Matrix Model Cosmology in Two Space-Time Dimensions

Allen Stern – University of Alabama

Citation of published version:

Stern, A. (2014): Matrix Model Cosmology in Two Space-Time Dimensions. *Physical Review D*, 90(12). DOI: https://doi.org/10.1103/PhysRevD.90.124056

PHYSICAL REVIEW D **90,** 124056 (2014)

Matrix model cosmology in two space-time dimensions

A. Stern*

Department of Physics, University of Alabama, Tuscaloosa, Alabama 35487, USA
(Received 10 October 2014; published 18 December 2014)

We examine solutions to the classical Ishibashi-Kawai-Kitazawa-Tsuchiya matrix model equations in three space-time dimensions. Closed, open and static two-dimensional universes naturally emerge from such models in the commutative limit. We show that tachyonic modes are a generic feature of these cosmological solutions.

DOI: 10.1103/PhysRevD.90.124056 PACS numbers: 04.60.Kz, 11.15.-q, 11.30.Cp

I. INTRODUCTION

Matrix models promise to be a convenient tool for studying nonperturbative aspects of string theory [1,2]. Space-time geometry, field theory and gravity can dynamically emerge from such models [3,4], and thus can have implications in cosmology [5–8]. The matrix model approach to cosmology has the advantage of including nonperturbative string theory effects and possibly resolving cosmological singularities [6]. Previously, numerical simulations were used to show how a three-spatial dimensional expanding universe can emerge from a ten-dimensional matrix model [8]. Moreover, solutions to the classical equations of motion have been found which resemble expanding universes [5,8], and can support many desirable features, such as a big bounce and an early inflationary phase with graceful exit [7].

While the appropriate framework for matrix models is a ten-dimensional supersymmetric theory [either Ishibashi-Kawai-Kitazawa-Tsuchiya (IKKT) [1] or Banks-Fischler-Shenker-Susskind [2]], an examination of simpler systems may prove beneficial. With this in mind, we shall restrict our attention to the bosonic sector of the IKKT matrix model in three space-time dimensions. We write down the standard matrix equations in Sec. II, and focus on its classical solutions. Evidence for nontrivial solutions to the Lorentzian matrix model equations is seen by going to the commutative limit, where the classical equations of motion coincide with those of a closed Nambu string, and are easily solved. Among the solutions is the cylindrically symmetric solution, which corresponds to a closed two-dimensional surface with an initial and final singularity. (Such singularities are not expected to appear in the analogous matrix model solution.) This solution is the Lorentzian space counterpart to the minimum area catenoid in Euclidean space, whose corresponding matrix model solution is nontrivial because it does not correspond to a finite dimensional Lie algebra [9]. Similarly, the matrix model solution (assuming it exists) in the Minkowski space background which gives the cylindrically symmetric closed two-dimensional surface in the commutative limit, is nontrivial for the same reason. However, in this article our aim is not in finding an explicit expression for the matrix solution. We shall instead be examining stability questions, more specifically, in the commutative limit. Perturbations about the solution can be expressed in terms of an Abelian gauge field and scalar field (or non-Abelian gauge fields and N scalar fields if one expands about a stack of N coinciding branes). This is possible thanks to the use of a Seiberg-Witten map [10] on the noncommutative space associated with the solution. We obtain the map up to first order in the noncommutativity parameter in order to obtain the lowest order effects in the action. Gauge transformations correspond to area preserving coordinate transformations on the two-dimensional surface, while the scalar field is associated with perturbations normal to the surface. At leading order, the perturbed action yields the usual description of a scalar field, which is decoupled to the (nondynamical) gauge field. We find that the scalar field is tachyonic, and thus that the system is unstable with respect to perturbations normal to the surface.

The system can be generalized with the inclusion of a cubic term in the matrix model action, and we do this in Sec. III. This term is the matrix analogue of a topological term. It preserves the symmetries of the three-dimensional matrix model and introduces a free parameter v in the theory. Three different types of nontrivial cylindrically symmetric solutions to the matrix equations can result from this model. Two are well known, and they are associated with finite dimensional Lie algebras. One is the noncommutative de Sitter solution with the associated algebra beings $so(2,1)$ [11,12]. This solution is the Lorentzian space analogue of the fuzzy sphere [13–19]. Another solution is the noncommutative cylinder, whose associated algebra generates the two-dimensional Euclidean group [20–22]. Neither of these two solutions are present when the cubic term is removed, corresponding to the $v \to 0$ limit. Evidence for the existence of a third class of solutions can be seen by once again going to the commutative limit. In that limit, we obtain solutions which are deformations of the cylindrically symmetric solution to the Nambu string described above. Like with the previous string solution, they are not associated with any finite dimensional Lie

*astern@ua.edu

algebra, and so their matrix model analogues are nontrivial. Here one gets a continuous family of cylindrically symmetric solutions (parametrized by v) which can describe closed, stationary or open space-times, the choice depending on the value of v. A string energy-momentum tensor can be defined for this system, and the energies of all the solutions can be compared. In this regard, we find that the solutions corresponding to open and stationary space-times are energetically favored. We also perform perturbations about the different solution, and again express them in terms of an Abelian gauge field and scalar field. At leading order, the action now reveals a coupling between the gauge field and scalar, which is not present when $v = 0$. We find that the effective mass squared for the scalar field is negative for all of the solutions (and moreover, it can be scale dependent). Thus as before, these systems are unstable with respect to perturbations normal to the surface.

Possible generalizations and cures of the instabilities are discussed in Sec. IV.

II. LORENTZIAN MATRIX MODEL

A. Classical equations and the commutative limit

We consider the IKKT Lorentzian matrix model in three space-time dimensions. As here we shall only be concerned with the bosonic sector, the dynamical degrees of freedom are contained in three infinite-dimensional Hermitian matrices, which we denote by Y^μ, $\mu = 0, 1, 2$. For the action $S(Y)$, we have the usual quartic or Yang-Mills term

$$S(Y) = -\frac{1}{4g^2}\mathrm{Tr}[Y_\mu, Y_\nu][Y^\mu, Y^\nu], \tag{2.1}$$

g being a constant, with resulting equations of motion

$$[[Y_\mu, Y_\nu], Y^\nu] = 0. \tag{2.2}$$

Here we raise and lower indices with the flat metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1)$. The equations of motion (2.2) are invariant under (i) Lorentz transformations $Y^\mu \to L^\mu{}_\nu Y^\nu$, where L is a 3×3 Lorentz matrix, (ii) translations in the three-dimensional Minkowski space $Y^\mu \to Y^\mu + v^\mu \mathbb{1}$, where $\mathbb{1}$ is the unit matrix, and (iii) unitary "gauge" transformations, $Y^\mu \to UY^\mu U^\dagger$, where U is an infinite-dimensional unitary matrix.

Evidence for a nontrivial solution to (2.2) is seen by going to the commutative limit of the matrix model. The commutative limit corresponds to the replacement of the matrices Y^μ, $\mu = 0, 1, 2$, by space-time coordinates y^μ, $\mu = 0, 1, 2$, and the replacement of the commutator of functions of Y^μ with some Poisson bracket of functions of y^μ. For this we can introduce a noncommutativity parameter θ, with the commutative limit corresponding to $\theta \to 0$. To lowest order in θ, $[f(Y), h(Y)] \to i\theta\{f(y), h(y)\}$, $\{,\}$ denoting the Poisson bracket, which is required to satisfy the usual properties such as the Jacobi identity. In the commutative limit, the equations of motion (2.2) take the form

$$\{\{y_\mu, y_\nu\}, y^\nu\} = 0. \tag{2.3}$$

B. A classical string solution

Throughout this article we shall be considering solutions corresponding to cylindrically symmetric surfaces, with x^0 (time) along the central axis. If $y^\mu = x^\mu$ denotes such a solution, we can write

$$(x^1)^2 + (x^2)^2 = a^2(x^0). \tag{2.4}$$

The function a is the radius of any time slice and it plays the role of the scale factor. In addition to a, we introduce the function h of x^0 in the following expression for the Poisson brackets of the coordinates:

$$\begin{aligned}\{x^1, x^2\} &= h(x^0)a(x^0)a'(x^0)\\ \{x^2, x^0\} &= -h(x^0)x^1\\ \{x^0, x^1\} &= -h(x^0)x^2,\end{aligned} \tag{2.5}$$

where the prime denotes differentiation with respect to x^0. The Poisson brackets are consistent with the constraint (2.4) and satisfy the Jacobi identity. They solve the equations of motion (2.3) provided that the two functions a and h satisfy

$$((aa'h)' + h)h = 0 \qquad (2a'h + ah')ah = 0. \tag{2.6}$$

For $h \neq 0$, we get the following equation for the scale factor:

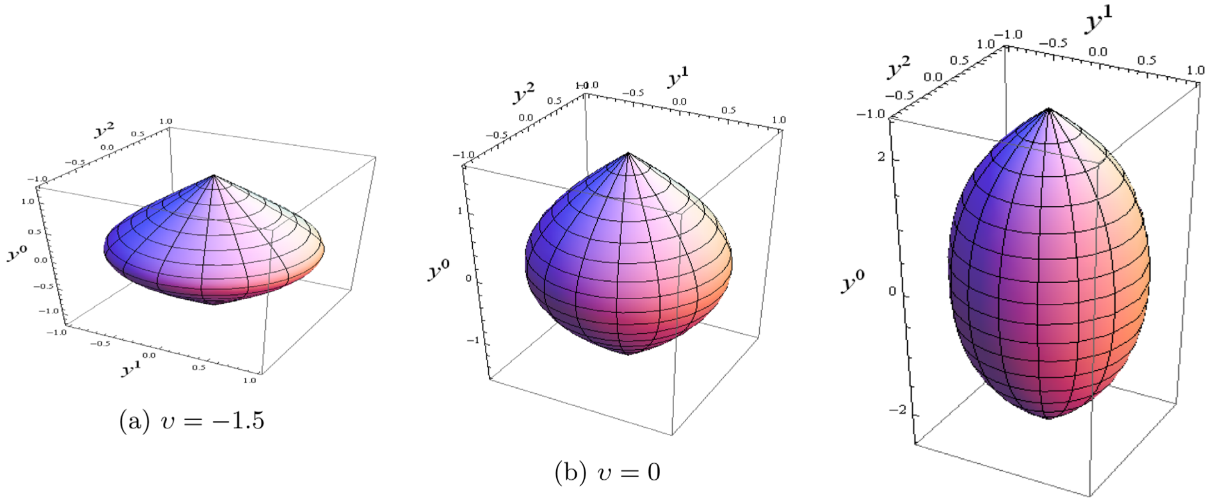
$$\frac{a''}{a} = \left(\frac{a'}{a}\right)^2 - \frac{1}{a^2}. \tag{2.7}$$

The integral of motion is $a/\sqrt{1-a'^2}$, which we shall see later is associated with the energy of a bosonic string. The equations (2.6) are easily solved by

$$a(x^0) = \cos x^0 \qquad h(x^0) = \sec^2 x^0, \tag{2.8}$$

where $-\frac{\pi}{2} \leq x^0 \leq \frac{\pi}{2}$. They are consistent with the boundary values $a(0) = 1$ and $a'(0) = 0$. The corresponding surface is a closed two-dimensional space-time with an initial and final singularity at $x^0 = -\frac{\pi}{2}$ and $x^0 = \frac{\pi}{2}$, respectively. Singularities are not expected to appear in the associated matrix model solution. The surface is pictured in Fig. 1(b).

The above solution is the Lorentzian space counterpart to the minimum area catenoid in Euclidean space. The matrix model analogue of the catenoid in Euclidean space is nontrivial, as it is not associated with any finite dimensional Lie algebra [9]. The same holds for the matrix model solution in the Minkowski space background, and methods



(a) $\upsilon = -1.5$

(b) $\upsilon = 0$

(c) $\upsilon = 0.4$

FIG. 1 (color online). Parametric plots of the closed universe solutions in the three-dimensional embedding space (with time along the vertical direction) for three different values of $\upsilon < \frac{1}{2}$. Subfigure (b) is the solution (2.8). Subfigures (a) and (c) are solutions for $\upsilon = -1.5$ and $\upsilon = 0.4$, respectively, when a cubic term is present in the action [c.f. sec. 3].

similar to those used in [9] can be applied to obtain it. Our interest here is to instead examine stability questions about the solution. For this we restrict our attention to the commutative limit.

In order to examine stability, consider a family of two-dimensional closed surfaces $y^\mu = y^\mu_\epsilon(\tau, \sigma)$ embedded in three-dimensional Minkowski space, where ϵ parametrizes the different surfaces, while τ and σ, $0 \le \sigma < 2\pi$, are the time and space parameters, respectively, spanning any given surface. Let the surface with $\epsilon = 0$ correspond to the classical solution (2.8), $y^\mu_0(\tau, \sigma) = x^\mu(\tau, \sigma)$. An explicit parametrization of the solution $x^\mu(\tau, \sigma)$ is

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} \tau \\ a(\tau)\cos\sigma \\ a(\tau)\sin\sigma \end{pmatrix}, \tag{2.9}$$

which only holds for the restricted time domain $-\frac{\pi}{2} \le \tau \le \frac{\pi}{2}$. We recover (2.5) upon defining the Poisson brackets of any two functions $\mathcal{F}$ and $\mathcal{G}$ of τ and $e^{i\sigma}$ according to

$$\{\mathcal{F}, \mathcal{G}\}(\tau, e^{i\sigma}) = h(\tau)(\partial_\tau \mathcal{F} \partial_\sigma \mathcal{G} - \partial_\sigma \mathcal{F} \partial_\tau \mathcal{G}). \tag{2.10}$$

[More generally, we can use (2.10), with h replaced by a general function of both parameters, to define the Poisson brackets on any of the closed two-dimensional surfaces $y^\mu = y^\mu_\epsilon(\tau, \sigma)$].

In addition to being a solution of (2.3), (2.9) also solves the equations of motion for a classical closed bosonic string. More generally, (2.3) contain the string equations of motion. For this one can introduce the induced metric

$$\mathsf{g}_{\mathsf{ab}}(\tau, \sigma) = \partial_{\mathsf{a}} y^\mu \partial_{\mathsf{b}} y_\mu, \qquad \mathsf{a} = \tau, \sigma, \tag{2.11}$$

on the two-dimensional surface or world sheet. The standard Nambu-Goto action is

$$S_{\mathrm{NG}} = -\mathcal{T} \int d\tau d\sigma \sqrt{-\mathsf{g}}, \tag{2.12}$$

where g is the determinant of the induced metric, and the constant $\mathcal{T}$ denotes the string tension. The equations of motion resulting from (2.12) are

$$\Delta y_\mu = 0, \tag{2.13}$$

where $\Delta = -\frac{1}{\sqrt{-\mathsf{g}}} \partial_{\mathsf{a}} \sqrt{-\mathsf{g}} \mathsf{g}^{\mathsf{ab}} \partial_{\mathsf{b}}$ is the Laplace-Beltrami operator on the world sheet, g^{ab} denotes the components of the inverse induced metric, $\mathsf{g}^{\mathsf{ab}} \mathsf{g}_{\mathsf{bc}} = \delta^{\mathsf{a}}_{\mathsf{c}}$. The string equations (2.13) are identical to (2.3) when the Poisson structure on the world sheet involves the metric tensor, specifically [9]

$$\{\mathcal{F}, \mathcal{G}\}(\tau, e^{i\sigma}) = \frac{1}{\sqrt{-\mathsf{g}}} (\partial_\tau \mathcal{F} \partial_\sigma \mathcal{G} - \partial_\sigma \mathcal{F} \partial_\tau \mathcal{G}), \tag{2.14}$$

which leads to $\{y^\mu, y^\nu\}\{y_\mu, y_\nu\} = -2$. So in comparing with (2.10), we get the condition

$$h = \frac{1}{\sqrt{-\mathsf{g}}}, \tag{2.15}$$

which is in fact satisfied for the solution (2.8).

The string equations of motion (2.13) imply the existence of a conserved current p^{a}_μ on the world sheet, $\partial_{\mathsf{a}} p^{\mathsf{a}}_\mu = 0$, where

$$p^{\mathsf{a}}_\mu = -\mathcal{T} \sqrt{-\mathsf{g}} \mathsf{g}^{\mathsf{ab}} \partial_{\mathsf{b}} y_\mu. \tag{2.16}$$

From p^{a}_μ one can construct the stress-energy tensor in the three-dimensional embedding space, $z = (z^0, z^1, z^2)$

$$T^{\mu\nu}(z) = \int d\sigma d\tau p^{\mathsf{a}\nu} \partial_{\mathsf{a}} y^\mu \delta^3(z - y(\sigma,\tau)), \tag{2.17}$$

satisfying $\frac{\partial}{\partial z^\mu} T^{\mu\nu}(z) = 0$, and thus define an energy E of the string at any given time z^0,

$$E = \int dz^1 dz^2 T^{00}(z). \tag{2.18}$$

Upon evaluating (2.18) for a solution of the form (2.9), one gets

$$E = 2\pi\mathcal{T} \frac{a(z^0)}{\sqrt{1 - a'(z^0)^2}}, \tag{2.19}$$

which using (2.8) gives $2\pi\mathcal{T}$. Compared to the vacuum, the solution (2.8) is energetically disfavored (assuming $\mathcal{T}$ to be positive).

Alternatively, we can address the issue of stability from the perspective of the matrix model (or at least, its commutative limit), which we do next.

C. Stability analysis using the Seiberg-Witten map

Here we consider small perturbations about the above classical solution. For this we shall utilize the commutative limit of the matrix model action (2.1). It is

$$S_c(y) = \frac{1}{4g_c^2} \int d\mu(\tau,\sigma)\{y_\mu, y_\nu\}\{y^\mu, y^\nu\}, \tag{2.20}$$

where g_c is the limiting value of g and $d\mu(\tau,\sigma)$ is an invariant integration measure on the two-dimensional surface. The latter is defined such that $\int d\mu(\tau,\sigma)\{\mathcal{F},\mathcal{G}\}\mathcal{H} = \int d\mu(\tau,\sigma)\mathcal{F}\{\mathcal{G},\mathcal{H}\}$, for arbitrary functions $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ on the world sheet. So upon assuming Poisson brackets (2.10), we can use the measure $d\mu(\tau,\sigma) = d\tau d\sigma/h(\tau)$. Along with Lorentz and translational invariance, the action is invariant under the commutative analogue of the unitary gauge transformations. Here infinitesimal gauge variations have the form $\delta y^\mu = \theta\{\Lambda, y^\mu\}$, where θ again denotes the noncommutativity parameter and Λ is an infinitesimal function on the world sheet. The equations (2.3) follow from extremizing S_c with respect to variations of y^μ. For this we do not have to require the condition (2.15) to hold for arbitrary configurations $y^\mu(\tau,\sigma)$.

Next we wish to evaluate the action (2.20) for small perturbations about the solution $x^\mu(\tau,\sigma)$, defined by (2.4) and (2.5). For this we set

$$y^\mu = x^\mu + \theta A^\mu \tag{2.21}$$

where the noncommutativity parameter θ can be also be regarded as a perturbation parameter and A^μ are three functions on the world sheet. (A^μ are replaced by $3N$ fields if one instead expands about a stack of N coinciding branes.) The perturbations (2.21) induce nonvanishing fluctuations in the induced metric tensor g_{ab} at first order in θ, and thus A_μ affect the space-time geometry. These functions transform as noncommutative gauge potentials up to first order in θ. Infinitesimal gauge variations of A_μ are given by

$$\delta A_\mu = \{\Lambda, x_\mu\} + \theta\{\Lambda, A_\mu\}. \tag{2.22}$$

Using the Poisson brackets (2.10), gauge variations at zeroth order in θ are along the tangential directions of the surface, $\delta A_\mu = h(\tau)(\partial_\tau\Lambda\partial_\sigma x_\mu - \partial_\sigma\Lambda\partial_\tau x_\mu) + \mathcal{O}(\theta)$. These leading order gauge variations are equivalent to area preserving[1] infinitesimal reparametrizations of the surface $y(\tau,\sigma)$:

$$(\tau,\sigma) \to (\tau - \theta h(\tau)\partial_\sigma\Lambda, \sigma + \theta h(\tau)\partial_\tau\Lambda). \tag{2.23}$$

If we now include first order terms, and use the parametrization (2.9), the gauge variations can be written as

$$\begin{aligned} \delta A^0 &= -h(\tau)\partial_\sigma\Lambda + \theta h(\tau)(\partial_\tau\Lambda\partial_\sigma A^0 - \partial_\sigma\Lambda\partial_\tau A^0) \\ \delta A_\pm &= h(\tau)(\pm i a(\tau)\partial_\tau\Lambda - a'(\tau)\partial_\sigma\Lambda)e^{\pm i\sigma} \\ &\quad + \theta h(\tau)(\partial_\tau\Lambda\partial_\sigma A_\pm - \partial_\sigma\Lambda\partial_\tau A_\pm), \end{aligned} \tag{2.24}$$

where $A_\pm = A_1 \pm A_2$.

Using a Seiberg-Witten map [10], the noncommutative potentials A_μ can be reexpressed in terms of commutative gauge potentials, denoted by $(\mathcal{A}_\tau, \mathcal{A}_\sigma)$, on the surface, along with their derivatives. Known expressions for the Seiberg-Witten map on the Moyal plane [23] do not apply in this case since the map must be consistent with the Poisson bracket relations (2.5). Moreover, since the noncommutative potentials A_μ have three components and the commutative potentials have only two, an additional degree of freedom, associated with a scalar field ϕ should be included in the map. Thus $A_\mu = A_\mu[\mathcal{A}_\tau, \mathcal{A}_\sigma, \phi]$. Using the Seiberg-Witten map, commutative gauge transformation, $(\mathcal{A}_\tau, \mathcal{A}_\sigma) \to (\mathcal{A}_\tau + \partial_\tau\lambda, \mathcal{A}_\sigma + \partial_\sigma\lambda)$, for arbitrary functions λ of τ and σ, should induce noncommutative gauge transformations on A_μ: $A_\mu[\mathcal{A}_\tau, \mathcal{A}_\sigma, \phi] \to A_\mu[\mathcal{A}_\tau + \partial_\tau\lambda, \mathcal{A}_\sigma + \partial_\sigma\lambda, \phi]$. For infinitesimal gauge transformations, the latter are given by (2.22), with Λ a function of λ, along with commutative potentials and their derivatives, $\Lambda = \Lambda[\lambda, \mathcal{A}_\tau, \mathcal{A}_\sigma]$.

The Seiberg-Witten map can be obtained order by order in an expansion in θ,

[1] It can be checked that the determinant of the induced metric is gauge invariant up to first order in θ.

$$A_\mu = A_\mu^{(0)} + \theta A_\mu^{(1)} + \mathcal{O}(\theta^2)$$
$$\Lambda = \Lambda^{(0)} + \theta\Lambda^{(1)} + \mathcal{O}(\theta^2). \tag{2.25}$$

Since we wish to expand the action S_c, and hence also y^μ, up to second order in θ, we need to obtain the Seiberg-Witten map for A_μ up to first order. Except for the inclusion of the scalar field, the zeroth order expression for the map is uniquely determined from the zeroth order terms in (2.24). At lowest order in θ, $\Lambda^{(0)} = \lambda$, while the contributions to $A_\mu^{(0)}$ from the commutative gauge potentials are along the tangent directions to the surface, i.e., $A_\mu^{(0)} = h(\tau)(\mathcal{A}_\tau\partial_\sigma x_\mu - \mathcal{A}_\sigma\partial_\tau x_\mu)$ + the scalar field contribution. The scalar field must then be associated with perturbations normal to the surface; i.e. its contribution to $A_\mu^{(0)}$ is proportional to ϕn_μ, $n_\mu = (-a(\tau)a'(\tau), x^1, x^2)$. Thus at zeroth order we may write

$$\begin{aligned} A^{(0)0} &= h(\tau)(-\mathcal{A}_\sigma + a'(\tau)a(\tau)\phi) \\ A_\pm^{(0)} &= h(\tau)e^{\pm i\sigma}(\pm i a(\tau)\mathcal{A}_\tau - a'(\tau)\mathcal{A}_\sigma + a(\tau)\phi) \\ \Lambda^{(0)} &= \lambda. \end{aligned} \tag{2.26}$$

To obtain the first order result we demand consistency with (2.24). This gives

$$\begin{aligned} A^{(1)0} &= h(\tau)\left(\frac{1}{2}\partial_\tau(h(\tau)\mathcal{A}_\sigma^2) + a'(\tau)h(\tau)\mathcal{A}_\tau\partial_\sigma(a(\tau)\phi) - \mathcal{A}_\sigma\partial_\tau(a'(\tau)h(\tau)a(\tau)\phi)\right) \\ A_\pm^{(1)} &= h(\tau)e^{\pm i\sigma}\Big(\mp i\partial_\tau(a(\tau)h(\tau)\mathcal{A}_\tau)\mathcal{A}_\sigma \mp i a(\tau)h(\tau)\mathcal{A}_\tau\mathcal{F}_{\tau\sigma} \pm i h(\tau)a(\tau)\mathcal{A}_\tau\phi + h(\tau)\mathcal{A}_\tau\partial_\sigma(a(\tau)\phi) \\ &\quad - \mathcal{A}_\sigma\partial_\tau(h(\tau)a(\tau)\phi) + \frac{1}{2}\partial_\tau(a'(\tau)h(\tau)\mathcal{A}_\sigma^2) - \frac{1}{2}a(\tau)h(\tau)\mathcal{A}_\tau^2\Big) \\ \Lambda^{(1)} &= -h(\tau)\mathcal{A}_\sigma\partial_\tau\lambda \end{aligned} \tag{2.27}$$

where $\mathcal{F}_{\tau\sigma} = \partial_\tau\mathcal{A}_\sigma - \partial_\sigma\mathcal{A}_\tau$ is the $U(1)$ gauge field on the surface.

To obtain the lowest order action for the scalar field and gauge field on the two-dimensional space-time manifold, we substitute (2.21) and (2.25)–(2.27) into (2.20) and keep only up to quadratic terms in the perturbation parameter θ. After some work, we get

$$\begin{aligned} S_c(y) = &\frac{\theta^2}{g_c^2}\int d\tau d\sigma\sqrt{-\mathsf{g}}\left(\frac{1}{4}\mathcal{F}^{\mathsf{ab}}\mathcal{F}_{\mathsf{ab}} - \frac{1}{2}\partial^{\mathsf{a}}\phi\partial_{\mathsf{a}}\phi - \frac{1}{2}m^2\phi^2\right) \\ &+ S_c(x), \end{aligned} \tag{2.28}$$

where g is again the determinant of the induced metric g_{ab} and m denotes a background-dependent mass for the scalar field. The indices $\mathsf{a}, \mathsf{b}\ldots$ are raised and lowered using the induced metric g_{ab}. This is the usual expression for the action of a scalar field and gauge field, except for the sign in front of the electric field contribution. However, the electric field, which is nondynamical in two dimensions, is decoupled from the scalar and therefore of no concern for dynamics. The explicit expression for the action in terms of the scale factor $a(\tau) = \cos\tau$ is

$$\begin{aligned} S_c(y) = &\frac{\theta^2}{g_c^2}\int d\tau d\sigma\left(-\frac{1}{2a(\tau)^2}\mathcal{F}_{\tau\sigma}^2 + \frac{1}{2}(\partial_\tau\phi)^2 - \frac{1}{2}(\partial_\sigma\phi)^2 \right.\\ &\left. + \frac{1}{a(\tau)^2}\phi^2\right) + S_c(x), \end{aligned} \tag{2.29}$$

where the action evaluated for the classical solution $y^\mu = x^\mu$ is $S_c(x) = -\frac{1}{2g_c^2}\int d\tau d\sigma \cos^2\tau = -\frac{\pi^2}{2g_c^2}$. From (2.29), the scalar field is tachyonic. The system is thus unstable with respect to perturbations normal to the surface. The tachyonic mass squared m^2 is scale dependent. By comparing (2.28) to (2.29), one gets that

$$m^2 = -\frac{2}{a(\tau)^4} \tag{2.30}$$

The tachyonic mass is inversely proportional to the scale squared and is singular in the limit τ tends to $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

III. ADDING A CUBIC TERM

A. Modified matrix equations and two well-known solutions

More solutions to the matrix model are possible upon including a cubic term in the action, which introduces a free parameter to the theory. With this in mind, we replace (2.1) by

$$S(Y) = \frac{1}{g^2}\mathrm{Tr}\left(-\frac{1}{4}[Y_\mu, Y_\nu][Y^\mu, Y^\nu] + \frac{2}{3}i\alpha\epsilon_{\mu\nu\lambda}Y^\mu Y^\nu Y^\lambda\right), \tag{3.1}$$

where α is the free parameter. Our convention for the Levi-Cevita tensor is $\epsilon_{012} = 1$. The equations of motion now read

$$[[Y_\mu, Y_\nu], Y^\nu] + i\alpha\epsilon_{\mu\nu\lambda}[Y^\nu, Y^\lambda] = 0. \tag{3.2}$$

They preserve the symmetries (i)–(iii) of (2.2).

There are two well-known solutions to these equations, and they are associated with finite dimensional Lie algebras. One is the noncommutative de Sitter solution [11,12]. For this one sets $Y^\mu = X^\mu$, where X^μ are the generators of the $2+1$ Lorentz group

$$[X^\mu, X^\nu] = i\alpha\epsilon^{\mu\nu\lambda}X_\lambda. \tag{3.3}$$

An irreducible representation results upon setting the Casimir of the algebra $X^\mu X_\mu$ equal to a constant times the identity. This solution is the Lorentzian space analogue of the fuzzy sphere [13]–[19].

Another solution is the noncommutative cylinder [20–22]. It is given by $Y^\mu = X^\mu$, where X^μ now generate the two-dimensional Euclidean group,

$$[X_0, X_\pm] = \pm 2\alpha X_\pm \qquad [X_+, X_-] = 0, \tag{3.4}$$

with $X_\pm = X_1 \pm iX_2$. The algebra possesses two central elements X_+X_- and $\exp(\frac{\pi i}{\alpha}X_0)$, whose eigenvalues determine the irreducible representations. The eigenvalue of X_+X_- is the radius squared of the noncommutative cylinder, while the eigenvalues of the 'time' operator X_0 are regularly spaced.

We argue that there can be a third solution to (3.2), which is not associated with a finite dimensional Lie algebra and is just a deformation of the previously proposed solution to (2.2). For this we again examine the commutative limit.

B. Solutions in the commutative limit

We again introduce the noncommutativity parameter θ, with the commutative limit corresponding to $\theta \to 0$. In order that the both terms in (3.2) survive in the limit, we need that α goes to zero and is of order θ as $\theta \to 0$:

$$\alpha \to \upsilon\theta, \qquad \upsilon \text{ finite} \tag{3.5}$$

Then (3.2) becomes

$$\{\{y_\mu, y_\nu\}, y^\nu\} + \upsilon\epsilon_{\mu\nu\rho}\{y^\nu, y^\rho\} = 0, \tag{3.6}$$

which generalizes (2.3).

We denote solutions to (3.6) by $y^\mu = x^\mu$. The commutative analogues of (3.3) and (3.4) are examples of such solutions, and they can be expressed in terms of the functions a and h appearing in (2.4) and (2.5). The commutative limit of (3.3) is

$$a^2(x^0) = \frac{1}{\upsilon^2} + (x^0)^2 \qquad h(x^0) = \upsilon, \tag{3.7}$$

while the commutative limit of (3.4) is

$$a = \frac{1}{2\upsilon} \qquad h = 2\upsilon. \tag{3.8}$$

The solutions (3.7) and (3.8) represent the $2D$ de Sitter universe and static universe, respectively. The υ dependence was inserted in $a(x^0)$ in (3.7) and (3.8) in order that the condition (2.15) is satisfied, however this is not a necessary condition to solve (3.6). Both solutions are singular in the limit $\upsilon \to 0$. Also in both cases they lead to linear Poisson brackets.

More generally, one has a solution to (3.6) if the two functions a and h in (2.4) and (2.5) satisfy

$$((aa'h)' + h - 2\upsilon)h = 0 \qquad (2ha' + ah' - 2\upsilon a')ah = 0. \tag{3.9}$$

These equations generalize (2.6). They are satisfied for (3.7) and for (3.8). In addition to these two solutions, one can obtain solutions to (3.9) which are deformations of (2.8). Upon imposing the condition (2.15), we now get the following equation for the scale factor:

$$\frac{a''}{a} = \left(\frac{a'}{a}\right)^2 - \frac{1}{a^2} + \frac{2\upsilon}{a}(1 - a'^2)^{\frac{3}{2}}. \tag{3.10}$$

This yields the integral of the motion $a/\sqrt{1-a'^2} - \upsilon a^2$, and as was the case with $\upsilon = 0$, it can be associated with the energy of a bosonic string, as we shall see later. This integral of the motion leads to the following Friedmann-type equation for the scale factor,

$$\left(\frac{a'}{a}\right)^2 - \frac{1}{a^2} = -\frac{1}{(\mathcal{E} + \upsilon a^2)^2}, \tag{3.11}$$

$\mathcal{E}$ being the integration constant.[2] The solutions to (3.11) resemble familiar cosmological space-times. Solutions can be expressed in terms of inverse elliptic integrals. For the boundary condition, let us assume that a has a turning point at $x^0 = 0$. Then the resulting solutions can describe closed,

[2] In comparing with the usual expression for cosmological evolution (in four space-time dimensions), the right-hand side of (3.11) behaves like a negative scale-dependent energy density. The latter is most significant at small scales. Moreover, from (3.10) one can also identify a scale-dependent pressure term. It too can be negative, thus mimicking dark energy. Of course, it would be more appropriate to compare the results with cosmological solutions to Einstein gravity in two space-time dimensions. However, Einstein gravity does not exist in two space-time dimensions; the Einstein tensor identically vanishes, meaning that the theory cannot support a nonvanishing energy-momentum source (except for a cosmological term). On the other hand, interpretations may be possible in the context of alternative formulations of gravity in two space-time dimensions [24,25], including an interesting $\epsilon \to 0$ limit of Einstein gravity in $2 + \epsilon$ dimensions [26].

stationary or open space-times, the choice depending on the value of υ. Closed two-dimensional space-times, having initial and final singularities at some $x^0 = \pm\tau_0$, occur for $\upsilon < \frac{1}{2}$ (including negative υ). An example, discussed in the previous section, is the case of $\upsilon = 0$, whose solution is given by the simple expression (2.8). It, as well as two other examples of solutions for $\upsilon < \frac{1}{2}$, is exhibited in Fig. 1. The case of $\upsilon = \frac{1}{2}$ coincides with the static or cylindrical space-time solution (3.8), and is shown in Fig. 2(a). Open universe solutions are recovered for $\upsilon > \frac{1}{2}$, examples of which are shown in Figs. 2(b) and 2(c). The case $\upsilon = 1$ coincides with the de Sitter solution (3.7), shown in Fig. 2(c). There are simple expressions for the solutions when $\upsilon = 0$, $\frac{1}{2}$ and 1, pictured in Figs. 1(b), 2(a) and 2(c).

In summary, we have found solutions to (3.6) of the form (2.4) and (2.5). In parametric form they were given by (2.9) and (2.10). For generic values of υ, there are three distinct solutions. They are the de Sitter universe (3.7), the static universe (3.8) and deformations of (2.8), which are solved by inverse elliptic integrals. At some special values of υ, there are fewer than three distinct solutions. For $\upsilon = 0$ there is only one solution, (2.8), corresponding to a closed universe. For $\upsilon = \frac{1}{2}$ and 1, there are two solutions, (3.7) and (3.8), corresponding to the static universe and de Sitter universe, respectively. With the exceptions of $\upsilon = \frac{1}{2}$ and 1, the solutions given in terms of inverse elliptic integrals yield a nonlinear Poisson bracket algebra (2.5). So except for these two cases, the corresponding matrix solutions are nontrivial. They are not investigated here.

We address the question of stability of these solutions first from the perspective of classical strings and then from the perspective of the matrix models.

C. Classical string perspective

As with the case of $\upsilon = 0$, the parametric expression (2.9) solves the equations of motion for a classical closed bosonic string, in addition to solving (3.6). However when $\upsilon \neq 0$, we must add a term, which we denote by $S_{\rm NS}$, to the standard Nambu-Goto action:

$$S_{\rm string} = S_{\rm NG} + S_{\rm NS},$$
$$S_{\rm NS} = -\frac{\upsilon\mathcal{T}}{3}\int \epsilon_{\mu\nu\rho} y^\mu dy^\nu \wedge dy^\rho. \qquad (3.12)$$

It can be regarded as a coupling to a Neveu-Schwarz field of the from $B_{\mu\nu} \propto \epsilon_{\mu\nu\lambda} y^\lambda$. Both terms in the action (2.12) are reparametrization invariant, and respect the Poincaré symmetry in $2+1$ space-time. The Nambu string equations (2.13) are now modified to

$$\Delta y_\mu + 2\upsilon n_\mu = 0, \qquad (3.13)$$

where $n_\mu = \frac{1}{2\sqrt{-\mathrm{g}}}\epsilon^{\mathrm{ab}}\epsilon_{\mu\nu\rho}\partial_{\mathrm{a}} y^\nu \partial_{\mathrm{b}} y^\rho$ is a spacelike unit vector normal to the world sheet and $\epsilon^{\tau\sigma} = -\epsilon^{\sigma\tau} = 1$. The string equations (3.13) are identical to the equations (3.6) when the Poisson structure on the world sheet involves the metric tensor according to (2.14).

Once again, the string equations of motion imply the existence of a conserved current on the world sheet. In comparing with (2.16), it has an additional term,

$$p^{\mathrm{a}}_\mu = -\mathcal{T}\sqrt{-\mathrm{g}}\mathrm{g}^{\mathrm{ab}}\partial_{\mathrm{b}} y_\mu + \upsilon\mathcal{T}\epsilon^{\mathrm{ab}}\epsilon_{\mu\nu\rho} y^\nu \partial_{\mathrm{b}} y^\rho. \qquad (3.14)$$

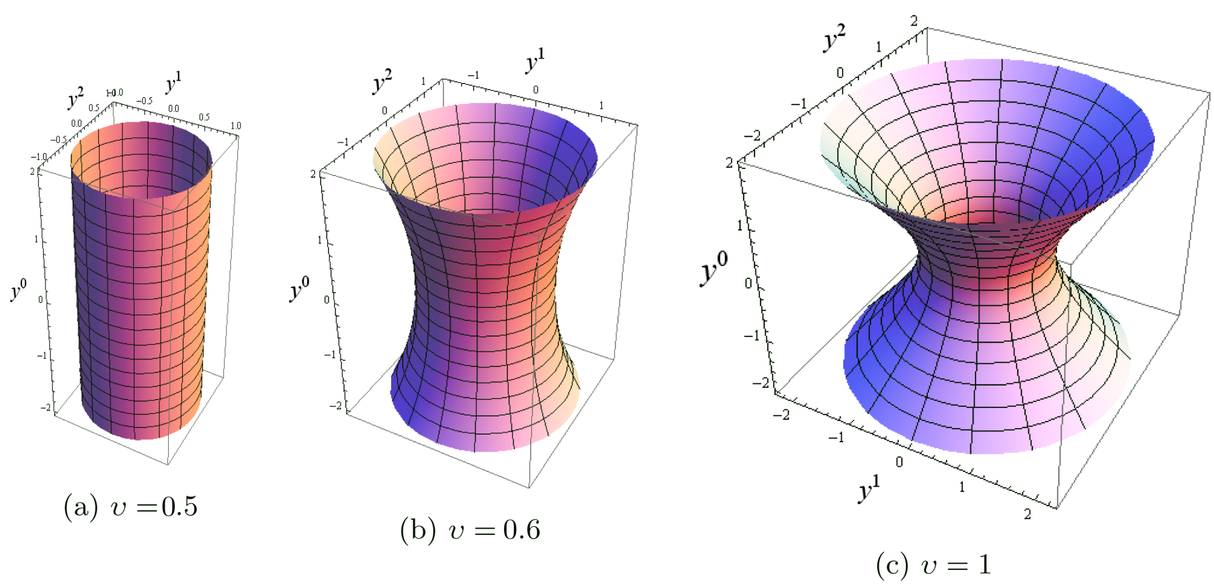


FIG. 2 (color online). Parametric plots of (a) the static universe solution $\upsilon = \frac{1}{2}$ and open universe solutions for two different values (b) $\upsilon = 0.6$ and (c) $\upsilon = 1$ of $\upsilon > \frac{1}{2}$ in the three-dimensional embedding space (with time along the vertical direction) (c) is the de Sitter solution. The boundary conditions are $a(0) = 1$ and $a'(0) = 0$.

Then the string energy (2.18) evaluated for any solution $y^\mu = x^\mu$ of the form (2.9), also acquires an extra term

$$E = 2\pi\mathcal{T}a(z^0)\left(\frac{1}{\sqrt{1-a'(z^0)^2}} - \upsilon a(z^0)\right). \tag{3.15}$$

It is proportional to the integration constant $\mathcal{E}$ appearing in (3.11), $E = 2\pi\mathcal{T}\mathcal{E}$. From the choice of boundary conditions used in Figs. 1 and 2, $\mathcal{E} = 1 - \upsilon$.

Let us compare the string energies of the three different types of solutions. The string energy vanishes when evaluated for the de Sitter solution (3.7), $E|_{\mathrm{dS}^2} = 0$, and so this solution is degenerate with the vacuum. For the case of the cylindrical space-time solution it is instead proportional to υ, $E|_{R\times S} = 2\pi\mathcal{T}\upsilon$. The remaining family of solutions are associated with inverse elliptic integrals, and describe closed universes for $\upsilon < \frac{1}{2}$, the static universe for $\upsilon = \frac{1}{2}$ and open universes for $\upsilon > \frac{1}{2}$. Let us again adopt the boundary conditions used in Figs. 1 and 2, i.e., $a(0) = 1$ and $a'(0) = 0$. The energy for this family of solutions is $E|_{\mathrm{elliptic}^{-1}} = 2\pi\mathcal{T}(1-\upsilon)$. We plot the string energies associated with the three different types of solutions in Fig. 3. The cylindrical space-time solution is the lowest energy configuration in the region $\upsilon < 0$. On the other hand, the dS^2 solution has the minimum energy for $0 < \upsilon \leq 1$, and it is degenerate with the vacuum solution. The family of open universe solutions is the minimum energy configuration for $\upsilon > 1$. The closed universe solution never has the least energy solution, except for the case $\upsilon = 0$, when it is the only nontrivial of the three types of solutions to survive. However it is unstable with respect to decay to the vacuum.

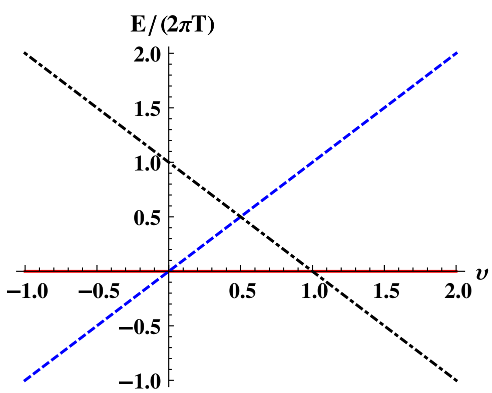


FIG. 3 (color online). Plots of the string energy (divided by $2\pi\mathcal{T}$) for the three types of solutions as a function of υ. The energy for the de Sitter solution is zero (red solid line). The energy for the cylindrical solution is given by the blue dashed line. (The de Sitter solution and cylinder solution are singular at $\upsilon = 0$.) The energy of the family of solutions given by inverse elliptic integrals is given by the black dashed-dotted line. It corresponds to closed universes for $\upsilon < \frac{1}{2}$, the static universe for $\upsilon = \frac{1}{2}$ and open universes for $\upsilon > \frac{1}{2}$. The boundary conditions are $a(0) = 1$ and $a'(0) = 0$.

D. Matrix model perspective

Instead of relying on the string action (3.12), we can analyze stability using the commutative limit of the matrix action (3.1):

$$\begin{aligned} S_c^{\mathrm{total}}(y) &= S_c(y) + S_c^{(3)}(y), \\ S_c^{(3)}(y) &= -\frac{\upsilon}{3g_c^2}\int d\mu(\tau,\phi)\epsilon_{\mu\nu\lambda}y^\mu\{y^\nu, y^\lambda\}, \end{aligned} \tag{3.16}$$

where $S_c(y)$ was given in (2.20) and $d\mu(\tau,\sigma)$ is again an invariant integration measure on the world sheet. Its resulting equations of motion are (3.6), and now we don't have to impose (2.15) for this purpose. We wish to evaluate this action for small perturbations about the three types of solutions to (3.9). The perturbations can again be expressed in terms of noncommutative potentials A_μ, as in (2.21), which can then be rewritten as functions of commutative gauge potentials, $(\mathcal{A}_\tau, \mathcal{A}_\sigma)$ and a scalar field ϕ on the surface using the Seiberg-Witten map (2.25)–(2.27). Upon substituting the map into (3.16), we get the action for small fluctuations in terms of $(\mathcal{A}_\tau, \mathcal{A}_\sigma)$ and ϕ. For all three types of solutions, we recover the terms appearing in (2.28) describing the scalar field and electromagnetism on the two-dimensional space-time manifold (with $m^2 = 0$ for the cylindrical solutions). As before the kinetic energy terms for the gauge and scalar field appear with opposite sign. Now when $\upsilon \neq 0$ we obtain a coupling between the scalar and gauge fields,

$$\begin{aligned} S_c^{\mathrm{total}}(y) = \frac{\theta^2}{g_c^2}\int d\tau d\sigma\sqrt{-\mathsf{g}}\bigg(&\frac{1}{4}\mathcal{F}^{\mathsf{ab}}\mathcal{F}_{\mathsf{ab}} - \frac{1}{2}\partial^{\mathsf{a}}\phi\partial_{\mathsf{a}}\phi - \frac{1}{2}m^2\phi^2 \\ &+ \frac{2\upsilon}{\sqrt{-\mathsf{g}}}\phi\mathcal{F}_{\tau\sigma}\bigg) + S_c^{\mathrm{total}}(x). \end{aligned} \tag{3.17}$$

The explicit expression in terms of the scale factor $a(\tau)$ is

$$\begin{aligned} S_c^{\mathrm{total}}(y) = \frac{\theta^2}{g_c^2}\int d\tau d\sigma\bigg(&-\frac{1}{2a(\tau)\sqrt{1-a'(\tau)^2}}\mathcal{F}_{\tau\sigma}^2 \\ &+ \frac{a(\tau)}{2\sqrt{1-a'(\tau)^2}}(\partial_\tau\phi)^2 - \frac{\sqrt{1-a'(\tau)^2}}{2a(\tau)}(\partial_\sigma\phi)^2 \\ &+ \left(\frac{1}{a(\tau)\sqrt{1-a'(\tau)^2}} - 2\upsilon\right)\phi^2 + 2\upsilon\phi\mathcal{F}_{\tau\sigma}\bigg) \\ &+ S_c^{\mathrm{total}}(x), \end{aligned} \tag{3.18}$$

where the action evaluated for any of the three classical solutions $y^\mu = x^\mu$ can be written as

$$S_c^{\mathrm{total}}(x) = -\frac{\pi}{g_c^2}\int d\tau a(\tau)\left(4\tau\upsilon a'(\tau) + \sqrt{1-a'(\tau)^2}\right). \tag{3.19}$$

The latter quantity is divergent for the solutions describing open and cylindrical space-times. The results agree with

(2.29) in the limit $\upsilon \to 0$. Upon comparing (3.17) and (3.18), the mass squared of the scalar field is

$$m^2 = -\frac{2}{a(\tau)\sqrt{1-a'(\tau)^2}}\left(\frac{1}{a(\tau)\sqrt{1-a'(\tau)^2}} - 2\upsilon\right). \tag{3.20}$$

Evaluating it for the three types of solutions, one finds that it vanishes for the case of cylindrical space-time solutions (3.8), it has a positive value of $2\upsilon^2$ for the case of de Sitter solutions (3.7), while its value *and sign* are scale dependent for the case of general solutions to (3.11) expressed in terms of inverse elliptic integrals. In the latter case, $m^2 = 2(\upsilon^2 - \frac{\mathcal{E}^2}{a(\tau)^4})$, where $\mathcal{E}$ is again the integration constant appearing in (3.11).

One additional step is needed to do the stability analysis due to the coupling of the scalar field to the nondynamical gauge field, which is present when $\upsilon \neq 0$. The gauge field can be eliminated using its equation of motion, $\frac{\mathcal{F}_{\tau\sigma}}{a(\tau)\sqrt{1-a'(\tau)^2}} = 2\upsilon\phi + \text{constant}$. After substituting back into the action, the mass squared for the scalar field (3.20) gets modified to

$$m^2_{\text{eff}} = -2\left\{\upsilon^2 + \left(\frac{1}{a(\tau)\sqrt{1-a'(\tau)^2}} - \upsilon\right)^2\right\}. \tag{3.21}$$

It is negative definite. *Thus the scalar field is tachyonic for all solutions*. Moreover, the effective mass squared is scale dependent for the family of solutions given in terms of inverse elliptic integrals. The results for the three cases are

$$m^2_{\text{eff}} = \begin{cases} -4\upsilon^2, & \text{cylindrical solution} \\ -2\upsilon^2, & \text{dS}^2\,\text{solution} \\ -2\left(\upsilon^2 + \frac{\mathcal{E}^2}{a(\tau)^4}\right), & \text{inverse elliptic integral solution.} \end{cases} \tag{3.22}$$

Recall that the cylindrical solution and dS^2 solution are singular in the limit $\upsilon \to 0$, so the scalar field can never be massless. The result for the inverse elliptic integral solution agrees with what we found previously (2.30) in the absence of the cubic term. Thus all of the solutions examined here were found to be unstable with respect to perturbations normal to the surface.

IV. CONCLUDING REMARKS

The results obtained here indicate that instabilities may be a common feature of the cosmological solutions to simple matrix models. Additional terms may be included in the IKKT matrix model action in order to cure the above instabilities. A quartic term, with a suitably adjusted coefficient, was shown to stabilize the cylindrical space-time solution [22]. The quadratic term $\text{Tr}(Y_\mu Y^\mu)$ is sufficient to cure the instability of the noncommutative dS^2 solution [27]. (A quadratic term was also included to the Banks-Fischler-Shenker-Susskind model in [5] and it played a role of a cosmological term in the Friedmann equations.) Similar such stabilization mechanisms should be possible for the inverse elliptic integral solutions, which were associated with closed and open universes. The tachyonic mass was found to be scale dependent for the inverse elliptic integral solutions, and is larger at smaller distance scales. The inclusion of extra terms in this case could produce a transition to a stable solution at a certain distance scale. This may be of use for inflationary models, as it is analogous to the transition from an inflationary to noninflationary phase.

Of course, it is of interest to generalize the $2D$ solutions studied here to higher dimensions, and see whether realistic cosmological space-times arise from such models. It may be a nontrivial problem, however, to insure full rotation invariance in an arbitrary number of dimensions. In that case we want to replace (2.4) by $(x^1)^2 + (x^2)^2 + \cdots + (x^d)^2 = a^2(x^0)$, $d > 2$, but the generalization of the Poisson brackets (2.5) to more than two spatial embedding dimensions is not obvious. On the other hand, one can examine solutions which are obtained by taking products of the lower dimensional noncommutative spaces examined here. We plan to explore these issues in coming works.

ACKNOWLEDGMENTS

I am very grateful to A. Chaney, D. Grumiller, B. Harms, L. Lu, A. Pinzul and H. Steinacker for valuable discussions.

Note added.—The work presented here is similar in spirit to that of Klammer and Steinacker [7], where four-dimensional cosmological solutions were found to have many desirable features. Our work differs in that we do not need to construct alternatives to the standard induced metric in two space-time dimensions and we require no Wick rotations. The instabilities which we find here are independent of the choice of metric tensor.

[1] N. Ishibashi, H. Kawai, Y. Kitazawa, and A. Tsuchiya, Nucl. Phys. **B498**, 467 (1997).
[2] T. Banks, W. Fischler, S. H. Shenker, and L. Susskind, Phys. Rev. D **55**, 5112 (1997).
[3] H. Aoki, S. Iso, H. Kawai, Y. Kitazawa, and T. Tada, Prog. Theor. Phys. **99**, 713 (1998); H. Aoki, S. Iso, H. Kawai, Y. Kitazawa, A. Tsuchiya, and T. Tada, Prog. Theor. Phys. Suppl. **134**, 47 (1999).
[4] H. Steinacker, Classical Quantum Gravity **27**, 133001 (2010).
[5] D. Z. Freedman, G. W. Gibbons, and M. Schnabl, AIP Conf. Proc. **743**, 286 (2005).
[6] B. Craps, S. Sethi, and E. P. Verlinde, J. High Energy Phys. 10 (2005) 005.
[7] D. Klammer and H. Steinacker, Phys. Rev. Lett. **102**, 221301 (2009).
[8] S. W. Kim, J. Nishimura, and A. Tsuchiya, Phys. Rev. Lett. **108**, 011601 (2012); Phys. Rev. D **86**, 027901 (2012); J. High Energy Phys. 10 (2012) 147.
[9] J. Arnlind and J. Hoppe, Phys. Lett. B **723**, 397 (2013).
[10] N. Seiberg and E. Witten, J. High Energy Phys. 09 (1999) 032.
[11] P.-M. Ho and M. Li, Nucl. Phys. **B590**, 198 (2000); **B596**, 259 (2001).
[12] D. Jurman and H. Steinacker, J. High Energy Phys. 01 (2014) 100.
[13] J. Madore, Classical Quantum Gravity **9**, 69 (1992).
[14] H. Grosse and P. Presnajder, Lett. Math. Phys. **33**, 171 (1995).
[15] U. Carow-Watamura and S. Watamura, Commun. Math. Phys. **212**, 395 (2000).
[16] G. Alexanian, A. Pinzul, and A. Stern, Nucl. Phys. **B600**, 531 (2001).
[17] B. P. Dolan, D. O'Connor, and P. Presnajder, J. High Energy Phys. 03 (2002) 013.
[18] A. P. Balachandran, S. Kurkcuoglu, and S. Vaidya, *Lectures on Fuzzy and Fuzzy SUSY Physics* (World Scientific, Singapore, 2007).
[19] S. Iso, Y. Kimura, K. Tanaka, and K. Wakatsuki, Nucl. Phys. **B604**, 121 (2001).
[20] M. Chaichian, A. Demichev, P. Presnajder, and A. Tureanu, Eur. Phys. J. C **20**, 767 (2001); Phys. Lett. B **515**, 426 (2001).
[21] A. P. Balachandran, T. R. Govindarajan, A. G. Martins, and P. Teotonio-Sobrinho, J. High Energy Phys. 11 (2004) 068.
[22] A. Stern, Phys. Rev. D **89**, 104051 (2014).
[23] See for example, K. Okuyama, J. High Energy Phys. 03 (2000) 016; H. Liu, Nucl. Phys. **B614**, 305 (2001); S. Mukhi and N. V. Suryanarayana, J. High Energy Phys. 05 (2001) 023; B. Jurco, P. Schupp, and J. Wess, Nucl. Phys. **B604**, 148 (2001); Y. Okawa and H. Ooguri, Phys. Rev. D **64**, 046009 (2001); H. Liu and J. Michelson, Phys. Lett. B **518**, 143 (2001); J. M. Grimstrup, Acta Phys. Pol. B **34**, 4855 (2003); A. Pinzul and A. Stern, Int. J. Mod. Phys. A **20**, 5871 (2005); K. Ulker and B. Yapiskan, Phys. Rev. D **77**, 065006 (2008); A. Stern, Phys. Rev. D **80**, 067703 (2009).
[24] R. Jackiw, arXiv:gr-qc/9511048.
[25] D. Grumiller, W. Kummer, and D. V. Vassilevich, Phys. Rep. **369**, 327 (2002).
[26] D. Grumiller and R. Jackiw, in *Recent Developments in Theoretical Physics*, edited by S. Gosh and G. Kar (World Scientific, Singapore, 2010).
[27] A. Chaney, L. Lu, and A. Stern (to be published).