Nonlinear Models as Gauge Theories

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Nonlinear models as gauge theories

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In the usual formulation of nonlinear models (such as chiral models), there is invariance under a nonlinear realization of a group G_F which becomes linear when restricted to a subgroup H_F. We formulate them so that they become gauge theories for a local group $\mathcal{H}_C$. It is the local version of a global group H_C. When the gauge transformations are unrestricted at spatial infinity, only H_C singlets are observable, and the usual formulation is recovered. When the gauge transformations are required to reduce to identity at spatial infinity, the usual formulation is no longer recovered. In particular, (1) nonsinglets under H_C become observable, (2) the classical vacuum becomes degenerate under suitable conditions as in Yang-Mills theories, (3) the spontaneous symmetry breakdown of G_F seems complete. (In the usual formulations, G_F is broken down only to H_F.) It is shown that the instanton and meron solutions of Yang-Mills theories are also solutions of certain nonlinear models. It is also shown that in a certain class of nonlinear models in (Euclidean) $(3+1)$-dimensional space-time, there are no instanton solutions for any choice of the groups.

I. INTRODUCTION

Some years ago, models based on "nonlinear realizations of a group G_F which become linear for a subroup H_F" were popular.[1,2] We will generically call such models as nonlinear models. The group G_F was usually a chiral group such as $SU(3)\times SU(3)$, and the group H_F was usually its parity-conserving diagonal subgroup such as SU(3). These models were successful in explaining the low-energy behavior of strong interactions. There seems to be less interest in these models now as realistic models.[3] This is because of the following reasons: (1) They are apparently not theories with invariance under a gauge group. There are indications that elementary-particle interactions should be described by gauge theories. (2) They may not be renormalizable.

In this paper, we formulate these models so that there is invariance under a gauge group $\mathcal{H}_C$. The group $\mathcal{H}_C$ is the local version of a global group H_C. The global group H_C is isomorphic to H_F, but has a different action on the dynamical variables. The standard formulation is recovered by choosing a specific gauge. Our approach is very close to that of Callan *et al.* and Volkov.[1] A similar formulation of O(3) nonlinear σ models has been carried out by D'Adda *et al.*[4] (See also Ref. 5.) Thus nonlinear models can be regarded as gauge theories as well. The renormalizability of these models is still in doubt.

It will be seen below that the group G_F resembles the (global) flavor group, and the group H_C the (gauged) color group. Further the spontaneous breakdown of G_F to H_F in nonlinear models is guaranteed without the necessity of Higgs potentials. (In the normal quantum chromodynamics, the mechanism which spontaneously breaks the chiral flavor symmetry is at best not direct.[6] In this respect, nonlinear chiral models may be superior.) We may also note that nonlinear models are models for current algebras. Current algebras have good empirical support.

In summary, (1) nonlinear models are gauge theories of a group $\mathcal{H}_C$ which is the local version of a global group H_C, (2) they have global invariance under a group G_F. The latter contains a subgroup H_F isomorphic to H_C and G_F is spontaneously broken to H_F, (3) they are models for current algebras. These features seem desirable in particle physics and suggest that such models merit careful study.

Fermions can be introduced in nonlinear models by the normal prescriptions of gauge theories. An alternative method is to supersymmetrize the models.[7,4] We will also briefly indicate this generalization.

Consider a normal gauge theory with a gauge group $\mathcal{R}$ and the associated global group R. If $\mathcal{R}$ is unrestricted at spatial infinity, then only singlets under R are gauge invariant and hence observable in the classical theory. In addition, the classical vacuum is not degenerate. However, it is often assumed in gauge theories that gauge transformations must reduce to identities at spatial infinity. This assumption has a profound effect on the classical theory. Nonsinglets under R become gauge invariant and observable. Further, in $(d+1)$-dimensional space-time, the classical vacuum becomes infinitely degenerate if the homotopy group $\pi_d(R)$ is nontrivial. The instanton solutions then describe quantum-mechanical tunnelling between these vacuums. These are known results[8] and their proofs will be recalled in the text.

The situation is similar for nonlinear models.

If the gauge group $\mathcal{H}_C$ is unrestricted at spatial infinity, there is complete "color confinement" (that is, only singlets under the global group H_C are gauge invariant and hence observable). Also, there is no vacuum degeneracy due to the *gauge* group.[9] *The formulation of nonlinear models developed here is equivalent to the older formulations*[1,2] *(at least at the classical level) only if* $\mathcal{H}_C$ *is unrestricted at infinity. The remarks about the symmetry breakdown from* G_F *to* H_F *are also accurate only under this condition.* These older formulations set up the Lagrangian directly in terms of H_C (and $\mathcal{H}_C$) singlets, and hence do not show nontrivial gauge properties. If the elements of $\mathcal{H}_C$ are required to reduce to identity at spatial infinity, the older formulations are no longer recovered at the classical level. Nonsinglets under H_C become observable. The vacuum becomes infinitely degenerate if $\pi_d(H_C)$ is not trivial. Thus nonlinear models show topological features which resemble those of normal gauge theories. Further, heuristic arguments show that the spontaneous breakdown of G_F is complete.

When $\mathcal{H}_C$ is restricted by boundary conditions and the vacuum becomes degenerate, there is the possibility of instanton[10] and meron[11] solutions which tunnel between these vacuums in nonlinear models.[12] (We are interested in such solutions in Euclidean space-time.[10]) It will be shown that there are no instanton solutions in $(3+1)$-dimensional space-time for a class of these models. For another such class, there are such solutions in $(3+1)$-dimensional space-time. The method of Atiyah *et al.*[13] can in fact be adapted to construct some of the solutions. Further, the possible existence of "meron solutions" in some nonlinear models will be established by showing that the known meron solutions of Yang-Mills theories[11] are also solutions of a nonlinear model with $G_F = SU(2)\times SU(2)$, $H_C =$ diagonal SU(2). Instantons can of course exist also in other space-times. In particular, the Belavin-Polyakov[14] solutions of the nonlinear σ model in $(1+1)$-dimensional space-time can be interpreted as solutions which tunnel between its degenerate gauge vacuums.[4] As in the usual gauge theories, the presence of these tunnelling solutions forces us to redefine the quantum ground state and introduces a term in the effective Lagrangian which violates discrete symmetries.

In Sec. II, we formulate nonlinear models so that they become special sorts of gauge theories. The extension of our approach to supersymmetric nonlinear models is indicated. In Sec. III, we discuss the consequences of restricting the gauge group at infinity. The possibility of instanton and meron solutions is studied in the final section.

II. FORMULATION OF NONLINEAR MODELS

A. General remarks

Let $G=\{g_0\}$ be a compact, connected Lie group which is given as a group of unitary matrices. Let $H=\{h_0\}$ be a (closed, connected) subgroup of G.[15] The group G can be the chiral $SU(3)\times SU(3)$ and the group H its parity-conserving diagonal SU(3) subgroup.

The (Hermitian) generators of the Lie algebra L_G of G are $L(\sigma)$, $\sigma=1,2,\ldots,[G]$, where $[M]$ means the dimension of M. Their normalization is

$$\mathrm{Tr}L(\rho)L(\sigma)=\delta_{\rho\sigma}. \tag{2.1}$$

For $\alpha\leq[H]$, the generators $L(\alpha)$ are taken to span the Lie algebra L_H of H and are called $T(\alpha)$:

$$L(\alpha)=T(\alpha),\quad \alpha\leq[H]. \tag{2.2}$$

The remaining generators are called $S(i)$:

$$L(i)=S(i),\quad i\geq[H]+1. \tag{2.3}$$

Note the commutation relations

$$[L(\rho),L(\sigma)]=i\eta_{\rho\sigma\lambda}L(\lambda), \tag{2.4}$$

where

$$[T(\alpha),T(\beta)]=iC_{\alpha\beta\gamma}T(\gamma), \tag{2.5}$$

$$[T(\alpha),S(i)]=i\overline{C}_{\alpha ij}S(j), \tag{2.6}$$

$$[S(i),S(j)]=i[D_{ij\alpha}T(\alpha)+\overline{D}_{ijk}S(k)]. \tag{2.7}$$

(Of course, since $\mathrm{Tr}S(j)[T(\alpha),S(i)]=\mathrm{Tr}T(\alpha)[S(i),S(j)]$, we have the equality $\overline{C}_{\alpha ij}=D_{ij\alpha}$.)

Let $\mathcal{G}=\{g\}$ denote the local group associated with G. An element g is thus a field on a $(d+1)$-dimensional space-time M^{d+1} with values in G:

$$M^{d+1}\ni x\xrightarrow{g} g(x)\in G. \tag{2.8}$$

The Lagrangian density $\mathcal{L}$ in any nonlinear model is a function of g and $\partial_\mu g$ (Ref. 16):

$$\mathcal{L}=\mathcal{L}(g,\partial_\mu g). \tag{2.9}$$

It is required to be invariant under certain transformations which we now describe.

There are two types of such transformations on g:

(1) A local gauge transformation,

$$g(x)\to g(x)h(x),\quad h(x)\in H. \tag{2.10}$$

This local transformation group on $\mathcal{G}$ is the gauge group $\mathcal{H}_C$.[15] Its global version is the group[15] H_C:

$$g(x)\to g(x)h_0,\quad h_0\in H,$$
$$h_0 \text{ independent of } x. \tag{2.11}$$

The transformation groups $\mathcal{H}_C$ and H_C act on $\mathcal{G}$ *to the right.*

(2) A global transformation

$$g(x) \to g_0 g(x), \quad g_0 \in G,$$
$$g_0 \text{ independent of } x. \tag{2.12}$$

This global transformation group is the group G_F.[15] Its associated subgroup[15] H_F has the action

$$g(x) \to h_0 g(x), \quad h_0 \in H. \tag{2.13}$$

The transformation groups G_F and H_F act on $\mathcal{G}$ *to the left.*

The Lagrangian density is required to be invariant under both (2.10) and (2.12). This has the following consequences:

(a) The Lagrangian density can be regarded as a function of fields with values in the space of left cosets G/H.[17] This is as in normal formulations. For instance, at least locally,[1,18] we can write

$$g(x) = e^{i\xi_i(x)S(i)} h(x) \equiv k(x)h(x). \tag{2.14}$$

The gauge invariance (2.10) implies that

$$\mathcal{L}(g, \partial_\mu g) = \mathcal{L}(k(x), \partial_\mu k(x)), \tag{2.15}$$

which recovers a canonical form of $\mathcal{L}$.[1]

(b) The global symmetry G_F acts on ξ_i by the rule

$$\xi_i(x) \xrightarrow{g_0} \xi_i'(x), \tag{2.16}$$

where

$$g_0 e^{i\xi_i(x)S(i)} = e^{i\xi_i'(x)S(i)} h'(x) \tag{2.17}$$

for a suitable $h'(x) \in H$. The transformation (2.16) is in general nonlinear. But it becomes linear when restricted to H_F. This follows from (2.6) which implies that

$$h_0 S(i) h_0^{-1} = D(h_0)_{ji} S(j), \tag{2.18}$$

where $\{D(h_0)\}$ is a representation of H. Now,

$$h_0 \exp[i\xi_i(x)S(i)] = \exp\{i\xi_i(x)[h_0 S(i) h_0^{-1}]\} h_0. \tag{2.19}$$

With the identification $h'(x) = h_0$, we thus find,

$$\xi_i'(x) = D(h_0)_{ij} \xi_j(x), \tag{2.20}$$

which is a linear transformation law.[19]

Thus, by (a) and (b), $\mathcal{L}$ is a function of ξ_i which is invariant under a nonlinear realization of G_F which becomes linear when restricted to H_F. This is the standard definition of $\mathcal{L}$ in nonlinear models.[1]

Thus nonlinear models can be regarded as defined by Lagrangian densities $\mathcal{L}$ of the form (2.9) with gauge invariance under $\mathcal{H}_C$ and global invariance under G_F. The standard formulation is recovered by the gauge choice $g(x) = k(x)$.[18]

B. Construction of nonlinear models (no Fermions)

The construction of $\mathcal{L}$ proceeds as follows. Let

$$\omega_\mu(g) = g^\dagger \partial_\mu g. \tag{2.21}$$

[$\omega_\mu(g)dx^\mu$ is the Maurer-Cartan form.[20]] We have

$$\omega_\mu(gh) = h^\dagger \omega_\mu(g) h + h^\dagger \partial_\mu h. \tag{2.22}$$

Thus under the action of $\mathcal{H}_C$ on $\mathcal{G}$, ω_μ transforms as a gauge potential. The associated field strength is zero:

$$F_{\mu\nu}(\omega) = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + [\omega_\mu, \omega_\nu] \equiv 0. \tag{2.23}$$

Hence ω_μ is a "trivial" connection.

Consider next the projections of ω_μ into the Lie algebra L_H and its orthogonal complement:

$$A_\mu(g) = T(\alpha)\,\mathrm{Tr}\,T(\alpha)\omega_\mu(g) \equiv T(\alpha)A_\mu^\alpha(g), \tag{2.24}$$
$$B_\mu(g) = S(i)\,\mathrm{Tr}\,S(i)\omega_\mu(g) \equiv S(i)B_\mu^i(g), \tag{2.25}$$
$$\omega_\mu(g) = A_\mu(g) + B_\mu(g). \tag{2.26}$$

Since $h^\dagger \partial_\mu h \in L_H$, we have the identities

$$T(\alpha)\,\mathrm{Tr}\,T(\alpha) h^\dagger \partial_\mu h = h^\dagger \partial_\mu h, \tag{2.27}$$
$$\mathrm{Tr}\,S(i) h^\dagger \partial_\mu h = 0. \tag{2.28}$$

It follows that

$$\begin{aligned} A_\mu(gh) &= T(\alpha)\,\mathrm{ad}h_{\beta\alpha}\,\mathrm{Tr}\,T(\beta)\omega_\mu(g) + h^\dagger \partial_\mu h \\ &= h^\dagger A_\mu(g) h + h^\dagger \partial_\mu h, \end{aligned} \tag{2.29}$$

$$\begin{aligned} B_\mu(gh) &= S(i) D(h)_{ji}\,\mathrm{Tr}\,S(j)\omega_\mu(g) \\ &= h^\dagger B_\mu(g) h. \end{aligned} \tag{2.30}$$

Here $\{\mathrm{ad}h\}$ is the adjoint representation of H:

$$hT(\alpha)h^\dagger = \mathrm{ad}h_{\beta\alpha} T(\beta). \tag{2.31}$$

The matrices $\mathrm{ad}h$ and $D(h)$ are real, and by (2.1), orthogonal.

In summary, we find that

(i) $A_\mu(g)$ transforms like a gauge potential under the transformation $g \to gh$;

(ii) $B_\mu(g)$ transforms like a vector in the representation $\{D(h)\}$ under the same transformation.

(iii) It is also evident that ω_μ, A_μ, and B_μ are invariant under the action of the global G_F.

Thus, Lagrangian densities constructed out of A_μ and B_μ which are invariant under the local $\mathcal{H}_C$ will be invariant under (2.10) and (2.12). Two such typical structures are (up to overall constants)

$$\mathcal{L}^{(1)} = \mathrm{Tr}\,B_\mu B_\mu, \tag{2.32}$$
$$\mathcal{L}^{(2)} = +\tfrac{1}{4}\,\mathrm{Tr}\,F_{\mu\nu}(A)F_{\mu\nu}(A). \tag{2.33}$$

(Note the resemblance of $\mathcal{L}^{(2)}$ to the conventional Yang-Mills Lagrangian density. It will be important in Sec. IV B.) With the gauge choice $g(x) = k(x)$, these reduce to the Lagrangian densities discussed earlier.[1] We will also see that for G_F

$=SU(2)$, $H_F=U(1)$, (2.32) is the O(3) nonlinear σ model. This example is worked out fully by D'Adda *et al.*[4]

We may note here that $D_\mu(A)g^\dagger$ transforms simply under $\mathcal{H}_C$ and G_F:

$$D_\mu(A)g^\dagger \xrightarrow{g\to gh} h^\dagger D_\mu(A)g^\dagger\,, \tag{2.34}$$

$$D_\mu(A)g^\dagger \xrightarrow{g\to g_0 g} [D_\mu(A)g^\dagger]\,g_0^\dagger\,. \tag{2.35}$$

Here $D_\mu(A)$ is the covariant derivative $\partial_\mu + A_\mu$. Thus

$$\mathcal{L}^{(3)} = -\mathrm{Tr}\{[D_\mu(A)g^\dagger]\,[D_\mu(A)g^\dagger]^\dagger\} \tag{2.36}$$

is also invariant under (2.10) and (2.12). But this is not different from (2.32). To see this, let us define

$$B_\mu(g) = -g^\dagger I_\mu(g) g\,. \tag{2.37}$$

By (2.26), $A_\mu(g)$ is the gauge transform of $I_\mu(g)$ by the group element g:

$$A_\mu(g) = g^\dagger I_\mu(g) g + g^\dagger \partial_\mu g\,. \tag{2.38}$$

Therefore,

$$D_\mu(A) = g^\dagger D_\mu(I) g \tag{2.39}$$

and

$$D_\mu(A)g^\dagger = g^\dagger D_\mu(I) = g^\dagger I_\mu(g) = -B_\mu(g)g^\dagger\,. \tag{2.40}$$

Since $B_\mu(g)$ is anti-Hermitian, this shows that $\mathcal{L}^{(1)} = \mathcal{L}^{(3)}$.

Example: the nonlinear σ model[4]

Let $G=SU(2)$ and $H_C=U(1)$. We choose $L(\rho)$ as follows:

$$T(1) = \sigma_1/\sqrt{2}\,,\quad S(i) = \sigma_i/\sqrt{2}\,,\quad i=2,3 \tag{2.41}$$

Here σ_ρ are Pauli matrices. The choice is consistent with (2.1). Let us define the triplet of ϕ fields by

$$(1/2)^{1/2} g\sigma_1 g^\dagger = (1/2)^{1/2}\sigma_\alpha \phi_\alpha \equiv \Phi\,. \tag{2.42}$$

We have

$$\mathrm{Tr}\Phi\Phi = \phi_\alpha\phi_\alpha = \tfrac{1}{2}\,\mathrm{Tr}\sigma_1^2 = 1\,. \tag{2.43}$$

We now show that $\mathcal{L}^{(1)}$ is $-\frac{1}{2}\partial_\mu\phi_\alpha\partial_\mu\phi_\alpha \equiv -\frac{1}{2}\mathrm{Tr}(\partial_\mu\Phi)^2$ so that it is the conventional nonlinear σ model. We have

$$\mathcal{L}^{(1)} = \mathrm{Tr} I_\mu I_\mu\,. \tag{2.44}$$

Now note that

$$[\tfrac{1}{2}\sigma_1, [\tfrac{1}{2}\sigma_1, S(i)]] = \epsilon_{1ij}\epsilon_{1kj}S(k) = S(i)\,. \tag{2.45}$$

Hence

$$\begin{aligned} I_\mu &= -g\{[\tfrac{1}{2}\sigma_1, [\tfrac{1}{2}\sigma_1, B_\mu]]\}g^\dagger \\ &= \tfrac{1}{2}[\Phi, \partial_\mu\Phi] \\ &= \tfrac{1}{2}i\epsilon_{\alpha\beta\gamma}\phi_\alpha\partial_\mu\phi_\beta\sigma_\gamma \end{aligned} \tag{2.46}$$

as a simple calculation shows. Since

$$\phi_\alpha\partial_\mu\phi_\alpha = 0 \tag{2.47}$$

due to (2.43), it follows that

$$\mathcal{L}^{(1)} = -\tfrac{1}{2}(\partial_\mu\phi_\alpha)(\partial_\mu\phi_\alpha)\,. \tag{2.48}$$

Of course, since $\mathcal{L}^{(i)}$ are functions of fields with values in G/H (due to invariance under $\mathcal{H}_C$),[21] and G/H is parametrized by Φ for the groups considered, these $\mathcal{L}^{(i)}$ are necessarily functions only of Φ.

For possible later use, we express $\mathcal{L}^{(2)}$ in terms of Φ. From (2.38),

$$\mathcal{L}^{(2)} = +\tfrac{1}{4}\mathrm{Tr}F_{\mu\nu}(I)F_{\mu\nu}(I)\,. \tag{2.49}$$

It follows from (2.46) that[22]

$$\partial_\mu I_\nu - \partial_\nu I_\mu = [\partial_\mu\Phi, \partial_\nu\Phi] \tag{2.50}$$

$$= \Phi\,\mathrm{Tr}\Phi[\partial_\mu\Phi, \partial_\nu\Phi]\,, \tag{2.51}$$

$$[I_\mu, I_\nu] = -\tfrac{1}{2}\Phi\,\mathrm{Tr}\Phi[\partial_\mu\Phi, \partial_\nu\Phi]\,, \tag{2.52}$$

$$F_{\mu\nu}(I) = \tfrac{1}{2}\Phi\,\mathrm{Tr}\Phi[\partial_\mu\Phi, \partial_\nu\Phi]\,, \tag{2.53}$$

$$\mathcal{L}^{(2)} = \tfrac{1}{16}(\mathrm{Tr}\{\Phi[\partial_\mu\Phi, \partial_\nu\Phi]\})^2\,. \tag{2.54}$$

C. Construction of nonlinear models with Fermions

(a) Fermion fields can be introduced in a conventional way in these models.[1,2] Thus if $\Gamma = \{d(h)\}$ is a unitary representation of H, and q is a (multicomponent) fermion field which transforms as

$$q \to d(h)^\dagger q\,, \tag{2.55}$$

then for example

$$\mathcal{L}^{(4)} = -\bar{q}\,[\gamma_\mu D_\mu(A) + m]\,q \tag{2.56}$$

is invariant under both (i) $g\to gh$, $q\to d(h)^\dagger q$ and (ii) $g\to g_0 g$ [cf. (2.10) and (2.12)]. Here D_μ is of course to be evaluated in the representation Γ. Such Lagrangian densities have been studied previously.[1,2]

(b) A more interesting generalization consists in promoting g to a superfield "with values in the group G." (That is, $g=\exp[iL(\alpha)f_\alpha]$ where f_α are even Hermitian superfields.) The derivative ∂_μ in ω_μ, A_μ, and B_μ is then to be replaced by the supersymmetric derivative d_α. This gives their supersymmetric analogs ω_α, A_α, and B_α. The gauge group $\mathcal{H}_C$ is to be replaced by the corresponding supergauge group. Its elements depend on points in superspace and has "values in the group H_C." Lagrangians can be con-

structed as before, with well-known modifications required by supersymmetry. Such a generalization of $\mathcal{L}^{(1)}$ in $(1+1)$-dimensional space-time for $G=\mathrm{SU}(2)$ and $H_C=\mathrm{U}(1)$ leads to the supersymmetric nonlinear σ model of D'Adda *et al.* and Witten.[7,14]

A detailed study of such supersymmetric models has not been carried out.

III. VACUUM STRUCTURE OF NONLINEAR MODELS

A. General considerations on observables in gauge theories[8]

Let us consider any theory (not necessarily nonlinear) which is invariant under a gauge group $\mathcal{R}$. (The associated global group is R.) If $\eta(\vec{x},t)$ denotes a trajectory (a solution of the equations of motion) in this theory, then so is its gauge transform $(r\circ\eta)(\vec{x},t)$, where r is a space-time-dependent gauge transformation ($r\in\mathcal{R}$). Now, there are nontrivial gauge transformations $\tilde{r}$ which reduce to identity at some time zero, say. Thus for a given set of initial data $\eta(\vec{x},0)$, we have many possible trajectories $(\tilde{r}\circ\eta)(\vec{x},t)$. The Cauchy problem is therefore ill defined unless we restrict our considerations to those functions $\mathcal{O}$ of η which are invariant under $\tilde{\mathcal{R}}=\{\tilde{r}\}$. Here $\tilde{\mathcal{R}}$ is the connected component of $\mathcal{R}$. The observables of the theory are by definition the set $\mathcal{O}$. If suitable initial data are specified from this set $\mathcal{O}$, then their time evolution can be uniquely specified. Thus requirements of determinism define observables.

We will see below that the two groups $\mathcal{R}$ and $\tilde{\mathcal{R}}$ may or may not coincide depending on the asymptotic conditions we put on elements of $\mathcal{R}$. Such conditions have an effect on the structure of the observables in the classical theory. In particular, if $\mathcal{R}\neq\tilde{\mathcal{R}}$, the factor group $\mathcal{R}/\tilde{\mathcal{R}}$ can act nontrivially on the classical vacuum. This, of course, is the source of the vacuum degeneracy of certain Yang-Mills theories.

1. *No asymptotic conditions at infinity on the gauge group*

Here we impose no conditions at spatial infinity on elements of $\mathcal{R}$. Then it is known that $\mathcal{R}=\tilde{\mathcal{R}}$.[23] The observables are thus invariant under $\mathcal{R}$. Further the global group R of constant gauge transformations is a subgroup of $\mathcal{R}$. Hence classical observables are singlets under R.[24] Finally, since $\mathcal{R}/\tilde{\mathcal{R}}$ is trivial, there is no vacuum degeneracy due to the gauge group.

2. *Asymptotic conditions at infinity on the gauge group*

In recent formulations of gauge theories,[25] one assumes that at spatial infinity, the Yang-Mills potentials vanish and the gauge transformations reduce to identity. Then in $(d+1)$-dimensional space-time, $\mathcal{R}\neq\tilde{\mathcal{R}}$ if the homotopy group $\pi_d(R)$ is not trivial. In this event $\mathcal{R}/\tilde{\mathcal{R}}$ is an infinite cyclic group generated by an element T, $\mathcal{R}/\tilde{\mathcal{R}}=\{T^n\}_{n=-\infty}^{+\infty}$. The observables need be invariant only under $\tilde{\mathcal{R}}$. They can change under T^n.

Further, it is obvious that the set of constant gauge transformations R is not a subgroup of $\tilde{\mathcal{R}}$. In fact, $\tilde{\mathcal{R}}\cap R=$ identity. As a consequence, there are $\tilde{\mathcal{R}}$-invariant objects which are not R singlets.[24] For instance, if q is a quark field, let[26]

$$q'(x)=V(x)q(x)\,, \tag{3.1}$$

where $V(x)$ is the representative in the representation Γ (cf. Sec. IIC) of

$$U(x)=P\left[\exp\left(\int_{-\infty}^{x_3}dx_3'A_3(x_1,x_2,x_3',x_4)\right)\right]. \tag{3.2}$$

Here P denotes path ordering. Regardless of the boundary conditions, the gauge transformation $A_\mu\to r^\dagger A_\mu r+r^\dagger\partial_\mu r$ induces the transformation

$$U(x)\to r^\dagger(x_1,x_2,-\infty,x_4)U(x)r(x)\,. \tag{3.3}$$

Thus, q' is gauge invariant with (2.55) (with h replaced by r) and the boundary condition. But if r is global $=r_0$, then clearly

$$q'(x)\to d(r_0)^\dagger q'(x)\,. \tag{3.4}$$

All observable configurations related by T^n have the same energy since T is a gauge transformation which leaves the Lagrangian invariant. That is, there is an energy degeneracy. In particular, the vacuum is degenerate.

B. Nonlinear models

1. *No asymptotic conditions at infinity on the gauge group*

Remarks similar to IIIA1 apply to nonlinear models with $\mathcal{R}=\mathcal{H}_C$ and $R=H_C$. Let us first examine the simple case $G=\mathrm{SO}(3)$, $H_C=\mathrm{U}(1)$, and the Lagrangian density $\mathcal{L}^{(1)}$ (cf. Sec. II). The observables are invariant under arbitrary gauge transformations $g\to gh$, $h\in\mathcal{H}_C$ with no conditions at infinity on h. They are thus functions of the field ϕ [(2.42)] with values in G/H. The latter is a singlet under the global H_C. The classical vacuums correspond to $\phi=$ constant. The different vacuums are related by the action of the global group G_F, $g\to g_0g$ which rotates the constant ϕ to different directions. This degeneracy is like the vacuum degeneracy due to the Higgs potential. It is not due to the topology of $\mathcal{H}_C$ unlike the vacuum degeneracy of some Yang-Mills theories.

Such considerations generalize to any nonlinear model. An observable is invariant under $g\to gh$

for arbitrary space-time dependence of h. (We assume that no field besides g is present.) It is thus $\mathcal{H}_C$ invariant, and H_C singlet, and a function of a field with values in G/H. (In the preceding discussion, ϕ is such a field.) Further (2.15) is now generally valid. Thus, with no conditions on $\mathcal{H}_C$, the present formulation is equivalent to older formulations.[1,2]

Let us next examine the vacuum structure of any nonlinear model. The classical vacuums correspond to[27]

$$g = g_0 h\,, \tag{3.5}$$

where $g_0 \in G$ is constant and $h(x) \in H$ has any space-time dependence. [Then, e.g., ϕ in (2.42) is constant.] For then

$$A_\mu(g_0 h) = h^\dagger \partial_\mu h \tag{3.6}$$

is a "pure gauge" and the Lagrangians of Sec. II vanish. By a gauge transformation of $\mathcal{H}_C$, we can first reduce (3.5) to constant g_0. Two such vacuums g_0 and g_0' are now gauge equivalent if $g_0' = g_0 h_0$, $h_0 \in H$ being constant. Thus the vacuums are in one-to-one correspondence with G/H.

In quantizing such a theory, this degeneracy is removed by orienting the vacuum along a particular left coset, say H. This is as in Higgs models. In terms of g_0, this amounts to setting $g_0 = e$. [The field ϕ of (2.42) is then oriented in the first direction.] The remaining symmetry of the vacuum is H_F since $h_0 g_0 = h_0 = g_0 h_0$ is gauge equivalent to g_0. [The little group of $\phi = (1,0,0)$ is U(1).] Thus the global symmetry G_F is spontaneously broken down to H_F. There are as many Goldstone bosons as the dimension of G/H.[28] These are the pions of the chiral model with $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$ and H = diagonal SU(2).

2. *Asymptotic conditions at infinity on the gauge group*

Considerations similar to IIIA2 now apply to nonlinear models with $\mathcal{R} = \mathcal{H}_C$, $\tilde{\mathcal{R}} = \tilde{\mathcal{H}}_C$, and $R = H_C$. If $\pi_d(H_C)$ is not {identity}, $\mathcal{H}_C/\tilde{\mathcal{H}}_C = \{T^n\}_{n=-\infty}^{n=+\infty}$ and observables can respond nontrivially to T^n. There are also observables which are not invariant under H_C. For instance [cf. (3.1) and (3.2)],

$$g'^\dagger(x) = U(x) g^\dagger(x) \tag{3.7}$$

is such an observable.

A field configuration $g_0 h$ is a vacuum of such a theory[27] (cf. Sec. IIIB1). Gauge equivalence under $\tilde{\mathcal{H}}_C$ means that the inequivalent vacuums are given by $\{g_0 T^n\}_{n=-\infty}^{+\infty}$ when $\mathcal{H}_C/\tilde{\mathcal{H}}_C = \{T^n\}_{n=-\infty}^{+\infty}$ is not trivial.

In quantizing the theory, the degeneracy due to g_0 is removed by giving it a specific orientation, say g_0 = identity e. Quantum theory thus spontaneously breaks the global symmetry (2.12). The breakdown seems complete. We can see this as follows. The symmetry of the vacuum is given by those global transformations $s_0 \in G_F$ such that

$$s_0 T^n = T^n \tilde{h}, \quad \tilde{h} \in \tilde{\mathcal{H}}_C\,, \tag{3.8}$$

since T^n and $T^n\tilde{h}$ are gauge equivalent. At spatial infinity, T^n and $\tilde{h} \to e$. But s_0 is global, so $s_0 \equiv e$.

Note that this situation is strikingly different from IIIB1 where the vacuum has the symmetry of H_F. We hope to study the structure of this symmetry breakdown elsewhere.

It should now be clear that at the classical level, with the above conditions at infinity, the present formulation of nonlinear models is not equivalent to the older ones. For instance, since the observable g' [(3.7)] is not H_C invariant, it can not be expressed in terms of the H_C-invariant ϕ for the SO(3) model of Sec. II.

IV. INSTANTONS AND MERONS IN NONLINEAR MODELS

When the vacuum becomes degenerate as in IIIB2, it becomes of interest to investigate tunnelling between these vacuums. Instanton solutions in the Euclidean $(d+1)$-dimensional space-time describe such tunnelling. (This is our definition of instantons. We do not use self-duality properties in defining them.) Meron solutions[11,12] are also relevant in semiclassical calculations (Callan *et al.*[6]). We now discuss instantons and merons in nonlinear models.

A. Instantons

1. *Lagrangian density* $\mathcal{L}^{(1)} = \mathrm{Tr} B_\mu B_\mu$ *[Eq. (2.32)]*

We wish to show that for this Lagrangian density, there is *no* instanton solution for any gauge group in (3+1)-dimensional space-time.

In the vacuums of these theories (in Minkowski space), $B_\mu = 0$. By (2.26), this means that $\omega_\mu = A_\mu$ and so by (2.23) that $F_{\mu\nu}(A) = 0$. Hence there is an $h \in \mathcal{H}$ which fulfills

$$g^\dagger \partial_\mu g = h^\dagger \partial_\mu h\,. \tag{4.1}$$

Let $g = \bar{g} h$. Then $\partial_\mu \bar{g} = 0$ or $\bar{g} = g_0$ = constant. Thus the vacuums are given by (3.5).

It follows that in Euclidean 4-space R^4, we need a solution g with the behavior

$$g \to g_0 h \text{ as } |x| \equiv (x_1^2 + x_2^2 + x_3^2 + x_4^2)^{1/2} \to \infty\,. \tag{4.2}$$

The degree of mapping defined by h from $\partial R^4 = S^3$ into H gives the instanton number k. For the latter, we have

$$k \propto \int_{\partial R^4} d\sigma_\mu \epsilon_{\mu\nu\lambda\rho} \mathrm{Tr}(A_\nu \partial_\lambda A_\rho + \tfrac{2}{3} A_\nu A_\lambda A_\rho)$$
$$\equiv \int_{\partial R^4} d\sigma_\mu \mathcal{J}_\mu(A)\,. \tag{4.3}$$

By (4.2),

$$k \propto \int_{\partial R^4} d\sigma_\mu \mathcal{J}_\mu(\omega). \tag{4.4}$$

But

$$\partial_\mu \mathcal{J}_\mu(\omega) \propto \epsilon_{\mu\nu\lambda\rho} \mathrm{Tr} F_{\mu\nu}(\omega) F_{\lambda\rho}(\omega) = 0. \tag{4.5}$$

Applying Stokes's theorem to shrink ∂R^4 to a point, we find $k=0$. Hence the result.

This proof, valid for $d=3$, need not be true for $d \neq 3$. Thus the SO(3) model of Sec. II has degenerate classical vacuums for $d=1$ since $\pi_1[\mathrm{U}(1)] = Z$. It is readily inferred from the work of D'Adda *et al.*[4] [cf. their Eq. (15)] that the Belavin-Polyakov solutions[14] describe tunnelling between these vacuums. A standard reasoning[29] then shows that the effective quantum Lagrangian density contains an additional piece

$$\theta \mathrm{Tr} T(3) F_{12}(A) \tag{4.6}$$

dependent on the angle θ. It violates discrete symmetries.

2. Lagrangian density $\mathcal{L}^{(2)} = \frac{1}{4}\mathrm{Tr} F_{\mu\nu}(A) F_{\mu\nu}(A)$ [Eq. (2.33)]

We show here that the self- and anti-self-dual solutions of Atiyah *et al.*[13] can be written in the form (2.24). By a well-known inequality (cf. Ref. 10, Sec. 3.5), they extremize the Euclidean action of $\mathcal{L}^{(2)}$ and solve the nonlinear model as well. Thus the argument in IV A1 must fail for $\mathcal{L}^{(2)}$. This happens as follows: The vacuums are now defined by $F_{\mu\nu}(A)=0$ instead of by $B_\mu = 0$. So (4.1) is replaced by the weaker statement

$$T(\alpha) \mathrm{Tr} T(\alpha) g^\dagger \partial_\mu g = h^\dagger \partial_\mu h \tag{4.7}$$

for some h. Setting $g = \bar{g} h$, we find

$$\mathrm{Tr} T(\alpha) \bar{g}^\dagger \partial_\mu \bar{g} = 0. \tag{4.8}$$

It is easy to find nonconstant $\bar{g}$ which fulfill (4.8) for suitable G and H. For instance, let $G = \mathrm{U}(n)$,

$$H = \mathrm{U}(k) = \left\{ \begin{bmatrix} U_k & 0 \\ 0 & 1_l \end{bmatrix} \right\}, \qquad H' = \mathrm{U}(l) = \left\{ \begin{bmatrix} 1_k & 0 \\ 0 & U_l \end{bmatrix} \right\}, \tag{4.9}$$

where $k+l=n$ (a subscript ρ indicates that the matrix is $\rho \times \rho$). Then (4.8) is fulfilled if $\bar{g} \in \mathrm{U}(l)$. Note that $A_\mu(gh') = A_\mu(g)$ for space-time dependent $h'(x) \in H'$. So $\mathcal{L}^{(2)}$ is invariant under the gauge group of $H \times H'$ and not just of H.

The Atiyah *et al.* construction for Sp(n) of instantons of topological number k is in terms of a $(k+n) \times n$ matrix N with quaternionic entries. The potential is

$$N^\dagger \partial_\mu N. \tag{4.10}$$

The conditions on N are

(a) $N^\dagger N = 1$. (4.11)

(b) There is a $(k+n) \times k$ matrix M with the following properties:

(i) $M = P - Qx$, (4.12)

where

$$x = x_\mu \tau_\mu, \tag{4.13}$$

and

$$\tau_\mu = (1, i\sigma_i), \quad \sigma_i = \text{Pauli matrices}. \tag{4.14}$$

Here P and Q are independent of x.

(ii) $M^\dagger M = \Delta$, (4.15)

where Δ is "real" (commutes with τ_μ), and invertible. (If τ_μ is replaced by $\tau_\mu^\dagger$, the alternative duality property is obtained.)

(iii) $N^\dagger M = 0$. (4.16)

Thus if

$$\bar{N} = M \Delta^{-1/2}, \tag{4.17}$$

the matrix

$$g = [N, \bar{N}] \tag{4.18}$$

is in Sp($n+k$). We can now imbed Sp(n) in Sp($n+k$) by the identification

$$\mathrm{Sp}(n) = \left\{ \begin{bmatrix} s_n & 0 \\ 0 & 1 \end{bmatrix} \right\}. \tag{4.19}$$

For its generators $T(\alpha)$, we have

$$T(\alpha) = \begin{bmatrix} t(\alpha) & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathrm{Tr} T(\alpha) T(\beta) = \mathrm{Tr} t(\alpha) t(\beta) = \delta_{\alpha\beta}. \tag{4.20}$$

Here $\{t(\alpha)\}$ are the Sp(n) generators in the $n \times n$ space where $N^\dagger \partial_\mu N$ lives. Since

$$\mathrm{Tr} T(\alpha) g^\dagger \partial_\mu g = \mathrm{Tr} t(\alpha) N^\dagger \partial_\mu N, \tag{4.21}$$

the components of the potential are of the required form (2.24). Thus the Sp(n) k-instanton solutions of Atiyah *et al.* solve the nonlinear model defined by $\mathcal{L}^{(2)}$ for $G = \mathrm{Sp}(n+k)$, $H = \mathrm{Sp}(n)$. As remarked before, $\mathcal{L}^{(2)}$ is actually invariant under the gauge group of $\mathrm{Sp}(n) \times \mathrm{Sp}(k)$.

The solutions of Atiyah *et al.* for the other classical groups can be adapted to nonlinear models in a similar fashion.

B. Merons

The meron solutions M_μ of SU(2) Yang-Mills theories in four-dimensional Euclidean space-time are of the form[11]

$$M_\mu = \tfrac{1}{2} t^\dagger \partial_\mu t, \tag{4.22}$$

where t is a specific element of SU(2) [cf. Ref. 11,

and Eqs. (4.36) and (4.37) below]. They fulfill

$$D_\mu(M)F_{\mu\nu}(M)=0 \tag{4.23}$$

except for isolated points in space-time.

Consider a nonlinear model with

$$G=\mathrm{SU}(2)\times\mathrm{SU}(2)=\left\{g=\begin{pmatrix} g_1 & 0 \\ 0 & g_2\end{pmatrix}\right\}$$

where g_i are 2×2 unitary unimodular matrices. Let H be its diagonal SU(2) subgroup $\{h=\left(\begin{smallmatrix} s & 0 \\ 0 & s\end{smallmatrix}\right)\}$. We show that in such a model with the Lagrangian density $\mathcal{L}^{(2)}$, (i) the potentials $A_\mu(g)$ are of the form dictated by (4.22), that is, that they are "half a gauge," (ii) any solution of $D_\mu(A)F_{\mu\nu}(A)=0$ also solves the field equations due to $\mathcal{L}^{(2)}$. It follows that for suitable g (specifically for $g_1=t$, $g_2=1$), the meron solutions also solve these nonlinear models.

(i) We let

$$g=\begin{pmatrix} g_1 & 0 \\ 0 & 1\end{pmatrix}. \tag{4.24}$$

(We can always reduce g to this form by a suitable gauge transformation $g\to gh$.) We choose for the generators of H,

$$T(\alpha)=\frac{1}{2}\begin{bmatrix} \sigma_\alpha & 0 \\ 0 & \sigma_\alpha\end{bmatrix}, \qquad \sigma_\alpha=\text{Pauli matrices}. \tag{4.25}$$

So

$$\mathrm{Tr}\,T(\alpha)T(\beta)=\delta_{\alpha\beta}. \tag{4.26}$$

It follows that

$$A_\mu=T(\alpha)\,\mathrm{Tr}\,T(\alpha)g^\dagger\partial_\mu g =\frac{1}{2}\begin{bmatrix} g_1^\dagger\partial_\mu g_1 & 0 \\ 0 & g_1^\dagger\partial_\mu g_1\end{bmatrix}, \tag{4.27}$$

where we used $\frac{1}{2}\sigma_\alpha\,\mathrm{Tr}\,\sigma_\alpha g_1^\dagger\partial_\mu g_1=g_1^\dagger\partial_\mu g_1$. Thus each block is of the form (4.22). Of course, the equation $D_\mu(A)F_{\mu\nu}(A)=0$ splits into two identical equations, one for each block.

(ii) We have

$$\delta\mathcal{L}^{(2)}=\mathrm{Tr}\{F_{\mu\nu}(\partial_\mu\delta A_\nu+[A_\mu,\ \delta A_\nu])\}, \tag{4.28}$$

which after a partial integration becomes

$$\mathrm{Tr}(D_\mu F_{\mu\nu})\delta A_\nu, \tag{4.29}$$

where the identity $\mathrm{Tr}A[B,C]=\mathrm{Tr}B[C,A]$ has been used. The variations of δA_μ are not arbitrary, they have to be induced by varying g. But in any case (4.29) shows that the action is stationary if $D_\mu F_{\mu\nu}=0$. This proves the result.

As an aside, note that the most general variation of g is

$$\delta g=iL(\rho)\epsilon_\rho g, \tag{4.30}$$

where ϵ_ρ are space-time-dependent parameters and $L(\rho)$'s span the Lie algebra of G (Sec. II A). From this and the form of $A_\mu(g)$, one gets the following field equation for $\mathcal{L}^{(2)}$ from (4.29):

$$\partial_\nu\mathrm{Tr}\{(D_\mu F_{\mu\nu})g^\dagger L(\rho)g\}=0. \tag{4.31}$$

[Use $\{\mathrm{Tr}F_{\mu\nu}T(\alpha)\}T(\alpha)=F_{\mu\nu}$.] This is just the conservation law for the Noether current of the symmetry transformation (4.30) with ϵ_ρ constant. Equation (4.31) is the same as

$$\mathrm{Tr}\{(D_\mu F_{\mu\nu})D_\nu(g^\dagger L(\rho)g)\}=0. \tag{4.32}$$

It is instructive to rewrite the simplest meron solutions in terms of conventional normalized fields [like ϕ in (2.42)]. Let τ_μ be as in (4.14) and

$$\gamma_\mu=\begin{bmatrix} 0 & \tau_\mu \\ \tau_\mu^\dagger & 0\end{bmatrix}. \tag{4.33}$$

Then the conventional field ψ is given by

$$g\gamma_4 g^\dagger=\gamma_\alpha\psi_\alpha. \tag{4.34}$$

It fulfills

$$\psi_\alpha\psi_\alpha=1, \tag{4.35}$$

and is the field of the "SO(4) nonlinear σ model."[30]

The two simplest meron solutions have[11]

$$t=t_1=\tau_\alpha\hat{x}_\alpha \tag{4.36}$$

and

$$t=t_2=\tau_\alpha^\dagger\hat{x}_\alpha, \tag{4.37}$$

where

$$\hat{x}_\alpha=\frac{x_\alpha}{(x_\mu x_\mu)^{1/2}}. \tag{4.38}$$

It follows that

$$\psi_\alpha(x)=\hat{x}_\alpha \tag{4.39}$$

for $t=t_1$ and

$$\psi_4(x)=\hat{x}_4, \quad \psi_k(x)=-\hat{x}_k \tag{4.40}$$

for $t=t_2$.

de Alfaro *et al.* solve a nonlinear σ model which seems different from ours.[12,30] They too find the solutions (4.39) and (4.40).

We can similarly construct the ψ fields from all the known meron solutions of the SU(2) Yang-Mills theory.[11] General covariance can also be introduced in $\mathcal{L}^{(2)}$ by following the last paper of Ref. 11. Such a generalization is of interest for discussing merons as shown in that paper.

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[9]There is a degeneracy due to the global group G_F. It is removed by spontaneous symmetry breakdown.

[10]See, for example, S. Coleman, lectures delivered at the 1977 International School of Subnuclear Physics, Ettore Majorana; Harvard Univ. Report No. HUTP-78/A004, 1978 (unpublished).

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[12]For work on instantons and merons in nonlinear σ models, see Ref. 4; V. de Alfaro, S. Fubini and G. Furlan, Nuovo Cimento 48A, 485 (1978) and references therein. This paper of de Alfaro *et al.* contains a meron solution for a nonlinear σ model in (3+1)-dimensional space-time. This model is defined by a differential equation containing fourth-order derivatives and seems different from ours discussed in Sec. IV B. The equation defining the latter contains only second-order derivatives. Still, two of our solutions [cf. Eqs. (4.39) and (4.40)] are identical to theirs.

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[15]Our notation is as follows. The space-time–independent (global) groups are indicated by capital letters: G, H, H_C, H_F. Space-time–independent group elements are distinguished by the subscript zero. A set of fields on space-time with values in a group is indicated by a script letter: $\mathcal{G}$, $\mathcal{H}_C$. The Lagrangian is composed of elements of $\mathcal{G}$. The groups $\mathcal{H}_C$, H_C, G_F, and H_F are transformation groups on $\mathcal{G}$.

[16]Systematic methods for the canonical quantization of such Lagrangian theories involving group elements are available from the work of A. P. Balachandran, S. Borchardt, and A. Stern, Phys. Rev. D 17, 3247 (1978). Note in particular that the techniques developed in this paper do not depend on the way the group elements are parametrized.

[17]In this section, it is assumed that there are no asymptotic restrictions on $\mathcal{H}_C$. Otherwise, it may not be possible to gauge away the $h(x)$ in (2.14). Cf. Sec. III.

[18]Gauge choices such as in (2.15) are the analogs of gauge-fixing conditions in the usual Yang-Mills theories. We note here that they may not exist globally in nonlinear models. For instance let $G=\mathrm{SU}(2)$, $H=\mathrm{U}(1)=\{e^{iT(1)\theta}\}$. Then it is not possible to write a smooth decomposition $g_0=k_0h_0$ for all $g_0\in G$, where $h_0\in H$ and k_0 labels the left cosets. This is so because $G/H=$ two-sphere S^2 and SU(2) is not topologically $S^2\times\mathrm{U}(1)$. [For instance, SU(2) is simply connected and $S^2\times\mathrm{U}(1)$ is infinitely connected.] In general this situation arises if, topologically, $G\neq(G/H)\times H$. See in this connection I. M. Singer, Commun. Math. Phys. 60, 7 (1978); V. N. Gribov, Lecture at Proceedings of the Twelfth Winter School of the Leningrad Nuclear Research Institute, 1977 (unpublished); S. Mandelstam, talk presented at the Washington Meeting of the American Physical Society, 1977 (unpublished); A. P. Balachandran, H. S. Mani, R. Ramachandran, and Pankaj Sharan, Phys. Rev. Lett. 40, 988 (1978); C. Montonen, Nucl. Phys. B140, 333 (1978), and references therein; M. S. Narasimhan and T. R. Ramdas, Tata Institute of Fundamental Research report, 1978 (unpublished).

[19]This argument is due to Coleman, Wess, and Zumino (Ref. 1).

[20]See, for example, Y. Choquet-Bruhat, C. De Witt-Morette, and M. Dillard-Bleick, *Analysis, Manifolds, and Physics* (North-Holland, Amsterdam, 1977), p. 199.

[21]This statement is incorrect if there are asymptotic conditions on $\mathcal{H}_C$. See Ref. 17.

[22]We can show (2.51) from (2.50) as follows. If $\partial_\mu\Phi$ and $\partial_\nu\Phi$ are parallel (point in the same direction in the Lie algebra L_G), both (2.50) and (2.51) are zero. If they are not parallel, Φ, $\partial_\mu\Phi$, and $\partial_\nu\Phi$ span L_G (since $\mathrm{Tr}\Phi\partial_\mu\Phi=\mathrm{Tr}\Phi\partial_\nu\Phi=0$). But $[\partial_\mu\Phi, \partial_\nu\Phi]$ has zero trace with $\partial_\mu\Phi$ or $\partial_\nu\Phi$. Hence it is proportional to Φ, and (2.51) follows.

[23]S. Coleman, in *New Phenomena in Subnuclear Physics*, Proceedings of the 14th Course of the International

School of Subnuclear Physics, Erice, 1975, edited by A. Zichichi (Plenum, New York, 1977).

[24]See in this connection, K. Cahill, Indiana University Report No. IUHET 34, 1978 (unpublished). In electrodynamics, local charge density is gauge invariant and hence observable. It is also energetically possible to separate positive and negative charges far apart in (3+1)-dimensional space-time. Thus in (3+1)-dimensional electrodynamics, we can sensibly talk of observing local charges. We can also approximate localized charged systems by neutral systems arbitrarily well by separating positive and negative charges. [Cf. S. Doplicher, R. Haag, and J. E. Roberts, Commun. Math. Phys. 23, 199 (1971).] It follows that absence of gauge conditions at infinity does not imply electric charge confinement. In non-Abelian theories, local charge density is gauge variant and not an observable. Hence the situation is not clear.

[25]Cf. Jackiw and Rebbi (Ref. 8), Callan *et al.* (Ref. 6), Coleman (Ref. 10).

[26]P. A. M. Dirac, Can. J. Phys. 33, 650 (1955); I. Bialynicki-Birula, J. Math. Phys. 3, 1094 (1962); S. Mandelstam, Ann. Phys. (N.Y.) 19, 1 (1962).

[27]For the Lagrangian $\mathcal{L}^{(2)}$ [Eq. (2.33)], for example, the vacuum degeneracy can be larger (see Sec. IV A2). We ignore this possibility in this section.

[28]There are of course no Goldstone bosons in (1+1)-dimensional space-time. See S. Coleman, Commun. Math. Phys. 31, 259 (1973).

[29]Cf. C. G. Callan, R. Dashen, and D. Gross, Phys. Lett. 63B, 334 (1976); Jackiw and Rebbi (Ref. 8); Coleman (Ref. 10).

[30]The Lagrangian $\mathcal{L}^{(2)}$ in terms of ψ_α is $\mathcal{L}^{(2)}=\frac{1}{4}[(\partial_\mu\psi_\alpha\partial_\nu\psi_\alpha)^2-(\partial_\mu\psi_\alpha\partial_\mu\psi_\alpha)^2]$. This may be compared with the expression of de Alfaro *et al.* (Ref. 12). The following is a direct method to derive this form of $\mathcal{L}^{(2)}$. Use the gauge (4.24) for g and the definition (4.34) to find

$$\mathcal{L}^{(2)}=\frac{1}{16}\mathrm{Tr}[\tau_\alpha^\dagger\tau_\beta\tau_\rho^\dagger\tau_\sigma]\{\partial_\mu\psi_\alpha\partial_\nu\psi_\beta\partial_\mu\psi_\rho\partial_\nu\psi_\sigma - \partial_\mu\psi_\alpha\partial_\nu\psi_\beta\partial_\nu\psi_\rho\partial_\mu\psi_\sigma\}.$$

Straightforward simplification leads to the answer.