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Superconducting extended objects and applications to the phase structure of quantum chromodynamics

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In a previous work the dynamics of relativistic extended objects (i.e., strings, shells, etc.) coupled to Abelian or non-Abelian gauge fields was developed. The extended objects possessed an electriclike current which was defined in the associated Lie algebra of the gauge group under consideration. In the present paper, the interaction between the extended objects and gauge fields is slightly modified so that the objects behave like superconductors. By this we mean (a) the electrical conductivity is infinite and (b) for objects other than strings, a magnetic shielding or Meissner effect (with zero penetration depth) is present. Both (a) and (b) are features which occur in the classical description of the system. We also develop the dynamics for a system which is dual to the one described above. That is, instead of possessing an electric current, the objects here carry a magnetic current (Abelian or non-Abelian). Furthermore, the magnetic conductivity is infinite, and for objects other than strings an electric shielding or "dual" Meissner effect is present. The systems developed here contain Dirac's extended electron model and the MIT bag model as special cases. The former coincides with the description of an electrically charged shell. In the latter, we verify that the dynamics of a cavity within a (magnetic) superconducting vacuum is identical to that of a glueball in the MIT bag. This agrees with the view that the true quantum-chromodynamic (QCD) vacuum may be in a magnetic superconducting phase, and that the "dual" Meissner effect may be relevant for the confinement question. We also examine the possibility of the QCD vacuum being in an electric (or conventional) superconducting phase and a mixed superconducting phase, and comment on the confinement question for these two cases.

I. INTRODUCTION

It has been speculated recently[1,2] that the vacuum in quantum chromodynamics (QCD) has features similar to that of a magnetic superconductor, whereby magnetic supercurrents build up an electric Meissner effect.[3] It was suggested by Mandelstam[4] that such a mechanism could be responsible for quark confinement. The physical picture of the corresponding vacuum is a "dual" version of a vacuum which exhibits magnetic flux confinement.[5]

In the phenomenological MIT bag model,[6] which is supposed to be a low-energy approximation of QCD,[7] one regards the physical QCD vacuum as a perfect dielectric substance.[8-10] Owing to the relativistic invariance of the theory one can regard the vacuum as a perfect paramagnet as well. Physical hadrons are then viewed as bubbles (or cavities) formed in the QCD vacuum. The physics is not very different from that of a liquid under constant pressure at the boiling point.[11]

In this paper we attempt to study some of the dynamical features of the above model. We begin by giving a general description of relativistic extended objects which behave as superconductors. The discussion may be applied to either Abelian or non-Abelian gauge theories, in a description of either (conventional) electric or magnetic superconductors. By relativistic extended objects we mean objects whose free dynamics is given by the generalized Nambu action[12]; examples are strings, surfaces (or membranes), and four-dimensional regions of space-time. To these objects we attach an electric or magnetic current density. The objects can be viewed as condensates formed out of electric or magnetic point charges. Superconductivity manifests itself in two properties: (a) infinite conductivity and (b) the Meissner effect (or the "dual" Meissner effect, in the case of magnetically charged objects). Property (a) appears for all extended objects, while property (b) appears only for shells and four-dimensional regions. In all cases, the classical equations of motion, as well as the corresponding action principle, will be given. The current densities are associated with dynamical quantities in the theory. Properties (a) and (b) are found to result from the *classical* equations of motion for the system.

In the case of electrically charged objects, the above model is a slight specialization of a previous work.[13] ("Superconducting" strings can also be discussed in terms of a Kaluza-Klein dimensional reduction. See Ref. 14 in this context.) There we developed the general theory of extended objects in interaction with a (non-Abelian) gauge field. In the models of Ref. 13 properties (a) and (b) were not necessarily present (with the exception of the Abelian string).

In the case of magnetically charged objects, our system corresponds to a generalization of Dirac's (1948) treatment of magnetic monopoles.[15] Dirac utilized dynamical (open) strings (although unphysical) in the description, and the magnetic charges appeared only at the end points of the strings. Here we show how a magnetic charge density can be introduced along the interior of the string as well as at the end points. The theory presented here is similar to that of Englert and Windey,[1] but we believe our description of the classical dynamics is more complete.

Two previous works are special cases of the systems presented here. They are (1) Dirac's extended electron model[16] and (2) the MIT bag model.[6] Example (1) coincides with the system of a superconducting shell interacting with an electromagnetic field. Example (2) coincides with the system of a cavity formed in a dual superconducting vacuum. The equations of motion of the cavity agree precisely with the MIT bag equations of a glueball. In Ref. 2 we have already shown this at the level of the Lagrangian. The gluon fields are trapped in the cavity because of the dual Meissner effect and are electrically confined. What we add to the conventional wisdom concerning the physical picture of confined gluons in the MIT bag model is the explicit appearance of magnetic currents at the surface of the cavity. The presence of these magnetic supercurrents will generate, through the dynamics described by our theory, the MIT bag boundary conditions.

Here we can also examine the dual versions of examples (1) and (2). The former represents an extended model for a magnetic charge, while the latter describes a bag with "magnetic" confinement of quarks and gluons. In the dual version of (2), the vacuum is in a (conventional) superconducting phase. In the language of 't Hooft,[17] it corresponds to being in the Higgs mode. The currents at the surface of the cavity are now electric in nature. They are seen to screen the color charge of the quarks and gluons inside the bag. With regard to example (2) we also examine the possibility of a "mixed" superconducting phase in QCD.

This paper is organized as follows: In Sec. II we review the equations of motion for an electrically charged object and discuss the superconductivity condition. The Lagrangian formulation is also given. In Sec. III we repeat the procedure for a magnetically charged object. We show that the electrically charged shell of Sec. II agrees with Dirac's extended electron in Sec. IV. The dual version will also be examined. The objects considered in Secs. II–IV are all closed (i.e., have no boundaries). Open extended objects of dimension four are considered in Sec. V. There we show the equivalence with the MIT bag model and examine the possibility of different superconducting phases in QCD.

II. ELECTRICALLY CHARGED EXTENDED OBJECTS

We begin by reviewing the equations of motion for an electrically charged extended object in the presence of an arbitrary gauge field, as was developed in Ref. 13.

A. Equations of motion

Here we shall assume the gauge group $\mathcal{G}$ to be given by a set of unitary matrices. The corresponding Hermitian generators will be denoted by $T(\alpha)$ ($\alpha = 1, 2, \cdots, N$). They are normalized by

$$\mathrm{Tr}\, T(\alpha) T(\beta) = \delta_{\alpha\beta}. \tag{2.1}$$

We define

$$\mathcal{A}_\mu(x) \equiv \mathcal{A}^\alpha_\mu(x) T(\alpha) \tag{2.2}$$

and

$$\begin{aligned}\mathcal{F}_{\mu\nu}(x) &\equiv \mathcal{F}^\alpha_{\mu\nu}(x) T(\alpha) \\ &= \partial_\mu \mathcal{A}_\nu(x) - \partial_\nu \mathcal{A}_\mu(x) - ie[\mathcal{A}_\mu(x), \mathcal{A}_\nu(x)],\end{aligned} \tag{2.3}$$

where $\mathcal{A}^\alpha_\mu(x)$ and $\mathcal{F}^\alpha_{\mu\nu}(x)$ are the Yang-Mills potentials and field strengths, respectively. The coupling of a relativistic point particle to a gauge theory is given by the Wong equations[18]:

$$\partial_0 \frac{\delta L_0}{\delta(\partial_0 x^\mu)} = -e\, \mathrm{Tr}\mathcal{F}_{\mu\nu}(z) J^0(\sigma^0) \partial_0 z^\mu, \tag{2.4}$$

$$\mathcal{D}_\mu \mathcal{F}^{\mu\nu}(x) = e \int d\sigma^0 \delta^4(x - z(\sigma^0)) J^0(\sigma^0) \partial_0 z^\nu, \tag{2.5}$$

where z^μ and J^0 correspond to the position and "isospin" of the particle. They are both functions of the time parameter σ^0, although Eqs. (2.4) and (2.5) do not depend on the explicit choice of parametrization used. J^0 actually is a vector in the internal space, i.e.,

$$J^0 = J^0_\alpha T(\alpha), \tag{2.6}$$

and transforms under the adjoint action of the gauge group. Furthermore, L_0 is the free-particle Lagrangian

$$L_0 = -m(-\partial_0 z^\mu \partial_0 z_\mu)^{1/2}, \quad \partial_0 \equiv \partial/\partial\sigma^0$$

and $\mathcal{D}_\mu$ is the covariant derivative

$$\mathcal{D}_\mu \equiv \frac{\partial}{\partial x^\mu} - ie[\mathcal{A}_\mu(x),\ ]. \tag{2.7}$$

We now imagine the situation where point particles condense to form an n-dimensional extended object Σ ($n = 2$ and 3 define a string and shell, respectively).[19] The coordinates will once again be denoted by $z_\mu(\sigma)$, where σ now corresponds to a set of n parameters $\sigma=(\sigma^0,\sigma^1,\ldots,\sigma^{n-1})$. The dynamics for the free system is given by the generalized Nambu action[12]

$$\mathcal{S}_0=\int_\Sigma d^n\sigma\,\mathcal{L}_0\,,\quad \mathcal{L}_0=-\Upsilon(-\det g)^{1/2}\,,$$
$$g_{ab}=\partial_a z^\mu\partial_b z_\mu\,,\quad a,b=0,1,\ldots,n-1. \tag{2.8}$$

Here $\partial_a=\partial/\partial\sigma^a$ and Υ is a constant associated with the "surface tension" of the condensate. The isospin J^0 of the particle is now replaced by an "isocurrent"

$$J^a=J^a_\alpha T(\alpha),\quad a=0,1,\ldots,\ n-1\,. \tag{2.9}$$

It is a function of the set of parameters σ. In the case where $\mathcal{G}=\mathrm{U}(1)$, J^a (actually eJ^a) can be interpreted as the electric current flowing on the extended object.

In analogy to (2.4) and (2.5), the following equations of motion were postulated for this sytem[13]:

$$\partial_a\frac{\delta\mathcal{L}_0}{\delta\partial_a z^\mu}=-e\,\mathrm{Tr}\mathcal{F}_{\mu\nu}(z)J^a(\sigma)\,\partial_a z^\nu\,, \tag{2.10}$$

$$\mathcal{D}_\mu\mathcal{F}^{\mu\nu}(x)=e\int_\Sigma d^n\sigma\,\delta^4(x-z(\sigma))J^a(\sigma)\partial_a z^\nu\,. \tag{2.11}$$

In addition to (2.10) and (2.11) it was found necessary to impose the following two equations:

$$\partial_a J^a-ie[\mathcal{A}_\nu(z),J^a]\partial_a z^\nu=0\,, \tag{2.12}$$

$$\mathrm{Tr}F_{ab}J^b=0\,, \tag{2.13}$$

where

$$F_{ab}\equiv\mathcal{F}_{\mu\nu}(z)\partial_a z^\mu\partial_b z^\nu\,. \tag{2.14}$$

Equation (2.12) states that the current is (covariantly) conserved for (non-) Abelian gauge theories. It follows as a consistency condition from the identity $\mathcal{D}_\mu\mathcal{D}_\nu\mathcal{F}^{\mu\nu}=0$. Equation (2.13) implies that static objects do not radiate energy (cf. Ref. 20). Equation (2.13) follows from the reparametrization invariance of the free action (2.8). For the purposes of this paper we will consider a stronger version of (2.13), namely,

$$F_{ab}=0\,. \tag{2.15}$$

To give a physical interpretation of (2.15) let us first specialize to strings. If we make the choice $z^0=\sigma^0$, (2.15) can be written

$$F_{01}=\mathcal{F}_{0i}(z)\partial_1 z^i+\mathcal{F}_{ij}(z)\partial_0 z^i\partial_1 z^j=0\,. \tag{2.16}$$

It follows that in the rest frame of any infinitesimal segment of the string the tangential electric field vanishes. Since J^1 is in general nonzero, the electrical conductivity is infinite. For shells, in addition to (2.16) we have

$$F_{02}(\sigma)=\mathcal{F}_{0i}(z)\partial_2 z^i+\mathcal{F}_{ij}(z)\partial_0 z^i\partial_2 z^j=0\,, \tag{2.17}$$

$$F_{12}(\sigma)=\mathcal{F}_{ij}(z)\partial_1 z^i\partial_2 z^j=0\,, \tag{2.18}$$

where we have again chosen $z^0=\sigma^0$. Now Eqs. (2.16) and (2.17) imply that in the rest frame of any segment of the shell there is no tangential electric field. Thus again the object has an infinite conductivity. In addition, in order to satisfy Eq. (2.18) the magnetic field normal to the shell surface must vanish. Thus no magnetic flux can penetrate the shell. This is analogous to the Meissner effect for superconductors. (Here, of course, there is no penetration depth.) To conclude, (2.15) implies that the extended object is a perfect conductor or, loosely speaking, a superconductor. Furthermore, if there are closed paths C_i in Σ which are not deformable to a point, then (2.15) implies the existence of certain conserved quantities. These quantities are represented by the Wilson loop operators

$$W(C_i)=\mathrm{Tr}P\exp\left(ie\int_{C_i}d\sigma^a A_\mu(z)\partial_a z^\mu\right)\,, \tag{2.19}$$

where P stands for the usual path ordering.

B. Lagrangian formalism

We now indicate how to obtain Eqs. (2.10)–(2.12) and (2.15) from an action principle. For the total action we take

$$\mathcal{S}=\mathcal{S}_0+\mathcal{S}_\mathcal{F}+\mathcal{S}_I\,, \tag{2.20}$$

where $\mathcal{S}_0$ is the generalized Nambu action (2.8), $\mathcal{S}_\mathcal{F}$ is the free-field action

$$\mathcal{S}_\mathcal{F}=-\tfrac{1}{4}\int d^4x\,\mathrm{Tr}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\,, \tag{2.21}$$

and $\mathcal{S}_I$ gives the interaction

$$\mathcal{S}_I=\int_\Sigma d^n\sigma\,\mathcal{L}_I\,. \tag{2.22}$$

As mentioned in Ref. 13 there is no unique interaction Lagrangian for the system. Here we choose[21]

$$\mathcal{L}_I=\frac{e}{2}\,\mathrm{Tr}\chi^{ab}F_{ab}\,, \tag{2.23}$$

where F_{ab} is defined in (2.14). The variable χ^{ab} is antisymmetric in a and b. It is defined by

$$\chi^{ab}=\epsilon^{ab}u\lambda u^\dagger\quad(n=2) \tag{2.24a}$$

for the string and

$$\chi^{ab}=\epsilon^{abc}u\lambda_c u^\dagger\quad(n=3) \tag{2.24b}$$

for the shell. Here u, λ, and λ_c are new dynamical variables, out of which the current J^a will be constructed. λ and λ_c take values in the Lie algebra associated to $\mathcal{G}$; i.e., $\lambda=\lambda^\alpha T(\alpha)$ and $\lambda_c=\lambda_c^\alpha T(\alpha)$. u is an element of $\mathcal{G}$. It transforms under the left action of the gauge group. Consequently, (2.23) is gauge invariant.

We now discuss the variational problem.

Variation of $\mathcal{A}_\nu$. By varying $\mathcal{A}_\nu$ in (2.23), we find

$$\delta\mathcal{L}_I = -e\,\mathrm{Tr} D_a \chi^{ab} \delta\mathcal{A}_\nu(z)\partial_b z^\nu , \tag{2.25}$$

where

$$D_a \equiv \partial_a - ie[\mathcal{A}_\mu(z),\ ]\partial_a z^\mu . \tag{2.26}$$

It follows that

$$\mathcal{D}_\mu \mathcal{F}^{\mu\nu}(x) = e\int_\Sigma d^n\sigma\, \delta^4(x - z(\sigma))\, D_a \chi^{ab}\partial_b z^\nu . \tag{2.27}$$

If we now make the identification

$$J^a = D_b \chi^{ba} , \tag{2.28}$$

(2.27) is identical to the field equation (2.11).

Variation of u. The most general variations of u are of the form

$$\delta u = i\epsilon_\alpha T(\alpha) u , \tag{2.29}$$

where ϵ_α corresponds to N real infinitesimal quantities. Also

$$\delta u^\dagger = -iu^\dagger \epsilon_\alpha T(\alpha) . \tag{2.30}$$

Substituting into (2.23), we find

$$\delta\mathcal{L}_I = \frac{i}{2} e\,\mathrm{Tr}\epsilon_\alpha T(\alpha)[\chi^{ab}, F_{ab}] . \tag{2.31}$$

Linear independence of the $T(\alpha)$'s then implies

$$[\chi^{ab}, F_{ab}] = 0 . \tag{2.32}$$

Note that (2.32) is equivalent to (2.12). This follows by taking the covariant divergence of (2.28),

$$D_a J^a = D_a D_b \chi^{ba} \equiv -ie[F_{ab}, \chi^{ab}] . \tag{2.33}$$

Variation of z^ν. Variations of z^ν in (2.23) lead to

$$\delta\mathcal{L}_I = e\,\mathrm{Tr}\chi^{ab}[\mathcal{F}_{\mu\nu}(z)\partial_a z^\mu \partial_b \delta z^\nu + \tfrac{1}{2}\partial_\nu \mathcal{F}_{\rho\mu}(z)\partial_a z^\rho \partial_b z^\mu \delta z^\nu] . \tag{2.34}$$

After an integration by parts and an application on the Bianchi identity, (2.34) can be simplified to

$$\delta\mathcal{L}_I = e\,\mathrm{Tr} D_a \chi^{ab} \mathcal{F}_{\mu\nu}(z)\partial_b z^\mu \delta z^\nu , \tag{2.35}$$

where we have also used (2.32). Minimizing the total action then leads to the equation of motion (2.10).

Variation of λ (λ_c). Upon considering the variations $\delta\lambda = \delta\lambda^\alpha T(\alpha)$ $[\delta\lambda_c = \delta\lambda_c^\alpha T(\alpha)]$ in (2.23), we find

$$u F_{ab} u^\dagger = 0 , \tag{2.36}$$

which is identical to the condition (2.15). The weaker condition (2.13) can be obtained by just allowing for the minimal set of variations in λ and λ_c consistent with the reparametrization symmetry in $\mathcal{S}_I$. Under a reparametrization $\sigma \to f(\sigma)$, λ is a scalar and λ_c transforms as the components of a one-form.

III. MAGNETICALLY CHARGED EXTENED OBJECTS

A. Equations of motion

Here we wish to rewrite the equations of motion of Sec. II to describe the dynamics of extended objects with a magnetic charge.[7] We begin with the Abelian case.

1. Abelian case

We now endow the extended object Σ with a magnetic current K^a. We replace J^a and the electromagnetic field $\mathcal{F}_{\mu\nu}$ by K^a and a dual field $^*\mathcal{G}_{\mu\nu}$, respectively, in (2.10) and (2.11) and obtain the following equations for this system:

$$\partial_a \frac{\delta\mathcal{L}_0}{\delta\partial_a z^\mu} = -g\, {}^*\mathcal{G}_{\mu\nu}(z) K^a \partial_a z^\nu , \tag{3.1}$$

$$\partial_\mu {}^*\mathcal{G}^{\mu\nu}(x) = g\int_\Sigma d^n\sigma\, \delta^4(x - z(\sigma)) K^a \partial_a z^\nu , \tag{3.2}$$

where g is a unit magnetic charge. As is well known, we cannot set $^*\mathcal{G}^{\mu\nu} = {}^*\mathcal{F}^{\mu\nu}$, where $^*\mathcal{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\rho\sigma}$, since the Bianchi identity would imply that $K^a = 0$. Instead we can apply a trick similar to that used by Dirac.[15] Let

$$\begin{aligned} {}^*\mathcal{G}^{\mu\nu}(x) &= {}^*\mathcal{F}^{\mu\nu}(x) - M^{\mu\nu}(x) , \\ M^{\mu\nu}(x) &= -g\int_\Sigma d^n\sigma\, \delta^4(x - z(\sigma))\chi^{ab}\partial_a z^\mu \partial_b z^\nu , \end{aligned} \tag{3.3}$$

where χ^{ab} is again antisymmetric in the indices a and b. Unlike Dirac, χ^{ab} will not be treated as a constant, but rather as a dynamical variable. Now from the Bianchi identity

$$\partial_\mu {}^*\mathcal{G}^{\mu\nu}(x) = -\partial_\mu M^{\mu\nu}(x) . \tag{3.4}$$

After an integration by parts the right-hand side becomes

$$g\int d^n\sigma\, \delta^4(x - z(\sigma))\partial_a \chi^{ab}(\sigma)\partial_b z^\nu , \tag{3.5}$$

where we have assumed that the extended object is closed. If we make the identification

$$K^a(\sigma) = \partial_b \chi^{ba}(\sigma) , \tag{3.6}$$

then Eq. (3.2) becomes an identity. From (3.6) it follows that

$$\partial_a K^a = 0 , \tag{3.7}$$

which is the analog of the Abelian version of (2.12). Along with Eqs. (3.1), (3.2), and (3.7) we shall require that

$$\partial_\mu \mathcal{G}^{\mu\nu} = 0\,, \tag{3.8}$$

which states that no electric sources are present, and

$$*G_{ab}K^b = 0\,, \quad *G_{ab} \equiv *\mathcal{G}_{\mu\nu}(z)\partial_a z^\mu \partial_b z^\nu\,. \tag{3.9}$$

Equation (3.9) is the analog of the Abelian version of (2.13). It can again be shown to follow as a consistency condition (cf. Ref. 13).

Upon considering open extended objects, additional terms would have to be included in (3.5). In the case of strings, these terms would be of the form

$$\begin{aligned} &+g\int d\sigma^0 \delta^4(x - z(\sigma))\chi^{01}(\sigma)\partial_0 z^\nu\,|_{\sigma^1\to 2\pi}\,, \\ &-g\int d\sigma^0 \delta^4(x - z(\sigma))\chi^{01}(\sigma)\partial_0 z^\nu\,|_{\sigma^1\to 0}\,, \end{aligned} \tag{3.10}$$

where $0 \leq \sigma^1 \leq 2\pi$. Consequently, two magnetic point sources are present at the end points $\sigma^1 = 0$ and 2π, in addition to the continuous distribution along the string.

The above is a slight modification of Dirac's work. In the latter, the strings were utilized in a description of magnetic monopoles. The monopole sources were located at the end points of the strings, as in (3.10). However, for Dirac the string interior was neutral, i.e., $K^a = 0$. This follows from the restriction $\chi^{ab} = \epsilon^{ab}$ which was imposed in Dirac's formulation.

2. *Non-Abelian case*

We now generalize the equations of the previous section to non-Abelian theories. The Yang-Mills field strengths $\mathcal{F}_{\mu\nu}$ associated with gauge group $\mathcal{G}$ are defined in (2.3). Again we define the dual field $*\mathcal{G}^{\mu\nu}$ and $M^{\mu\nu}$ according to (3.3). Only here the quantity $\chi^{ab}(\sigma)$ is Lie algebra-valued. Upon computing the covariant divergence of $*\mathcal{G}^{\mu\nu}(x)$ and applying the Bianchi identity we obtain

$$\mathfrak{D}_\mu *\mathcal{G}^{\mu\nu}(x) = -\mathfrak{D}_\mu M^{\mu\nu}(x)\,. \tag{3.11}$$

For closed extended objects this reduces to

$$\mathfrak{D}_\mu *\mathcal{G}^{\mu\nu}(x) = g\int_\Sigma d^n\sigma\, \delta^4(x - z(\sigma))K^a(\sigma)\partial_a z^\nu\,, \tag{3.12}$$

where

$$K^a(\sigma) = D_b\chi^{ba}(\sigma)\,. \tag{3.13}$$

Note here the current is not conserved (not even covariantly). Applying the identity $D_aD_b = -(ie/2)[F_{ab}(z),\ ]$, we find

$$D_aK^a(\sigma) = -\frac{ie}{2}[F_{ab}(z), \chi^{ab}(\sigma)]\,. \tag{3.14}$$

In addition to (3.12) we shall require that

$$\mathfrak{D}_\mu \mathcal{G}^{\mu\nu}(x) = 0\,, \tag{3.15}$$

which is the non-Abelian version of (3.8). Unlike (3.8), Eq. (3.15) leads to an additional consistency condition. From $\mathfrak{D}_\mu\mathfrak{D}_\nu \mathcal{G}^{\mu\nu} = 0$, it follows that

$$[\mathcal{F}_{\mu\nu}, \mathcal{G}^{\mu\nu}] = 0 \tag{3.16}$$

or

$$[\mathcal{F}_{\mu\nu}, *M^{\mu\nu}] = 0\,. \tag{3.17}$$

The latter equation is equivalent to

$$[*F_{ab}(z), \chi^{ab}(\sigma)] = 0\,, \quad *F_{ab} = *\mathcal{F}_{\mu\nu}\partial_a z^\mu \partial_b z^\nu\,, \tag{3.18}$$

which is to be compared with (2.32) of the previous section.

The non-Abelian generalization of the Lorentz force Eq. (3.1) is

$$\partial_a \frac{\delta \mathcal{L}_0}{\delta(\partial_a z^\mu)} = -g\,\mathrm{Tr}\, *\mathcal{G}_{\mu\nu}(z)K^a(\sigma)\partial_a z^\nu\,. \tag{3.19}$$

The generalization of the consistency condition (3.9) is

$$\mathrm{Tr}\, *G_{ab}(z)K^b(\sigma) = 0\,. \tag{3.20}$$

In the discussion of electrically charged objects it was possible to strengthen the analogous condition via a suitable action principle. A similar possibility exists in the case of magnetically charged objects. In the next section we discuss a Lagrangian formulation of the system which yields

$$*G_{ab}(z) = 0\,. \tag{3.21}$$

Extended objects satisfying (3.21) will exhibit the gross features of dual superconductors. For a string (3.21) implies that in the rest frame of any infinitesimal segment of the string there is no tangential magnetic field. Since the magnetic current need not be zero, the magnetic conductivity is infinite. For a shell we have, in addition, the property that the electric field normal to any time slice of the surface vanishes. Thus electric fields cannot penetrate the shell. Consequently, we have a dual Meissner effect.

B. Lagrangian formalism

We shall proceed directly with the non-Abelian case. For the total action we take

$$\mathcal{S} = \mathcal{S}_\mathcal{G} + \mathcal{S}_0\,. \tag{3.22}$$

Here $\mathcal{S}_0$ is given by (2.8) and the field contribution $\mathcal{S}_\mathcal{G}$ is of the form

$$\mathcal{S} = -\tfrac{1}{4}\int dx^4\, \mathrm{Tr}\,\mathcal{G}_{\mu\nu}\mathcal{G}^{\mu\nu}\,, \tag{3.23}$$

where the field tensor $\mathcal{G}^{\mu\nu}$ is defined in (3.3). We shall define the quantity $\chi^{ab}(\sigma)$ according to (2.24). As in Sec. II the dynamical variables are $\mathcal{A}^{\mu}$, u, z^{μ}, and λ (λ_c). We now verify that their variation in (3.22) leads to the equations of motion of subsection A.

Variations of $\mathcal{A}^{\mu}$. From (3.22)

$$\delta S_{\mathcal{G}} = -\int d^4x \, \mathrm{Tr}\mathcal{G}^{\mu\nu}\mathfrak{D}_{\mu}\,\delta\mathcal{A}_{\nu}\,. \tag{3.24}$$

Consequently, we obtain the field equation (3.15).

Variations of u. Variations are of the form (2.29). They lead to

$$\delta S_{\mathcal{G}} = -\tfrac{1}{2}\int d^4x \, \mathrm{Tr}\, {}^*\mathcal{G}_{\mu\nu}(x)\delta M^{\mu\nu}(x)$$
$$= \frac{ig}{2}\int d^4\sigma \, \mathrm{Tr}\, {}^*\mathcal{G}_{\mu\nu}(z)[\epsilon_{\alpha}T(\alpha), \chi^{ab}]\partial_a z^{\mu}\partial_b z^{\nu}\,. \tag{3.25}$$

Consequently,

$$[{}^*G_{ab}(z), \chi^{ab}] = 0\,, \tag{3.26}$$

which is equivalent to (3.18).

Variations of z^{μ}. Variations in $S_{\mathcal{G}}$ are

$$\delta S_{\mathcal{G}} = -\tfrac{1}{2}\int d^4x \, \mathrm{Tr}\, {}^*\mathcal{G}_{\mu\nu}(x)\delta M^{\mu\nu}(x)$$
$$= g\int d^n\sigma \, \mathrm{Tr}\chi^{ab}\partial_b z^{\nu}\left[\frac{1}{2}\frac{\partial\, {}^*\mathcal{G}_{\rho\nu}(z)}{\partial z^{\mu}}\partial_a z^{\rho}\delta z^{\mu} + {}^*\mathcal{G}_{\mu\nu}(z)\partial_a\delta z^{\mu}\right] \tag{3.27}$$

Integrating by parts and using (3.26), we obtain

$$\delta S_{\mathcal{G}} = g\int d^n\sigma \, \mathrm{Tr}\big[\tfrac{1}{2}\chi^{ab}\mathfrak{D}_{\mu}\, {}^*\mathcal{G}_{\rho\nu}(z)\partial_a z^{\rho} - \chi^{ab}\mathfrak{D}_{\rho}\, {}^*\mathcal{G}_{\mu\nu}(z)\partial_a z^{\rho} - K^b\, {}^*\mathcal{G}_{\mu\nu}(z)\big]\partial_b z^{\nu}\,\delta z^{\mu}\,, \tag{3.28}$$

where we have assumed that the extended object is closed. The field equation (3.15) can be written in the form

$$\mathfrak{D}_{\rho}\, {}^*\mathcal{G}_{\mu\nu} + \mathfrak{D}_{\mu}\, {}^*\mathcal{G}_{\nu\rho} + \mathfrak{D}_{\nu}\, {}^*\mathcal{G}_{\rho\mu} = 0\,. \tag{3.29}$$

It follows that the first two terms in brackets in Eq. (3.28) cancel. We thus find the equation of motion (3.19).

Variations of $\lambda(\lambda_c)$. Full variations lead to

$$\delta S_{\mathcal{G}} = -\tfrac{1}{2}g\int d^4x \, \mathrm{Tr}\, {}^*\mathcal{G}_{\mu\nu}(x)\delta M^{\mu\nu}(x)$$
$$= \tfrac{1}{2}g\int_{\Sigma} d^n\sigma \, \mathrm{Tr}\, {}^*G_{ab}\delta\chi^{ab}\,. \tag{3.30}$$

Upon minimizing the action

$$u^{\dagger}\, {}^*G_{ab}u = 0\,, \tag{3.31}$$

which is identical to (3.21). The weaker condition (3.20) can once again be obtained by just allowing for the minimal set of variations in λ (λ_c) consistent with the reparametrization symmetry.

IV. THE CLASSICAL SELF-ENERGY PROBLEM AND DIRAC EXTENDED ELECTRON MODEL

As is well known, there are no finite-energy solutions to both the field equations and Lorentz force equation for a charged point particle. The same situation exists for charged strings (here, however, the divergence becomes less severe). On the other hand, a self-consistent solution can be found for charged shells. In the case of electromagnetism, such a solution was found by Dirac.[16] This solution was interpreted by Dirac to be a possible extended model for the electron. It was hoped that the muon could be viewed as an excited state of this solution. A semiclassical analysis, however, failed to support this view.

Although Dirac's extended electron model is not a realistic description of charged leptons, it is of theoretical interest in that it is a prototype of relativistic extended models. Here we show that Dirac's system is identical to the system described in Sec. II when we specialize to shells in an electromagnetic field. Thus the Dirac extended electron is a perfect conductor. We also indicate how to generalize Dirac's model to describe a non-Abelian charged shell and an extended model for a magnetic monopole.

A. Dirac's extended electron

Dirac's system is given by the following three equations: (a) the free-field equation away from the shell surface, (b) $F_{ab}(\sigma) = 0$ on the surface, and (c) the shell equation of motion:

$$\Upsilon\partial_a n_{\mu} g^{-1}{}_{ab}\partial_b z^{\mu} = \tfrac{1}{4}\mathfrak{F}_{\mu\nu}(y)\,\mathfrak{F}^{\mu\nu}(y)\big|_{\sigma^3\to 0^+}\,, \tag{4.1}$$

where $\mathfrak{F}_{\mu\nu}$ is the electromagnetic field, and g_{ab} is defined in (2.8). Here $n_{\mu}(\sigma)$ is a unit spacelike vector normal to the three-surface:

$$n_{\mu}(\sigma) = \frac{1}{3!}(-\det g)^{-1/2}\epsilon_{\mu\nu\rho\sigma}\epsilon^{abc}\partial_a z^{\nu}\partial_b z^{\rho}\partial_c z^{\sigma}\,, \tag{4.2}$$

and the coordinates y^{μ} are defined by

$$y^{\mu}(\sigma, \sigma^3) = z^{\mu}(\sigma) + \sigma^3 n^{\mu}(\sigma)\,. \tag{4.3}$$

The variable σ^3 parametrizes small distances away from the surface. Note that $z^{\mu}(\sigma) = y^{\mu}(\sigma, 0)$.

The above system, at first glance, differs from ours in that (4.1) contains no explicit dependence on the electric current $J^a(\sigma)$. In fact, Dirac's system contains no reference at all to $J^a(\sigma)$. This difference can be understood by the fact that the currents on a shell are constrained by certain components of the fields near the three-surface. This simply follows from Gauss's law (cf. the Appendix). Consequently, when $n=3$ we should be able to eliminate J^a from our equation of motion (2.10).

In the Appendix we prove the following relations:

$$0=n_\mu(\sigma)[\,{}^*\mathcal{F}^{\mu\nu}(y)\,|_{\sigma^3\to 0^+}-{}^*\mathcal{F}^{\mu\nu}(y)\,|_{\sigma^3\to 0^-}]\,, \tag{4.4a}$$

$$eJ^a(\sigma)\partial_a z^\nu = (-\det g)^{1/2} n_\mu(\sigma)[\mathcal{F}^{\mu\nu}(y)\,|_{\sigma^3\to 0^+}-\mathcal{F}^{\mu\nu}(y)\,|_{\sigma^3\to 0^-}]\,. \tag{4.4b}$$

Equation (4.4a) states that no magnetic charge is present on the surface, while (4.4b) gives the relation between the electric currents and $\mathcal{F}^{\mu\nu}$ at the surface. From (4.4b), $\mathcal{F}^{\mu\nu}(y)$ is discontinuous at the shell surface. Consequently, the quantity $\mathcal{F}^{\mu\nu}(z)$ [appearing in (2.10), (2.13), and (2.15)] is ambiguous. To correct this situation we attach the following definition to the shell system:

$$\mathcal{F}^{\mu\nu}(z)\equiv\tfrac{1}{2}[\mathcal{F}^{\mu\nu}(y)\,|_{\sigma^3\to 0^+}+\mathcal{F}^{\mu\nu}(y)\,|_{\sigma^3\to 0^-}]\,. \tag{4.5}$$

Substituting (4.5) into (2.10), we find

$$-\Upsilon\partial_a[(-\det g)^{1/2}g^{-1}{}_{ab}\partial_b z_\mu] = \frac{e}{2}[\mathcal{F}_{\mu\nu}(y)\,|_{\sigma^3\to 0^+}+\mathcal{F}_{\mu\nu}(y)\,|_{\sigma^3\to 0^-}]J^a(\sigma)\partial_a z^\nu\,. \tag{4.6}$$

In Ref. 13 it was shown that when contracted with a tangent vector $\partial_b z^\mu$, the left-hand side of (4.6) vanishes identically. We can therefore project (4.6) along the normal direction $n^\mu(\sigma)$ without any loss in generality. In so doing we obtain

$$-\Upsilon n^\mu(\sigma)\partial_a[(-\det g)^{1/2}g^{-1}{}_{ab}\partial_b z_\mu] = \frac{e}{2}[f_\nu(y)\,|_{\sigma^3\to 0^+}+f_\nu(y)\,|_{\sigma^3\to 0^-}]J^a(\sigma)\partial_a z^\nu\,, \tag{4.7}$$

where

$$f_\nu(y)\equiv\mathcal{F}_{\mu\nu}(y)n^\mu(\sigma)\,. \tag{4.8}$$

Now applying (4.4), we find

$$\Upsilon\partial_a n^\mu g^{-1}{}_{ab}\partial_b z_\mu = \tfrac{1}{2}f^2(y)\,|_{\sigma^3\to 0^+}-\tfrac{1}{2}f^2(y)\,|_{\sigma^3\to 0^-}\,. \tag{4.9}$$

Here we have used (4.2) to simplify the left-hand side of (4.7). Now the current no longer appears in the shell equation of motion. To show that (4.9) agrees with Dirac's equation (4.1) we need to apply the condition $F_{ab}=0$. It may be written

$$n_\mu[\,{}^*\mathcal{F}^{\mu\nu}(y)\,|_{\sigma^3\to 0^+}+{}^*\mathcal{F}^{\mu\nu}(y)\,|_{\sigma^3\to 0^-}]=0\,. \tag{4.10}$$

From (4.4a),

$$n_\mu\,{}^*\mathcal{F}^{\mu\nu}(y)\,|_{\sigma^3\to 0^\pm}=0\,. \tag{4.11}$$

Consequently,

$$f^2(y)=\tfrac{1}{2}\mathcal{F}^2(y) \tag{4.12}$$

(where $\mathcal{F}^2\equiv\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$) and the right-hand side of (4.9) becomes

$$\tfrac{1}{4}[\mathcal{F}^2(y)\,|_{\sigma^3\to 0^+}-\mathcal{F}^2(y)\,|_{\sigma^3\to 0^-}]\,, \tag{4.13}$$

which completes the proof [Dirac sets $\mathcal{F}^{\mu\nu}(y)\,|_{\sigma^3\to 0^-}=0$].

Thus we have shown that for the case of electromagnetism and $n=3$, $J^a(\sigma)$ can be eliminated from Eq. (2.10) and the result is Dirac's extended electron model. In terms of the currents $J^a(\sigma)$ his spherically symmetric solution can be written

$$z=(\sigma^0, R\sin\sigma^1\cos\sigma^2, R\sin\sigma^1\sin\sigma^2, R\cos\sigma^1)\,,$$
$$J=\left(\frac{Q}{4\pi e}\sin\sigma^1, 0, 0\right),\quad 0\leq\sigma^1\leq\pi\,,\quad 0\leq\sigma^2<2\pi\,. \tag{4.14}$$

Equation (4.14) represents a static spherical bubble of radius R with total charge Q. From the field equation (2.11), R must satisfy the relation

$$\Upsilon R^3=\left(\frac{Q}{8\pi}\right)^2. \tag{4.15}$$

Note that for any closed-shell solution the net magnetic charge inside the shell must be zero. This follows since magnetic fields cannot penetrate the surface (cf. Sec. II). Thus the extended electron model provides a mechanism for magnetic charge confinement.

B. Non-Abelian generalization

It is straightforward to generalize the preceding model to non-Abelian gauge theories. As shown in Appendix Eqs. (4.4a) and (4.4b) hold in both the Abelian and non-Abelian cases. All that is required in generalizing to arbitrary gauge fields is that we replace (4.13) by

$$\tfrac{1}{4}\,\mathrm{Tr}[\mathcal{F}^2(y)\,|_{\sigma^3\to 0^+}-\mathcal{F}^2(y)\,|_{\sigma^3\to 0^-}]\,. \tag{4.13$'$}$$

C. Magnetic analog

In describing a magnetically charged surface we replace $\mathcal{F}^{\mu\nu}$ and J^a by ${}^*\mathcal{G}^{\mu\nu}$ and K^a, respectively, in (4.4). Consequently,

$$0=n_\mu(\sigma)[\mathcal{G}^{\mu\nu}(y)\,|_{\sigma^3\to 0^+}-\mathcal{G}^{\mu\nu}(y)\,|_{\sigma^3\to 0^-}]\,, \tag{4.16a}$$

$$gK^a(\sigma)\partial_a z^\nu=\frac{1}{3!}\epsilon_{\mu\eta\rho\sigma}\epsilon^{abc}\partial_a z^\eta\partial_b z^\rho\partial_c z^\sigma \times[\,{}^*\mathcal{G}^{\mu\nu}(y)\,|_{\sigma^3\to 0^+}-{}^*\mathcal{G}^{\mu\nu}(y)\,|_{\sigma^3\to 0^-}]\,, \tag{4.16b}$$

where we have used (4.2). Equation (4.16a) states that no electric charges are present on the surface, while Eq. (4.16b) gives a relation between the magnetic current and the fields $\mathcal{F}^{\mu\nu}$ ($\mathcal{G}^{\mu\nu}=\mathcal{F}^{\mu\nu}$, outside the surface) near the surface. Equation (4.16b) can be used to eliminate K^a from the equation of motion (3.19). Now using the definition

$$\mathcal{G}^{\mu\nu}(z)\equiv\tfrac{1}{2}[\mathcal{F}^{\mu\nu}(y)\,|_{\sigma^3\to 0^+}+\mathcal{F}^{\mu\nu}(y)\,|_{\sigma^3\to 0^-}] \tag{4.17}$$

in analogy to (4.5), we find

$$\Upsilon\partial_a n^\mu g^{-1}{}_{ab}\partial_b z_\mu = -\tfrac{1}{4}\,\mathrm{Tr}[\mathcal{F}^2(y)\,|_{\sigma^3\to 0^+}-\mathcal{F}^2(y)\,|_{\sigma^3\to 0^-}]\,. \tag{4.18}$$

Note that this equation of motion is identical to its electrical counterpart except for an overall sign. A more substantial difference between the electric and magnetic shell is seen in the boundary conditions. For the magnetic shell, we find

$$n_\mu\mathcal{F}^{\mu\nu}(y)\,|_{\sigma^3\to 0}=0\,, \tag{4.19}$$

contrary to (4.11). It implies that a closed magnetically charged shell shields electric non-Abelian charge. It resembles the color confinement condition in the MIT bag model.[6,2] We shall discuss this connection in detail in the following section.

V. FOUR-DIMENSIONAL CONDENSATES AND THE POSSIBLE PHASES OF QCD

In this section we generalize the previous considerations to study superconducting extended objects of dimension $n=4$. Connections to questions concerning the QCD vacuum and possible phase structure will be shown.

A. Generalization to four-dimensions

We first take up the generalizations of Eqs. (2.15) and (3.21) to four dimensions. Equations (2.15) and (3.21) were shown to give extended objects features resembling superconductors and dual superconductors, respectively. They have a simple interpretation in four dimensions. Assuming $\partial(z)/\partial(\sigma)\neq 0$, Eq. (2.15) [Eq. (3.21)] implies that all fields $\mathcal{F}_{\mu\nu}$ [$\mathcal{G}_{\mu\nu}$] vanish inside the object. From the respective field equations it follows that the currents J^a [K^a] must also vanish inside the extended object.

In the derivation of the equations of motion in Secs. II and III it was assumed that the extended objects had no boundaries. Since this assumption is unrealistic when $n=4$, we need to rewrite the equations so as to include surface contributions.

We first reexamine the case of electrically charged objects.

Electrically charged objects. Let us assume the boundary $\partial\Sigma$ of the extended object to be given by the equation

$$\sigma^3=0\,. \tag{5.1}$$

(We also assume $\sigma^3<0$ inside the extended object.) The boundary can then be parametrized by

$$\boldsymbol{s}=(\boldsymbol{s}^0,\boldsymbol{s}^1,\boldsymbol{s}^2)=(\sigma^0,\sigma^1,\sigma^2)\,. \tag{5.2}$$

Upon varying $\mathcal{A}_\mu$ in the action (2.20) we find the following field equation:

$$\mathcal{D}_\mu\mathcal{F}^{\mu\nu}(x)=e\int_{\partial\Sigma}d^3\boldsymbol{s}\,\delta^4(x-z(\boldsymbol{s},0))\tilde{J}^i\partial_i z^\nu\,,\quad i=0,1,2\,. \tag{5.3}$$

In deriving (5.3) we have used the fact that the current J^a vanishes inside the extended object. $\tilde{J}^i$ corresponds to a current which exists solely on the boundary. In terms of the variables $\chi^{ab}(\sigma)$,[22]

$$\tilde{J}^i=\chi^{i3}(\boldsymbol{s},0)\,. \tag{5.4}$$

Next we consider variations of z^μ in (2.20). The Nambu free action $\mathcal{S}_0$ is associated with the volume of the extended object. For $n=4$, it can be written

$$\mathcal{S}_0=-\Upsilon\int_\Sigma\frac{\partial(z)}{\partial(\sigma)}d^4\sigma\,. \tag{5.5}$$

In Ref. 20 it was shown that $\mathcal{S}_0$ could be reduced to an integral on the boundary

$$\mathcal{S}_0=-\frac{\Upsilon}{4!}\epsilon_{\alpha\beta\gamma\delta}\epsilon^{ijk}\int_{\partial\Sigma}d^3\boldsymbol{s}\,z^\alpha\partial_i z^\beta\partial_j z^\gamma\partial_k z^\delta\,. \tag{5.6}$$

Here $z^\mu=z^\mu(\boldsymbol{s},0)$. Varying z^μ leads to the following equation of motion for the boundary:

$$\frac{\Upsilon}{3!}\epsilon_{\alpha\beta\gamma\delta}\epsilon^{ijk}\partial_i z^\beta\partial_j z^\gamma\partial_k z^\delta=e\,\mathrm{Tr}\,\mathcal{F}_{\mu\nu}(z)\tilde{J}^i\partial_i z^\mu\,. \tag{5.7}$$

Magnetically charged objects. In analogy to (5.3) and (5.7), the equations describing a magnetically charged four-dimensional object with the boundary (5.1) are

$$\mathcal{D}_\mu{}^*\mathcal{G}^{\mu\nu}(x)=g\int_{\partial\Sigma}d^3\boldsymbol{s}\,\delta^4(x-z(\boldsymbol{s},0))\tilde{K}^i\partial_i z^\nu\,, \tag{5.8}$$

$$\frac{\Upsilon}{3!}\epsilon_{\nu\beta\gamma\delta}\epsilon^{ijk}\partial_i z^\beta\partial_j z^\gamma\partial_k z^\delta=g\,\mathrm{Tr}{}^*\mathcal{G}_{\mu\nu}(z)\tilde{K}^i\partial_i z^\mu\,, \tag{5.9}$$

We also have (3.15) if no electric charges are present. Once again $z^\mu=z^\mu(\boldsymbol{s},0)$. Equation (5.8) follows from the definition of $\mathcal{G}^{\mu\nu}(x)$ [cf. Eq. (3.3)] and the definition for the surface current

$$\tilde{K}^i=\chi^{i3}(\boldsymbol{s},0)\,. \tag{5.10}$$

Also we have used $K^a=0$. Equation (5.9) follows from variations of z^ν in (3.22).

Notice that the field action $\mathcal{S}_\mathcal{G}$ [cf. Eq. (3.23)] can be simplified for $n=4$. The condition $G_{ab}=0$ can be regarded as an equation for the variable χ_{ab}. Since no derivatives of χ_{ab} appear in $\mathcal{S}_\mathcal{G}$, χ_{ab} can be eliminated. The result is

$$\mathcal{S}_{\mathcal{G}} = -\tfrac{1}{4}\int_V d^4x \,\mathrm{Tr}\,\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\,, \tag{5.11}$$

where V is the region which excludes the extended object. Here we have used $\mathcal{G}_{\mu\nu}(x)=\mathcal{F}_{\mu\nu}(x)$, $x\in V$, and $\partial(z)/\partial(\sigma)\neq 0$, everywhere.

B. The magnetic vacuum and the MIT bag

We first examine the example where the dual superconductor occupies all of Minkowski space. In this case there is no region V, so the total action is

$$\mathcal{S}_{\mathrm{VAC}} = \mathcal{S}_0 = -\Upsilon\int_{\mathrm{R}^4} d^4x\,, \tag{5.12}$$

which is, of course, divergent. If a bubble (or cavity) should arise in this "vacuum," (5.12) would be replaced by

$$-\Upsilon\int_{\mathrm{R}^4-V} d^4x - \tfrac{1}{4}\int_V d^4x\,\mathrm{Tr}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\,, \tag{5.13}$$

where V is the region of the cavity. Subtracting off the infinite vacuum action gives

$$\mathcal{S}_{\mathrm{cavity}} = \int_V d^4x\,(\Upsilon - \tfrac{1}{4}\,\mathrm{Tr}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu})\,. \tag{5.14}$$

If we identify Υ as minus the bag constant, (5.14) is precisely the action for a glueball in an MIT bag. The bag can therefore be identified as a cavity in a dual superconducting vacuum. By varying $\mathcal{a}_\mu$ and the boundary ∂V one obtains the bag-model equations

$$\mathfrak{D}_\mu\mathcal{F}^{\mu\nu}(x)=0 \quad \text{inside } V\,, \tag{5.15}$$

$$n_\mu\mathcal{F}^{\mu\nu}(x)=0 \quad \text{at } \partial V\,, \tag{5.16}$$

$$\Upsilon = \tfrac{1}{4}\,\mathrm{Tr}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} \quad \text{at } \partial V\,, \tag{5.17}$$

where n_μ is a unit spacelike vector normal to the surface ∂V. Note that Eqs. (5.15)–(5.17) contain no reference to the surface currents $\tilde{K}^i$ (just as Dirac's extended electron model could be written without reference to J^a). Alternatively, the bag-model equations can be expressed in terms of the magnetic currents. We show below that Eqs. (3.15), (3.21), (5.8), and (5.9) are equivalent to the bag-model equations.

As in subsection A, the boundary $\partial\Sigma$ can be given by $\sigma^3=0$, with $\sigma^3>0$ and <0 describing the interior and exterior of the bag, respectively. We again parametrize the boundary by $\mathfrak{s}$ [cf. Eq. (5.2)]. From (3.21), it is clear that (3.15) is identical to (5.15). Also,

$$\mathcal{G}_{\mu\nu}(z)\big|_{\sigma^3\to 0^-}=0\,. \tag{5.18}$$

Applying (5.18) to (4.16), we find

$$0 = n_\mu\mathcal{G}^{\mu\nu}\big|_{\sigma^3\to 0^+}\,, \tag{5.19}$$

$$g\tilde{K}^i\partial_i z^\nu = \frac{1}{3!}\,\epsilon_{\mu\eta\rho\sigma}\epsilon^{ijk}\partial_i z^\eta\partial_j z^\rho\partial_k z^\sigma\,{}^*\mathcal{G}^{\mu\nu}\big|_{\sigma^3\to 0^+}\,. \tag{5.20}$$

Since

$$\mathcal{G}^{\mu\nu}\big|_{\sigma^3\to 0^+} = \mathcal{F}^{\mu\nu}\big|_{\sigma^3\to 0^+}\,,$$

(5.19) is equivalent to the electric confinement condition (5.16). Equation (5.20) relates the fields near the boundary to $\tilde{K}^i$. Equation (5.9) gives the dynamics for the boundary. Upon substituting (5.20) into the right-hand side of (5.9), we obtain

$$\frac{1}{2\times 3!}\,\epsilon_{\lambda\eta\rho\sigma}\epsilon^{ijk}\partial_i z^\eta\partial_j z^\rho\partial_k z^\sigma\,\mathrm{Tr}\,{}^*\mathcal{F}^{\lambda\mu}\,{}^*\mathcal{F}_{\mu\nu}\big|_{\sigma^3\to 0^+}\,, \tag{5.21}$$

where $\mathcal{G}_{\mu\nu}$ evaluated at $\sigma^3=0$ is defined in (4.17). In terms of the normal vector n_μ [cf. Eq. (4.2)] (5.9) can then be written

$$\Upsilon n_\nu = \tfrac{1}{2}n_\lambda\mathrm{Tr}\,{}^*\mathcal{F}^{\lambda\mu}\,{}^*\mathcal{F}_{\mu\nu}\big|_{\sigma^3\to 0^+} \tag{5.22}$$

or

$$-\Upsilon = \tfrac{1}{2}\mathrm{Tr}(n_\lambda{}^*\mathcal{F}^{\lambda\mu})(n^\nu\,{}^*\mathcal{F}_{\nu\mu})\big|_{\sigma^3\to 0^+}\,. \tag{5.23}$$

But since $n_\mu\mathcal{F}^{\mu\nu}\big|_{\sigma^3\to 0^+}$, (5.23) can be written

$$\Upsilon = \tfrac{1}{4}\,\mathrm{Tr}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\big|_{\sigma^3\to 0^+}\,. \tag{5.24}$$

We thus recover the pressure balance condition of the MIT bag model [cf. Eq. (5.17)].

In Ref. 2 we argued that the action (5.14) already gives a clear indication that the magnetic supercurrents can be eliminated entirely from the dynamical description of cavities in a superconducting vacuum. Here we have shown, in detail, that the dynamics of the induced supercurrents at the boundary of the cavity yields the bag-model equation (5.17). The physical content of this fact, as well as the formal derivation given above, is very similar to the case of the Dirac extended electron model of Sec. IV.

The inclusion of matter fields into the above picture of bags was briefly discussed in Ref. 2, where we concluded that the complete MIT bag model essentially emerges. We believe, however, that our picture of hadrons as cavities in a superconducting vacuum may lead to some new aspects of the dynamics of gluons and quarks when quantum-mechanical considerations are taken into account (see Ref. 2 for some remarks on this issue).

We notice that the equations of motion (5.19) and (5.24) are consistent with the conservation of the energy-momentum tensor $T_{\mu\nu}$ as derived from the "effective" action (5.14). This conservation law does not, however, uniquely determine the boundary conditions at ∂V (cf. subsection C).

C. Other possible phases in QCD

In subsection B we have argued that the MIT bag model emerges naturally in a picture of hadrons as cavities in a dual superconducting vacuum. The matter fields, due to the dual Meissner effect, are electrically confined [cf. (5.15)]. The dynamics of the confined fields is given by perturbative QCD. Gluon fields cannot penetrate into the vacuum, but instead, induce magnetic currents $\tilde{K}^i$ to flow on the boundary of the cavity.

In line with the general analysis of 't Hooft[17] on the phase structure of Yang-Mills theories, we can also study a vacuum structure which gives rise to magnetic confinement of matter fields. This can arise through the Meissner effect in conventional superconductors. In this case, surface currents will be electric in nature and will lead to a different set of boundary conditions on the gluon fields.

1. Magnetic confining phase

In analogy with subsection B, we now study the example of a cavity in a (conventional) superconducting vacuum. The cavity will be associated with the perturbative phase of QCD. Because $\mathcal{F}_{\mu\nu}=0$ in the superconducting region, the gluon fields are again confined to the cavity. Here they induce electric currents $\tilde{J}^i$ to flow on boundary. The dynamics of this system is given in subsection A. In subsection B it was shown that the dynamics of the MIT bag could be described with or without reference to the currents $\tilde{K}^i$. We can similarly write the equations of motion for this system with or without reference to $\tilde{J}^i$. Let us once again describe the boundary by the equation $\sigma^3=0$, with $\sigma^3>0$ in the cavity and $\sigma^3<0$ in the superconducting vacuum. Then

$$\mathcal{F}_{\mu\nu}(z)\big|_{\sigma^3\to 0^-}=0\,. \tag{5.25}$$

The analogs of (5.19) and (5.20) are

$$0=n_\mu {}^*\mathcal{F}^{\mu\nu}\big|_{\sigma^3\to 0^+}\,, \tag{5.26}$$

$$e\tilde{J}^i\partial_i z^\nu=\frac{1}{3!}\epsilon_{\mu\eta\rho\sigma}\epsilon^{ijk}\partial_i z^\eta\partial_j z^\rho\partial_k z^\sigma\mathcal{F}^{\mu\nu}\big|_{\sigma^3\to 0^+}\,, \tag{5.27}$$

respectively (see the Appendix). Equation (5.26) is the magnetic confinement condition. Equation (5.27) relates the fields at the surface to the current $\tilde{J}^i$, and allows us to eliminate $\tilde{J}^i$ from Eq. (5.7). Substituting (5.27) into the right-hand side of (5.7) yields

$$\frac{1}{2\times 3!}\epsilon_{\lambda\eta\rho\sigma}\epsilon^{ijk}\partial_i z^\eta\partial_j z^\rho\partial_k z^\sigma\,\mathrm{Tr}\mathcal{F}^{\lambda\mu}\mathcal{F}_{\mu\nu}\big|_{\sigma^3\to 0^+}\,, \tag{5.28}$$

where $\mathcal{F}_{\mu\nu}$ evaluated at $\sigma^3=0$ is defined by (4.5). In analogy to (5.23), (5.7) can be written

$$-\Upsilon=-\tfrac{1}{2}\,\mathrm{Tr}(n_\lambda\mathcal{F}^{\lambda\mu})(n^\nu\mathcal{F}_{\nu\mu})\big|_{\sigma^3\to 0^+}\,. \tag{5.29}$$

Now from Eq. (5.26), we find

$$-\Upsilon=\tfrac{1}{4}\,\mathrm{Tr}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\big|_{\sigma^3\to 0^+}\,. \tag{5.30}$$

Thus the dual bag-model equations are

$$\mathcal{D}_\mu\mathcal{F}^{\mu\nu}(x)=0\ \text{ inside } V\,, \tag{5.31}$$

$$n_\mu {}^*\mathcal{F}^{\mu\nu}(x)=0\ \text{ at } \partial V\,, \tag{5.32}$$

$$\Upsilon=-\tfrac{1}{4}\,\mathrm{Tr}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\ \text{ at } \partial V\,. \tag{5.33}$$

Notice that (5.33) differs from (5.17) by a sign. A similar result was found in Sec. IV [cf. Eqs. (4.13) and (4.18)]. Unlike (5.18), Eq. (5.33) does not lead to the conclusion that the confined glueball in the superconducting vacuum is a color singlet. Instead, from the Meissner effect, the sum of the glueball charge and the surface electric charge is zero. Therefore, gluons (and quarks, if matter fields are added to the system) are screened by the electric surface currents.[23] In the sense of 't Hooft[17] the magnetic confinement discussed in the present section corresponds to the Higgs phase of QCD.

2. Mixed phase

We now study the final possibility of a "mixed" phase of QCD, i.e., a phase where both electric and magnetic supercurrents are present. We again require that all fields $\mathcal{G}^{\mu\nu}$ vanish inside the superconducting region. The equations for this system are (5.3) with $\mathcal{F}_{\mu\nu}$ replaced by $\mathcal{G}^{\mu\nu}$, (5.8), and[24]

$$\frac{\Upsilon}{3!}\epsilon_{\nu\beta\gamma\delta}\epsilon^{ijk}\partial_i z^\beta\partial_j z^\gamma\partial_k z^\delta$$
$$=e\,\mathrm{Tr}\mathcal{G}_{\mu\nu}(z)\tilde{J}^i\partial_i z^\mu+g\,\mathrm{Tr}\,{}^*\mathcal{G}_{\mu\nu}(z)\tilde{K}^i\partial_i z^\mu\,. \tag{5.34}$$

Equation (5.34) is the general equation of motion for the boundary between the superconducting and perturbative regions. We can once again rewrite this equation so that it contains no reference to the surface currents. Applying (5.20) and (5.27) we find

$$-\Upsilon=\tfrac{1}{2}\,\mathrm{Tr}(n_\lambda\mathcal{F}^{\lambda\mu})(n^\nu\mathcal{F}_{\nu\mu})+\tfrac{1}{2}\,\mathrm{Tr}(n_\lambda {}^*\mathcal{F}^{\lambda\mu})(n^\nu {}^*\mathcal{F}_{\nu\mu})\ \text{ at } \partial V\,. \tag{5.35}$$

Note in general neither $n^\mu\mathcal{F}_{\mu\nu}$ nor $n^\mu {}^*\mathcal{F}_{\mu\nu}$ vanish. Rather, from (5.20) and (5.27), they are proportional to $e\tilde{J}$ and $g\tilde{K}$, respectively. Thus, once again the fields in the bag need not be in a color singlet, and the color charge is shielded by the electric currents flowing at the boundary.

The (conventional) superconducting phase is obtained from the mixed phase in the limit $g\to 0$. To obtain the dual superconducting phase from the mixed phase let us write $\tilde{J}=(e'/e)\tilde{J}'$. The dual superconducting phase is obtained in the limit e'

$\to 0$.[25] Thus e' and g can serve as order parameters in the theory. Note at the quantum level a phase transition corresponding to $e' \to 0$ may sometimes be forbidden. For instance, if one quark exists inside a bubble in the mixed vacuum phase, the color-singlet condition would be violated when $e' \to 0$. On the other hand, there seem to be no such restrictions for phase transitions corresponding to $g \to 0$.

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APPENDIX

Here we prove the following two relationships for an electrically charged surface of dimension $n=3$:

$$n_\mu(\sigma)\{\mathfrak{F}^{\mu\nu}(y(\sigma))|_{\sigma^3\to 0^+} - \mathfrak{F}^{\mu\nu}(y(\sigma))|_{\sigma^3\to 0^-}\} = e(-\det g)^{-1/2}J^a(\sigma)\partial_a z^\nu(\sigma) \quad \text{(A1)}$$

and

$$n_\mu(\sigma)\{{}^*\mathfrak{F}^{\mu\nu}(y(\sigma))|_{\sigma^3\to 0^+} - {}^*\mathfrak{F}^{\mu\nu}(y(\sigma))|_{\sigma^3\to 0^-}\} = 0\,, \quad \text{(A2)}$$

where $n_\mu(\sigma)$ and the coordinates $y^\mu(\sigma)$ are defined in Eqs. (4.2) and (4.3). We begin with the Abelian case. The field equation is

$$\partial_\mu \mathfrak{F}^{\mu\nu}(x) = e\int d^3\sigma\, \delta^4(x - z(\sigma))\, J^a(\sigma)\partial_a z^\nu(\sigma)\,. \quad \text{(A3)}$$

Now let us define a four-volume R:

$$R = \{y^\mu(\sigma, \sigma^3)\,|\,\sigma^a_{(1)} \leq \sigma^a \leq \sigma^a_{(2)};\ -\epsilon \leq \sigma^3 \leq \epsilon\}$$
$$\epsilon \text{ real and positive}\,. \quad \text{(A4)}$$

Here $a = 0, 1, 2$. We will take ϵ to be "small." As shown in Fig. 1, R corresponds to a pill box[26] of thickness 2ϵ enclosing an element of the shell three-surface. We now integrate both sides of the Eq. (A3) over the volume R and apply Gauss's law

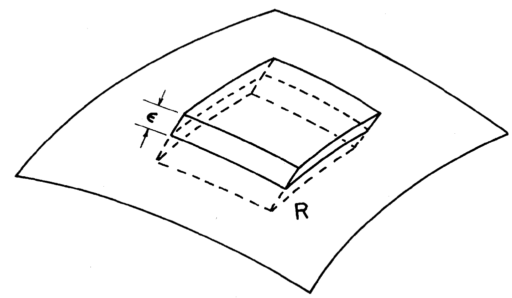

FIG. 1. Pill-box integration volume used in the derivation of Eq. (A1).

with the result

$$\int_{\partial R} ds_\mu \mathfrak{F}^{\mu\nu} = e\int_\Sigma d^3\sigma\, J^a(\sigma)\partial_a z^\nu\,. \quad \text{(A5)}$$

Here ∂R is the boundary of R and

$$\Sigma = \{\sigma\,|\,\sigma^a_{(1)} \leq \sigma^a \leq \sigma^a_{(2)};\ a = 0, 1, 2\}\,.$$

Each of the six sides of the pill box will contribute to the left-hand side of (A5). Now in the limit $\epsilon \to 0$ four of the sides will contribute terms which are of first order in ϵ, and these contributions can be dropped. The remaining two terms are

$$\int_\Sigma d^3\sigma(-\det g)^{+1/2} n_\mu(\sigma) \times \{\mathfrak{F}^{\mu\nu}(y(\sigma))|_{\sigma^3\to 0^+} - \mathfrak{F}^{\mu\nu}(y(\sigma))|_{\sigma^3\to 0^-}\}\,. \quad \text{(A6)}$$

Combining Eq. (A5) and (A6) and noting the arbitrariness of Σ (Ref. 27) leads to Eq. (A1).

To generalize the above derivation to the non-Abelian case we should add the term

$$-ie\int_R d^4x[\mathfrak{A}_\mu(x), \mathfrak{F}^{\mu\nu}(x)] \quad \text{(A7)}$$

to the left-hand side of (A5). In the limit $\epsilon \to 0$ (A7) is, however, of first order in ϵ (since the four-volume is linear in ϵ) and (A7) can therefore be dropped. Thus (A1) generalizes intact to the non-Abelian shell as well.

Equation (A1) relates the electric current to certain components of the fields at the three-surface. Because no magnetic current is present, in this case, the dual field components at the three-surface must obey Eq. (A2). In other words, if we apply the above procedure to the Bianchi identity $\mathfrak{D}_\mu {}^*\mathfrak{F}^{\mu\nu}(x) = 0$, (A2) follows.

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[22]For $n=4$, we can define $\chi^{ab}=\epsilon^{abcd}u\lambda_{cd}u^\dagger$ [cf. Eq. (2.24)].

[23]This result follows upon integrating (5.31) over V. Applying Stokes's Law, $0=\int_{\partial V} d^2x\, n^i \mathfrak{F}_{i0}+\int_V d^3x \times [\mathfrak{A}^i, \mathfrak{F}_{i0}]$, on any time slice. The first term corresponds to the surface current charge [from (2.27)] and the second term is the gluon charge. If matter fields ψ are added we replace (5.31) by $\mathfrak{D}_\mu \mathfrak{F}^{\mu\nu}=j^\nu[\psi]$ (inside V) and we find $0=\int_{\partial V} d^2x\, n^i \mathfrak{F}_{i0}+\int_V d^3x\, [\mathfrak{A}^i, \mathfrak{F}_{i0}] + \int_V d^3x\, j_0[\psi]$.

[24]This system can be obtained from an action principle. For an n-dimensional object $\mathcal{S}=\mathcal{S}_G+\mathcal{S}_0+(e/2) \times \int d^n\sigma \operatorname{Tr} \chi'^{ab} G_{ab}$. Note that we should distinguish in general between χ'^{ab} and the variable χ_{ab} which appears in the definition of $\mathcal{G}_{\mu\nu}$.

[25]We cannot instead take the limit $e\to 0$ since it would imply that all commutator terms can be ignored.

[26]The derivation presented here of Eqs. (A1) and (A2) makes use of familiar arguments from classical electrodynamics.

[27]A "smoothness" condition on the shell surface is assumed in such a way that the expressions above make sense.