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Non-constant non-commutativity in 2d field theories and a new look at fuzzy monopoles

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Abstract

We write down scalar field theory and gauge theory on two-dimensional non-commutative spaces $\mathcal{M}$ with non-vanishing curvature and non-constant non-commutativity. Usual dynamics results upon taking the limit of $\mathcal{M}$ going to (i) a commutative manifold $\mathcal{M}_0$ having non-vanishing curvature and (ii) the non-commutative plane. Our procedure does not require introducing singular algebraic maps or frame fields. Rather, we exploit the Kähler structure in the limit (i) and identify the symplectic two-form with the volume two-form. As an example, we take $\mathcal{M}$ to be the stereographically projected fuzzy sphere, and find magnetic monopole solutions to the non-commutative Maxwell equations. Although the magnetic charges are conserved, the classical theory does not require that they be quantized. The non-commutative gauge field strength transforms in the usual manner, but the same is not, in general, true for the associated potentials. We develop a perturbation scheme to obtain the expression for gauge transformations about limits (i) and (ii). We also obtain the lowest order Seiberg–Witten map to write down corrections to the commutative field equations and show that solutions to Maxwell theory on $\mathcal{M}_0$ are stable under inclusion of lowest order non-commutative corrections. The results are applied to the example of non-commutative AdS^2.

1. Introduction

Much work has been carried out for field theories on the non-commutative plane. This is the case of constant non-commutativity. On the other hand, not much is known for field theories on spaces with non-constant non-commutativity. Exceptional cases are when the non-commutativity is associated with certain Lie-algebra structures, such as the case with fuzzy spheres and fuzzy

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CP^n models (for a review, see [1]). Other cases have been discussed in [2,3]. Among the obstacles to constructing field theories on general non-commutative spaces are problems in defining differentiation, integration and a Dirac operator.[1] Recently, scalar field theory [5] and gauge theory [6] have been formulated on general curved non-commutative manifolds. Although the procedures in [5] and [6] differ, both make use of frame fields induced by a non-constant metric. They were associated with algebraic maps to the non-commutative plane in [5], and appear in the definition of derivations in [6]. A loss of generality may result in the non-commutative theory, however, as frame fields are only defined on local coordinate patches. In [5] this resulted in the possibility of singular algebraic maps in the non-commutative theory. Below we shall develop an alternative approach for two-dimensional non-commutative scalar and gauge theories which avoids the introduction of frame fields. We also avoid the problem of defining non-commutative derivatives by writing the theory algebraically from the start, expressing the underlying commutative theory in terms of a Poisson bracket algebra.

In writing down field theories on some non-commutative space $\mathcal{M}$, we shall require that the results be consistent with deformations of known theories. In particular, we insist that the field theories reduce to the standard form in the limit that $\mathcal{M}$ reduces (i) to a commutative manifold with non-vanishing curvature and (ii) to the non-commutative plane. As stated above, we restrict to two-dimensional field theories. In that case we can exploit the Kähler structure of the commutative space and identify the symplectic two-form with the volume two-form. The Lagrangian densities for the commutative theories can then be expressed in terms of Poisson brackets. In passing to the non-commutative theory we simply replace the Poisson bracket by an appropriate commutator and the classical measure by its non-commutative counterpart. This can be done without spoiling the symmetries of the underlying commutative space $\mathcal{M}_0$, if there are any, since the Poisson brackets can be constructed to preserve these symmetries. The resulting free non-commutative scalar field and Maxwell equations have a simple form. Concerning the latter, there are no propagating degrees of freedom, just as with commutative electrodynamics in two dimension, and solutions are characterized by a constant flux per unit area and action per unit area. The non-commutative field strength is covariant with respect to gauge transformations. However, the corresponding transformations of the potentials are nontrivial and geometry dependent.

We shall apply the results to find magnetic monopole solutions on the fuzzy sphere. Electrodynamics on the fuzzy sphere has been of considerable interest [7–14]. Ansätse for magnetic monopoles on the fuzzy sphere have been proposed [15], although they were not required to be deformations of monopoles on the commutative sphere. Non-commutative magnetic monopoles were expressed using the analogue of embedding coordinates in [12], and had the correct commutative limit. Here we obtain magnetic monopoles as solutions to the non-commutative Maxwell equations on the analogue of the projective plane. A nonsingular map from the fuzzy sphere to the non-commutative projective plane was given in [16]. (The coordinate singularity appears only in the commutative limit.) Since it is nonsingular, one can express the non-commutative potentials free of Dirac-string singularities. We find that the associated magnetic charges are conserved, although not for topological reasons, although they need not be quantized, at least at the classical level. Alternatively, we can get charge quantization, upon imposing additional constraints, but these charges have a singular commutative limit.

[1] On the other hand, such problems do not arise if one is only interested in doing particle mechanics on these spaces, as for example is done in [4].

It is common to realize the non-commutative algebra with a star product. The Groenewold–Moyal star product is often used, but since it is associated with constant non-commutativity, it is not very convenient to realize the algebra on $\mathcal{M}$. More appropriate is Konsevich's formality map [17] which was utilized in [6]. Alternatively, we shall rely upon the star product developed in [16] which is based on a non-linear deformation of coherent states on the complex plane [18]. An exact integral expression for this star product was given, which can be expanded about either limit (i) or (ii). An expansion for the measure can also be given by simply demanding that the result satisfies the usual properties of a trace. By applying these expansions we get corrections to the scalar and Maxwell actions about the two limits. Although approximation schemes for these actions have been given previously [5,6], the one presented here has the advantage of simplicity. Concerning the scalar field action, we get that the lowest order effects of non-commutativity are obtained by just replacing derivatives on $\mathcal{M}_0$ by 'covariant' ones. We also compute lowest order corrections to the commutative expression for gauge transformations of the potentials, and show how to Seiberg–Witten map [19] these corrections away. Using the Seiberg–Witten map we can also compute corrections to the commutative flux through any region, as well as to the Maxwell equations and its solutions. As expected from the exact theory, the flux per unit area and action per unit area are constants for the solutions, but their values are shifted from the commutative results. Because the shift is small (i.e. of the order of the non-commutativity parameter) we say that solutions to the commutative theory are stable under inclusion of the non-commutative corrections. As an example, we apply the techniques to the case where $\mathcal{M}$ is the non-commutative analogue of the Lobachevsky plane. This space is defined by a projection from non-commutative AdS^2. Here we show how to obtain corrections to the solutions to the commutative free scalar field theory. The solutions to the commutative Maxwell equations receive no first order corrections.

We review scalar field theory and gauge field theory in the commutative limit (i) in Section 2 and the non-commutative plane limit (ii) in Section 3. Field theories on spaces with non-vanishing curvature and non-constant non-commutativity are described in Section 4. In Section 5 we apply the results to the fuzzy sphere and obtain the analogue of magnetic monopole solutions. The first order corrections away from the two limits are computed in Section 6. In Section 7, we apply the results to the example of non-commutative Lobachevsky plane. Brief remarks are made in Section 8.

2. Curved space—commutative theory

Here we take advantage of the Kähler structure of any two-dimensional commutative manifold $\mathcal{M}_0$ to express scalar field and gauge field Lagrangians on any coordinate patch P_0 of $\mathcal{M}_0$ in terms of Poisson brackets. Let $g_{\mu\nu}$ denote the metric tensor associated with P_0, parameterized by real coordinates x^μ, $\mu = 1, 2$. Alternatively, we can define complex coordinates $z = x^1 + ix^2$ and $\bar{z} = x^1 - ix^2$. We introduce a function $\theta_0(z, \bar{z})$, which we assume is non-singular, and the commutative measure $d\mu_0(z, \bar{z})$ on P_0 by writing the area two form as

$$\sqrt{g}\, d^2x = \frac{i\, dz \wedge d\bar{z}}{\theta_0\,(z, \bar{z})} \equiv 2\pi\, d\mu_0(z, \bar{z}). \tag{2.1}$$

This can be identified with a symplectic two-form, with a corresponding Poisson bracket $\{\,,\,\}$. So if α and β are functions of z and $\bar{z}$ their Poisson bracket is

$$\{\alpha, \beta\} = -i\theta_0(z, \bar{z})(\partial\alpha\bar{\partial}\beta - \bar{\partial}\alpha\partial\beta), \tag{2.2}$$

where $\partial = \frac{\partial}{\partial z}$, $\bar\partial = \frac{\partial}{\partial \bar z}$. Its integral over P_0 with respect to the measure $d\mu_0(z,\bar z)$ vanishes provided α and β vanish sufficiently rapidly as the boundary of P_0 is approached. More generally, for some region σ in P_0, the integral is equal to boundary terms:

$$\int_\sigma d\mu_0(z,\bar z)\,\{\alpha,\beta\} = \int_{\partial\sigma} dz\,\beta\partial\alpha + \int_{\partial\sigma} d\bar z\,\beta\bar\partial\alpha = -\int_{\partial\sigma} dz\,\alpha\partial\beta - \int_{\partial\sigma} d\bar z\,\alpha\bar\partial\beta, \tag{2.3}$$

where $\partial\sigma$ is the boundary of σ.

In writing down scalar field theory we shall choose the conformal gauge. In that case, the free action for a massless scalar field ϕ is

$$S^0_\phi = i\int dz\wedge d\bar z\,\partial\phi\bar\partial\phi, \tag{2.4}$$

which can then be re-expressed in terms of Poisson brackets

$$S^0_\phi = 2\pi\int d\mu_0(z,\bar z)\,\theta_0(z,\bar z)^{-1}\{z,\phi\}\{\bar z,\phi\}. \tag{2.5}$$

It is not necessary to choose a gauge restricting the metic tensor in the case of gauge theories. For this introduce a potential one form $\mathtt{a} = dz\,a + d\bar z\,\bar a$ on P_0 which gauge transforms as

$$\mathtt{a} \to \mathtt{a} + d\lambda. \tag{2.6}$$

The invariant field strength two-form is

$$\mathtt{f} = if\,dz\wedge d\bar z = (\bar\partial a - \partial\bar a)\,dz\wedge d\bar z. \tag{2.7}$$

Using (2.3) the flux Φ^0_σ through any region σ can be expressed as an integral of Poisson brackets of a and $\bar a$

$$\Phi^0_\sigma = \int_\sigma \mathtt{f} = \int_{\partial\sigma} \mathtt{a} = 2\pi\int_\sigma d\mu_0(z,\bar z)\big(\{z,a\} + \{\bar z,\bar a\}\big), \tag{2.8}$$

having no dependence on the metric, since θ_0 appearing in the measure cancels with θ_0 appearing in the Poisson brackets. In two dimensions the standard quadratic field action depends on the metric only through its determinant, and like the flux, its integrand can be expressed solely in terms of the Poisson brackets of a and $\bar a$

$$S^0_f = \int_\sigma \frac{d^2x}{\sqrt{g}} f^2 = \frac{i}{4}\int_\sigma dz\wedge d\bar z\,\theta_0(z,\bar z) f^2 \tag{2.9}$$

$$= \frac{\pi}{2}\int_\sigma d\mu_0(z,\bar z)\big(\{z,a\} + \{\bar z,\bar a\}\big)^2. \tag{2.10}$$

In comparing with the free scalar field action (2.5), here we have not specified a gauge for the metric and θ_0 does not appear explicitly in the integrand, despite its implicit dependence. There are no propagating degrees of freedom for this system. Rather, the equations of motion imply that

$$f = \frac{C_0}{\theta_0(z,\bar z)}, \tag{2.11}$$

where C_0 is the constant associated with the flux per unit area

$$C_0 = \frac{\Phi^0_\sigma}{2\pi \int_\sigma d\mu_0(z,\bar{z})}. \tag{2.12}$$

The action per unit area of the solution is also a constant, namely $C_0^2/4$.

3. Flat space—non-commutative theory

Now we review field theory on the non-commutative plane. This is the case of constant non-commutativity and no curvature. The algebra is generated by the operator $\mathbf{z}$ and its Hermitean conjugate $\mathbf{z}^\dagger$, satisfying

$$\left[\mathbf{z}, \mathbf{z}^\dagger\right] = \bar{k}, \tag{3.1}$$

where $\bar{k}$ denotes the non-commutativity parameter. It is standardly realized using the Groenewold–Moyal star product $\star_M$

$$\star_M = \exp \frac{\bar{k}}{2}\left(\frac{\overleftarrow{\partial}}{\partial z}\frac{\overrightarrow{\partial}}{\partial \bar{z}} - \frac{\overleftarrow{\partial}}{\partial \bar{z}}\frac{\overrightarrow{\partial}}{\partial z}\right), \tag{3.2}$$

where the complex coordinates z and $\bar{z}$ are now symbols of $\mathbf{z}$ and $\mathbf{z}^\dagger$, respectively. Upon defining the star commutator of any two functions $\mathcal{A}$ and $\mathcal{B}$ on the complex plane according to $[\mathcal{A},\mathcal{B}]_{\star_M} \equiv \mathcal{A} \star_M \mathcal{B} - \mathcal{B} \star_M \mathcal{A}$ one has, for example, $[z,\bar{z}]_{\star_M} = \bar{k}$. The standard convention for the integration measure is

$$d\mu_M(z,\bar{z}) = \frac{i}{2\pi \bar{k}} dz \wedge d\bar{z} \tag{3.3}$$

$\bar{k}$ has units of length-squared and hence the measure is dimensionless, unlike the commutative measure in (2.1). A well known identity results from the star product $\star_M$

$$\int d\mu_M(z,\bar{z})\, \mathcal{A} \star_M \mathcal{B} = \int d\mu_M(z,\bar{z}) \mathcal{A}\mathcal{B}, \tag{3.4}$$

where $\mathcal{A}$ and $\mathcal{B}$ vanish sufficiently rapidly at infinity, and from this, the cyclic property of trace easily follows.

The free scalar field action on the non-commutative plane is well known

$$\begin{aligned} S^M_\phi &= -\frac{2\pi}{\bar{k}} \int d\mu_M(z,\bar{z}) [z,\phi]_{\star_M} \star_M [\bar{z},\phi]_{\star_M} \\ &= -\frac{i}{\bar{k}^2} \int dz \wedge d\bar{z}\, [z,\phi]_{\star_M} [\bar{z},\phi]_{\star_M}, \end{aligned} \tag{3.5}$$

and from the fact that derivatives in z and $\bar{z}$ are realized by $\partial = -\frac{1}{\bar{k}}[\bar{z},]_{\star_M}$ and $\bar{\partial} = \frac{1}{\bar{k}}[z,]_{\star_M}$, respectively, (3.5) is identical to the free scalar field action on the commutative plane.

For gauge theories on the non-commutative plane we replace the potential one form a by $dz\,\mathcal{A} + d\bar{z}\,\bar{\mathcal{A}}$. Infinitesimal gauge variations by Λ of $\mathcal{A}$ and $\bar{\mathcal{A}}$ are given by

$$\delta\mathcal{A} = -\frac{1}{\bar{k}}[\bar{z},\Lambda]_{\star_M} - i[\mathcal{A},\Lambda]_{\star_M}, \qquad \delta\bar{\mathcal{A}} = \frac{1}{\bar{k}}[z,\Lambda]_{\star_M} - i[\bar{\mathcal{A}},\Lambda]_{\star_M}. \tag{3.6}$$

The field strength two-form is $i\mathcal{F}_M\, dz \wedge d\bar{z}$, where

$$i\mathcal{F}_M = \frac{1}{\bar{k}}[z, \mathcal{A}]_{\star_M} + \frac{1}{\bar{k}}[\bar{z}, \bar{\mathcal{A}}]_{\star_M} + i[\mathcal{A}, \bar{\mathcal{A}}]_{\star_M}, \tag{3.7}$$

which transforms covariantly under gauge transformations,

$$\delta\mathcal{F}_M = -i[\mathcal{F}_M, \Lambda]_{\star_M}, \tag{3.8}$$

$\mathcal{F}_M$ can be also be expressed as

$$\mathcal{F}_M = [\mathcal{Z}, \bar{\mathcal{Z}}]_{\star_M} + \frac{1}{\bar{k}^2}[z, \bar{z}]_{\star_M}, \tag{3.9}$$

where

$$\mathcal{Z} = -\frac{i}{\bar{k}}\bar{z} + \mathcal{A}, \qquad \bar{\mathcal{Z}} = \frac{i}{\bar{k}}z + \bar{\mathcal{A}}, \tag{3.10}$$

which also transform covariantly,

$$\delta\mathcal{Z} = -i[\mathcal{Z}, \Lambda]_{\star_M}, \qquad \delta\bar{\mathcal{Z}} = -i[\bar{\mathcal{Z}}, \Lambda]_{\star_M}. \tag{3.11}$$

The standard gauge theory action on the non-commutative plane is

$$\begin{aligned} S_f^M &= \frac{\pi\bar{k}}{2}\int d\mu_M(z, \bar{z})\mathcal{F}_M \star_M \mathcal{F}_M \\ &= \frac{i}{4}\int dz \wedge d\bar{z}\, \mathcal{F}_M^2. \end{aligned} \tag{3.12}$$

When $\bar{k} \to 0$, the star commutator goes to $i\bar{k}$ times the Poisson bracket (2.2), with $\theta_0(z, \bar{z})$ equal to one, and so (3.12) reduces to S_f^0 with the flat metric. The free field equations following from variations of $\mathcal{A}$ and $\bar{\mathcal{A}}$ are

$$[\mathcal{Z}, \mathcal{F}_M]_{\star_M} = [\bar{\mathcal{Z}}, \mathcal{F}_M]_{\star_M} = 0. \tag{3.13}$$

They are solved for $\mathcal{F}_M$ proportional to the identity. For the case of a pure gauge solutions ($\mathcal{F}_M = 0$), $\mathcal{Z}$ and $\bar{\mathcal{Z}}$ are given by

$$\begin{aligned} \mathcal{Z}_{pg} &= -\frac{i}{\bar{k}}\bar{U} \star_M \bar{z} \star_M U, \\ \bar{\mathcal{Z}}_{pg} &= \frac{i}{\bar{k}}\bar{U} \star_M z \star_M U, \end{aligned} \tag{3.14}$$

where U are unitary functions on the complex plane with regard to the Groenewold–Moyal star product, $\bar{U} \star_M U = U \star_M \bar{U} = 1$.

It is straightforward to couple the scalar field to gauge theories on the non-commutative plane. For this replace (3.5) by

$$\begin{aligned} S_{\phi,\mathcal{Z}}^M &= -2\pi\bar{k}\int d\mu_M(z, \bar{z})[\bar{\mathcal{Z}}, \phi]_{\star_M} \star_M [\mathcal{Z}, \phi]_{\star_M} \\ &= -i\int dz \wedge d\bar{z}\, [\bar{\mathcal{Z}}, \phi]_{\star_M}[\mathcal{Z}, \phi]_{\star_M}. \end{aligned} \tag{3.15}$$

Gauge invariance follows from (3.11) and $\delta\phi = -i[\phi, \Lambda]_{\star_M}$. The coupled field equations resulting from variations of the combined action $S_f^M + S_{\phi,\mathcal{Z}}^M$ are

$$\begin{aligned}
&[\mathcal{Z}, \mathcal{F}_M]_{\star_M} + 2\big[\phi, [\mathcal{Z}, \phi]_{\star_M}\big]_{\star_M} = 0,\\
&[\bar{\mathcal{Z}}, \mathcal{F}_M]_{\star_M} - 2\big[\phi, [\bar{\mathcal{Z}}, \phi]_{\star_M}\big]_{\star_M} = 0,\\
&\big[\mathcal{Z}, [\bar{\mathcal{Z}}, \phi]_{\star_M}\big]_{\star_M} + \big[\bar{\mathcal{Z}}, [\mathcal{Z}, \phi]_{\star_M}\big]_{\star_M} = 0.
\end{aligned} \tag{3.16}$$

4. Curved space—non-commutative theory

In the previous section, field theories on the non-commutative plane can be expressed in terms of commuting inner derivatives $\partial = -\frac{1}{\not{k}}[\bar{z}, \,]_{\star_M}$ and $\bar{\partial} = \frac{1}{\not{k}}[z, \,]_{\star_M}$, satisfying the usual Leibniz rule. This, however, is not in general possible for non-constant non-commutativity. Fortunately, two dimensional non-commutative field theories can be expressed purely algebraically, without relying on the notion of derivatives. Non-constant non-commutativity in two dimensions means we replace (3.1) by

$$\big[\mathbf{z}, \mathbf{z}^\dagger\big] = \Theta\big(\mathbf{z}, \mathbf{z}^\dagger\big) \tag{4.1}$$

for some function $\Theta(\mathbf{z}, \mathbf{z}^\dagger)$. (4.1) defines an algebra associated with some non-commutative manifold $\mathcal{M}$. In addition to being a function of the generators $\mathbf{z}$ and $\mathbf{z}^\dagger$ of the algebra, Θ depends on an additional parameter, the non-commutativity parameter, which we again denote by $\not{k}$. The Groenewold–Moyal star product is not very convenient to realize this algebra since then z and $\bar{z}$ appearing in its definition (3.2) cannot be symbols of $\mathbf{z}$ and $\mathbf{z}^\dagger$. A more convenient associative star product was developed in [16] which has an exact integral expression, and will be reviewed in Subsection 6.1. Here we denote it by $\star$, and so if z and $\bar{z} \in \mathbb{C}$ are symbols of $\mathbf{z}$ and $\mathbf{z}^\dagger$, respectively, then

$$[z, \bar{z}]_\star \equiv z \star \bar{z} - \bar{z} \star z = \theta(z, \bar{z}), \tag{4.2}$$

where $\theta(z, \bar{z})$ denotes the symbol of $\Theta(\mathbf{z}, \mathbf{z}^\dagger)$. Now in general we will not have the analogue of the identity (3.4). On the other hand, the appropriate integration measure $d\mu(z, \bar{z})$ will be required to satisfy

$$\int d\mu(z, \bar{z})\, [\mathcal{A}, \mathcal{B}]_\star = 0 \tag{4.3}$$

for any functions $\mathcal{A}$ and $\mathcal{B}$ that fall off sufficiently rapidly at infinity. (4.3) corresponds to the cyclic property of the trace. In Subsection 6.2 we shall use this property and the definition of the $\star$ to perturbatively construct the measure. We assume that like the measure $d\mu_M(z, \bar{z})$ on the non-commutative plane, $d\mu(z, \bar{z})$ is dimensionless.

To recover the systems of the previous two sections we will need to examine two limits:

(i) *The commutative limit*. This is $\not{k} \to 0$. We assume that $\Theta(\mathbf{z}, \mathbf{z}^\dagger)$, and consequently $\theta(z, \bar{z})$, is linear in $\not{k}$ in this limit,

$$\theta(z, \bar{z}) \to \not{k}\theta_0(z, \bar{z}), \tag{4.4}$$

where $\theta_0(z, \bar{z})$ is a dimensionless function which is independent of $\not{k}$. We shall identify it with $\theta_0(z, \bar{z})$ appearing in (2.1) and (2.2). As is usual, we require that the star product goes to the point-wise product, and the star commutator goes to i times the Poisson bracket in this limit. The commutative limit of the measure $d\mu(z,\bar z)$ is $d\mu_0(z,\bar z)/\hbar$, where $d\mu_0(z,\bar z)$ was given in (2.1) ($\hbar$ is introduced since $d\mu_0(z,\bar z)$ has units of length-squared).

(ii) *The non-commutative plane limit*. This is

$$\theta(z,\bar z)\to\hbar. \tag{4.5}$$

The combination of both of the above limits gives the commutative plane.

Ordering ambiguities occur in deforming the free scalar actions (2.4) and (3.5) of the previous two sections to this general case. Moreover, here we need that $\Theta(\mathbf{z},\mathbf{z}^\dagger)$ is nonsingular. We can choose the ordering such that the general scalar field action is

$$S_\phi=-2\pi\int d\mu(z,\bar z)\,\theta(z,\bar z)_\star^{-1}\star[z,\phi]_\star\star[\bar z,\phi]_\star, \tag{4.6}$$

where $\theta(z,\bar z)_\star^{-1}$ is defined by $\theta(z,\bar z)_\star^{-1}\star\theta(z,\bar z)=\theta(z,\bar z)\star\theta(z,\bar z)_\star^{-1}=1$. One easily recovers (2.4) from (4.6) in limit (i), and (3.5) in limit (ii). The field equation following from variations of ϕ in (4.6) read

$$\big[[\bar z,\phi]_\star\star\theta_\star^{-1},z\big]_\star+\big[\theta_\star^{-1}\star[z,\phi]_\star,\bar z\big]_\star=0. \tag{4.7}$$

It has the trivial solutions $\phi=z$ and $\phi=\bar z$, as well as the constant solution, but the analytic and anti-analytic solutions of the commutative theory, $\phi=\phi_+(z)$ and $\phi=\phi_-(\bar z)$, are not in general present in the non-commutative theory.

The situation is more straightforward for gauge theories. Since the commutative action (2.10) could be expressed without explicit reference to θ_0 the above ordering ambiguity does not arise, and, moreover, no space–time gauge was necessary in writing down (2.10). Upon again introducing potentials $\mathcal{A}$ and $\bar{\mathcal{A}}$ we can define the field strength by

$$i\mathcal{F}=\frac{1}{\hbar}[z,\mathcal{A}]_\star+\frac{1}{\hbar}[\bar z,\bar{\mathcal{A}}]_\star+i[\mathcal{A},\bar{\mathcal{A}}]_\star, \tag{4.8}$$

and generalize the commutative flux (2.8) in some region σ on the complex plane to

$$\Phi_\sigma=2\pi\hbar\int_\sigma d\mu(z,\bar z)\mathcal{F}. \tag{4.9}$$

From (4.3), it is zero if $\mathcal{A}$ and $\bar{\mathcal{A}}$ vanish on the boundary $\partial\sigma$ of σ. The action (3.12) on the non-commutative plane can be generalized to

$$S_f=\frac{\pi\hbar}{2}\int d\mu(z,\bar z)\,\mathcal{F}\star\mathcal{F}. \tag{4.10}$$

In the commutative limit (i), $\mathcal{F}\to\theta_0 f$, and (4.10) reduces to the commutative Maxwell action (2.10). In the non-commutative limit (ii), the field strength (4.8) becomes (3.7) and we recover (3.12) from (4.10). The non-commutative Maxwell equations following from variations of $\mathcal{A}$ and $\bar{\mathcal{A}}$ in (4.10) (ignoring boundary terms) are

$$[\mathcal{Z},\mathcal{F}]_\star=[\bar{\mathcal{Z}},\mathcal{F}]_\star=0, \tag{4.11}$$

where $\mathcal{Z}$ and $\bar{\mathcal{Z}}$ were defined in (3.10). They are again solved for $\mathcal{F}$ proportional to the identity. Then from (4.9) and (4.10), respectively, the flux per unit area and action per unit area for any such solutions are constants, just as in the commutative case. Now the area of any region σ on the complex plane is given by $\int_\sigma d\mu(z,\bar z)$. We have not found a simple expression for pure gauge solutions, analogous to those on the non-commutative plane (3.14), although an expansion about the commutative answer can be obtained. We do this in Subsection 6.4.

The issue of gauge invariance of the action (4.10) is more complicated than it was for the previous two limits. Applying (4.3), gauge invariance of the action follows if the field strength transforms covariantly, i.e. variations are of the form

$$\delta\mathcal{F} = -i[\mathcal{F}, \Lambda]_\star, \tag{4.12}$$

for infinitesimal Λ. Although the field strength transforms in a simple manner, the same is not, in general, true for the potentials $\mathcal{A}$ and $\bar{\mathcal{A}}$. For example, something analogous to (3.6) will not work because θ does not have zero star-commutator with Λ. The gauge symmetry of the action is therefore hidden. Here it does not help to introduce the quantities $\mathcal{Z}$ and $\bar{\mathcal{Z}}$ defined in (3.10) and express $\mathcal{F}$ according to

$$\mathcal{F} = [\mathcal{Z}, \bar{\mathcal{Z}}]_\star + \frac{1}{\hbar^2}\theta(z, \bar{z}), \tag{4.13}$$

in analogy to (3.9). Since $\theta(z, \bar{z})$ is not covariant, neither can be $\mathcal{Z}$ and $\bar{\mathcal{Z}}$. In Subsection 6.4 we develop a perturbation scheme for determining how $\mathcal{A}$ and $\bar{\mathcal{A}}$, or equivalently $\mathcal{Z}$ and $\bar{\mathcal{Z}}$, gauge transform away from limits (i) and (ii).

The scalar field action coupled to gauge theories on the non-commutative plane (3.15) can be generalized to arbitrary $\theta(z, \bar{z})$ by

$$S_{\phi,\mathcal{Z}} = -2\pi\hbar^2 \int d\mu(z, \bar{z})\, \theta(z, \bar{z})_\star^{-1} \star [\bar{\mathcal{Z}}, \phi]_\star \star [\mathcal{Z}, \phi]_\star. \tag{4.14}$$

For gauge invariance we need that

$$\begin{aligned}
\delta[\mathcal{Z}, \phi]_\star &= -\frac{i}{\hbar}\big[[\mathcal{Z}, \phi]_\star, \Lambda\big]_\star \star \theta, \\
\delta[\bar{\mathcal{Z}}, \phi]_\star &= -\frac{i}{\hbar}\theta \star \big[[\bar{\mathcal{Z}}, \phi]_\star, \Lambda\big]_\star.
\end{aligned} \tag{4.15}$$

It is then clear that the scalar field ϕ cannot, in general, transform covariantly. After developing a perturbation scheme for the gauge transformation of the potentials, one can then use (4.15) to do the same for ϕ. The coupled fields equations (3.16) are then generalized to

$$\begin{aligned}
&[\mathcal{Z}, \mathcal{F}]_\star + 2\hbar\big[\phi, [\mathcal{Z}, \phi]_\star \star \theta_\star^{-1}\big]_\star = 0, \\
&[\bar{\mathcal{Z}}, \mathcal{F}]_\star - 2\hbar\big[\phi, \theta_\star^{-1} \star [\bar{\mathcal{Z}}, \phi]_\star\big]_\star = 0, \\
&\big[[\mathcal{Z}, \phi]_\star \star \theta_\star^{-1}, \bar{\mathcal{Z}}\big]_\star + \big[\theta_\star^{-1} \star [\bar{\mathcal{Z}}, \phi]_\star, \mathcal{Z}\big]_\star = 0.
\end{aligned} \tag{4.16}$$

5. Magnetic monopoles on the fuzzy sphere

Fuzzy spaces are standardly defined to be non-commutative theories with finite dimensional matrix representations. So in that case the generators $\mathbf{z}$ and $\mathbf{z}^\dagger$ of the algebra in (4.1) are represented $N \times N$ matrices. This also applies to the fields ϕ, $\mathcal{Z}$ and $\bar{\mathcal{Z}}$ which are polynomials functions of $\mathbf{z}$ and $\mathbf{z}^\dagger$. The star can be replaced by ordinary matrix multiplication, and so the star commutator can be replaced by the matrix commutator. Integration corresponds to taking the trace. Specializing to gauge theories, one gets that the field strength is traceless and the total flux vanishes $\mathrm{Tr}\,\mathcal{F} = 0$. Furthermore, the constant solution, i.e. $\mathcal{F}$ proportional to the identity, to the non-commutative Maxwell equations (4.11) collapses to the trivial solution, i.e. $\mathcal{F} = 0$. This indicates the absence of any magnetic monopole solutions in a fuzzy physics. In deriving (4.11) one assumed arbitrary variations of the gauge fields in the Maxwell action (4.10). The negative result for monopoles can be avoided if we restrict variations of $\mathcal{Z}$ and $\bar{\mathcal{Z}}$ to block diagonal matrices. In that case $\mathcal{F}$ has solutions which are block diagonal matrices, with the individual blocks being proportional to identity matrices, and their combined trace equal to zero. We shall use this technique to construct fuzzy magnetic monopole solutions in Subsection 5.2.[2] In Subsection 5.1 we review the commutative case.

5.1. Commutative case

We shall fix the radius of the sphere to be 1, so in terms of embedding coordinates x_i, $i = 1, 2, 3$, $x_1^2 + x_2^2 + x_3^2 = 1$. Poisson brackets which preserve the $SO(3)$ symmetry are

$$\{x_i, x_j\} = \epsilon_{ijk} x_k. \tag{5.1}$$

In defining gauge theory, one can introduce 3-potentials a_i, $i = 1, 2, 3$, which gauge transform like [20]

$$a_i \to a_i + \{x_i, \lambda\}, \tag{5.2}$$

for some function λ on the sphere. A constraint should be imposed on a_i as there are only two independent gauge potentials on the surface. It should not restrict the gauge transformations (otherwise it would be a gauge condition), and be invariant under rotations. This is the case for

$$a_i x_i = 0. \tag{5.3}$$

From this one gets the identity

$$x_j \{x_i, a_j\} = \epsilon_{ijk} x_j a_k, \tag{5.4}$$

in addition to

$$x_i \{x_i, a_j\} = 0. \tag{5.5}$$

A gauge invariant scalar is

$$b = -\epsilon_{ijk} x_i \{x_j, a_k\}, \tag{5.6}$$

which can be interpreted as the magnetic flux density normal to the surface. The Maxwell action on the sphere is

$$S_f^0 = \frac{1}{4\pi} \int d\Omega\, b^2, \tag{5.7}$$

where the integral is over the solid angle Ω.

To recover the formalism of Section 2, we can stereographically project to the complex plane, where the north pole is mapped to infinity thus corresponding to a coordinate singularity

$$z = \frac{x_1 - i x_2}{1 - x_3}, \qquad \bar{z} = \frac{x_1 + i x_2}{1 - x_3}. \tag{5.8}$$

[2] Although the procedures differ, reducible representations were also necessary for describing monopoles in [12].

The Poisson structure is then projected to

$$\{z,\bar z\} = -i\theta_0\big(|z|^2\big), \qquad \theta_0\big(|z|^2\big) = \frac{1}{2}\big(1+|z|^2\big)^2, \tag{5.9}$$

while the potentials a_i are mapped to the one form $\mathtt{a} = dz\,a + d\bar z\,\bar a$, where

$$2ia = (1-x_3)(a_1+ia_2) + (x_1+ix_2)a_3, \tag{5.10}$$

and $\bar a$ is the complex conjugate of a. The inverse map is

$$a_1+ia_2 = i\big(a+\bar z^2\bar a\big), \qquad a_1-ia_2 = -i\big(\bar a+z^2 a\big), \qquad a_3 = i(za-\bar z\bar a). \tag{5.11}$$

From (5.2) one recovers the gauge transformations (2.6). The magnetic flux density (5.6) is mapped to[3]

$$b = \{z,a\} + \{\bar z,\bar a\}. \tag{5.12}$$

Thus, by doing the sterographic projection of the Maxwell action (5.7) we are able to recover the expression (2.10).

The magnetic monopole solutions are $b = C_0 = \frac{g_0}{4\pi}$, or

$$f = \frac{g_0}{2\pi}\frac{1}{(1+|z|^2)^2}, \tag{5.13}$$

where g_0 is the magnetic charge. The Maxwell action (5.7) evaluated for this solution is $(\frac{g_0}{4\pi})^2$. Potentials can be given after removing the point at infinity, the location of the Dirac string,

$$\mathtt{a} = \frac{ig_0}{4\pi}\frac{\bar z\,dz - z\,d\bar z}{1+|z|^2}. \tag{5.14}$$

5.2. *Non-commutative case*

In going to the fuzzy sphere, we replace real coordinates x_i by Hermitean operators $\mathbf{x}_i$, satisfying commutation relations:

$$[\mathbf{x}_i,\mathbf{x}_j] = i\hbar\epsilon_{ijk}\mathbf{x}_k, \tag{5.15}$$

as well as $\mathbf{x}_i\mathbf{x}_i = \mathbb{1}$, $\mathbb{1}$ being the unit operator. When the non-commutativity parameter $\hbar$ has values $\hbar_J = 1/\sqrt{J(J+1)}$, $J = \frac{1}{2}, 1, \frac{3}{2}, \ldots$, $\mathbf{x}_i$ have finite dimensional representations. For a given J, one can set $\mathbf{x}_i = \hbar_J\mathbf{J}_i$, $\mathbf{J}_i$ being the angular momentum matrices associated with the $(2J+1)$-dimensional irreducible representation Γ^J acting on Hilbert space H^J with states $|J,M\rangle$, $M = -J, -J+1, \ldots, J$. The commutative limit is $J\to\infty$ corresponding to infinite-dimensional representations.

A non-singular fuzzy stereographic projection was given in [16].[4] It is defined up to an operator ordering ambiguity to be

$$\mathbf{z} = (\mathbf{x}_1 - i\mathbf{x}_2)(1-\mathbf{x}_3)^{-1}, \qquad \mathbf{z}^\dagger = (1-\mathbf{x}_3)^{-1}(\mathbf{x}_1+i\mathbf{x}_2). \tag{5.16}$$

[3] To prove this substitute (5.10) in (5.12) to get $(1-x_3)(\{z,a\}+\{\bar z,\bar a\}) = b + \epsilon_{ij3}\{x_i,a_j\} - a_3$ and use the identity $x_3 b = a_3 - \epsilon_{ij3}\{x_i,a_j\}$, which follows from (5.4) and (5.5).

[4] A singular one was given in [21].

This is a non-singular map because 1 is excluded from the spectrum of $\mathbf{x}_3$, except in the commutative limit $J \to \infty$. For the top state $M = J$, $\mathbf{x}_3$ has eigenvalue $J/\sqrt{J(J+1)}$, which approaches 1 in the limit, and we thereby recover the coordinate singularity. Using properties of angular momentum matrices, $\Theta(\mathbf{z}, \mathbf{z}^\dagger)$ is represented by a diagonal matrix

$$\Theta\big(\mathbf{z}, \mathbf{z}^\dagger\big)|J, M\rangle = \theta_{J,M}|J, M\rangle, \tag{5.17}$$

with elements

$$\theta_{J,M} = \frac{J(J+1) - M(M+1)}{(\sqrt{J(J+1)} - M - 1)^2} - \frac{J(J+1) - M(M-1)}{(\sqrt{J(J+1)} - M)^2}. \tag{5.18}$$

When evaluated on the top state one has $\theta_{J,J} = -2J/(\sqrt{J(J+1)} - J)^2$, which goes like $-8J$ in the limit $J \to \infty$.

In constructing gauge theories, as stated previously, the field strength expressed in terms of potentials, (4.8) or (4.13), is traceless, and as a result the total flux vanishes, i.e. $\mathrm{Tr}\,\mathcal{F} = 0$. This implies that there can be no magnetic monopoles in this formalism, and furthermore that the constant solution to the non-commutative Maxwell equations (4.11) collapses to the trivial solution, i.e. $\mathcal{F} = 0$. This is not surprising since also in the commutative theory, if we insist on writing the field strength in terms of potentials globally, there can be no magnetic monopoles. The monopole potential in (5.14) is defined only after removing the point at infinity from the domain of the commutative theory. In the non-commutative theory, this point is approached by the top state as $J \to \infty$. So let us similarly remove it from the domain of the non-commutative theory. Equivalently, we can restrict *variations* of the fields $\mathcal{Z}$ and $\bar{\mathcal{Z}}$ to be block diagonal matrices, one block being $2J \times 2J$ and the other being 1×1, the latter associated with the top state. Then the equations of motion (4.11) will only hold for the diagonal blocks. Solutions to the non-commutative Maxwell equations (4.11) for $\mathcal{F}$ will then also be block diagonal matrices $\mathcal{F}_{2J\times 2J}$ and $\mathcal{F}_{1\times 1}$, which are proportional to the identity, and since $\mathrm{Tr}\,\mathcal{F} = 0$,

$$\mathcal{F}_{2J\times 2J} = \frac{g}{2J}\mathbb{1}_{2J\times 2J}, \tag{5.19}$$

$$\mathcal{F}_{1\times 1} = -g, \tag{5.20}$$

$\mathbb{1}_{2J\times 2J}$ being the $2J \times 2J$ identity matrix. The trace over only $\mathcal{F}_{2J\times 2J}$ is g, which by analogy with the commutative theory defines the magnetic charge, while $\mathcal{F}_{1\times 1}$ corresponds to the compensating flux of the Dirac string. So (5.20) is the non-commutative analogue of the Dirac string. The action (4.10) evaluated for this solution gives

$$\frac{\pi g^2(2J+1)}{4J\sqrt{J(J+1)}}. \tag{5.21}$$

In comparing with the commutative answer of $(\frac{g_0}{4\pi})^2$, we need that g grows like $\sqrt{J}$ in the commutative limit, i.e.

$$g \to \frac{\sqrt{J}g_0}{(2\pi)^{3/2}}, \quad \text{as } J \to \infty, \tag{5.22}$$

in order to recover a finite charge g_0 in the commutative theory.

By equating (5.20) with the last row and column of the matrix associated with (4.13) we get

$$g = \sum_{M=-J}^{J-1} \big(\big|\langle J, M|\mathcal{Z}|J, J\rangle\big|^2 - \big|\langle J, M|\bar{\mathcal{Z}}|J, J\rangle\big|^2\big) + \frac{2J^2(J+1)}{(\sqrt{J(J+1)} - J)^2}. \tag{5.23}$$

Note that only off-block diagonal matrix elements of $\mathcal{Z}$ and $\bar{\mathcal{Z}}$ are present in the result. These are non-dynamical fields (they were not varied in obtaining the field equations), and so *the magnetic flux is a constant of the motion*. This is a result of the dynamics, and not topology. In the commutative limit, the last term in (5.23) goes to infinite like $8J^3$, and thus the sum must go like $-8J^3$ in order for g to have the limit in (5.22). Alternatively, we can use (4.8) to write the charge in terms of matrix elements of $\mathcal{A}$ and $\bar{\mathcal{A}}$

$$g = \sum_{M=-J}^{J-1} \left(\left|\langle J, M|\mathcal{A}|J, J\rangle\right|^2 - \left|\langle J, M|\bar{\mathcal{A}}|J, J\rangle\right|^2\right) - i\frac{J\sqrt{2(J+1)}}{\sqrt{J(J+1)}-J}\left(\langle J, J|\mathcal{A}|J, J-1\rangle - \langle J, J-1|\bar{\mathcal{A}}|J, J\rangle\right). \tag{5.24}$$

Again, only the (non-dynamical) off-block diagonal matrix elements appear, and so g is a constant of the motion.

In the above, although the charge is a constant of motion, we get no quantization, at least at the classical level.[5] On the other hand, quantization does occur if we impose the stronger condition that the fields $\mathcal{Z}$ and $\bar{\mathcal{Z}}$, and not just their variations, are block diagonal. Then the sum in (5.23) vanishes and g is just equal to the last term. We can also allow for more general block diagonal matrices. So let $\mathcal{Z}$ and $\bar{\mathcal{Z}}$ be reducible to $(2J+1-N)\times(2J+1-N)$ and $N\times N$ matrices, $1 \leqslant N \leqslant 2J$. Solutions to the non-commutative Maxwell equations (4.11) for $\mathcal{F}$ will then be block diagonal matrices $\mathcal{F}_{(2J+1-N)\times(2J+1-N)}$ and $\mathcal{F}_{N\times N}$, where

$$\begin{aligned}
\mathcal{F}_{(2J+1-N)\times(2J+1-N)} &= \frac{g}{2J+1-N}\mathbb{1}_{(2J+1-N)\times(2J+1-N)}, \\
\mathcal{F}_{N\times N} &= -\frac{g}{N}\mathbb{1}_{N\times N}.
\end{aligned} \tag{5.25}$$

So now the non-commutative analogue of the Dirac string includes N states. By equating the trace of either of the matrices in (5.25) with the corresponding trace of (4.13) we get

$$g = -\frac{1}{\bar{k}_J^2}\sum_{n=1}^{N}\theta_{J,J+1-n}. \tag{5.26}$$

Using (5.18) we thereby get quantized magnetic charges, depending on J and N. Examples are

$$\begin{aligned}
J &= \frac{1}{2}, & N &= 1, & g &= \frac{3}{(\sqrt{3}-1)^2}, \\
J &= 1, & N &= 1, & g &= \frac{4}{(\sqrt{2}-1)^2}, \\
& & N &= 2, & g &= 2, \\
J &= \frac{3}{2}, & N &= 1, & g &= \frac{45}{(\sqrt{15}-3)^2}, \\
& & N &= 2, & g &= \frac{30(8\sqrt{15}-31)}{95\sqrt{15}-368},
\end{aligned}$$

[5] Quantized magnetic charges were obtained in [12] upon requiring the gauge fields to be associated with reducible $SU(2)$ representations.

$$N = 3, \qquad g = \frac{45}{(\sqrt{15}+1)^2}. \tag{5.27}$$

These charges do not have a well defined limit when $J \to \infty$. Here it does not help that we have an additional parameter N. For N equals one, or close to one, g goes like $8J^3$ as $J \to \infty$, which is too divergent when compared to (5.22). On the other side is $N = 2J$, where $g = \theta_{J,-J}/\bar{k}^2$. So when N equals $2J$, or close to $2J$, g goes like $J/2$ as $J \to \infty$, which is also too divergent when compared to (5.22). It follows that off-block diagonal matrices must be present for $\mathcal{Z}$ and $\bar{\mathcal{Z}}$ in order to recover the usual commutative limit, even though these matrices are non-dynamical.

6. Non-commutative corrections

In this section we do an expansion in $\bar{k}$ to obtain non-commutative corrections to the commutative scalar and gauge field actions. We also obtain corrections to the corresponding actions on the non-commutative plane. For the latter we do an expansion in derivatives of $\theta(z,\bar{z})$. The two expansions are not independent as is explained below. We will also give an expansion for gauge transformations of $\mathcal{A}$ and $\bar{\mathcal{A}}$ about the two limits. The expression in general depends on the star product and $\theta(z,\bar{z})$, which in the commutative limit is related to the determinant of the metric. So for this gauge theory, motion along a fibre depend on the geometry of the base manifold. The non-commutative gauge theory can be Seiberg–Witten [19] (also see [22–24]) mapped to the commutative theory, leading to corrections to the commutative flux (2.8) and the Maxwell action (2.10). We then get the corrections to the commutative solution (2.11). In Section 7 we apply the techniques to the example of non-commutative AdS^2. There are a number of obstacles in using the approximation scheme developed here for analyzing magnetic monopoles in fuzzy gauge theories, which we comment on in Section 8.

6.1. Star product

We now review the star product in [16] which can be expressed in terms of the symbols z and $\bar{z} \in \mathbb{C}$ of operators $\mathbf{z}$ and $\mathbf{z}^\dagger$, respectively, and which easily reproduces the star commutator (4.2). It is based on an overcomplete set of unit vectors $\{|z\rangle\}$ spanning an infinite-dimensional Hilbert space. The states $|z\rangle$ are, in general, non-linear deformations of standard coherent states on the complex plane. They diagonalize $\mathbf{z}$,[6]

$$\mathbf{z}|z\rangle = z|z\rangle. \tag{6.1}$$

The covariant symbols of operators $A, B, \ldots$ are given by $\mathcal{A}(z,\bar{z}) = \langle z|A|z\rangle$, $\mathcal{B}(z,\bar{z}) = \langle z|B|z\rangle, \ldots$, and their star product by $[\mathcal{A}\star\mathcal{B}](z,\bar{z}) = \langle z|AB|z\rangle$. The expression for the star product was obtained in [16]. Given the symbols $\mathcal{A}(z,\bar{z})$ and $\mathcal{B}(z,\bar{z})$, then $[\mathcal{A}\star\mathcal{B}](z,\bar{z})$ can be written as

$$\mathcal{A}(z,\bar{z}) \int d\mu(\eta,\bar{\eta}) {:}\exp \frac{\overleftarrow{\partial}}{\partial z}(\eta - z){:} \big|\langle z|\eta\rangle\big|^2 {:}\exp(\bar{\eta} - \bar{z}) \frac{\overrightarrow{\partial}}{\partial \bar{z}}{:} \mathcal{B}(z,\bar{z}), \tag{6.2}$$

where $d\mu(z,\bar{z})$ is the appropriate measure on the complex plane satisfying the partition of unity $\int d\mu(z,\bar{z})|z\rangle\langle z| = \mathbb{1}$. The colons in (6.2) denote an ordered exponential, with the derivatives

[6] It is problematic to construct such states for fuzzy manifolds. Alternatively, for the case of the fuzzy sphere we found a set of states $\{|z\rangle\}$ where the difference of $\mathbf{z}|z\rangle$ and $z|z\rangle$ was proportional to the top state.[16] As a result, the expressions which follow for the star product and measure get modified for the fuzzy sphere.

ordered to the right in each term in the Taylor expansion of $\exp(\eta - z)\frac{\overrightarrow{\partial}}{\partial z}$, and to the left in each term in the Taylor expansion of $\exp\frac{\overleftarrow{\partial}}{\partial \zeta}(\eta - z)$; i.e.

$$\begin{aligned} &:\exp(\bar\eta - \bar z)\frac{\overrightarrow{\partial}}{\partial \bar z}: = 1 + (\bar\eta - \bar z)\frac{\overrightarrow{\partial}}{\partial \bar z} + \frac{1}{2}(\bar\eta - \bar z)^2\frac{\overrightarrow{\partial^2}}{\partial \bar z^2} + \cdots, \\ &:\exp\frac{\overleftarrow{\partial}}{\partial z}(\eta - z): = 1 + \frac{\overleftarrow{\partial}}{\partial z}(\eta - z) + \frac{1}{2}\frac{\overleftarrow{\partial^2}}{\partial z^2}(\eta - z)^2 + \cdots. \end{aligned} \tag{6.3}$$

When $\theta(z,\bar z) = \langle z|\Theta(\mathbf{z},\mathbf{z}^\dagger)|z\rangle$ equals $\hbar$, $|z\rangle$ are the standard coherent states and the star product reduces to the Voros product, which can be transformed to (3.2). A derivative expansion of the above star product was performed in [25]. There one obtained the following leading three terms acting between functions of z and $\bar z$:

$$\star = 1 + \frac{\overleftarrow{\partial}}{\partial z}\theta(z,\bar z)\frac{\overrightarrow{\partial}}{\partial \bar z} + \frac{1}{4}\left[\frac{\overleftarrow{\partial^2}}{\partial z^2}\frac{\overrightarrow{\partial}}{\partial \bar z}\theta(z,\bar z)^2\frac{\overrightarrow{\partial}}{\partial \bar z} + \frac{\overleftarrow{\partial}}{\partial z}\theta(z,\bar z)^2\frac{\overleftarrow{\partial}}{\partial z}\frac{\overrightarrow{\partial^2}}{\partial \bar z^2}\right] + \cdots. \tag{6.4}$$

At lowest order in $\hbar$, we assume that $\theta(z,\bar z)$ is given by (4.4). Since we interpret $\hbar$ as the non-commutativity parameter, then the lowest order in the derivative expansion (6.4) gives the commutative product, and from the first order term, one gets that the star commutator goes to i times the Poisson bracket. Moreover, after expanding $\theta(z,\bar z)$ in $\hbar$,

$$\theta(z,\bar z) = \hbar\theta_0(z,\bar z) + \hbar^2\theta_1(z,\bar z) + \cdots, \tag{6.5}$$

the derivative expansion of the star can also be regarded as an expansion in $\hbar$. So one can use (6.4) to expand about the commutative field theory or the field theory on the non-commutative plane.

6.2. Measure

We next expand the integration measure about its (i) commutative limit $d\mu_0(z,\bar z)/\hbar$ and (ii) the non-commutative plane limit $d\mu_M(z,\bar z)$. For this we require that the trace property (4.3) holds order-by-order for functions $\mathcal{A}$ and $\mathcal{B}$ that fall off sufficiently rapidly at infinity. Using (6.4) we then find

$$d\mu(z,\bar z) = \frac{i}{2\pi}\frac{dz\wedge d\bar z}{\theta(z,\bar z)}\left(1 + \frac{1}{2}\partial\bar\partial\theta(z,\bar z) + \cdots\right). \tag{6.6}$$

We can regard this as a derivative expansion, and thus a perturbation about the non-commutative plane limit (ii). Eq. (6.6) can also be regarded as an expansion in $\hbar$, and thus a perturbation about the commutative limit (i). For the latter, apply (6.5) to get

$$d\mu(z,\bar z) = \frac{d\mu_0(z,\bar z)}{\hbar}\left\{1 + \hbar\left(\frac{1}{2}\partial\bar\partial\theta_0 - \frac{\theta_1}{\theta_0}\right) + \cdots\right\}. \tag{6.7}$$

More generally, if real functions $\mathcal{A}$ and $\mathcal{B}$ and their derivatives are non-vanishing on the boundary $\partial\sigma$ of some region σ. Then the integral of their star commutator can be expressed on the boundary. The generalization of (2.3) gives

$$\int_\sigma d\mu(z,\bar z)\,[\mathcal{A},\mathcal{B}]_\star = \frac{i}{2\pi}\int_\sigma dz\wedge d\bar z\bigg\{\partial\mathcal{A}\bar\partial\mathcal{B} + \frac{1}{2}\partial\bar\partial\theta\partial\mathcal{A}\bar\partial\mathcal{B} + \frac{1}{4\theta}\big(\partial^2\mathcal{A}\bar\partial\big(\theta^2\bar\partial\mathcal{B}\big) + \partial\big(\theta^2\partial\mathcal{A}\big)\bar\partial^2\mathcal{B}\big) + \cdots - (\mathcal{A}\rightleftharpoons\mathcal{B})\bigg\}$$
$$= \frac{i}{2\pi}\int_\sigma dz\wedge d\bar z\,(\bar\partial\mathcal{J}_{[\mathcal{A},\mathcal{B}]} - \partial\bar{\mathcal{J}}_{[\mathcal{A},\mathcal{B}]})$$
$$= \frac{i}{2\pi}\int_{\partial\sigma}(dz\,\mathcal{J}_{[\mathcal{A},\mathcal{B}]} + d\bar z\,\bar{\mathcal{J}}_{[\mathcal{A},\mathcal{B}]}), \tag{6.8}$$

with

$$\mathcal{J}_{[\mathcal{A},\mathcal{B}]} = \frac{1}{2}\mathcal{B}\partial\mathcal{A} + \frac{1}{4}\big(\partial\mathcal{B}(\theta\partial\bar\partial\mathcal{A} - \partial\theta\bar\partial\mathcal{A}) - \theta\partial^2\mathcal{B}\bar\partial\mathcal{A}\big) + \cdots - (\mathcal{A}\rightleftharpoons\mathcal{B}),$$
$$\bar{\mathcal{J}}_{[\mathcal{A},\mathcal{B}]} = \frac{1}{2}\mathcal{B}\bar\partial\mathcal{A} + \frac{1}{4}\big(\bar\partial\mathcal{B}(\theta\partial\bar\partial\mathcal{A} - \bar\partial\theta\partial\mathcal{A}) - \theta\bar\partial^2\mathcal{B}\partial\mathcal{A}\big) + \cdots - (\mathcal{A}\rightleftharpoons\mathcal{B}). \tag{6.9}$$

6.3. Field theory

We can now apply the previous expansions about limit (i) and (ii) to the scalar field and gauge theory actions. Starting with the real scalar field, we get

$$[z,\phi]_\star = \theta\bigg(\bar\partial\phi + \frac{1}{2}\partial\theta\bar\partial^2\phi + \cdots\bigg),$$
$$[\bar z,\phi]_\star = -\theta\bigg(\partial\phi + \frac{1}{2}\bar\partial\theta\partial^2\phi + \cdots\bigg). \tag{6.10}$$

After substituting into (4.6), and dropping boundary terms, we can write S_ϕ, up to first order, by simply replacing the ordinary derivatives in the commutative action (2.4) with 'covariant' ones

$$S_\phi = i\int dz\wedge d\bar z\,\mathcal{D}_\theta\phi\,\bar{\mathcal{D}}_\theta\phi, \tag{6.11}$$

where $\mathcal{D}_\theta$ and $\bar{\mathcal{D}}_\theta$ are defined by[7]

$$\mathcal{D}_\theta\phi = \partial\phi\bigg(1 - \frac{1}{2}\theta^{-1}\partial\theta\bar\partial\theta\bigg) - \frac{1}{2}\theta\partial^2\bar\partial\phi + \cdots,$$
$$\bar{\mathcal{D}}_\theta\phi = \bar\partial\phi\bigg(1 - \frac{1}{2}\theta^{-1}\partial\theta\bar\partial\theta\bigg) - \frac{1}{2}\theta\bar\partial^2\partial\phi + \cdots. \tag{6.12}$$

The result is quite simple when compared to previous approaches [5]. From the lowest order term we recover the scalar field action in the conformal gauge, S^0_ϕ. Conformal invariance is broken by the non-commutative corrections. The lowest order corrections can be expressed in terms of the determinant of the classical metric using (2.1) and (6.5). The field equation following from variations of ϕ in (6.11) gives the conservation law

$$\bar\partial\mathcal{I}_\phi + \partial\bar{\mathcal{I}}_\phi = 0, \tag{6.13}$$

[7] The geometric meaning of these derivatives is not immediately obvious. They may be associated with an automorphism between the star product and the commutative product [26].

with currents

$$\mathcal{I}_\phi = \partial\phi\big(1-\theta^{-1}\partial\theta\bar{\partial}\theta\big) - \frac{1}{2}\big[\theta\partial^2\bar{\partial}\phi + \partial\bar{\partial}(\theta\partial\phi)\big] + \cdots,$$
$$\bar{\mathcal{I}}_\phi = \bar{\partial}\phi\big(1-\theta^{-1}\partial\theta\bar{\partial}\theta\big) - \frac{1}{2}\big[\theta\bar{\partial}^2\partial\phi + \partial\bar{\partial}(\theta\bar{\partial}\phi)\big] + \cdots. \tag{6.14}$$

For gauge theories we can use (6.4) to compute lowest order corrections to the field strength $\mathcal{F}$ in (4.8)

$$\mathcal{F} = \frac{\theta}{\hbar}\left(\mathcal{F}_M - \frac{i}{2}\big(\partial\theta\bar{\partial}^2\mathcal{A} - \bar{\partial}\theta\partial^2\bar{\mathcal{A}}\big) + \cdots\right), \tag{6.15}$$

where $\mathcal{F}_M$ is the Moyal–Weyl field strength. As a result we have corrections to the non-commutative plane limit, as well as the to commutative limit. For the latter, note that there are terms of order $\hbar$ in $\mathcal{F}_M$, as well as in $\theta(z,\bar{z})$ and its derivatives. The lowest order non-commutative corrections to the flux can be read off from (4.9). Using (6.8) and (6.9) one gets the boundary terms

$$\Phi_\sigma = \int_{\partial\sigma}\left\{dz\left(\mathcal{A} - i\hbar\mathcal{A}\partial\bar{\mathcal{A}} + \frac{1}{2}\partial\theta\bar{\partial}\mathcal{A} + \cdots\right)\right.$$
$$\left. + d\bar{z}\left(\bar{\mathcal{A}} + i\hbar\bar{\mathcal{A}}\bar{\partial}\mathcal{A} + \frac{1}{2}\bar{\partial}\theta\partial\bar{\mathcal{A}} + \cdots\right)\right\}. \tag{6.16}$$

Corrections to the gauge field action are

$$S_f = \frac{i}{4\hbar}\int dz\wedge d\bar{z}\,\theta\left\{\mathcal{F}_M^2\left(1+\frac{1}{2}\partial\bar{\partial}\theta\right) - i\mathcal{F}_M\big(\partial\theta\bar{\partial}^2\mathcal{A} - \bar{\partial}\theta\partial^2\bar{\mathcal{A}}\big)\right.$$
$$\left. + \theta^{-1}\partial(\theta\mathcal{F}_M)\bar{\partial}(\theta\mathcal{F}_M) + \cdots\right\}. \tag{6.17}$$

When $\theta(z,\bar{z}) = \hbar$ we easily recover the expression for the non-commutative plane. A Seiberg–Witten map should be utilized to compare with the commutative case, which we do in Subsection 5.5.

6.4. Gauge transformations

Although the field strength transforms covariantly under gauge transformations, this is not the case the potentials. Here we compute corrections to gauge transformations of $\mathcal{A}$ and $\bar{\mathcal{A}}$ from (i) the commutative limit (2.6) and (ii) the non-commutative plane limit (3.6). For this write infinitesimal gauge variations as

$$\delta\mathcal{A} = \partial\Lambda + \Delta, \qquad \delta\bar{\mathcal{A}} = \bar{\partial}\Lambda + \bar{\Delta}, \tag{6.18}$$

and substitute into (4.12) using (6.15) to get

$$\bar{\partial}\left(\Delta + \left(\frac{1}{2}\partial\theta - i\hbar\mathcal{A}\right)\partial\bar{\partial}\Lambda + \theta\mathcal{F}_M\partial\Lambda + i\hbar\bar{\mathcal{A}}\partial^2\Lambda\right)$$
$$-\,\partial\left(\bar{\Delta} + \left(\frac{1}{2}\bar{\partial}\theta + i\hbar\bar{\mathcal{A}}\right)\partial\bar{\partial}\Lambda + \theta\mathcal{F}_M\bar{\partial}\Lambda - i\hbar\mathcal{A}\bar{\partial}^2\Lambda\right) = 0, \tag{6.19}$$

up to first order in $\bar{k}$. Then we can write down Δ and $\bar{\Delta}$, up to the divergence of some arbitrary function σ

$$\Delta = -\left(\frac{1}{2}\partial\theta - i\bar{k}\mathcal{A}\right)\partial\bar{\partial}\Lambda - \theta\mathcal{F}_M\partial\Lambda - i\bar{k}\bar{\mathcal{A}}\partial^2\Lambda + \partial\sigma,$$
$$\bar{\Delta} = -\left(\frac{1}{2}\bar{\partial}\theta + i\bar{k}\bar{\mathcal{A}}\right)\partial\bar{\partial}\Lambda - \theta\mathcal{F}_M\bar{\partial}\Lambda + i\bar{k}\mathcal{A}\bar{\partial}^2\Lambda + \bar{\partial}\sigma. \tag{6.20}$$

The divergence of σ can be absorb in a re-definition of the gauge parameter $\Lambda \to \hat{\Lambda} = \Lambda + \sigma$, yielding

$$\delta\mathcal{A} = \partial\hat{\Lambda} - i[\mathcal{A}, \hat{\Lambda}]_{\star_M} - \frac{1}{2}\partial\theta\partial\bar{\partial}\hat{\Lambda} + (\bar{k} - \theta)\mathcal{F}_M\partial\hat{\Lambda} + \cdots, \tag{6.21}$$

$$\delta\bar{\mathcal{A}} = \bar{\partial}\hat{\Lambda} - i[\bar{\mathcal{A}}, \hat{\Lambda}]_{\star_M} - \frac{1}{2}\bar{\partial}\theta\partial\bar{\partial}\hat{\Lambda} + (\bar{k} - \theta)\mathcal{F}_M\bar{\partial}\hat{\Lambda} + \cdots. \tag{6.22}$$

From this we easily recover the expressions for limit (i) by setting $\theta(z,\bar{z}) = \bar{k} = 0$, and limit (ii) after setting $\theta(z,\bar{z}) = \bar{k}$ with $\hat{\Lambda} = \Lambda$ or $\sigma = 0$. Gauge transformations close after including the first order corrections. If one goes beyond the leading order, the closure of gauge transformations should put restrictions on σ. The corresponding gauge variations of $\mathcal{Z}$ and $\bar{\mathcal{Z}}$ are

$$\delta\mathcal{Z} = -i[\mathcal{Z}, \hat{\Lambda}]_{\star_M} - \frac{1}{2}\partial\theta\partial\bar{\partial}\hat{\Lambda} + (\bar{k} - \theta)\mathcal{F}_M\partial\hat{\Lambda} + \cdots, \tag{6.23}$$

$$\delta\bar{\mathcal{Z}} = -i[\bar{\mathcal{Z}}, \hat{\Lambda}]_{\star_M} - \frac{1}{2}\bar{\partial}\theta\partial\bar{\partial}\hat{\Lambda} + (\bar{k} - \theta)\mathcal{F}_M\bar{\partial}\hat{\Lambda} + \cdots. \tag{6.24}$$

Setting $\mathcal{A}$ and $\bar{\mathcal{A}}$ equal to zero in (6.21) and (6.22) gives a first order expression for pure gauge potentials

$$\mathcal{Z}_{pg} = -\frac{i}{\bar{k}}\bar{z} + \partial\hat{\Lambda} - \frac{1}{2}\partial\theta\partial\bar{\partial}\hat{\Lambda} + \cdots,$$
$$\bar{\mathcal{Z}}_{pg} = \frac{i}{\bar{k}}\bar{z} + \bar{\partial}\hat{\Lambda} - \frac{1}{2}\bar{\partial}\theta\partial\bar{\partial}\hat{\Lambda} + \cdots, \tag{6.25}$$

which generalizes the infinitesimal version of pure gauge solutions (3.14) to the non-commutative plane.

6.5. Seiberg–Witten map

We now construct the lowest order map from the gauge theory written on a coordinate patch P_0 of commutative manifold $\mathcal{M}_0$ to the non-commutative gauge theory. (A more geometrical treatment can be found in [26].) We write the non-commutative potentials and gauge parameter as functions of the commutative ones, $\mathcal{A}[a,\bar{a}]$, $\bar{\mathcal{A}}[a,\bar{a}]$, $\hat{\Lambda}[\lambda,a,\bar{a}]$, and require that the right-hand sides of (6.21) and (6.22) be equal to $\mathcal{A}[a+\partial\lambda, \bar{a}+\bar{\partial}\lambda] - \mathcal{A}[a,\bar{a}]$ and $\bar{\mathcal{A}}[a+\partial\lambda, \bar{a}+\bar{\partial}\lambda] - \bar{\mathcal{A}}[a,\bar{a}]$, respectively. The first order equation is then solved by

$$\mathcal{A}[a,\bar{a}] = a(1-\theta f) + \frac{i\bar{k}}{2}(a\partial\bar{a} - \bar{a}\partial a) - \frac{1}{2}\partial\theta\bar{\partial}a,$$
$$\bar{\mathcal{A}}[a,\bar{a}] = \bar{a}(1-\theta f) - \frac{i\bar{k}}{2}(\bar{a}\bar{\partial}a - a\bar{\partial}\bar{a}) - \frac{1}{2}\bar{\partial}\theta\partial\bar{a},$$
$$\hat{\Lambda}[\lambda,a,\bar{a}] = \lambda + \frac{i\bar{k}}{2}(a\bar{\partial}\lambda - \bar{a}\partial\lambda), \tag{6.26}$$

where f is the commutative curvature, $if = \bar\partial a - \partial\bar a$. Eq. (6.26) reduces to the standard Seiberg–Witten map to the Moyal plane when $\theta = \hbar$. The map to the non-commutative curvature up to first order in $\hbar$ is

$$\mathcal{F}[a,\bar a] = \frac{\theta}{\hbar}\left(f\left(1 - \frac{1}{2}\partial\bar\partial\theta\right) + i\bar\partial(\theta f a) - i\partial(\theta f\bar a)\right). \tag{6.27}$$

Substituting into the non-commutative flux (6.16) gives, up to first order in $\hbar$,

$$\begin{aligned}\Phi_\sigma &= \int_{\partial\sigma}\left\{dz\, a(1-\theta f+\cdots) + d\bar z\,\bar a(1-\theta f+\cdots)\right\}\\ &= \Phi_\sigma^0 + \Phi_\sigma^1 + \cdots,\end{aligned} \tag{6.28}$$

where Φ_σ^0 is the commutative flux (2.8) and

$$\Phi_\sigma^1 = -\hbar\int_{\partial\sigma}\theta_0 f(dz\,a + d\bar z\,\bar a). \tag{6.29}$$

After substituting (6.27) into the action (4.10) we get the lowest order correction to the commutative action (2.10)

$$\begin{aligned}S_f &= \frac{i}{4\hbar}\int_\sigma dz\wedge d\bar z\left(\theta f^2\left(1 - \frac{1}{2}\partial\bar\partial\theta - \theta f\right) + \partial(\theta f)\bar\partial(\theta f)\right)\\ &= S_f^0 + S_f^1 + \cdots,\end{aligned} \tag{6.30}$$

where S_f^0 is the commutative Maxwell action (2.10) and

$$S_f^1 = \frac{i\hbar}{4}\int_\sigma dz\wedge d\bar z\left(f^2\left(\theta_1 - \frac{1}{2}\theta_0\partial\bar\partial\theta_0 - \theta_0^2 f\right) + \partial(\theta_0 f)\bar\partial(\theta_0 f)\right), \tag{6.31}$$

and we have dropped boundary terms. As in the commutative theory, there are no propagating degrees of freedom. The field equations resulting from variations of a and $\bar a$ in (6.30) imply that

$$f\left(1 - \frac{1}{2}\partial\bar\partial\theta - \frac{3}{2}\theta f\right) - \partial\bar\partial(\theta f) = \frac{C}{\theta}, \tag{6.32}$$

C being a constant. If we apply the expansion (6.5), then (6.32) gives the solution for the corrected field strength

$$f = f_0 + f_1 + \cdots, \tag{6.33}$$

where the lowest order agrees with the commutative solution $f_0 = C_0/\theta_0$ and

$$f_1 = \frac{\hbar C_0}{\theta_0}\left(\frac{1}{2}\partial\bar\partial\theta_0 - \frac{\theta_1}{\theta_0}\right), \tag{6.34}$$

and we also expanded the constant

$$C = C_0\hbar - \frac{3}{2}C_0^2\hbar^2 + \cdots. \tag{6.35}$$

From (2.12), C_0 was equal to the flux per unit area of the solution in the commutative theory. The flux per unit area remains a constant, but its value gets shifted from the commutative result

$$\frac{\Phi_\sigma}{2\pi\hbar \int_\sigma d\mu(z,\bar{z})} = C_0(1 - \hbar C_0 + \cdots), \tag{6.36}$$

where we used the corrected flux (6.28) and measure (6.6). The action per unit area of the solution, $S_f = S^0_{f_0+f_1} + S^1_{f_0}$, also remains a constant, its value being shifted from the commutative result by the same factor appearing in (6.36),

$$\frac{C_0^2}{4}(1 - \hbar C_0 + \cdots). \tag{6.37}$$

Since these shifts are small for small $\hbar$ we can say that the solutions are stable under the inclusion of non-commutative corrections.

7. Non-commutative AdS^2

We now apply the results of the previous section to obtain the lowest order non-commutative corrections to the scalar and gauge theory actions on the Lobachevsky plane and show that the solution to the commutative Maxwell action receives no such corrections. For other approaches to the non-commutative AdS^2 and the Lobachevsky plane, see [27,28].

7.1. Lobachevsky plane

We first review the commutative theory. Here we write down the AdS^2 measure on the disc, which defines the Lobachevsky plane. We start with AdS^2 embedded in three-dimensional Euclidean space with coordinates x_i, $i = 0, 1, 2$, and the constraint

$$x_0^2 - x_1^2 - x_2^2 = 1, \quad x_0 \geqslant 1. \tag{7.1}$$

A natural Poisson structure on it is

$$\{x_0, x_1\} = x_2, \qquad \{x_2, x_0\} = x_1, \qquad \{x_1, x_2\} = -x_0, \tag{7.2}$$

as it preserves the $SO(2,1)$ symmetry. We parameterize the Lobachevsky plane by complex coordinates z and $\bar{z}$, with $0 \leqslant |z|^2 < 1$. The projection from AdS^2 to the disc is given by

$$z = \frac{x_1 - ix_2}{x_0 + 1}, \qquad \bar{z} = \frac{x_1 + ix_2}{x_0 + 1}. \tag{7.3}$$

The Poisson brackets (7.2) are projected to

$$\{z, \bar{z}\} = -i\theta_0\big(|z|^2\big), \qquad \theta_0\big(|z|^2\big) = \frac{1}{2}\big(1 - |z|^2\big)^2, \tag{7.4}$$

with the associated measure given by (2.1). The boundary $|z| = 1$ corresponds to x_0 and $\sqrt{x_1^2 + x_2^2}$ going to infinity in the AdS^2 space. From (7.4) it follows that the non-commutativity will tend to zero as the boundary is approached in the non-commutative version of the theory. This is fortunate because of the known difficulties associated with defining boundaries in non-commutative field theory [29,30]. We note that the Lobachevsky plane differs from the disc, and hence the non-commutative version differs from the fuzzy disc [31], since the metric and $\{z, \bar{z}\}$ are not constants in the interior.

The commutative Maxwell action (2.10) is

$$S_f^0 = \frac{i}{8}\int dz \wedge d\bar{z}\,\left(1-|z|^2\right)^2 f^2, \tag{7.5}$$

and the solutions of the corresponding free field equations are of the form

$$f = \frac{2C_0}{(1-|z|^2)^2}, \tag{7.6}$$

which diverges as the boundary $|z|^2 = 1$ is approached. A possible gauge choice for the potential one-form is

$$\mathtt{a} = iC_0\frac{\bar{z}\,dz - z\,d\bar{z}}{1-|z|^2}. \tag{7.7}$$

The flux going through a disc of radius $r = |z| < 1$ is $\Phi_r = 4\pi C_0/(1-\frac{1}{r^2})$.

7.2. Non-commutative case

For non-commutative AdS^2 we replace the embedding coordinates x_i, $i = 0, 1, 2$, by hermitean operators $\mathbf{x}_i$, $i = 0, 1, 2$, satisfying

$$\mathbf{x}_0^2 - \mathbf{x}_1^2 - \mathbf{x}_2^2 = 1, \tag{7.8}$$

with commutation relations

$$[\mathbf{x}_0, \mathbf{x}_1] = i\bar{k}\mathbf{x}_2, \qquad [\mathbf{x}_2, \mathbf{x}_0] = i\bar{k}\mathbf{x}_1, \qquad [\mathbf{x}_1, \mathbf{x}_2] = -i\bar{k}\mathbf{x}_0. \tag{7.9}$$

Thus the $SO(2,1)$ symmetry of the commutative theory is undeformed. Here, of course, there are no finite-dimensional unitary representations of the algebra.

Next we write down an operator analogue of the projection (7.3) to the Lobachevsky plane. Up to operator ordering ambiguities we have

$$\mathbf{z} = (\mathbf{x}_1 - i\mathbf{x}_2)(\mathbf{x}_0+1)^{-1}, \qquad \mathbf{z}^\dagger = (\mathbf{x}_0+1)^{-1}(\mathbf{x}_1 + i\mathbf{x}_2), \tag{7.10}$$

which is non-singular if -1 is not in the spectrum of $\mathbf{x}_0$. We can find the commutation relations for $\mathbf{z}$ and $\mathbf{z}^\dagger$ in a manner similar to what was done for the projective coordinates of the fuzzy sphere [16]. For this define $\chi^{-1} = \frac{1}{2}(1+\mathbf{x}_0)$, which from (7.9) commutes with $|\mathbf{z}|^2 = \mathbf{z}\mathbf{z}^\dagger$. After some algebra one can show that

$$\Theta\left(\mathbf{z}, \mathbf{z}^\dagger\right) = \bar{k}\chi\left(1 - |\mathbf{z}|^2 - \frac{\chi}{2}\left(1 + \frac{\bar{k}}{2}|\mathbf{z}|^2\right)\right), \tag{7.11}$$

and from the constraint (7.8) one also has

$$\mathbf{z}\chi^{-2}\mathbf{z}^\dagger + \chi^{-1}\mathbf{z}^\dagger\mathbf{z}\chi^{-1} - 2\chi^{-1}\left(\chi^{-1} - 1\right) = 0. \tag{7.12}$$

Using the commutation relations it is then possible to solve for χ as a function of $|\mathbf{z}|^2$, and hence write $\Theta(\mathbf{z}, \mathbf{z}^\dagger)$ as a function of just $|\mathbf{z}|^2$, and of course, the commutativity parameter $\bar{k}$. We will not write down the exact expression, but instead give the expansion up to first order in $\bar{k}$. Actually, we only need the zeroth order expression for $\Theta(\mathbf{z}, \mathbf{z}^\dagger)$, its symbol being the classical answer (7.4) times $\bar{k}$, to write down the first order corrections to the real scalar field action (6.11) and the conserved currents $\mathcal{I}_\phi$ and $\bar{\mathcal{I}}_\phi$ in (6.14). Substituting (7.4) in the latter gives

$$\mathcal{I}_\phi = \mathcal{I}_\phi^0 + \mathcal{I}_\phi^1 + \cdots, \qquad \mathcal{I}_\phi^0 = \partial\phi,$$
$$\mathcal{I}_\phi^1 = -\frac{\hbar}{2}\bigl\{\bigl(6|z|^2-1\bigr)\mathcal{I}_\phi^0 - \bigl(1-|z|^2\bigr)\bigl(z\partial\mathcal{I}_\phi^0 + \bar{z}\bar{\partial}\mathcal{I}_\phi^0\bigr) + \bigl(1-|z|^2\bigr)^2\partial\bar{\partial}\mathcal{I}_\phi^0\bigr\}, \tag{7.13}$$

and $\bar{\mathcal{I}}_\phi$ is the complex conjugate. It follows that the solutions to the commutative theory are not preserved in the first order noncommutative theory. If ϕ_0 is a solution to the commutative theory, i.e. $\partial\bar{\partial}\phi_0 = 0$, then one can set

$$\phi = \phi_0 + \phi_1 \cdots, \tag{7.14}$$

and, in principle, iteratively solve for the non-commutative corrections. Substituting into (6.13) gives the following for the first order correction ϕ_1:

$$\partial\bar{\partial}\phi_1 = \frac{\hbar}{2}\left\{3(z\partial\phi_0 + \bar{z}\bar{\partial}\phi_0) + \frac{1}{2}\bigl(z^2\partial^2\phi_0 + \bar{z}^2\bar{\partial}^2\phi_0\bigr)\right\}. \tag{7.15}$$

To obtain the leading non-commutative corrections to gauge theory we need to compute θ beyond zeroth order. This is since θ_1 appears in the lowest order corrections to the Maxwell action (6.31). We first expand χ and substitute in (7.12) to get

$$\chi = \bigl(1-|\mathbf{z}|^2\bigr)\left(1 + \frac{\hbar}{4}\bigl(1-3|\mathbf{z}|^2\bigr) + \mathcal{O}\bigl(\hbar^2\bigr)\right). \tag{7.16}$$

Then from (7.11)

$$\Theta\bigl(\mathbf{z},\mathbf{z}^\dagger\bigr) = \frac{\hbar}{2}\bigl(1-|\mathbf{z}|^2\bigr)^2\left(1 - \frac{\hbar}{2}|\mathbf{z}|^2 + \mathcal{O}\bigl(\hbar^2\bigr)\right), \tag{7.17}$$

whose symbol $\theta(z,\bar{z})$ is

$$\theta(z,\bar{z}) = \frac{\hbar}{2}(1 - z\star\bar{z})_\star^2 \star \left(1 - \frac{\hbar}{2}z\star\bar{z} + \mathcal{O}\bigl(\hbar^2\bigr)\right), \tag{7.18}$$

where $\alpha_\star^2 = \alpha\star\alpha$. If we apply the expansion of the star product, given in (6.4), up to order $\hbar$ this leads to

$$\theta(z,\bar{z}) = \frac{\hbar}{2}\bigl(1-|z|^2\bigr)^2 - \frac{\hbar^2}{2}\bigl(1-|z|^2\bigr)^3 + \mathcal{O}\bigl(\hbar^3\bigr), \tag{7.19}$$

and so θ_1 in (6.5) is

$$\theta_1\bigl(|z|^2\bigr) = -\frac{1}{2}\bigl(1-|z|^2\bigr)^3. \tag{7.20}$$

This leads to

$$\frac{1}{2}\partial\bar{\partial}\theta_0 - \frac{\theta_1}{\theta_0} = \frac{1}{2}, \tag{7.21}$$

and as a result the commutative measure only gets corrected at first order by an overall constant factor

$$d\mu(z,\bar{z}) = \frac{i}{\pi\hbar}\frac{dz\wedge d\bar{z}}{(1-|z|^2)^2}\left(1 + \frac{\hbar}{2} + \mathcal{O}\bigl(\hbar^2\bigr)\right). \tag{7.22}$$

If one assumes a non-commutative analogue of the relationship (2.1) between the measure and the metric, then the components of the metric tensor also scale by the factor $1 + \bar{k}/2$, up to first order. From (7.21) it also follows that the solution (7.6) to the commutative gauge theory is unchanged at this order; i.e. (7.6) satisfies the first order corrected field equation (6.32), although the constant C_0 is modified from its commutative value. In terms of this shifted constant, the flux per unit area in (6.36) then goes to $C_0[1 - \bar{k}(C_0 + \frac{1}{2})]$, and the action per unit area is shifted by the same small factor.

8. Concluding remarks

In Section 6 we found that the flux per unit area of solutions to non-commutative Maxwell theory gets shifted from the commutative answer. If the result can be applied to the example of monopoles on the fuzzy sphere that would imply a shift in the magnetic monopole charge. However, it is not straightforward to apply the techniques of Section 6 to the case of the fuzzy sphere. Among the problems are: (a) As mentioned earlier, non-linear coherent states $|z\rangle$ satisfying (6.1) are not readily available for fuzzy manifolds. Alternatives states have been found for the fuzzy sphere, and they lead to a modified star product and integration measure [5,16]. (b) The Seiberg–Witten map is problematic in the case of fuzzy gauge theories since mapping from the non-commutative theory to the commutative one would mean connecting finite-dimensional spaces with an infinite dimensional one [23]. (c) A constraint must be introduced to insure that off block-diagonal matrix components of the potential remain non-dynamical when $\bar{k}$ is small. This constraint was necessary in the full non-commutative theory in order to obtain solutions with non-vanishing magnetic charge. We also found that the off block-diagonal matrix components should not vanish if we were to recover a finite value for the magnetic charge g_0 in the limit. If a proper treatment does reveal a shift in the magnetic charge due to non-commutative corrections, one expects that there should also be a shift in the Dirac quantization condition. For this it would be useful to analyze the quantum mechanics of a particle in a general monopole background (5.25) (similar to what was carried out in [12]). This would require finding a gauge invariant coupling to the non-commutative potentials. Constraints should then be found on the charges which reduce to the Dirac quantization condition in the commutative limit.

In the approach taken here we took advantage of the Kähler structure in two dimensions to construct the fundamental commutator (4.1) from the determinant of the metric tensor on P_0. As the commutative gauge theory action depends on the metric only through its determinant, no other ingredient was needed to write down the non-commutative version of the theory. This was not the case for the scalar field theory. There we chose a space–time gauge to get rid of additional degrees of freedom in the metric. More constraints will have to be imposed in order to generalize this approach to other field theories. In particular, frame fields and spin connections must be dealt with in the case of fermions, and also for gravity. Concerning the latter, the non-commutativity was taken to be non-dynamical, and in fact a constant, in most previous treatments of non-commutative gravity [32–35]. However, here since the non-commutativity is connected to the metric tensor, a consistent approach would mean that $\Theta(\mathbf{z}, \mathbf{z}^\dagger)$ is elevated to a dynamical field, which should lead to a more realistic model for quantum gravity (at least in two dimensions).

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