Noncommutative Static Strings from Matrix Models

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Noncommutative static strings from matrix models

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We examine the noncommutative cylinder solution to a matrix model with a Minkowski background metric. It can be regarded as the noncommutative analogue of a static circular string. Perturbations about the solution yield a tachyonic scalar field (and an additional tachyonic fermion in the full supersymmetric version of the model) in the commutative limit. The tachyonic mode is attributed to the fact that the circular string is unstable under uniform adiabatic deformations. We obtain a stabilizing term which, when added to the matrix model, removes the tachyonic mass.

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I. INTRODUCTION

Matrix models were introduced to capture nonperturbative aspects of string theory [1], [2]. Space-time geometry, field theory and even gravity can dynamically emerge from matrix models [3], [4]. The resulting space-time geometry is a feature of the solutions to matrix models. In the generic case, it is noncommutative, with a straightforward commutative limit. A well-studied example is the fuzzy sphere, which has finite dimensional representations and a commutative limit corresponding to the sphere [5], [6], [7], [8], [9], [10]. Perturbations about this solution yields a gauge theory on the fuzzy sphere [11]. The background metric in this case is Euclidean. Counterparts in Minkowski space-time are the fuzzy de Sitter and anti-de Sitter solutions, which have infinite dimensional representations and have been studied in [12], [13]. Here we exam the noncommutative cylinder solution, which also has infinite dimensional representations. This solution, whose commutative limit is the cylinder (or static circular string), holds in either Euclidean or Minkowski space, although we specialize to the latter where the noncompact direction corresponds to the time. The noncommutative cylinder has novel features, such as a discrete spectrum for the time operator, [14], [15], [16] and it appears in a noncommutative version of Bañados, Teitelboim, Zanelli (BTZ) black hole geometry [17]. Here we shall show that the field theory resulting from perturbations about the noncommutative cylinder solution describe a tachyonic scalar, and this is due to the fact that the static circular string is unstable under uniform adiabatic deformations. We propose the addition of a term to the matrix model which removes this instability.

The matrix model setting for this article is a three-dimensional version of the Ishibashi, Kawai, Kitazawa and Tsuchiya (IKKT) model, with a cubic term included in the bosonic action. It is identical to the matrix model used in [13]. The target space for the bosonic sector is thus spanned by three infinite dimensional Hermitean matrices, which transform covariantly under the action of the $2+1$ Lorentz group. The fuzzy de Sitter and anti-de Sitter solutions of [13] are preserved under the homogeneous symmetry transformations of the target space. On the other hand, the noncommutative cylinder is an example of a symmetry breaking solution, where $2+1$ Lorentz symmetry of the target space is broken to $SO(2)\times\tilde{T}$. $SO(2)$ corresponds to rotations in the plane, while $\tilde{T}$ refers to discrete time translations. Perturbations about the solution lead to a nontrival noncommutative field theory, which in the commutative limit describes a scalar field coupled to a $U(1)$ gauge field on the cylinder. The gauge field has no dynamics, since it lives on a two dimensional space-time, and instead induces a tachyonic mass to the scalar field. As stated above, this can be traced back to the instability of the classical string solution. We show that the instability is cured with the addition of explicit symmetry breaking terms to the matrix model action. The full supersymmetric theory also yields a tachyonic fermionic field in the commutative limit. This tachyonic instability is also easily eliminated with a supersymmetry breaking contribution to the action.

The outline of this article is the following: We begin with closed classical string solutions in Sec. II. The matrix model analogues are discussed in Sec. III, while perturbations about the noncommutative cylinder solution are given in Sec. IV. We consider the generalization to $U(N)$ gauge theory and the fermionic sector of the $\mathcal{N}=1$ supersymmetric extension in Sec. V. The stabilizing term is discussed in Sec. VI, and a topological model is considered in Sec. VII. Some concluding remarks are made in Sec. VIII. We review the Moyal-Weyl star product on a noncommutative cylinder in Appendix A, while our spinor conventions are given in Appendix B.

II. STATIC STRING SOLUTION

We start with a closed string in $2+1$ Minkowski space-time, and denote the embedding coordinates by x^μ, $\mu=0$, 1, 2. ξ^a, $a=0$, 1, will parametrize the string world sheet

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$\Sigma = R^1 \times S^1$, where ξ^0 $(-\infty < \xi^0 < \infty)$ is the time coordinate and ξ^1 $(-\pi \le \xi^1 < \pi)$ the space coordinate. For the action we take the standard Nambu-Goto form plus an additional interaction term which we denote by $S_{\rm NS}$

$$S_{\rm string} = -\mathcal{T}\int d^2\xi\sqrt{-{\rm g}} + S_{\rm NS}, \qquad (2.1)$$

where $\sqrt{\rm g}$ is the determinant of the induced metric ${\rm g}_{ab} = \partial_a x^\mu \partial_b x_\mu$ on Σ, $\partial_a = \frac{\partial}{\partial \xi^a}$ and the constant $\mathcal{T}$ denotes the string tension. Our convention for the background space-time metric is $\eta = {\rm diag}(-1, 1, 1)$. The second term in (2.1) is

$$S_{\rm NS} = -\frac{\mathcal{T}}{6\rho}\int \epsilon_{\mu\nu\rho} x^\mu dx^\nu \wedge dx^\rho, \qquad (2.2)$$

where the constant ρ has units of length. It can be regarded as a coupling to a Neveu-Schwarz field of the from $B_{\mu\nu} \propto \epsilon_{\mu\nu\lambda}x^\lambda$. Both terms in the action (2.1) are reparametrization invariant, and respect the Poincaré symmetry of $2+1$ Minkowski space-time.

Upon extremizing the total action with respect to variations in coordinates x^μ one gets

$$\Delta x_\mu + \frac{1}{\rho} n_\mu = 0, \qquad (2.3)$$

where $\Delta = -\frac{1}{\sqrt{-{\rm g}}}\partial_a\sqrt{-{\rm g}}{\rm g}^{ab}\partial_b$ is the Laplace-Beltrami operator on Σ, ${\rm g}^{ab}$ denotes the components of the inverse induced metric, ${\rm g}^{ab}{\rm g}_{bc} = \delta^a_c$, and $n_\mu = \frac{1}{2\sqrt{-{\rm g}}}\epsilon^{ab}\epsilon_{\mu\nu\rho}\partial_a x^\nu \partial_b x^\rho$ is a space-like unit vector normal to Σ. (Our conventions are $\epsilon^{01} = \epsilon_{012} = 1$.) The equations of motion imply the existence of a conserved current p^a_μ on the world sheet

$$\partial_a p^a_\mu = 0, \qquad p^a_\mu = -\mathcal{T}\sqrt{-{\rm g}}{\rm g}^{ab}\partial_b x_\mu + \frac{\mathcal{T}}{2\rho}\epsilon^{ab}\epsilon_{\mu\nu\rho}x^\nu\partial_b x^\rho. \qquad (2.4)$$

From p^a_μ one can then construct the stress-energy tensor in the three-dimensional embedding space,

$$T^{\mu\nu}(y) = \int d^2\xi p^{a\nu}\partial_a x^\mu \delta^3(y - x(\xi)), \qquad (2.5)$$

satisfying $\frac{\partial}{\partial y^\mu}T^{\mu\nu}(y) = 0$.

The equations of motion (2.3) are satisfied by [13]

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix} = 2\rho \begin{pmatrix} \tan\xi^0 \\ \sec\xi^0\cos\xi^1 \\ \sec\xi^0\cos\xi^1 \end{pmatrix}, \qquad (2.6)$$

leading to

$$-(x^0)^2 + (x^1)^2 + (x^2)^2 = 4\rho^2. \qquad (2.7)$$

It corresponds to two-dimensional de Sitter space ${\rm dS}^2$.[1] Here we must restrict the range of ξ^0 to $-\frac{\pi}{2} \le \xi^0 \le \frac{\pi}{2}$. This is a zero energy configuration. For this we define the energy of the string at a given time y^0 as $\int dy^1 dy^2 T^{00}(y)$. Moreover, all components of the stress-energy tensor vanish in this case. Equation (2.6) is thus degenerate with the vacuum solution.

Another solution to the system, which is of interest in this article, is the closed static string

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} \xi^0 \\ \rho\cos\xi^1 \\ \rho\sin\xi^1 \end{pmatrix}, \qquad (2.8)$$

where the world sheet is a cylinder of radius ρ,

$$(x^1)^2 + (x^2)^2 = \rho^2. \qquad (2.9)$$

The energy of this string configuration is nonzero; More specifically, the energy equals the string tension times one-half the circumference, $\int dy^1 dy^2 T^{00}(y) = \pi\rho\mathcal{T}$.

Next we define Poisson brackets on the world sheet Σ. We choose [18]

$$\{\xi_0, e^{i\xi^1}\} = \frac{ie^{i\xi^1}}{\sqrt{-{\rm g}}}. \qquad (2.10)$$

More generally, the Poisson bracket of any two functions f and h of the string parameters ξ is

$$\{f(\xi), h(\xi)\} = \frac{1}{\sqrt{-{\rm g}}}\epsilon^{ab}\partial_a f \partial_b h. \qquad (2.11)$$

With this definition the equations of motion (2.3) can be reexpressed in terms of the Poisson algebra

$$\{\{x_\mu, x_\nu\}, x^\nu\} + \frac{1}{2\rho}\epsilon_{\mu\nu\rho}\{x^\nu, x^\rho\} = 0. \qquad (2.12)$$

The ${\rm dS}^2$ solution (2.6) corresponds to the $SO(2,1)$ Poisson structure,

$$\{x_\mu, x_\nu\} = \frac{1}{2\rho}\epsilon_{\mu\nu\lambda}x^\lambda, \qquad (2.13)$$

and thus respect the $2+1$ Lorentz symmetry of the background space. On the other hand, for the static string solution (2.8), one gets the algebra of the two-dimensional Euclidean group

[1] ${\rm AdS}^2$ solutions result if the background space-time metric is changed to $\eta = {\rm diag}(-1, -1, 1)$, but we will not consider that possibility. Our focus instead will be on the static solution (2.8).

$$\{x^0, x^1\} = -\frac{1}{\rho} x^2, \qquad \{x^0, x^2\} = \frac{1}{\rho} x^1, \qquad \{x^1, x^2\} = 0. \tag{2.14}$$

This solution is invariant only under rotations in the plane and continuous time translations.

III. NONCOMMUTATIVE STATIC STRING

The equations of motion (2.12), and the solutions (2.13) and (2.14), have a straightforward noncommutative generalization. For this we replace the three embedding coordinates x^μ by infinite-dimensional self-adjoint matrices X^μ, and Poisson brackets by commutators. Then the noncommutative version of (2.12) is

$$[[X_\mu, X_\nu], X^\nu] + i\alpha\epsilon_{\mu\nu\lambda}[X^\nu, X^\lambda] = 0, \tag{3.1}$$

where α is a constant with units of length. Equation (2.12) can be regarded as the commutative limit of this matrix equation. For this we should introduce some noncommutativity parameter, say θ, such that $X^\mu \to x^\mu$ when $\theta \to 0$, and to lowest order in θ, $[f(X), h(X)] \to i\theta\{f(x), h(x)\}$. In order to recover (2.12) we also need that α vanishes in the limit $\theta \to 0$, specifically, $\alpha \to \frac{\theta}{2\rho}$. The equations of motion (3.1) are invariant under: (i) Lorentz transformations $X^\mu \to \Lambda^\mu{}_\nu X^\nu$, where Λ is a 3×3 Lorentz matrix, (ii) translations in the three-dimensional Minkowski space $X^\mu \to X^\mu + a^\mu \mathbb{1}$, where $\mathbb{1}$ is the unit matrix, and (iii) unitary "gauge" transformations, $X^\mu \to U X^\mu U^\dagger$, where U is an infinite dimensional unitary matrix. In Sec. V, we include the fermionic sector so that the system has an additional $\mathcal{N} = 1$ supersymmetry.

The equations of motion (3.1) can be obtained from an action principle, with the matrix model action given by [11]

$$S(X) = \frac{1}{g^2} \mathrm{Tr}\left(-\frac{1}{4}[X_\mu, X_\nu][X^\mu, X^\nu] + \frac{2}{3} i\alpha\epsilon_{\mu\nu\lambda} X^\mu X^\nu X^\lambda \right). \tag{3.2}$$

Here we introduce the coupling constant g and Tr is an invariant trace. The first term is the standard IKKT kinetic energy, while the second term is the matrix analogue of (2.2). Extremizing $S(X)$ with respect to variations in X_μ gives (3.1).

The matrix analogue of the dS^2 solution (2.13) is $X^\mu = X^\mu_{(\mathrm{dS})}$, where $X^\mu_{(\mathrm{dS})}$ are defined by the commutation relations

$$[X^\mu_{(\mathrm{dS})}, X^\nu_{(\mathrm{dS})}] = i\alpha\epsilon^{\mu\nu\lambda} X_{(\mathrm{dS})\lambda}. \tag{3.3}$$

The commutation relations are preserved under the action of the $2+1$ Lorentz group, and moreover, they define the $so(2,1)$ Lie algebra. $X^\mu_{(\mathrm{dS})} X_{(\mathrm{dS})\mu}$ is a Casimir of the algebra, and so an irreducible representation is obtained by setting $X^\mu_{(\mathrm{dS})} X_{(\mathrm{dS})\mu} = 4\rho^2 \mathbb{1}$. This solution has been studied previously in [12] [13], while its Euclidean space counterpart, the fuzzy sphere, has been known for a long time [5–11].

The focus in this article is the matrix analogue of the static string solution (2.14). It is given by $X^\mu = X^\mu_{(0)}$, where $X^\mu_{(0)}$ satisfy

$$[X_{(0)0}, X_{(0)\pm}] = \pm 2\alpha X_{(0)\pm}, \qquad [X_{(0)+}, X_{(0)-}] = 0, \tag{3.4}$$

and $X_{(0)\pm} = X_{(0)1} \pm i X_{(0)2}$. This solution breaks the $2+1$ Lorentz symmetry of the target space to $SO(2) \times \tilde{T}$. $SO(2)$ corresponds to rotations in the plane, while $\tilde{T}$ refers to discrete time translations (whose action is defined below). The solution is known as the noncommutative or fuzzy cylinder. We note that this solution survives when the background metric is changed to a Euclidean metric. The algebra generated by $X_{(0)\mu}$ is the two dimensional Euclidean group. $X_{(0)+}X_{(0)-}$ is in the center of the algebra, and in an irreducible representation it is proportional to the identity,

$$X_{(0)+} X_{(0)-} = \rho^2 \mathbb{1}. \tag{3.5}$$

ρ can now be regarded as the radius of the noncommutative cylinder, and we can write $X_{(0)+} = \rho e^{i\hat{\phi}}$, where $e^{i\hat{\phi}}$ is a unitary operator. Another central element is $\exp(\frac{\pi i}{\alpha} X_{(0)0})$. In an irreducible representation it is a phase times the identity, $e^{i\phi_0}\mathbb{1}$. Irreducible representation are thus labeled by both ρ and ϕ_0, $\{\rho > 0, 0 \le \phi_0 < 2\pi\}$. The eigenvectors v_n of $X_{(0)0}$ satisfy

$$X_{(0)0} v_n = \tau_n v_n \qquad X_{(0)\pm} v_n = \rho v_{n\pm 1}, \tag{3.6}$$

where $\tau_n = (2n + \phi_0/\pi)\alpha$, n running over all integers. The eigenvectors v_n form a basis for the irreducible representations of the noncommutative cylinder. Here one sees that the action of the discrete time translation operator $\tilde{T}$ is just $\tau_n \to \tau_n + 2\alpha\Delta n$, where Δn is an integer.

IV. PERTURBATIONS ABOUT THE NONCOMMUTATIVE CYLINDER

Next we consider perturbations about the noncommutative cylinder,

$$X_\mu = X_{(0)\mu} + 2\alpha A_\mu, \tag{4.1}$$

where A_μ are fields on the noncommutative cylinder in some irreducible representation. A_μ are thus functions of $e^{i\hat{\phi}}$ and $X_{(0)0}$. It is useful to define the field strengths

$$F_{+-} = \frac{1}{4\alpha^2}[X_+, X_-],$$
$$F_{\pm 0} = \frac{1}{4\alpha^2}[X_\pm, X_0] \pm \frac{1}{2\alpha} X_\pm, \tag{4.2}$$

where $X_\pm = X_1 \pm iX_2$. They transform covariantly under unitary gauge transformations and vanish when $A_\mu = 0$. A_μ and $F_{\mu\nu}$ are defined to be dimensionless. Upon substituting (4.1) and (4.2) into the action (3.2), one gets

$$S(X) = \frac{16\alpha^4}{g^2}\,\mathrm{Tr}\bigg\{\frac{1}{8}(F_{+-})^2 + \frac{1}{2}F_{+0}F_{-0} - \frac{1}{2}(A_+F_{-0} - A_-F_{+0} - A_0F_{+-}) - \frac{1}{4\alpha}(X_{(0)+}[A_-,A_0] - X_{(0)-}[A_+,A_0] - X_{(0)0}[A_+,A_-]) - \frac{1}{2}A_+A_-\bigg\} + S(X_{(0)}). \tag{4.3}$$

The action evaluated for the solution, $S(X_{(0)})$, is singular.

We next replace the potentials and field strengths by their corresponding Weyl symbols, and their matrix products by the Moyal-Weyl star product on the cylinder. The latter was given in [14], and is reviewed in Appendix A. So now A_μ denote functions of a phase $e^{i\phi}$ and τ_n, and the Weyl symbols of the field strengths (4.2) are given by

$$F_{+-} = \frac{\rho}{2\alpha}([e^{i\phi}, A_-]_\star - [e^{-i\phi}, A_+]_\star) + [A_+, A_-]_\star$$
$$F_{\pm 0} = \frac{\rho}{2\alpha}[e^{\pm i\phi}, A_0]_\star + [A_\pm, A_0]_\star + (i\partial_\phi \pm 1)A_\pm, \tag{4.4}$$

where $[,]_\star$ denotes the star commutator. In obtaining the action we replace the trace in (4.3) by $\frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi \sum_{n=0,\pm1,\pm2,\ldots}$, where the sum is over all eigenvalues τ_n of $X_{(0)0}$. Thus

$$S(X) = \frac{8\alpha^4}{\pi g^2}\int_{-\pi}^{\pi} d\phi \sum_{n=0,\pm1,\pm2,\ldots}\bigg\{\frac{1}{8}(F_{+-})^2_\star + \frac{1}{2}F_{+0}\star F_{-0} - \frac{1}{2}(A_+\star F_{-0} - A_-\star F_{+0} - A_0\star F_{+-}) - \frac{\rho}{4\alpha}([e^{i\phi}, A_-]_\star - [e^{-i\phi}, A_+]_\star)A_0 - \frac{i}{2}\partial_\phi A_+\star A_- - \frac{1}{2}A_+\star A_-\bigg\} + S(X_{(0)}), \tag{4.5}$$

where $F_\star^2 = F\star F$. The resulting equations of motion are

$$D_+F_{-0} + D_-F_{+0} = -iF_{+-},$$
$$D_+F_{+-} - 2D_0F_{+0} = 0, \tag{4.6}$$

where the noncommutative covariant derivatives are defined by

$$D_0F = \partial_\phi F + i[A_0, F]_\star$$
$$D_\pm F = i\left[\frac{\rho}{2\alpha}e^{\pm i\phi} + A_\pm, F\right]_\star. \tag{4.7}$$

The dynamical degrees of freedom A_μ, $\mu = 0, 1, 2$, are noncommutative gauge potentials in the $2+1$ dimensional target space. The system can also be expressed in terms of gauge potentials on the noncommutative cylinder. Of course, there are only two of the latter, which we denote by a_τ and a_ϕ. This means that A_μ contains one additional degree of freedom, which we call b and assume to be in the adjoint representation of the noncommuative gauge group. In order to make the identification, we compare noncommutative gauge transformations of A_μ with those of a_τ, a_ϕ and b. Infinitesimal gauge variations of the former have the form $\delta A_\mu = D_\mu\lambda$, λ being an infinitesimal function of τ_n and $e^{i\phi}$. Consequently, $\delta F_{\mu\nu} = -i[\lambda, F_{\mu\nu}]_\star$. For the latter we want

$$\delta a_\phi = \partial_\phi\lambda + i[a_\phi, \lambda]_\star$$
$$\delta e^{i(\phi - 2\alpha a_\tau)} = i[e^{i(\phi-2\alpha a_\tau)}, \lambda]_\star$$
$$\delta b = i[b, \lambda]_\star. \tag{4.8}$$

It is evident that A_0 transforms as a_ϕ, and so we identify the two, $A_0 = a_\phi$. Therefore $A_\pm$ contain both a_τ and b. An identification which is consistent with the variations (4.8) is

$$2\alpha A_+ = (\rho + 2\alpha b)\star e^{i(\phi - 2\alpha a_\tau)} - \rho e^{i\phi}$$
$$2\alpha A_- = e^{-i(\phi-2\alpha a_\tau)}\star(\rho + 2\alpha b) - \rho e^{-i\phi}, \tag{4.9}$$

where $\rho e^{\pm i\phi}$ was subtracted off on the right-hand side in order to have the correct commutative limit $\alpha \to 0$.

We now examine the action in the commutative limit $\alpha \to 0$. The eigenvalues τ_n become continuous in this limit, $\tau_n \to \tau$, and so we recover the commutative cylinder, while the sum in (4.5) is replaced by $\frac{1}{2\alpha}\int d\tau$. The potentials (a_τ, a_ϕ) on the cylinder undergo standard $U(1)$ gauge variations, $\delta(a_\tau, a_\phi) = (\partial_\tau\lambda, \partial_\phi\lambda)$ in the limit, and b is gauge invariant at zeroth order in α. From (4.9) we get

$$A_\pm \to (b \mp i\rho a_\tau)e^{\pm i\phi}, \tag{4.10}$$

in addition to $A_0 = a_\phi$. Substituting in (4.4) and taking the commutative limit gives

$$F_{+-} \to -2\rho\partial_\tau b, \qquad F_{\pm 0} \to (i\partial_\phi b \mp \rho f_{\tau\phi})e^{\pm i\phi}, \tag{4.11}$$

where $f_{\tau\phi} = \partial_\tau a_\phi - \partial_\phi a_\tau$ is the $U(1)$ field strength on the cylinder. Finally, substituting (4.10) and (4.11) into the action (4.5) and taking the commutative limit gives

$$S(X) - S(X_{(0)}) \to \frac{4\alpha^3}{\pi g^2}\int_{-\pi}^{\pi} d\phi \int_{-\infty}^{\infty} d\tau \left(-\frac{\rho^2}{2}(f_{\tau\phi})^2 + \frac{\rho^2}{2}(\partial_\tau b)^2 - \frac{1}{2}(\partial_\phi b)^2 - \rho b f_{\tau\phi} \right). \tag{4.12}$$

It describes a scalar field coupled to a $U(1)$ gauge field on the cylinder. In comparing with the corresponding noncommutative limit for the fuzzy sphere [11], it is interesting to note that, unlike there, no explicit mass term appears for the scalar field. However, the $U(1)$ gauge field is not dynamical in one spatial dimension, and can be eliminated using the equations of motion,

$$f_{\tau\phi} = -\frac{b}{\rho} + \text{constant}. \tag{4.13}$$

The result is a tachyonic mass term for the scalar field. The tachyonic mass is $m^2 = -1/\rho^2$, which vanishes in the infinite radius limit.

Nonlocal interactions will appear upon including noncommutative corrections to the action (4.12), but they are unlikely to cure the tachyonic instability present at lowest order. Actually, the presence of a tachyon may be traced to the instability of the classical solution (2.8). The instability arises from uniform adiabatic excitations of the radial degree of freedom. For this, replace the constant ρ in (2.8) by $\rho + 2\alpha b(\xi^0, \xi^1)$. One can compute the energy for this string configuration using the previous definition (2.5) for the stress-energy tensor, where we identify $\xi^0 = \tau$ and $\xi^1 = \phi$. After expanding in α, one gets

$$\int dy^1 dy^2 T^{00}(y) = \pi\rho\mathcal{T}\left\{1 + 4\alpha^2\left((\partial_0 b)^2 + \frac{1}{\rho^2}(\partial_1 b)^2 - \frac{1}{\rho^2}b^2\right) + \mathcal{O}(\alpha^3)\right\} \tag{4.14}$$

Thus, since this expression is not positive definite, the static closed string is unstable, specifically due to uniform adiabatic deformations in the radius. In Sec. VI, we shall introduce a fourth order term to the matrix model which explicitly breaks the $SO(2,1)$ symmetry and eliminates the tachyonic mass.

V. NON-ABELIAN AND SUPERSYMMETRIC EXTENSIONS

Before discussing the cure to the above instability, we give two standard extensions of the model. In the first, we generalize to a stack of N coinciding branes, and in the second, we include the fermionic sector in an $\mathcal{N} = 1$ supersymmetric theory.

A. Generalization to $U(N)$

The usual generalization to $U(N)$ gauge theory is possible. For this one examines a solution to (3.1) of the form

$$X_\mu = X_{(0)\mu} \otimes \mathbb{1}_N, \tag{5.1}$$

$\mathbb{1}_N$ being the identity matrix in N dimensions and $X_{(0)\mu}$ defined by (3.4) and (3.5). This solution is associated with a stack of N coinciding branes. General perturbations about the solution are of the form $X_\mu = X_{(0)\mu} \otimes \mathbb{1}_N + 2\alpha A^{\rm a}_\mu \otimes T_{\rm a}$, $\rm a = 1, 2, 3, \ldots N^2$, where $T_{\rm a}$ are $N \times N$ Hermitian matrices generating $U(N)$. We will assume they are normalized according to $\mathrm{Tr} T_{\rm a} T_{\rm b} = \delta_{\rm a,b}$. The non-Abelian generalization of the action (4.5) on the noncommutative cylinder is straightforward. One can reexpress $A^{\rm a}_\mu$ in terms of non-Abelian gauge potentials $(a^{\rm a}_\tau, a^{\rm a}_\phi)$ on the noncommutative cylinder, along with additional fields $b^{\rm a}$ which transform in the adjoint representation of the noncommutative gauge group. In the commutative limit $\alpha \to 0$, the identification is

$$A_\pm \to (b^{\rm a} \mp i\rho a^{\rm a}_\tau)e^{\pm i\phi} \otimes T_{\rm a}, \tag{5.2}$$

in addition to $A_3 = a^{\rm a}_\phi \otimes T_{\rm a}$. This is the generalization of (4.10). Then

$$\begin{aligned} F_{+-} &\to -2\rho(\mathrm{D}_\tau b)^{\rm a} \otimes T_{\rm a} \\ F_{\pm 3} &\to ((\mathrm{D}_\phi b)^{\rm a} \mp \rho f^{\rm a}_{\tau\phi})e^{\pm i\phi} \otimes T_{\rm a}, \end{aligned} \tag{5.3}$$

where the Yang-Mills fields and the covariant exterior derivative of the scalar are defined, respectively, by $f^{\rm a}_{\tau\phi} = \partial_\tau a^{\rm a}_\phi - \partial_\phi a^{\rm a}_\tau - \mathrm{C}^{\rm a}_{\rm bc} a^{\rm b}_\tau a^{\rm c}_\phi$ and $(\mathrm{D}b)^{\rm a} = db^{\rm a} - \mathrm{C}^{\rm a}_{\rm bc} a^{\rm b} b^{\rm c}$. Here $\mathrm{C}^{\rm a}_{\rm bc}$ are the $u(N)$ structure constants, $[T_{\rm b}, T_{\rm c}] = i\mathrm{C}^{\rm a}_{\rm bc} T_{\rm a}$. Then the action on the cylinder (4.12) has the obvious $U(N)$ generalization

$$S(X) - S(X_{(0)}) \to \frac{4\alpha^3}{\pi g^2}\int_{-\pi}^{\pi} d\phi \int_{-\infty}^{\infty} d\tau \left(-\frac{\rho^2}{2} f^{\rm a}_{\tau\phi} f^{\rm a}_{\tau\phi} + \frac{\rho^2}{2}(\mathrm{D}_\tau b)^{\rm a}(\mathrm{D}_\tau b)^{\rm a} - \frac{1}{2}(\mathrm{D}_\phi b)^{\rm a}(\mathrm{D}_\phi b)^{\rm a} - \rho b^{\rm a} f^{\rm a}_{\tau\phi} \right). \tag{5.4}$$

Again, the gauge fields are not dynamical in one spatial dimension. The field equation for $f^{\rm a}_{\tau\phi}$ states that $f^{\rm a}_{\tau\phi} + \frac{1}{\rho} b^{\rm a}$ is covariantly constant, while the equations for $b^{\rm a}$ yield N^2 tachyons.

B. Fermionic sector

We now consider fermionic degrees of freedom in the $\mathcal{N} = 1$ suspersymmetric extension of the matrix model. They are denoted by the infinite dimensional Hermitian matrix Ψ, whose matrix elements Ψ_{AB}, $A, B = 1, 2, \ldots$ are complex Majorana spinors in $2+1$ Minkowski space. More precisely, $\Psi_{AB} = \Psi^*_{BA} = \xi^1_{AB} + i\xi^2_{AB}$, where ξ^1_{AB} and ξ^2_{AB} are real Majorana spinors in $2+1$ Minkowski space. The spinors are acted on by a real 2×2 representation of the γ-matrices, satisfying $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_2$. Our conventions for the spinors and gamma matrices are those of [19], which are reviewed in Appendix B. As usual, Ψ transforms in the adjoint representation of the infinite dimensional unitary group. We take the components of Ψ to have units of length to the three-halves power. The addition of the terms

$$S_F(X,\Psi) = \frac{1}{g^2}\,\mathrm{Tr}\left(\frac{1}{2}\bar{\Psi}\gamma^\mu[X_\mu,\Psi] + i\alpha\bar{\Psi}\Psi\right), \tag{5.5}$$

to the bosonic action (3.2) is consistent with $\mathcal{N}=1$ supersymmetry, where infinitesimal variations are given by

$$\delta X_\mu = \bar{\epsilon}\gamma_\mu\Psi, \qquad \delta\Psi = -\frac{1}{2}[X_\mu, X_\nu]\gamma^\mu\gamma^\nu\epsilon. \tag{5.6}$$

Here ϵ is an infinitesimal real Majorana spinor. To verify this, the supersymmetry variation of the first term in the trace in (5.5) gives

$$\frac{1}{2}\delta(\mathrm{Tr}\bar{\Psi}\gamma^\mu[X_\mu,\Psi]) = \mathrm{Tr}\bar{\Psi}[X^\mu,[X_\mu,X_\nu]]\gamma^\nu\epsilon, \tag{5.7}$$

after using the identities (B2) and (B4) in Appendix B. This result is canceled by the corresponding supersymmetric variation of the quartic term in the trace in (3.2), i.e., $-\frac{1}{4}\,\mathrm{Tr}[X_\mu, X_\nu]^2$. Similarly, the supersymmetric variation of the "mass" term in (5.5) gives

$$\delta\mathrm{Tr}\bar{\Psi}\Psi = -\epsilon^{\mu\nu\lambda}\mathrm{Tr}\bar{\epsilon}\gamma_\mu\Psi[X_\nu, X_\lambda], \tag{5.8}$$

using (B2) and (B3), which is canceled by the corresponding supersymmetric variation of the cubic term in the trace in (3.2), i.e., $\frac{2}{3}\epsilon_{\mu\nu\lambda}\mathrm{Tr}X^\mu X^\nu X^\lambda$.

The fermionic contributions (5.5) contribution to the action introduce a source term to the right-hand side of the equation of motion (3.1). The fuzzy de Sitter solution (3.3) and noncommutative cylinder solution (3.4) remain valid when $\Psi = 0$.

Next we examine perturbations (4.1) about the noncommutative cylinder solution with $\Psi = 0$. For the perturbations, it is convenient to rescale the fermion field, using $\Psi = \frac{1}{4}\sqrt{\rho\alpha^3}\psi$, where ψ is a function on the noncommutative cylinder in some irreducible representation. Then we can write the fermionic addition to the action (4.5) in terms of symbols of the fields on the noncommutative cylinder. We get

$$S_F(X,\Psi) = \frac{8\alpha^4}{\pi\rho g^2}\int_{-\pi}^{\pi} d\phi \sum_{n=0,\pm1,\pm2,\ldots}\left\{-i\bar{\psi}\star\left(\gamma^0 D_0\psi + \frac{1}{2}\gamma^- D_+\psi + \frac{1}{2}\gamma^+ D_-\psi\right) + i\bar{\psi}\star\psi\right\}, \tag{5.9}$$

where $\gamma^\pm = \gamma^1 \pm i\gamma^2$ and the covariant derivatives are given in (4.7). In the commutative limit $\alpha\to 0$, the covariant derivatives reduce to $D_0\psi \to \partial_\phi\psi$ and $D_\pm\psi \to \mp i\rho e^{\pm i\phi}\partial_\tau\psi$, and the fermionic action (5.9) reduces to

$$S_F(X,\Psi) \to \frac{4\alpha^3}{\pi g^2}\int_{-\pi}^{\pi} d\phi \int_{-\infty}^{\infty} d\tau\left\{\bar{\psi}\left(\tilde{\gamma}^\tau\partial_\tau\psi + \frac{1}{\rho}\tilde{\gamma}^\phi\partial_\phi\psi\right) + \frac{i}{\rho}\bar{\psi}\psi\right\}, \tag{5.10}$$

where the gamma matrices on the cylinder $(\tilde{\gamma}^\tau, \tilde{\gamma}^\phi)$ are defined by $\tilde{\gamma}^\tau = i\gamma^2\cos\phi - i\gamma^1\sin\phi$ and $\tilde{\gamma}^\phi = -i\gamma^0$. The action leads to the Dirac equation

$$\tilde{\gamma}^\tau\partial_\tau\psi + \frac{1}{\rho}\tilde{\gamma}^\phi\partial_\phi\psi + \frac{i}{\rho}\psi = 0 \tag{5.11}$$

on the cylinder. Using the conventions in the appendix, it can be checked that the fermionic degrees of freedom, like the bosonic degree of freedom, are tachyonic. Also, like with the bosonic field, the tachyonic mass is $m^2 = -1/\rho^2$, which vanishes in the infinite radius limit.

VI. EXPLICIT SYMMETRY BREAKING

Here we show how one can remove tachyonic mass term that appeared in the commutative limit of the model in Sec. IV. Our proposal is to add an explicit symmetry breaking term to the bosonic sector of the matrix model action (3.2). The term breaks Lorentz symmetry, as well as supersymmetry, but is consistent with the invariance group of the noncommutative cylinder solution, i.e., $SO(2)\times\tilde{T}$. One possibility is that we add the quadratic term $\mathrm{Tr}X_+X_-$ to (3.2). It can be checked that there can be up to two noncommuative cylinder solutions to the equations of motion following from this modification of the action. However, after perturbing around these solutions and taking the commutative limit one finds that the mass of the scalar field b is not modified. In order to modify the mass we instead need to add a higher order term to the matrix action. Below we examine a quartic symmetry breaking term. The total action is

$$S_{sb}(X) = S(X) + \frac{\beta}{4g^2}\,\mathrm{Tr}(X_+X_-)^2 = \frac{1}{g^2}\,\mathrm{Tr}\bigg(\frac{1}{8}[X_+,X_-]^2 + \frac{1}{2}[X_+,X_0][X_-,X_0] + \tilde{\alpha}X_0[X_+,X_-] + \frac{\beta}{4}(X_+X_-)^2\bigg), \tag{6.1}$$

where β is a real constant and we now denote the coefficient of the cubic term by $\tilde{\alpha}$ in order to distinguish it from the noncommutative parameter α appearing in the classical solution (3.4). The $2+1$ Lorentz symmetry of the background space is reduced to rotational symmetry on the plane due to the last term. (If one includes the fermionic contribution (5.5) to the action, then supersymmetry is broken as well.) We will see that $1+1$ space-time symmetry of the cylinder is preserved in the commutative limit.

The equations of motion now read

$$[[X_+,X_0],X_-] + [[X_-,X_0],X_+] = 2\tilde{\alpha}[X_-,X_+]$$
$$[[X_0,X_+],X_0] + \frac{1}{2}[[X_+,X_-],X_+] = 2\tilde{\alpha}[X_+,X_0] - \beta X_+X_-X_+. \tag{6.2}$$

They reduce to (3.1) upon setting $\beta = 0$ and $\tilde{\alpha} = \alpha$. The noncommutative de Sitter configuration (3.3) does not satisfy the equations of motion when $\beta \neq 0$. On the other hand, there exist two noncommutative cylinder solutions of the form (3.4), provided that $\tilde{\alpha}^2 + \beta\rho^2 > 0$ (and one when $\tilde{\alpha}^2 + \beta\rho^2 = 0$). ρ again denotes the radius of the cylinder. The noncommutative parameters $\alpha = \alpha_\pm$ for the two solutions are given by

$$\alpha_\pm = \frac{1}{2}\big(\tilde{\alpha} \pm \sqrt{\tilde{\alpha}^2 + \beta\rho^2}\big). \tag{6.3}$$

We can consider various limits. The two solutions reduce to the solution of the previous section ($\alpha_+ \to \tilde{\alpha}$) and the vacuum solution ($\alpha_- \to 0$) in the limit $\beta \to 0$. The solutions are degenerate, $\alpha_\pm \to \tilde{\alpha}/2$, in the limit $\tilde{\alpha}^2 + \beta\rho^2 \to 0$.

Perturbations (4.1) can be considered about either of the two noncommutative cylinder solutions, associated with some irreducible representation. The latter are again labeled by the radius ρ and phase angle ϕ_0. Upon substituting into the action (6.1) and using the previous definitions (4.4) for the noncommutative field strengths, we get

$$S_{sb}(X) = \frac{8\alpha^4}{\pi g^2}\int_{-\pi}^{\pi} d\phi \sum_{n=0,\pm1,\pm2,\ldots}\bigg\{\frac{1}{8}(F_{+-})^2_\star + \frac{1}{2}F_{+0}\star F_{-0} - \frac{1}{2}\bigg(A_+\star F_{-0} - A_-\star F_{+0} - \frac{\tilde{\alpha}}{\alpha}A_0\star F_{+-}\bigg) - \frac{\rho}{4\alpha}\big(e^{i\phi}[A_-,A_0]_\star - e^{-i\phi}[A_+,A_0]_\star\big) - \frac{i\tilde{\alpha}}{2\alpha}\partial_\phi A_+\star A_- + \frac{\beta}{16\alpha^2}\big(\rho(A_+\star e^{-i\phi} + A_-\star e^{i\phi}) + 2\alpha A_+\star A_-\big)^2_\star + \frac{1}{2}\bigg(\frac{\beta\rho^2}{4\alpha^2} - 1\bigg)A_+\star A_-\bigg\} + S_{sb}(X_{(0)}), \tag{6.4}$$

where α stands for either α_+ or α_-, depending upon which solution is expanded around. This action reduces to (4.5) when $\beta = 0$ and $\tilde{\alpha} = \alpha$. In Sec. IV, the commutative limit corresponded to $\alpha \to 0$. Now since there are two noncommutative cylinder solutions, we can have $\alpha_- \to 0$, $\alpha_+ \to 0$ or both. In the generic case where both go to zero, this means that the constants $\tilde{\alpha}$ and $\sqrt{\beta}$ are of order $\alpha \sim \alpha_\pm$, using (6.3).[2] The limiting values are constrained by the condition $\frac{\beta\rho^2}{4\alpha^2} - 1 + \frac{\tilde{\alpha}}{\alpha} = 0$. We assume that at least one cylindrical solution (6.3) exists when taking the limit, i.e., that α is real. Taking α, β and $\tilde{\alpha} \to 0$, while keeping $\tilde{\alpha}/\alpha$ and β/α^2 finite, yields

$$S_{sb}(X) - S_{sb}(X_{(0)}) = \frac{4\alpha^3}{\pi g^2}\int_{-\pi}^{\pi} d\phi \int_{-\infty}^{\infty} d\tau \bigg\{\frac{\rho^2}{2}(\partial_\tau b)^2 - \frac{1}{2}(\partial_\phi b)^2 - \frac{\rho^2}{2}(f_{\tau\phi})^2 - \bigg(1 + \frac{\beta\rho^2}{4\alpha^2}\bigg)\rho b f_{\tau\phi} + \frac{\beta\rho^2}{4\alpha^2}b^2\bigg\}. \tag{6.5}$$

The last term in the integral gives a new contribution to the mass-squared of the scalar field. Since the sign of β is not restricted, neither is the sign of this contribution. To determine the total effective mass of b, we should again eliminate the nondynamical gauge field using its equations of motion. Here we get

$$f_{\tau\phi} = -\bigg(1 + \frac{\beta\rho^2}{4\alpha^2}\bigg)\frac{b}{\rho} + \text{constant}, \tag{6.6}$$

which can be substituted back into the action (6.5). The last three terms in the integrand combine to give a mass term for the scalar, $-\frac{m^2}{2}b^2$, where

$$\rho^2 m^2 = -1 - \frac{\beta\rho^2}{\alpha^2} - \frac{1}{16}\bigg(\frac{\beta\rho^2}{\alpha^2}\bigg)^2. \tag{6.7}$$

[2]Another possibility is that we keep $\tilde{\alpha}$ finite, while β tends to zero. Then $\alpha_- \to -\rho^2\tilde{\alpha}\beta/4$, while α_+ remains finite.

Zero total mass occurs for $\frac{\tilde{\alpha}}{\alpha} = 3 \mp \sqrt{3}$, or equivalently, $\frac{\beta\rho^2}{4\alpha^2} = -2 \pm \sqrt{3}$. More generally, the tachyonic mass is eliminated for the following limiting values for $\frac{\tilde{\alpha}}{\alpha}$ and $\frac{\beta\rho^2}{4\alpha^2}$:

$$3-\sqrt{3} \le \frac{\tilde{\alpha}}{\alpha} \le 3+\sqrt{3}, \qquad -2-\sqrt{3} \le \frac{\beta\rho^2}{4\alpha^2} \le -2+\sqrt{3}. \quad (6.8)$$

The maximum value for the mass-squared in this model is $3/\rho^2$, which occurs for $\frac{\tilde{\alpha}}{\alpha} \to 3$, or equivalently $\frac{\beta\rho^2}{4\alpha^2} \to -2$.

Concerning the fermionic sector, one can, of course, eliminate the tachyonic mass for the fermion of the previous section by modifying the second term in the trace in (5.5), again violating supersymmetry.

VII. A BF MATRIX MODEL

In the previous section we allowed for the addition of a symmetry breaking term in the action. An interesting limit of this system corresponds to the kinetic energy term, i.e., the first term in the trace of (3.2), vanishing. For the example of a quadratic symmetry breaking term [instead of the quartic term in (6.1)], the action would reduce to

$$S_{\rm BF}(X) = \frac{1}{g^2}\,{\rm Tr}\left(\tilde{\alpha} X_0 [X_+, X_-] + \frac{\tilde{\beta}}{2} X_+ X_-\right) \qquad (7.1)$$

in the limit of zero kinetic energy. $\tilde{\beta}$ is the coefficient of the symmetry breaking term. The resulting matrix model is somewhat analogous to that considered in [20], which was utilized for the purpose of obtaining the BTZ black hole entropy. The major difference between the two systems is that time is a continuous parameter in the previous model, which thus describes a zero brane.

The equations of motion following from (7.1) have a noncommutative cylinder solution (3.4), where the noncommutative parameter is given by $\alpha = -\frac{\tilde{\beta}}{4\tilde{\alpha}}$. (As in the previous section, the symmetry breaking term does not allow for the fuzzy de Sitter solution.) As before, ρ and ϕ_0 label the irreducible representations of the solutions, and the spectrum of $X_{(0)0}$ is given by $\{\tau_n = (2n + \phi_0/\pi)\alpha, n = 0, \pm 1, \pm 2, \ldots\}$. Perturbations about this solution give

$$S_{\rm BF}(X) - S_{\rm BF}(X_{(0)}) = \frac{4\alpha^3\tilde{\alpha}}{\pi g^2}\int_{-\pi}^{\pi} d\phi \sum_{n=0,\pm1,\pm2,\ldots}\left\{A_0 \star F_{+-} - i\partial_\phi A_+ \star A_- + \frac{\tilde{\beta}}{4\alpha\tilde{\alpha}} A_+ \star A_-\right\}, \qquad (7.2)$$

where A_μ are again the symbols for the noncommutative potentials in $2+1$ target space. The field strengths $F_{\pm 0}$ do not appear in the action. In the noncommutative limit, $\alpha = -\frac{\tilde{\beta}}{4\tilde{\alpha}} \to 0$, one gets

$$S_{\rm BF}(X) - S_{\rm BF}(X_{(0)}) \to -\frac{4\alpha^3 c}{\pi g^2}\int_{-\pi}^{\pi} d\phi \int_{-\infty}^{\infty} d\tau \rho b f_{\tau\phi}, \quad (7.3)$$

where $f_{\tau\phi}$ and b are again the gauge and scalar field, respectively, with the identifications (4.9), and $c = 4\tilde{\alpha}^2/\tilde{\beta}$. Not surprisingly, the kinetic energies for the gauge and scalar field are absent, and all that remains is a BF action on the cylinder. The resulting system has a global degree of freedom, $\int_{-\pi}^{\pi} d\phi a_\phi$, which can be interpreted as the magnetic flux through the cylinder.

VIII. CONCLUDING REMARKS

We have shown that perturbations around the noncommuative cylinder in the matrix action (3.2) lead to a tachyonic scalar field. This was attributed to the instability of the circular static string under uniform adiabatic deformations. In Sec. VI we introduced a stabilizing term in the action which removes the tachyonic mass term for the scalar field. In these discussions we focused our attention predominantly on the commutative limit. The full noncommutative theory has nonlocal interactions. The effective theory for the scalar field is highly nontrivial since it also involves couplings to the nondynamical gauge fields. Field theories on the noncommutative cylinder have been investigated previously and have been shown to exhibit issues with unitarity [14]. It is of interest to investigate whether such problems can be resolved in the context of the matrix model discussed here.

There are various possible generalizations of the matrix action (3.2), including the cubic term, to higher space-time dimensions. One is that $\epsilon_{\mu\nu\lambda}$ in (3.1) is replaced by other Lie algebra structure constants. For the case of the $su(n)$ algebra, $\mu, \nu, \ldots$ take on $n^2 - 1$ values, leading to an $n^2 - 1$ dimensional target space. When the signature is Euclidean, the resulting theory has fuzzy CP^{n-1} solutions, which preserve the full $SU(n)$ symmetry of the target space [10]. Other solutions, which break the symmetry and thus are analogous to the noncommutative cylinder, should exist as well. Another possible generalization is that we rewrite the action in terms of the noncommutative analogue of antisymmetric, or Neveu-Schwarz fields $B_{\mu\nu}$, which here are infinite dimensional Hermitian matrices. In three space-time dimensions, we can take them to be just the space-time dual of the Hermitian matrices X^μ, $B_{\mu\nu} = \epsilon_{\mu\nu\rho}X^\rho$. The action (3.2) in terms of these matrices reads

$$S(B) = \frac{1}{g^2}\,{\rm Tr}\left(-\frac{1}{16}[B_{\mu\nu}, B_{\rho\sigma}]^2 - \frac{2}{3} i\alpha B_{\mu\nu} B^{\nu\rho} B_\rho{}^\mu\right). \qquad (8.1)$$

Expressed in this manner, the matrix action easily generalizes to any number of space-time dimensions. In $d+1$ space-time there are a total of $d(d+1)/2$ Hermitian matrices $B_{\mu\nu}$. Solutions, such as noncommutative de Sitter space and tensor products of noncommutative cylinders, fuzzy spheres and de Sitter spaces, appear in this model. Fuzzy CP^{n-1} solutions [10] may result from this model as well. We plan to pursue such generalizations in a later article.

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APPENDIX A: STAR PRODUCT ON THE NONCOMMUTATIVE CYLINDER

Following [14], one can define the analogue of a Moyal-Weyl star product for fields $\Phi(e^{i\hat{\phi}}, X_{(0)0})$ on the noncommutative cylinder in any given irreducible representation. For this, expand Φ in terms of the noncommutative analogue of Fourier modes

$$e^{i(k\hat{\phi}-\omega X_{(0)0})}, \quad k=0,\pm 1,\pm 2,\ldots, \quad -\frac{\pi}{2\alpha}<\omega\leq\frac{\pi}{2\alpha}, \tag{A1}$$

subject to the orthonormality conditions

$$\frac{\alpha}{\pi}\mathrm{Tr}e^{i(-k'\hat{\phi}+\omega' X_{(0)0})}e^{i(k\hat{\phi}-\omega X_{(0)0})}=\delta_{k,k'}\delta(\omega'-\omega). \tag{A2}$$

Thus

$$\Phi(e^{i\hat{\phi}}, X_{(0)0})=\sum_{k=0,\pm 1,\pm 2,\ldots}\int_{-\frac{\pi}{2\alpha}}^{\frac{\pi}{2\alpha}}d\omega\tilde{\Phi}_k(\omega)e^{i(k\hat{\phi}-\omega X_{(0)0})}. \tag{A3}$$

The Fourier coefficients $\tilde{\Phi}_k(\omega)$ are used to define the analogue of Weyl symbols $\Phi(e^{i\phi},\tau_n)$ on the cylinder

$$\Phi(e^{i\phi},\tau_n)=\sum_{k=0,\pm 1,\pm 2,\ldots}\int_{-\frac{\pi}{2\alpha}}^{\frac{\pi}{2\alpha}}d\omega\tilde{\Phi}_k(\omega)e^{i(k\phi-\omega\tau_n)}. \tag{A4}$$

Using [14] the analogue of the Moyal-Weyl star product of two Weyl symbols Φ and Φ' on the cylinder is given by

$$[\Phi\star\Phi'](e^{i\phi},\tau_n)=e^{i\alpha(\partial_{t'}\partial_\chi-\partial_t\partial_{\chi'})}\Phi(e^{i(\phi+\chi)},\tau_n+t)\Phi' \times(e^{i(\phi+\chi')},\tau_n+t')|_{t=t'=\chi=\chi'=0}. \tag{A5}$$

APPENDIX B: SPINOR CONVENTIONS

Here we review our conventions for spinors in $2+1$ Minkowski space, which are those of [19]. They were utilized in Sec. V B. For the γ-matrices satisfying $\{\gamma^\mu,\gamma^\nu\}=2\eta^{\mu\nu}\mathbb{1}_2$, we utilize the real 2×2 representation

$$[\gamma^\mu]^\alpha{}_\beta=\{-i\sigma_2,\sigma_1,\sigma_3\}, \tag{B1}$$

where σ_i are the Pauli matrices and $\alpha,\beta,\ldots=1,2$ denote the spinor indices. The gamma matrices act on two-dimensional spinors ψ^α. Spinor indices are raised and lowered with $\epsilon_{\alpha\beta}$: $\psi_\alpha=\psi^\beta\epsilon_{\beta\alpha}$, $\psi^\alpha=\epsilon^{\alpha\beta}\psi_\beta$. From the representation (B1), it follows in that $[\gamma^\mu]_{\alpha\beta}$ and $[\gamma^\mu]^{\alpha\beta}$ are symmetric matrices. Majorana spinors ψ^α satisfy $\bar{\psi}=\psi^\dagger i\gamma^0=\psi^T C$, where $\dagger$ and T denote the adjoint and transpose, respectively, and C is the charge conjugation matrix, satisfying $C\gamma^\mu C^{-1}=-\gamma^{\mu T}$. All real spinors are Majorana since we can take $C=i\gamma^0$. For any pair of real Majorana spinors ψ and χ one has the identities

$$\bar{\psi}\chi=\bar{\chi}\psi, \qquad \bar{\psi}\gamma^\mu\chi=-\bar{\chi}\gamma^\mu\psi. \tag{B2}$$

Additional identities for the gamma matrices are

$$\gamma^\mu\gamma^\nu=-\epsilon^{\mu\nu\rho}\gamma_\rho+\eta^{\mu\nu}\mathbb{1}_2, \tag{B3}$$

$$\gamma^\mu\gamma^\nu\gamma^\rho=-\epsilon^{\mu\nu\rho}\mathbb{1}_2-\eta^{\mu\rho}\gamma^\nu+\eta^{\nu\rho}\gamma^\mu+\eta^{\mu\nu}\gamma^\rho. \tag{B4}$$

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