-space noncommutativity, there can be no intrinsic notion of points for such theories. It has then been argued that charges become smeared in noncommutative gauge theory [1]. Gaussian distributions having width equal to the noncommutativity scale were utilized to model noncommutative sources, and, in particular, sources associated with noncommutative black holes [2]. However, once one introduces a star product realization of the noncommutative algebra on a commutative manifold, there can be no guarantee that the solutions to some noncommutative field equations will be free of singularities. Here it is demonstrated that such singularities can be associated with points on the commutative manifold. More precisely, a regular solution and star product can be defined on a commutative manifold with points removed. Thus although there is no intrinsic definition of points in noncommutative geometry, after introducing a star product there nevertheless can be a notion of point sources for noncommutative field theory.

The example discussed in this Letter is that of a static point charge for noncommutative $U(1)$ gauge theory. Constant noncommutativity is assumed and the Groenewald-Moyal star product is utilized. The solution is obtained perturbatively up to second order in the noncommutativity tensor. Only space-space components of the noncommutativity tensor affect the fields around the static point source. A magnetostatic potential is induced at first order in the noncommutativity tensor, while corrections to the electrostatic potential are induced at second order. These lowest order corrections are independent of the choice of star product. The solution is nontrivial in the sense that it is not obtained from a Seiberg-Witten map of the commutative Coulomb solution. The latter would instead induce a nonvanishing current density away from the point source.

There is some utility in applying the lowest order solution to the study of the noncommutative version of the hydrogen atom. As noncommutativity is well motivated from the perspective of quantum gravity and string theory, any noncommutative corrections are expected to occur at the Planck scale. Nevertheless, experimentally accessible scales should also be explored, especially in light of research on large extra dimensions which can potentially bring down the four-dimensional Planck scale. It is then reasonable to put experimental bounds on the noncommutative scale. With regards to the hydrogen atom, a debate in the literature concerns how to treat the nucleus in the noncommutative theory [3–5]. For a multiparticle system, the commutation relations for the different particles should, in principle, be derived starting from the noncommutative field theory. In [5] starting from a noncommutative version of QED the authors found that two particles of opposite charge have opposite noncommutativity, while the relative coordinates commute. This would then lead to no noncommutative corrections to the hydrogen atom spectrum. As pointed out in [4], the correct approach would have to include noncommutative QCD, which unfortunately is not well understood. There it was further argued that the nucleus should be treated as a commutative object since QCD effects dominate over any noncommutative physics [4]. Corrections then result in the Lamb shifts due to the noncommutativity of just the electron. It may be difficult to answer the debate conclusively in the absence of a consistent theory of noncommutative quarks and gluons. A pragmatic approach would be to instead set separate bounds on the noncommutativity of the electron and nucleus. We do this by presuming the nucleus to be a noncommutative point charge in the sense described above. New shifts result in the hydrogen atom spectra at lowest order in the noncommutativity parameter, including in the $1S$ hyperfine splitting.

Point sources in noncommutative electrodynamics.—Here we find it helpful to work in terms of SI units, with $c = 1$ (but not $\hbar = 1$), where the noncommutative gauge coupling constant $g_{\rm SI}$ has nontrivial units. Assuming constant noncommutativity $\theta^{\mu\nu} = -\theta^{\nu\mu}$, the $U(1)$ gauge field equations read

$$\partial^\mu F_{\mu\nu} - ig_{\rm SI}[A^\mu, F_{\mu\nu}]_\star = J_\nu, \tag{1}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig_{\rm SI}[A_\mu, A_\nu]_\star, \tag{2}$$

$[,]_\star$ being the star commutator associated with the Groenewald-Moyal star

$$\star = \exp\left\{\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu\overrightarrow{\partial}_\nu\right\}. \tag{3}$$

We can perturbatively solve these equations starting from the commutative Coulomb solution in three spatial dimensions

$$A_\mu^{(0)} = -\frac{q}{4\pi\epsilon_0 r}\delta_{\mu 0}, \tag{4}$$

$\mu, \nu, \ldots = 0, 1, 2, 3$. Here we included the permittivity constant ϵ_0 in the current, $J_\mu^{(0)} = (q/\epsilon_0)\delta_{\mu 0}\delta^3(x)$. We denote by

$$g = \frac{g_{\rm SI} q}{4\pi\epsilon_0} \tag{5}$$

a dimensionless factor and take (4) to be the zeroth order term in a Taylor expansion in $\theta^{\mu\nu}$: $A_\mu = A_\mu^{(0)} + A_\mu^{(1)} + A_\mu^{(2)} + \cdots$. Assume that the noncommutative current J_μ vanishes everywhere away from the origin at all orders in $\theta^{\mu\nu}$. Then (1) gives

$$\nabla^2 A_0^{(1)} = 0,$$

$$(\nabla^2 - \partial_0^2)A_i^{(1)} + \partial_0\partial_i A_0^{(1)} - \frac{gq}{4\pi\epsilon_0}\frac{\theta^{ij}x_j}{r^6} = 0, \qquad r \neq 0, \tag{6}$$

$i, j, \ldots = 1, 2, 3$, after extracting the first order terms and applying the Coulomb gauge $\nabla \cdot \vec{A} = 0$. A static first order solution is

$$A_i^{(1)} = \frac{gq}{16\pi\epsilon_0}\frac{\theta^{ij}x_j}{r^4}, \tag{7}$$

with $A_0^{(1)} = 0$. Equation (7) satisfies the Coulomb gauge condition due to the antisymmetry of θ^{ij} and implies the existence of a noncommutative magnetic field

$$B_i^{(1)} = \frac{1}{2}\epsilon_{ijk}F_{jk}^{(1)} = -\frac{gq}{16\pi\epsilon_0}\epsilon_{ijk}\left\{\frac{\theta^{jk}}{r^4} - 4\frac{\theta^{j\ell}x_\ell x_k}{r^6}\right\}. \tag{8}$$

We call (7) the inhomogeneous solution. It falls off faster than a magnetic dipole potential $\epsilon_{ijk}m_j^{(1)}x_k/r^3$, yet it cannot be expressed in terms of a magnetic quadrupole potential $\mathcal{M}_{ijk}^{(1)}x_j x_x/(2r^5)$, with constant coefficients $\mathcal{M}_{ijk}^{(1)}$. On the other hand, these potentials, as well as higher moment potentials, can be regarded as homogeneous terms which can be added to the first order result (7). The moments are arbitrary, except for being linear in $\theta^{\mu\nu}$. (For instance, one can have $m_i^{(1)} \propto \theta^{i0}$ or $\epsilon_{ijk}\theta^{jk}$.) Additional first order homogeneous terms can be introduced with a multimoment expansion for the time component of $A_\mu^{(1)}$:

$$A_0^{(1)} = -\frac{1}{4\pi\epsilon_0}\left\{\frac{q^{(1)}}{r} + \frac{p_i^{(1)}x_i}{r^3} + \frac{Q_{ij}^{(1)}x_i x_j}{2r^5} + \cdots\right\}, \tag{9}$$

where the constant coefficients $q^{(1)}, p_i^{(1)}, Q_{ij}^{(1)}, \ldots$ are undetermined, except that they are linear in $\theta^{\mu\nu}$.

The first order solution (7) can be reexpressed in terms of the zeroth order solution (4) and its derivatives:

$$A_i^{(1)} = -\frac{1}{4}g_{\rm SI}\theta^{ij}A_0^{(0)}\partial_j A_0^{(0)}. \tag{10}$$

This is not a Seiberg-Witten map [6] of $A_0^{(0)}$, as commutative gauge transformations $A_0^{(0)} \to A_0^{(0)} + \partial_0\lambda$ do not induce noncommutative gauge transformations in A_μ. The standard expression for the Seiberg-Witten map at first order.

$$A_\mu^{(0)} \to A_\mu^{\rm SW} = A_\mu^{(0)} - \tfrac{1}{2}g_{\rm SI}\theta^{\rho\sigma}A_\rho^{(0)}(\partial_\mu A_\sigma^{(0)} - 2\partial_\sigma A_\mu^{(0)}) + \cdots \tag{11}$$

instead leads to a nonvanishing first order current density away from the origin (in addition to a singular current density at the origin). (Homogeneous terms $\mathcal{H}_{\mathcal{A}_\mu^{(0)}}$, satisfying $\mathcal{H}_{\mathcal{A}_\mu^{(0)}+\partial_\mu\lambda} - \mathcal{H}_{\mathcal{A}_\mu^{(0)}} = \theta^{\rho\sigma}\partial_\rho\lambda\partial_\sigma\mathcal{H}_{\mathcal{A}_\mu^{(0)}}$ at first order, can be added to Eq. (11) [7–9].) Substituting (7) in (11) gives

$$A_\mu^{\rm SW} = -\frac{q}{4\pi\epsilon_0 r}\left(1 + g\frac{\theta^{0i}x_i}{r^3} + \cdots\right)\delta_{\mu 0}, \tag{12}$$

which is associated with the nonvanishing current density for $r \neq 0$,

$$\begin{aligned} J_0^{\rm SW} &= -\frac{qg}{\pi\epsilon_0}\frac{\theta^{0i}x_i}{r^6} + \cdots \\ J_i^{\rm SW} &= -\frac{qg}{4\pi\epsilon_0}\frac{\theta^{ij}x_j}{r^6} + \cdots, \qquad r \neq 0. \end{aligned} \tag{13}$$

It is straightforward to extend the inhomogeneous solution to higher orders. At second order in $\theta^{\mu\nu}$ the field Eq. (1) gives

$$\begin{aligned} \nabla^2 A_0^{(2)} + \frac{qg^2}{8\pi\epsilon_0}\left\{\frac{{\rm Tr}\theta^2}{r^7} - 7\frac{[\theta^2]^{ij}x_i x_j}{r^9}\right\} &= 0 \\ (\nabla^2 - \partial_0^2)A_i^{(2)} + \partial_0\partial_i A_0^{(2)} = 0, \qquad r \neq 0, \end{aligned} \tag{14}$$

in the Coulomb gauge. It is solved by

$$A_0^{(2)} = -\frac{qg^2}{16\pi\epsilon_0}\left\{\frac{{\rm Tr}\theta^2}{5r^5} - \frac{[\theta^2]^{ij}x_i x_j}{r^7}\right\}, \tag{15}$$

and $A_i^{(2)} = 0$. $A_0^{(2)}$ then falls off faster than an electric quadrupole potential, but cannot be expressed as an octopole potential.

The lowest order corrections to the Coulomb potential (7) and (15) were computed using the leading order of the star commutator. They are therefore independent of the choice of star product. [This refers to the fact that there are other (in fact, infinitely many) star product realizations of the noncommutative algebra. They form a huge gauge equivalence class [10] and can be used to construct novel gauge theories [11]. The selection of a particular star product in the equivalence class can be regarded as a gauge choice. The leading order of the star commutator is the same for all star products in the equivalence class and is proportional to the Poisson bracket. As the computations that led to (9) and (19) involved only the leading order of the star commutator, the expressions are the same for all star products in the equivalence class. The same cannot be said for higher order corrections to the commutative field theory].

Another look at the noncommutative hydrogen atom.— Now consider a "noncommutative" electron moving in the potential found above. Following [3] its quantum algebra is defined by

$$[\hat{x}_i, \hat{x}_j] = i\theta_{ij} \qquad [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij} \qquad [\hat{p}_i, \hat{p}_j] = 0, \tag{16}$$

along with the usual spin algebra. It is well known that this can be mapped to the standard Heisenberg algebra, spanned by $\hat{X}_i$ and $\hat{P}_j$, using

$$\hat{x}_i \to \hat{X}_i = \hat{x}_i + \frac{1}{2\hbar}\theta_{ij}\hat{p}_j \qquad \hat{p}_i \to \hat{P}_i = \hat{p}_i. \tag{17}$$

For the dynamics in a noncommutative gauge field we can adapt the standard Hamiltonian for a nonrelativistic electron

$$\hat{H} = \frac{1}{2m}[\hat{p}_i - qA_i(\hat{x})]^2 + qA_0(\hat{x}) - \frac{2\mu_B}{\hbar}\vec{S}\cdot\vec{B}(\hat{x}), \tag{18}$$

where $\mu_B = q\hbar/2m$. Alternatively, $\hat{H}$ can be realized in terms of differential operators acting on wave functions on $\mathbb{R}^3$, using the Groenewald-Moyal star (3). For example, the first term corresponds to $-\frac{\hbar^2}{2m}D_{\star i}D_{\star i}$. The covariant derivative $D_{\star i}$ must be the same as that entering in the field equations (1) and the definition field strength (2), here written in the fundamental representation; i.e., $D_{\star i} = \partial_i - ig_{\rm SI}A_i\star$. In comparing with (18) one gets the identification of $g_{\rm SI}$ with $q/\hbar$, or equivalently, the dimensionless coupling constant g defined in (5) with the fine structure constant $g = q^2/(4\pi\epsilon_0\hbar) = \alpha$.

Next we substitute the solution for A_μ and $\vec{B}$ found above, keeping only the first order correction. The result is

$$\hat{H} = \frac{1}{2m}\left(\hat{p}_i - \frac{\alpha^2\hbar}{4}\frac{\theta^{ij}\hat{x}_j}{\hat{r}^4}\right)^2 - \frac{\alpha\hbar}{\hat{r}} + \frac{\alpha^2\hbar}{4m}\epsilon_{ijk}S_i\left\{\frac{\theta^{jk}}{\hat{r}^4} - 4\frac{\theta^{j\ell}\hat{x}_\ell\hat{x}_k}{\hat{r}^6}\right\}, \tag{19}$$

where $\hat{r}^2 = \hat{x}_i\hat{x}_i$. Since we are only interested in the first order in θ, it does not matter if we express the vector potential and magnetic field as functions of the commuting or noncommuting coordinates, $\hat{X}_i$ or $\hat{x}_i$. This of course is not the case for the Coulomb potential. Following [3], $\hat{H}$ can be reexpressed in terms of $\hat{X}_i$ using (17). Thus

$$\hat{H} = \hat{H}^{(0)} + \hat{H}_1^{(1)} + \hat{H}_2^{(1)} + \hat{H}_3^{(1)} + \cdots,$$
$$\hat{H}^{(0)} = \frac{1}{2m}\hat{P}_i\hat{P}_i - \frac{\alpha\hbar}{\hat{R}} \qquad \hat{H}_1^{(1)} = -\frac{\alpha}{2}\frac{\vec{\theta}\cdot\vec{L}}{\hat{R}^3}$$
$$\hat{H}_2^{(1)} = \frac{\alpha^2\hbar}{4m}\frac{\vec{\theta}\cdot\vec{L}}{\hat{R}^4} \qquad H_3^{(1)} = \frac{\alpha^2\hbar}{2m}\left\{\frac{2(\vec{X}\cdot\vec{S})(\vec{X}\cdot\vec{\theta})}{\hat{R}^6} - \frac{\vec{\theta}\cdot\vec{S}}{\hat{R}^4}\right\}, \tag{20}$$

where $\hat{R}^2 = \hat{X}_i\hat{X}_i$, $\theta_{ij} = \epsilon_{ijk}\theta_k$, and the dots indicate higher orders. $\hat{H}_1^{(1)}$ was obtained in [3], while $\hat{H}_2^{(1)}$ and $\hat{H}_3^{(1)}$ are the new corrections following from $A_i^{(1)}$, and are due to the noncommutativity of the source. (It was argued in [5] that the relative coordinate $\hat{x}_i$ is commuting and that, as a result, the correction $\hat{H}_1^{(1)}$ to the Coulomb interaction is absent. On the other hand, the perturbations $\hat{H}_2^{(1)}$ and $\hat{H}_3^{(1)}$ persist when $\hat{x}_i$ is commuting, resulting in first order shifts in the hydrogen atom spectrum.) The latter contains couplings of the noncommutativity to both the orbital and spin angular momentum, respectively. Corrections to the Lamb shifts of the $\ell \neq 0$ states result from $\hat{H}_1^{(1)}$ and $\hat{H}_2^{(1)}$. The matrix elements are diagonalized by taking $\vec{\theta} = (0, 0, \theta)$. The former were computed in [3]. Similar expressions result for the latter. For the two $2P_{1/2}$ states,

$$\langle\hat{H}_1^{(1)}\rangle_{2P_{1/2}^{\pm 1/2}} = -\frac{\alpha\theta}{2}\left\langle\frac{L_z}{\hat{R}^3}\right\rangle_{2P_{1/2}^{\pm 1/2}} = \mp\frac{\alpha\hbar\theta}{72a_0^3} = \mp\frac{\alpha^4 m\theta}{72\lambda_e^2}, \tag{21}$$

$$\langle\hat{H}_2^{(1)}\rangle_{2P_{1/2}^{\pm 1/2}} = \frac{\alpha^2\hbar\theta}{4m}\left\langle\frac{L_z}{\hat{R}^4}\right\rangle_{2P_{1/2}^{\pm 1/2}} = \pm\frac{\alpha^2\hbar^2\theta}{144ma_0^4} = \pm\frac{\alpha^6 m\theta}{144\lambda_e^2}, \tag{22}$$

using spectroscopic notation $n\ell_j^{m_j}$. The new contribution (22) is down by a factor of α^2 and thus gives a much weaker bound on θ. According to [12] the current theoretical accuracy on the $2P$ Lamb shift is about 0.08 kHz. From the splitting (21), this then gives the bound $\theta \lesssim (6\ \mathrm{GeV})^{-2}$ (there was a computational error in [3]), while from (22) one gets $\theta \lesssim (30\ \mathrm{MeV})^{-2}$. As has been argued in [5], noncommutativity is not the same for all particles in noncommutative quantum mechanics. Here the $(6\ \mathrm{GeV})^{-2}$ bound is associated with the test charge (electron), while the $(30\ \mathrm{MeV})^{-2}$ bound is associated with the lowest order noncommutativity of the source (proton). Comparing the latter with the QCD scale $\Lambda_{\rm QCD} \sim 200$ MeV, one cannot here conclude that strong interactions dominate over any noncommutative effects of the source.

More interesting are the matrix elements of $\hat{H}_3^{(1)}$, as they induce new splittings in the $1S$ states, thus affecting the hyperfine structure. (Noncommutative corrections to the $1S$ hyperfine splitting were examined previously in [13] by expressing the dipole-dipole interaction in terms of the noncommutative coordinates $\hat{x}_i$. Those corrections, however, go like θ^2 at the lowest order.) Actually, with the restriction to static point sources, the $1S$ matrix elements are linearly divergent. To get a finite answer we take into account the finite size of the nucleus and insert the $\Lambda_{\rm QCD}$ cutoff. [This of course would not be valid for the muonium atom ($e^- \mu^+$). Relaxing the assumption of static sources, thereby taking into account recoil effects, may cure the ultraviolet divergence for that case.]

$$\begin{aligned}\langle \hat{H}_3^{(1)} \rangle_{1S_{1/2}^{\pm 1/2}} &= \frac{\alpha^2 \hbar}{2m} \left\langle \frac{S_i \theta_j}{\hat{R}^6} (2\hat{X}_i \hat{X}_j - \hat{R}^2 \delta_{ij}) \right\rangle_{1S_{1/2}^{\pm 1/2}} \\ &= -\frac{\alpha^2 \hbar \theta}{6m} \left\langle \frac{S_z}{\hat{R}^4} \right\rangle_{1S_{1/2}^{\pm 1/2}} \\ &= \mp \frac{\alpha^2 \hbar \theta}{3 m a_0^3 \Lambda_{\rm QCD}^{-1}} \\ &= \mp \frac{\alpha^5 m \theta}{3 \hbar \lambda_e \Lambda_{\rm QCD}^{-1}}, \end{aligned} \tag{23}$$

where again $\vec{\theta} = (0, 0, \theta)$, and we used $\langle \hat{X}_i \hat{X}_j / \hat{R}^n \rangle_{\ell=0} = \frac{1}{3} \delta_{ij} \langle 1/\hat{R}^{n-2} \rangle_{\ell=0}$. These terms should then mix with the usual $1S$ hyperfine matrix elements. According to [12] the current theoretical accuracy on the $1S$ shift is about 14 kHz. From the splitting (23), this gives $\theta \lesssim (4\ {\rm GeV})^{-2}$ for the noncommutativity of the proton, which is now well above the QCD scale. However, without having a consistent treatment of noncommutative QCD, the insertion of the QCD cutoff in this approach remains uncertain.

Concluding remarks.—We have found that the noncommutativity of the electron and the proton have distinct experimental signatures in the hydrogen spectrum. We further found the same order of magnitude for their bounds.

There appear to be a number of possibilities for generalizations of this work. (a) One is to obtain the exact solution for the noncommutative potential and also its dependence on the choice of star product. (b) Another is to drop the restriction of static sources. This will allow for the study of recoil effects in noncommutative quantum systems. As stated earlier, this appears necessary to remove the divergence in the correction to the $1S$ state of the noncommutative muonium atom. (c) A self-consistent dynamics for these point sources, at the classical as well as the quantum level, is then also of interest. The classical equations of motion would be analogous to the Wong equations in Yang-Mills theory [14,15]. (d) Generalizations to other gauge theories, including gravity, should be possible. For the case of gravity this should lead to yet another description of noncommutative black holes [16]. (e) More challenging perhaps would be an attempt to find analogous solutions in theories with nonconstant noncommutativity.

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