Frozen Solitons in a Two-Dimensional Ferromagnet

A. Stern – University of Alabama

Citation of published version:

Stern, A. (1987): Frozen Solitons in a Two-Dimensional Ferromagnet. *Physical Review Letters,* 59(14). DOI: https://doi.org/10.1103/PhysRevLett.59.1506

Frozen Solitons in a Two-Dimensional Ferromagnet

A. Stern
Department of Physics and Astronomy, University of Alabama, Tuscaloosa, Alabama 35487
(Received 6 May 1987)

The fundamental soliton solution of the O(3) nonlinear σ model in two space dimensions is examined. It is pointed out that the classical soliton is associated with an infinite moment of inertia and infinite moment of dilation, and argued that the quantum soliton is as well. It follows that the spin of the soliton is undefined, contrary to recent claims that it be fractional.

PACS numbers: 11.10.Lm, 75.70.−i

The O(3) nonlinear σ model is known to describe the system of an isotropic ferromagnet in the limit of 0 K. The model in two space dimensions contains topological solitons,[1-3] corresponding to certain metastable states of the ferromagnet. In this Letter, I examine some properties of the fundamental soliton (skyrmion) of the two-dimensional σ model. Solitons with different orientations and scales are degenerate in the classical energy. I point out, however, that the moments associated with changes in orientation and scale are both infinite. Further, I shall argue that the quantum expectation values of these moments are also infinite. As a result, the quantum ground state will not be a superposition of states with different orientations and scales. Rather, the rotation and dilation symmetries are spontaneously broken.

By adding a Hopf term to the nonlinear-σ-model Lagrangean, Wilczek and Zee[4] argued that it was possible to obtain a system with fractional spin and statistics. Here, however, I claim that this possibility is not realized for the fundamental soliton. As is usual in theories with a spontaneous symmetry breaking, the broken generators (namely, the spin and the generator of dilations) are undefined when acting on the quantum ground state.

The Lagrangean for the nonlinear σ model is

$$\mathcal{L}=\tfrac{1}{2}\,\partial_\mu\phi_a\partial^\mu\phi_a, \tag{1}$$

where $\phi_a=\phi_a(\mathbf{x},t)$, $a=1,2,3$, are the three spin components. They are subject to the constraint

$$\phi_a\phi_a=1, \tag{2}$$

defining a two-sphere S^2. The energy associated with (1) is

$$E[\phi]=\tfrac{1}{2}\int d^2x\{(\partial_0\phi_a)^2+(\partial_i\phi_a)^2\},\quad i=1,2. \tag{3}$$

The vacuum solutions to the equations of motion are $\phi_a=\phi_0^{(0)}=$const [subject to the constraint (2)]. Since Eq. (1) is invariant under a global SO(3) rotation $\phi_a\to R_{ab}\phi_b$, we can choose $\phi^{(0)}=(0,0,1)$.

For nonzero, but finite, energy configurations, I require that $\phi(\mathbf{x},t)\to\phi^{(0)}$, as $|\mathbf{x}|\to\infty$. This identification at spatial infinity essentially converts the domain of the map ϕ from $\mathbf{R}^2$ to the surface a two-sphere S^2. Thus, $\phi:S^2\to S^2$. Such maps are classified by the winding number

$$n=\frac{1}{8\pi}\int d^2x\,\epsilon_{abc}\epsilon_{ij}\phi_a\partial_i\phi_b\partial_j\phi_c, \tag{4}$$

which is an integer. Let $\phi^{(n)}$ denote a mapping associated with winding number n. A lower bound on the energy associated with $\phi^{(n)}$ is obtained from the identity $(\partial_i\phi_a\mp\epsilon_{abc}\epsilon_{ij}\phi_b\partial_j\phi_c)^2\ge 0$. From Eqs. (3) and (4), $E[\phi^{(n)}]\ge 4\pi|n|$. This bound is saturated if we set

$$\partial_i\phi_a=\pm\,\epsilon_{abc}\epsilon_{ij}\phi_b\partial_j\phi_c. \tag{5}$$

If we replace variables ϕ_a by two independent coordinates w_i, $i=1,2$,

$$w_i=2\phi_i/(1-\phi_3), \tag{6}$$

obtained via a stereographic projection, Eq. (5) reduces to the Cauchy-Riemann conditions,

$$\partial_1 w_1=\pm\,\partial_2 w_2,\quad \partial_2 w_1=\mp\,\partial_1 w_2. \tag{7}$$

Solutions $w=w_1+iw_2$ to Eq. (7) are analytic functions of $z=x_1+ix_2$ [z^* for the lower signs in Eq. (7)]. Isolated poles are allowed for $w(z)$. Such poles are mapped to the north pole of the two-sphere spanned by ϕ_a.

The general solution to (7) for arbitrary n is given in Ref. 3. Here we shall primarily be interested in the $n=1$ solution. It is

$$w(z)=(z-z_0)/\lambda, \tag{8}$$

where z_0 and λ are complex constants. They indicate that there are four zero-frequency modes of the classical solution, z_0 corresponding to two translation modes, while $\lambda=ae^{i\theta}$ corresponds to both a rotation and dilation of the classical configuration. Each zero-frequency mode is associated with a symmetry of the soliton. For example, with regard to θ, we note that the spin fields (ϕ_1,ϕ_2) are invariant under a combined spatial rotation (about the soliton center) and internal U(1) transformation. This is evident since for arbitrary θ, $\phi_i(\mathbf{x})=\phi_i^\theta(\mathbf{x})$, $i=1,2$, can be written

$$\phi_i^\theta(\mathbf{x})=R_{ij}\phi_j^{\theta=0}(\mathbf{x})=\phi_i^{\theta=0}(\mathbf{R}\mathbf{x}), \tag{9}$$

where $R_{11}=R_{22}=\cos\theta$, $R_{12}=-R_{21}=\sin\theta$. In terms of w, the invariance condition reads

$$\left[(z^*-z_0^*)\frac{\partial}{\partial z^*}-(z-z_0)\frac{\partial}{\partial z}\right]w+w=0. \qquad (10)$$

We are interested in the question of whether the above symmetries persist for the corresponding quantum ground soliton. In the quantum theory, we let $\Psi_{z_0,\lambda}$ denote a quantum state which is peaked about the $n=1$ classical ground state associated with a given z_0 and λ. The classical ground states are degenerate in energy, and so one might expect that there is a single quantum ground state which is a superposition of all $\Psi_{z_0,\lambda}$. We shall claim, however, that this is not the case. We conjecture that there are degenerate quantum ground states, which are parametrized by λ.

The first indication of the above arises when we attempt to quantize the collective motions induced by changes in z_0 and λ. Since these degrees of freedom connect classically degenerate ground states, they should be included in any semiclassical approximation of the quantum theory. The procedure amounts to our making z_0 and λ in (8) time dependent, substituting into the classical Lagrangean, and performing the spatial integrations. This procedure was adapted in Adkins, Nappi, and Witten[5] for three-dimensional skyrmions, and in Bowick, Karabali, and Wijewardhana[6] for the two-dimensional analogs. We find

$$L[\phi]=\int d^2x\,\mathcal{L}(\phi,\partial\phi)=8\int d^2x\frac{(\partial^\mu w^i)^2}{(4+|w|^2)^2}$$
$$=2\pi|\dot z_0|^2+\frac{I[\phi]}{2}\dot\theta^2+\frac{\mu[\phi]}{2}\dot a^2-4\pi. \qquad (11)$$

Equation (11) shows that the soliton's inertial mass agrees with its rest mass. It is easy to verify that the moments $I[\phi]$ and $\mu[\phi]$ are both logarithmically divergent. Upon inserting a cutoff R in the upper limit of the radial integral, one obtains

$$\frac{I[\phi]}{a^2}=\mu[\phi]=16\pi\left\{\ln\left[1+\frac{R^2}{4a^2}\right]-\frac{R^2}{4a^2+R^2}\right\}. \qquad (12)$$

The fact that the moment $I[\phi]$ is infinite is also seen by substitution into the conserved angular momentum

$$J=\int d^2x\,\epsilon_{ij}x_i\partial_j\phi_a\dot\phi_a. \qquad (13)$$

For the rotating soliton $\dot\theta\neq0$, J is divergent. Similarly upon substituting the expression for the dilating soliton into the generator of dilations

$$M=\int d^2x\,x_i\partial_i\phi^a\dot\phi^a, \qquad (14)$$

we find that the result is divergent for $\dot a\neq0$.

The above shows that infinite energy is required to rotate a rigid classical soliton. Likewise, infinite energy is required for a soliton to dilate and still retain the form given in Eq. (8).

Next, let us discuss the corresponding quantum system. We perturb about the classical solution to obtain various virtual field configurations $\tilde\phi$. Configurations $\tilde\phi$ associated with finite values for the moments $I[\tilde\phi]$ or $\mu[\tilde\phi]$ may allow for a quantum-mechanical tunneling between the various degenerate ground states of the theory.

We shall check whether the above is the case for the class of configurations which are rotationally invariant [i.e., satisfy Eq. (10)]. The general solution to (10) is

$$w(z,z^*)=f\left(\left|\frac{z-z_0}{\lambda}\right|\right)\frac{z-z_0}{\lambda}, \qquad (15)$$

where f is a real function. Only for the case of $f=$const is w an analytic function, and hence an energy minimum of the $n=1$ sector. The requirement that the configurations (15) have $n=1$ means that $f(r)$ satisfies the following boundary conditions:

$$f(r)\to r^\delta,\ \text{as } r\to\infty,\quad \to r^{\delta'},\ \text{as } r\to0, \qquad (16)$$

with $\delta,\delta'>-1$. The same set of conditions arises from our demanding finite energy. A stronger condition at spatial infinity results upon the requirement of finite moments $I[\tilde\phi]$ and $\mu[\tilde\phi]$. Then $\delta>0$.

Now consider the family of functions $f(r)=f_\epsilon(r)$, where ϵ is a "small" dimensionless parameter, $\epsilon\geq0$. The corresponding field configurations $\tilde\phi=\phi_\epsilon$ are given by (6) and (15). Further, take $f_{\epsilon=0}(r)=1$ so that the family of configurations ϕ_ϵ contains the classical solution ϕ_0. For $\epsilon>0$, assume that the functions $f_\epsilon(r)$ are defined such that the moments $I[\phi_\epsilon]$ and $\mu[\phi_\epsilon]$ are finite. Thus, we can take $f_\epsilon(r)\to r^\epsilon$ as $r\to\infty$, or more generally,

$$f_\epsilon(r)\to r^{\epsilon^p},\ \text{as } r\to\infty, \qquad (17)$$

where p is a positive constant.

We shall consider quantizing the soliton mode associated with collective coordinate ϵ. (For convenience, we hold other possible collective coordinates fixed. I believe that this will not effect the conclusions which follow.) One can then solve for the quantum Hilbert space $\mathcal{H}$, and, in principle, compute the various expectation values of $I[\phi_\epsilon]$ and $\mu[\phi_\epsilon]$ on this space. Finite results would then indicate that the quantum soliton can rotate and/or dilate, and that the true quantum ground state should be a superposition of states associated with all possible orientations and scales. I argue, however, that this is not the case, because to $\mathcal{H}$ is the null set.

Following the standard procedure, we promote the parameter ϵ to a dynamical variable $\epsilon(t)$. We assume that the potential energy is analytic near $\epsilon=0$; so it goes like ϵ^m for small ϵ, where m is an integer greater than one. The kinetic energy is given by the expression $16\pi k(\epsilon)\dot\epsilon^2$, where

$$k(\epsilon)=\int_0^\infty\frac{dr\,r^3(\partial f/\partial\epsilon)^2}{(4+f^2r^2)^2}. \qquad (18)$$

Now $k(\epsilon)$ must be divergent as $\epsilon \to 0$, because of the large-distance behavior of f_ϵ. To ascertain the rate of divergence, consider the contribution to $k(\epsilon)$ from $r \geq \tilde{R}$, where $\tilde{R} \gg 1$,

$$\int_{\tilde{R}}^{\infty} \frac{dr}{rf^2} \left[\frac{\partial \ln f}{\partial \epsilon} \right]^2. \tag{19}$$

After substituting the asymptotic form (17) into the integrand in (19) and integrating, we get $\tilde{R}^3/4\epsilon^{p+2}$ to lowest order in ϵ. (We assume that $\tilde{R} \ll e^{\epsilon^{-p}}$.) This indicates that $k(\epsilon)$, at best, is quadratically divergent in ϵ. This is only true in a limiting sense, however, since $p > 0$.

Actually, in a slightly different approach, it is possible to find a kinetic energy term which goes precisely like $\dot{\epsilon}^2/\epsilon^2$. This approach essentially involves quantizing a cutoff R of the classical solution. More specifically, we take

$$f_\epsilon(r) = \begin{cases} 1, & r < R, \\ \dfrac{2}{r}\cot[\alpha - \beta(r-R)], & R \leq r < R'. \end{cases} \tag{20}$$

where $\alpha = \tan^{-1} 2/R$, $\frac{1}{2}\beta = (4+R^2)^{-1}$, and $R' = R + \alpha/\beta$. With this choice, $f_\epsilon(r)$ and $\partial f_\epsilon/\partial r$ are continuous for all $r < R'$. For $r \geq R'$, $f_\epsilon(r)$ is infinite, corresponding to $\phi = (0,0,1)$. The classical solution is recovered for $R \to \infty$. Next we make the cutoff R time dependent and set $R(t) = \epsilon(t)^{-1}$. After substituting (20) into the Lagrangean and performing the spatial integration, we obtain

$$L = \tfrac{1}{2} b\dot{\epsilon}^2/\epsilon^2 - c\epsilon^2, \tag{21}$$

$$b = \tfrac{896}{3}\pi, \quad c = 8\pi[32\ln 2 - 19]$$

to lowest order in ϵ. (We dropped the constant mass term.)

Now let us canonically quantize the mode associated with ϵ. The Hamiltonian is $H = (2b)^{-1}\epsilon^2\rho^2 + c\epsilon^2$, where ρ is canonically conjugate to ϵ. The quantum Hamiltonian $\hat{H}$ has a factor ordering ambiguity, which is easily resolved (up to an overall additive constant) by our demanding hermiticity,

$$\hat{H} = (2b)^{-1}[\hat{\epsilon}^2\hat{\rho}^2 - 2i\hat{\epsilon}\hat{\rho}] + c\hat{\epsilon}^2, \tag{22}$$

$\hat{\epsilon}$ and $\hat{\rho}$ being the quantum operators associated with ϵ and ρ. Eigenfunctions $\psi_E(\epsilon)$ of $\hat{H}$ satisfy the modified Bessel equation

$$\left[\frac{\partial^2}{\partial \epsilon^2} + \frac{2}{\epsilon}\frac{\partial}{\partial \epsilon} + 2b\left[\frac{E}{\epsilon^2} - c\right]\right]\psi_E(\epsilon) = 0. \tag{23}$$

Equation (23) is identical to the radial Schrödinger equation for bound states in a $1/\epsilon^2$ potential, ϵ being the radial variable $(b,c > 0)$. However, for us the integration measure for ψ_E is $d\epsilon$, not $\epsilon^2 d\epsilon$. We require that ψ_E be normalizable. This, however, is not possible if the measure is taken to be $d\epsilon$. Consequently, the relevant Hilbert space is empty, which indicates that the collective mode associated with ϵ cannot be quantum-mechanically excited.

I now argue that this results persists if we consider the more general Lagrangean

$$L = \tfrac{1}{2} b\dot{\epsilon}^2/\epsilon^{p+2} - c\epsilon^2, \quad p \geq 0, \tag{24}$$

corresponding to the behavior found previously. The associated quantum Hamiltonian is

$$\hat{H} = (2b)^{-1}[\hat{\epsilon}^{p+2}\hat{\rho}^2 - i(p+2)\hat{\epsilon}^{p+1}\hat{\rho}] + c\hat{\epsilon}^2. \tag{25}$$

Eigenfunctions ψ_E of $\hat{H}$ diagonal in ϵ now satisfy the differential equation

$$\{\partial^2/\partial\epsilon^2 + (p+2)\epsilon^{-1}\partial/\partial\epsilon + 2b\epsilon^{-p}(E/\epsilon^2 - c)\}\psi_E(\epsilon) = 0. \tag{26}$$

As in the previous case we find that there exist no normalizable solutions to the relevant Schrödinger equation. Thus, we again have an empty Hilbert space $\mathcal{H}$.

$\mathcal{H}$ being null means that the collective mode associated with ϵ cannot be excited. Consequently, the quantum states of theory, obtained from the fluctuation about the classical solution in all other possible collective modes, must necessarily have infinite expectation values of the moments I and μ. As a result, no tunneling is allowed between states of different orientations and scales. The action of J and M [cf. Eqs. (13) and (14)] is undefined when acting on such states. We may thus say that the rotation and dilation symmetries are spontaneously broken, and that there are infinitely many degenerate quantum ground states, which are labeled by a and θ.

I note that the above result need not necessarily apply for solitons in higher topological sectors; i.e., $|n| > 1$. For example, for $n = 2$, we can have the classical solution $w(z) = [(z - z_0)/\lambda]^2$, which gives finite values for both I and μ. (It, however, does not correspond to the most general $n = 2$ solution. With regard to this, see Ref. 3.)

Of course, for the fundamental soliton, the moments I and μ can be made finite through the inclusion of a mass term for the fields; e.g., $m^2(1 - \phi_3^2)$. Such a term explicitly breaks the O(3) symmetry to U(1).

In Refs. 4 and 6, a "Hopf term" is added to the nonlinear-σ-model Lagrangean. It can be written $-\theta A_\mu j^\mu$, where j^μ is the topological current

$$j_\mu = (8\pi)^{-1}\epsilon_{\mu\nu\lambda}\epsilon^{abc}\phi_a\partial^\nu\phi_b\partial^\lambda\phi_c. \tag{27}$$

To define the U(1) potential A_μ, let us replace $\phi^a(x)$ by a new set of fields $g(x)$ which take values in SU(2). From the three independent degrees of freedom in g, one can define the two degrees of freedom $\phi^a(x)$ according to

$$\phi^a = \tfrac{1}{2}\,\mathrm{Tr}\,\tau^a g \tau^3 g^\dagger, \tag{28}$$

τ^a being the Pauli matrices. The extra degree of freedom in g can be considered a gauge degree of freedom[7] since

$$g(x) \to g(x) e^{-i\Lambda(x)\tau^3} \tag{29}$$

leaves $\phi^a(x)$ invariant. Now define

$$A_\mu(x) = \tfrac{1}{2} i\,\mathrm{Tr}\,\tau^3 g^\dagger \partial_\mu g. \tag{30}$$

Under (29), $A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda$. Some work shows that $12\pi j^\mu = \epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda$, which is consistent with Ref. 7. Further, the Hopf term can be written solely in terms of the field $g(x)$ and its derivatives:

$$S_\mathrm{H} = -\theta \int d^3x\, A_\mu j^\mu = \frac{-\theta}{6\pi}\int \mathrm{Tr}(g^\dagger dg)^3. \tag{31}$$

Any infinitesimal variation of (31) produces a surface term. Under $\delta g = i\epsilon_a \tau^a g$,

$$\delta S_\mathrm{H} = -\frac{\theta i}{2\pi}\int d\,\mathrm{Tr}\,\epsilon_a \tau^a (dg g^\dagger)^2. \tag{32}$$

Thus, the classical equations of motion are unchanged upon the inclusion of (32), and the minimal $n=1$ classical solutions are still of the form (8). Further, (32) does not affect the form of the generations J and M, as given in (13) and (14). As a result, the moments of inertia and dilation are still divergent, and I again expect that this result holds quantum mechanically as well. As a result, the symmetries associated with J and M are once again spontaneously broken. It thus appears that fractional spin does not result from the fundamental solitons of the nonlinear σ model.

It is a pleasure to thank A. P. Balachandran, E. Braaten, P. Salamonson, and B. Skagerstam for helpful discussions.

[1]T. Skyrme, Proc. Roy. Soc. London **262**, 237 (1961).

[2]L. D. Fadeev, Max-Planck Institut Report No. MPI-PAE/16, 1974 (unpublished).

[3]A. A. Belavin and A. M. Polyakov, Pis'ma Zh. Eksp. Teor. Fiz. **22**, 503 (1975) [JETP Lett. **22**, 245 (1975)].

[4]F. Wilczek and A. Zee, Phys. Rev. Lett. **51**, 2250 (1983). Also, see Y. S. Wu and A. Zee, Phys. Lett. **147B**, 325 (1984); G. Murthy, Phys. Rev. Lett. **58**, 755 (1987).

[5]G. Adkins, C. Nappi, and E. Witten, Nucl. Phys. **B228**, 552 (1983).

[6]M. Bowick, D. Karabali, and L. C. R. Wijewardhana, Nucl. Phys. **B271**, 417 (1986).

[7]A. P. Balachandran, A. Stern, and G. Trahern, Phys. Rev. D **19**, 2416 (1979).