

Matrix Model Approach to Cosmology

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Deposited 04/18/2019

Citation of published version:

Chaney, A., Lu, L., Stern, A. (2016): Matrix Model Approach to Cosmology. *Physical Review D*, 93(6). DOI: <https://doi.org/10.1103/PhysRevD.93.064074>



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Matrix model approach to cosmologyA. Chaney,^{*} Lei Lu,[†] and A. Stern[‡]*Department of Physics, University of Alabama, Tuscaloosa, Alabama 35487, USA*

(Received 20 January 2016; published 29 March 2016)

We perform a systematic search for rotationally invariant cosmological solutions to toy matrix models. These models correspond to the bosonic sector of Lorentzian Ishibashi, Kawai, Kitazawa and Tsuchiya (IKKT)-type matrix models in dimensions d less than ten, specifically $d = 3$ and $d = 5$. After taking a continuum (or commutative) limit they yield $d - 1$ dimensional Poisson manifolds. The manifolds have a Lorentzian induced metric which can be associated with closed, open, or static space-times. For $d = 3$, we obtain recursion relations from which it is possible to generate rotationally invariant matrix solutions which yield open universes in the continuum limit. Specific examples of matrix solutions have also been found which are associated with closed and static two-dimensional space-times in the continuum limit. The solutions provide for a resolution of cosmological singularities, at least within the context of the toy matrix models. The commutative limit reveals other desirable features, such as a solution describing a smooth transition from an initial inflation to a noninflationary era. Many of the $d = 3$ solutions have analogues in higher dimensions. The case of $d = 5$, in particular, has the potential for yielding realistic four-dimensional cosmologies in the continuum limit. We find four-dimensional de Sitter dS^4 or anti-de Sitter AdS^4 solutions when a totally antisymmetric term is included in the matrix action. A nontrivial Poisson structure is attached to these manifolds which represents the lowest order effect of noncommutativity. For the case of AdS^4 , we find one particular limit where the lowest order noncommutativity vanishes at the boundary, but not in the interior.

DOI: [10.1103/PhysRevD.93.064074](https://doi.org/10.1103/PhysRevD.93.064074)**I. INTRODUCTION**

Matrix models were introduced by Ishibashi, Kawai, Kitazawa and Tsuchiya (IKKT) [1] and Banks, Fischler, Shenker, and Susskind (BFSS) [2] as a way of addressing nonperturbative aspects of string theory. It is argued that space-time geometry, field theory, and gravity dynamically emerge from these models [3–7]. This holds the promise of including quantum gravity effects in descriptions of space-time. In particular, matrix models have the potential of resolving the singularities of general relativity [8].

Different directions have been pursued in order to recover a physical space-time starting from the IKKT model, which is the model of interest here. One proposal relies upon the trivial solutions to the classical Euclidean matrix equations where all bosonic matrices are diagonal, and hence commute. The diagonal matrix elements are interpreted as points of the ten-dimensional space-time. Effective actions for the space-time points were computed by integrating out the remaining (bosonic and fermionic) matrix degrees of freedom in the partition function [3]. Supersymmetry plays an important role in preventing interactions amongst the diagonal matrix elements. Recent results indicate that the rotational invariance of nine spatial dimensions in the IKKT matrix model may be spontaneously broken to $SO(3)$ acting on the three spatial

dimensions of space-time [9]. The partition function for the Lorentzian matrix model has also been investigated and there are indications that the model can lead to realistic cosmologies which include inflationary behavior in the early universe [10].

In addition to computing the quantum partition function, a number of authors have been investigating classical (or semiclassical) aspects of the IKKT model [2, 11–19]. In this approach one generally searches for nontrivial solutions to the classical matrix equations, where the bosonic matrices generate families of noncommutative algebras that carry isometries associated with space-time. The objective is then to find a certain limit, known as the commutative or continuum limit, in the parameter space of solutions where the algebra reduces to a commutative one which spans a smooth Poisson manifold. From a Lorentzian background metric we can get a Lorentzian induced metric on the Poisson manifold, which if four-dimensional can be a candidate for space-time. The idea that space-time could be associated with a noncommutative algebra and that this is due to gravitational effects, has been under consideration for some time [20]. While supersymmetry is important for the evaluation of the quantum partition function, the fermionic degrees of freedom do not play a role in the classical analysis outlined here. The questions of whether or not the classical results survive quantum fluctuations and backreactions from the fermionic fields remain open and nontrivial. In [21], it is claimed that the classical approximation becomes valid at late cosmological times.

Instead of working with the ten-dimensional supersymmetric system, many authors on matrix models have started

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from less ambitious toy models [11,12,15–19,22–24]. These models are lower-dimensional versions of the IKKT matrix model. While they are not derived from the quantum partition function of the full ten-dimensional supersymmetric theory, they provide a natural setting for the study of noncommutative gauge theories [25] about different backgrounds, which realize space-time uncertainty relations. The stability of these backgrounds has been under investigation and it was shown that supersymmetry can play an important role [26]. From a phenomenological perspective, some encouraging results have already been reported for the toy matrix models, which we discuss below. This suggests that there may exist limiting cases where it is possible to ignore the many serious challenges facing the compactification of the originally proposed ten-dimensional supersymmetric theory down to a realistic four-dimensional theory of space-time. That is, independent of ten-dimensional superstring theory, lower-dimensional matrix models might serve as reasonable effective theories of space-time, that include some quantum gravity effects. We remark that there is a trivial way to derive matrix models in *any* number of dimensions from classical strings. It is well known that the string action can be expressed in terms of Poisson brackets, which when “quantized” leads to a matrix model action. (See for example [15].)

Different approaches have been applied in attempts at recovering black hole and cosmological space-times [10,13,14,27–30]. Of course, there remain obstacles to obtaining definitive results starting with the quantum partition function of the ten-dimensional supersymmetric theory. Nevertheless, some encouraging results are seen from the classical (or semiclassical) analysis and from toy models. Concerning black holes, Schwarzschild and Reissner-Nordstrom metrics have been recovered from noncommutative solutions to the matrix equations [14]. The entropy formula for a black hole in two-spatial dimensions was recovered from a lower-dimensional matrix model with a simple state counting [23]. The application of matrix models to cosmology first appeared in a work by Álvarez and Meessen [27], where a Newtonian-type cosmology resulted from the BFSS model. Alternatively, because time and spatial coordinates are treated in the same manner in the IKKT model, a generally covariant theory can appear from this model in the continuum limit. As mentioned previously, indications of an expanding early universe are seen from evaluation of the partition function [10]. Similar configurations appear as solutions to the classical matrix equations [13,28]. In [19] it was shown how space-time singularities can be resolved in a toy low-dimensional matrix model. There we found classical solutions which were the Lorentzian analogues of fuzzy spheres. They are expressed in terms of $N \times N$ matrices, where time and space have a discrete spectra. The continuous, or commutative, limit corresponds to $N \rightarrow \infty$, and singularities on an otherwise smooth manifold appeared upon taking this limit. The manifold describes a

closed two-dimensional cosmological space-time and the singularities resemble cosmic singularities.

In this article we examine a toy model consisting of the bosonic sector of Lorentzian matrix models in dimensions $d = 3$ and 5 . The matrix models are of the IKKT type, as they are obtained by a reduction of a Yang-Mills theory to a zero-dimensional domain. Terms are included in the action which respect the Lorentz symmetry of the theory. One such term, which can be written down for any odd d , is totally antisymmetric in space-time indices and is analogous to a topological term. We shall search for “rotationally invariant” matrix solutions to the classical equations. In the commutative limit, these rotationally invariant configurations are associated with $d - 1$ space-time dimensional manifolds.

Rotational invariant matrices in $2 + 1$ space-time are easy to define. The dynamical degrees of freedom in this case are contained in three infinite-dimensional Hermitean matrices Y^μ , $\mu = 0, 1, 2$, with 0 being the time index and 1 and 2 being spatial indices. We can take rotationally invariant matrix configurations for (Y^0, Y^1, Y^2) to be those satisfying

$$[Y_+ Y_-, Y^0] = 0, \quad (1.1)$$

where

$$Y_\pm = Y^1 \pm iY^2. \quad (1.2)$$

The commutative limit of a matrix model is defined in analogous fashion to the classical limit of a quantum system. The former limit corresponds to replacing the matrices Y^μ by commuting space-time coordinates y^μ . y^0 and $y^i, i = 1, 2$, denote time and space coordinates, respectively. In addition, one replaces the commutator of functions of Y^μ by some Poisson bracket $\{, \}$ of the corresponding functions of y^μ . For this one introduces a noncommutativity parameter Θ , and defines the commutative limit by $\Theta \rightarrow 0$. To lowest order in Θ , $[\mathcal{F}(Y), \mathcal{G}(Y)] \rightarrow i\Theta\{\mathcal{F}(y), \mathcal{G}(y)\}$ for arbitrary functions \mathcal{F} and \mathcal{G} . Then (1.1) goes to

$$\{(y^1)^2 + (y^2)^2, y^0\} = 0 \quad (1.3)$$

in the limit. This restricts the spatial radius to a function of only the time y^0 coordinate

$$(y^1)^2 + (y^2)^2 = a^2(y^0), \quad (1.4)$$

which defines a rotationally invariant manifold embedded in three space-time dimensions. Similarly, for $d = 5$ we can write down an ansatz for matrices which in the commutative limit yield four-dimensional rotationally invariant space-time manifolds. They then are possible candidates for a realistic cosmology.

Exact solutions to matrix model equations of motion are notoriously difficult to obtain, which is true even for $d = 3$ [15]. Exceptional cases are those where the matrices define a finite-dimensional Lie-algebra. Well-known examples of

the latter are the fuzzy sphere [11,19,31–36] and two-dimensional noncommutative de Sitter space [16,18,37]. Here we show how one can generate a large class of rotationally invariant solutions, not necessarily associated with a Lie algebra, to three-dimensional matrix models using a simple recursion relation. While finding exact classical solutions to matrix models can be nontrivial, it is easy to obtain solutions in the commutative limit. Starting with a three- and five-space-time dimensional matrix models, we end up with two- and four-dimensional space-time manifolds, respectively, in the commutative limit. A large family of open, closed, and static space-time cosmologies can be recovered in this manner. Among the matrix solutions is the Lorentzian fuzzy sphere discussed above which yields a closed two-dimensional universe in the commutative limit. Other solutions to the toy matrix model are obtained which resolve singularities that are present in the commutative limit. In the case of the five-dimensional matrix model, we were not able to obtain exact solutions. However, a large family of solutions was obtained in the commutative limit. Among them is four-dimensional de Sitter (or anti-de Sitter) space which is endowed with a nontrivial Poisson structure.

We examine the bosonic sector of a three-dimensional Lorentzian IKKT-type matrix model in Sec. II. A totally antisymmetric cubic term is included in the action, which is consistent with the three-dimensional Lorentz symmetry. The dynamics is also invariant under unitary gauge transformation and translations. In the commutative limit of the theory it was possible to find a one-parameter family of rotationally invariant solutions to the equations of motion [18]. These solutions describe closed, open, and static two-dimensional space-time surfaces with some Poisson structure attached to the surface. For some special values of the parameter the Poisson brackets define three-dimensional Lie algebras, and in these cases one can easily find exact solutions to the Lorentzian matrix model. They correspond to noncommutative de Sitter, anti-de Sitter, and static space-times, and have been discussed previously [16–18,37–39]. In Sec. II we search for rotationally invariant matrix solutions in the generic case, i.e., solutions which are not in general associated with any three-dimensional Lie algebra. For this we use the definition of rotationally invariant matrices in (1.1). After restricting to such matrix configurations we can obtain recursion relations for the eigenvalues of matrices satisfying the equations of motion. The eigenvalues are discrete and define the spectra of the time and space (or radius) coordinates, Y^0 and $\sqrt{Y_+ Y_-}$, respectively. The recursion relations are trivially solved for noncommutative de Sitter and static space-times solutions. More generally, the recursion relations can be solved numerically and their spectra describe discrete versions of open space-time universes. For the noncommutative de Sitter solutions one recovers the principal, supplementary, and discrete series representations of $su(1, 1)$ [16]. One feature of the discrete

series is that there is a minimum (maximum) time eigenvalue which is associated with the minimum radial eigenvalue. In the commutative limit it corresponds to an initial (final) space-time singularity. Thus the discrete series solution provides a noncommutative resolution of a big bang (crunch) singularity.

There are two disadvantages to the approach described above for finding matrix solutions. (a) While the matrix analogues of open space-time universes are easy to obtain, the solutions which are matrix analogues of closed space-times universes is more difficult to obtain. The recursion relations which are derived in Sec. II have only infinite-dimensional matrix solutions, which may not be an appropriate assumption for modeling a closed noncommutative space-time. In this regard, we are unable to find any finite-dimensional matrices which solve the Lorentzian IKKT-type matrix model (containing only the additional cubic term in the action). (b) It was shown previously [18] that all the rotationally invariant solutions to that particular model have tachyonic-like excitations and so the stability of the solutions is not ensured. This is seen after taking the commutative limit.

The conclusions (a) and (b) change when additional terms are included in the matrix model action. This is the case for a quadratic or masslike term, which we consider in Sec. III. The inclusion of this term preserves the unitary gauge symmetry and Lorentz symmetry, but breaks translation invariance. The modified equations of motion yield a multiparameter family of rotationally invariant solutions in the commutative limit. They can be solved for numerically, and have novel features. Among them are solutions which exhibit a transition from a rapid inflation to a noninflationary phase. Another solution yields a closed universe which is associated with an $su(2)$ algebra (in contrast with the Lorentz group symmetry of the embedding coordinates). Its matrix analogue is a Lorentzian fuzzy sphere which has finite-dimensional representations and describes a noncommutative closed universe [19].

In Sec. IV we consider small perturbations about the rotationally invariant matrix solutions and then take the commutative limit of the action. The result is a scalar field theory on the space-time manifold associated with the commutative solution. The general analysis involves obtaining a nontrivial Seiberg-Witten map [40] on the noncommutative space defined by the solution, provided the quadratic term is included in the matrix action. We get that the effective mass squared of the scalar field can be positive ensuring the stability of the field theory in the commutative limit.

We examine a five-dimensional IKKT-type matrix model in Sec. V. This system is physically relevant since its commutative limit can yield four-dimensional space-times. A totally antisymmetric fifth order term is included in the bosonic sector of the matrix model action. The commutative limit has solutions describing four-dimensional cosmologies. One solution, which occurs in the limit that the Yang-Mills term vanishes, is four-dimensional de Sitter

space dS^4 . The Poisson structures on this space preserve three-dimensional rotation invariance. The Seiberg-Witten map for the four-dimensional de Sitter solution is applied to write the perturbative action in terms of commutative gauge fields and a scalar field. Magnetic monopoles are shown to emerge from the perturbations. In Sec. V we also define a notion of rotational symmetry for the five-dimensional matrix model. Finally, by changing the signature of the background metric we can obtain a four-sphere solution S^4 and a four-dimensional anti-de Sitter solution AdS^4 in the commutative limit of the five-dimensional matrix model. The four-sphere solution is distinct from the commutative limit of the fuzzy four-sphere appearing in earlier works [12,41,42]. The AdS^4 solution is also new. For generic values of the parameters of the theory, we find that the Poisson brackets are nonvanishing at the AdS^4 boundary. An exceptional case is an AdS^4 solution which follows from an action which consists only of a totally antisymmetric term. In that case the Poisson brackets vanish at the boundary, but not in the interior.

Concluding remarks are made in Sec. VI.

In Appendix A we give the result for the Seiberg-Witten map on a general two-dimensional manifold. The Seiberg-Witten map on four-dimensional de Sitter space appears in Appendix B.

II. TRANSLATIONAL INVARIANT LORENTZIAN IKKT-TYPE MODEL

Here we examine the bosonic sector of a Lorentzian IKKT-type matrix model in three space-time dimensions. The dynamics for the three infinite-dimensional Hermitean matrices Y^μ , $\mu = 0, 1, 2$ is determined from the action

$$S(Y) = \frac{1}{g^2} \text{Tr} \left(-\frac{1}{4} [Y_\mu, Y_\nu] [Y^\mu, Y^\nu] - \frac{2}{3} i\alpha \epsilon_{\mu\nu\lambda} Y^\mu Y^\nu Y^\lambda \right), \quad (2.1)$$

where a totally antisymmetric cubic term is added to the standard Yang-Mills term and α and g are constants. Our conventions are $\epsilon_{012} = 1$, and we raise and lower indices with the flat metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1)$. The resulting equations of motion are

$$[[Y_\mu, Y_\nu], Y^\nu] - i\alpha \epsilon_{\mu\nu\lambda} [Y^\nu, Y^\lambda] = 0. \quad (2.2)$$

They are invariant under:

- (i) Lorentz transformations $Y^\mu \rightarrow L^\mu_\nu Y^\nu$, where L is a 3×3 Lorentz matrix,
- (ii) translations in the three-dimensional Minkowski space $Y^\mu \rightarrow Y^\mu + v^\mu \mathbf{1}$, where $\mathbf{1}$ is the unit matrix, and
- (iii) unitary ‘‘gauge’’ transformations, $Y^\mu \rightarrow U Y^\mu U^\dagger$, where U is an infinite-dimensional unitary matrix.

Our interest shall be in constructing solutions to (2.2) which are rotationally invariant in the 1-2 plane. This will

of course require defining a notion of rotationally invariant matrix configurations which we put off to Sec. II B 1. We first review the much simpler problem of finding rotationally invariant solutions in the commutative limit of the matrix model equations.

A. Commutative limit

The commutative limit of the matrix equations was examined previously in [18] and a family of rotationally invariant solutions was obtained. We review them here. As stated in the Introduction, the commutative limit corresponds to replacing the matrices Y^μ by commuting space-time coordinates y^μ , and the commutator of functions of Y^μ is replaced by some Poisson bracket $\{, \}$ of the corresponding functions of y^μ . The Poisson brackets on the three-dimensional space spanned by y^μ are singular, and a function of the coordinates can be found which is central in the Poisson bracket algebra. Setting that function equal to a constant yields a two-dimensional surface \mathcal{M}_2 , upon which a nonsingular Poisson bracket can be defined. Similar arguments can be made to recover an even-dimensional manifold starting with a $d = \text{odd}$ dimensional matrix model. Say that τ and σ parametrize the two-dimensional surface, where τ is a timelike parameter and σ is spacelike. We will assume that any time slice of \mathcal{M}_2 is a circle, $0 \leq \sigma < 2\pi$. In terms of the three embedding coordinates the surface is defined by the functions $y^\mu = y^\mu(\tau, e^{i\sigma})$. Since \mathcal{M}_2 is a two-dimensional surface the Jacobi identity is automatically satisfied, and for any two functions $\mathcal{F}(\tau, e^{i\sigma})$ and $\mathcal{G}(\tau, e^{i\sigma})$ on \mathcal{M}_2 we can write

$$\{\mathcal{F}, \mathcal{G}\}(\tau, e^{i\sigma}) = h(\partial_\sigma \mathcal{F} \partial_\tau \mathcal{G} - \partial_\tau \mathcal{F} \partial_\sigma \mathcal{G}), \quad (2.3)$$

where in general h is some function of τ and $e^{i\sigma}$.

Since the matrix model action (2.1) and the equations of motion (2.2) can be expressed in terms of commutators, their commutative limit can be expressed in terms of Poisson brackets. In order that all terms survive in the commutative limit, we need that α vanishes in the limit, or more specifically, that it is proportional to Θ . We write as $\alpha \rightarrow +v\Theta$, with v finite. Then the commutative limit of the action is

$$S_c(y) = \frac{1}{g_c^2} \int_{\mathcal{M}_2} d\mu(\tau, \sigma) \left(\frac{1}{4} \{y_\mu, y_\nu\} \{y^\mu, y^\nu\} + \frac{v}{3} \epsilon_{\mu\nu\lambda} y^\mu \{y^\nu, y^\lambda\} \right), \quad (2.4)$$

where g_c is the commutative limit of the coupling g and $d\mu(\tau, \sigma)$ is the integration measure on \mathcal{M}_2 . The latter is required to be consistent with the cyclic trace identity,

$$\int_{\mathcal{M}_2} d\mu(\tau, \sigma) \{\mathcal{F}, \mathcal{G}\} \mathcal{H} = \int_{\mathcal{M}_2} d\mu(\tau, \sigma) \mathcal{F} \{\mathcal{G}, \mathcal{H}\}, \quad (2.5)$$

for arbitrary functions \mathcal{F} , \mathcal{G} , and \mathcal{H} on \mathcal{M}_2 . We can then take

$$d\mu(\tau, \sigma) = d\tau d\sigma/h. \quad (2.6)$$

The commutative limit of the equations of motion (2.2) is given by

$$\{\{y_\mu, y_\nu\}, y^\nu\} - v\epsilon_{\mu\nu\rho}\{y^\nu, y^\rho\} = 0. \quad (2.7)$$

The dynamics retains its invariance under (i) Lorentz transformations, (ii) translations, and (iii) gauge transformations. Infinitesimal gauge variations have the form $\delta y^\mu = \Theta\{\Lambda, y^\mu\}$, where Θ again denotes the noncommutativity parameter and Λ is an infinitesimal function on \mathcal{M}_2 .

The dynamical equations coincide with string equations of motion. Here the relevant string action is

$$S_{\text{string}} = -\mathcal{T} \left[\int_{\mathcal{M}_2} d\tau d\sigma \sqrt{-g} - \frac{v}{3} \int_{\mathcal{M}_2} \epsilon_{\mu\nu\rho} y^\mu dy^\nu \wedge dy^\rho \right], \quad (2.8)$$

where g is the determinant of the induced metric

$$g_{ab}(\tau, \sigma) = \partial_a y^\mu \partial_b y_\mu, \quad a = \tau, \sigma \quad (2.9)$$

on \mathcal{M}_2 and the constant \mathcal{T} denotes the string tension. The first term in (2.8) is the Nambu-Goto action, while the second corresponds to a coupling to a Neveu-Schwarz field of the form $B_{\mu\nu} \propto \epsilon_{\mu\nu\lambda} y^\lambda$. Both terms are reparametrization invariant, and respect Poincaré symmetry. They lead to the equations of motion

$$\Delta y_\mu - 2v n_\mu = 0. \quad (2.10)$$

Concerning the first term, $\Delta = -\frac{1}{\sqrt{-g}} \partial_a \sqrt{-g} g^{ab} \partial_b$ is the Laplace-Beltrami operator on the world sheet, and g^{ab} denotes the components of the inverse induced metric, $g^{ab} g_{bc} = \delta_c^a$. Concerning the second term, $n_\mu = \frac{1}{2\sqrt{-g}} \epsilon^{ab} \epsilon_{\mu\nu\rho} \partial_a y^\nu \partial_b y^\rho$ is a spacelike unit vector normal to the world sheet and $\epsilon^{\tau\sigma} = -\epsilon^{\sigma\tau} = 1$. The string equations (2.10) were shown to be identical to the equations (2.7) when the following condition is satisfied [15]:

$$h = \frac{1}{\sqrt{-g}}. \quad (2.11)$$

We denote solutions to the equations of motion by $y^\mu = x^\mu(\tau, e^{i\sigma})$, and focus on solutions with an $SO(2)$ isometry group, associated with rotations in the 1-2 plane. For this we write the ansatz

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} \tau \\ a(\tau) \cos \sigma \\ a(\tau) \sin \sigma \end{pmatrix}. \quad (2.12)$$

Here we have introduced a factor $a(\tau)$ which is the radius at any τ -slice. The ansatz (2.12) is consistent with (1.4). The invariant interval on the surface is

$$ds^2 = -(1 - a'(\tau)^2) d\tau^2 + a(\tau)^2 d\sigma^2, \quad (2.13)$$

the prime denoting differentiation in τ . This gives the Ricci scalar

$$R = \frac{2a''(\tau)}{a(\tau)(1 - a'(\tau)^2)^2}. \quad (2.14)$$

Rotational invariance in the 1-2 plane requires that we restrict h in (2.3) to being a function of only τ . In order to have a solution to (2.7), the functions a and h need to satisfy

$$\begin{aligned} ((aa'h)' + h - 2v)h &= 0, \\ (2ha' + ah' - 2va')ah &= 0. \end{aligned} \quad (2.15)$$

From these equations it follows that $h^2 g$ is a constant of integration, which is consistent with the condition (2.11). We can use (2.11) to eliminate $h(\tau)$ and obtain a second order equation for the scale factor

$$\frac{a''}{a} = \left(\frac{a'}{a}\right)^2 - \frac{1}{a^2} + \frac{2v}{a}(1 - a'^2)^{\frac{3}{2}}. \quad (2.16)$$

This yields the integral of the motion

$$\mathcal{E} = a/\sqrt{1 - a'^2 - va^2}, \quad (2.17)$$

which was shown in [18] to be associated with the energy of a bosonic string. From (2.17) we then get the following Friedmann-type equation for $a(\tau)$:

$$\left(\frac{a'}{a}\right)^2 - \frac{1}{a^2} = -\frac{1}{(\mathcal{E} + va^2)^2}. \quad (2.18)$$

Solutions to (2.18) can be expressed in terms of inverse elliptic integrals, which are plotted in Fig. 1. For all the solutions plotted there (except the limiting case of $v = \frac{1}{2}$) we assume that a has a turning point at $\tau = 0$, i.e., $a(0) = 1$ and $a'(0) = 0$. The solutions describe closed, stationary, and open space-times, the choice depending on values for v [18]. Closed two-dimensional space-times, having initial and final singularities, occur for $v < \frac{1}{2}$. The limiting case of $v = \frac{1}{2}$ gives the static or cylindrical space-time solution [17]. Open universe solutions correspond to $v > \frac{1}{2}$.

Exact expressions for the solutions exist for different values of v . They are as follows:

(a) For the case of $v = 0$, one has the simple expression

$$a(\tau) = \cos \tau, \quad -\frac{\pi}{2} \leq \tau \leq \frac{\pi}{2}, \quad (2.19)$$

where once again we assumed $a(0) = 1$ and $a'(0) = 0$, which leads to singularities at $\tau = \pm \frac{\pi}{2}$. It defines the surface $(x^1)^2 + (x^2)^2 = \cos^2(x^0)$, with metric given by

$$ds^2 = \cos^2 \tau (-d\tau^2 + d\sigma^2), \quad (2.20)$$

and Ricci curvature $R = -2 \sec^4(\tau)$, the latter being singular at $\tau = \pm \frac{\pi}{2}$. The Poisson brackets of the embedding coordinates are

$$\begin{aligned} \{x^0, x^1\} &= \frac{x^2}{\cos^2(x^0)}, \\ \{x^0, x^2\} &= -\frac{x^1}{\cos^2(x^0)}, \\ \{x^1, x^2\} &= \tan(x^0), \end{aligned} \quad (2.21)$$

where we used (2.3) and (2.11).

(b) For $v = \frac{1}{2}$ the solution is simply

$$a = 1. \quad (2.22)$$

The manifold is just a cylinder of unit radius with a flat metric tensor

$$ds^2 = -d\tau^2 + d\sigma^2. \quad (2.23)$$

Using (2.3) and (2.11) one now gets the Poisson brackets

$$\begin{aligned} \{x^0, x^1\} &= x^2, \\ \{x^0, x^2\} &= -x^1, \\ \{x^1, x^2\} &= 0, \end{aligned} \quad (2.24)$$

which define the three-dimensional Euclidean algebra.

(c) When $v = 1$ one gets

$$a(\tau)^2 = 1 + \tau^2, \quad (2.25)$$

corresponding to a de Sitter space-time,

$$(x^1)^2 + (x^2)^2 - (x^0)^2 = 1. \quad (2.26)$$

The invariant measure is given by

$$ds^2 = -\frac{d\tau^2}{a(\tau)^2} + a(\tau)^2 d\sigma^2, \quad (2.27)$$

yielding a constant positive Ricci curvature $R = 2$. The Poisson brackets on the surface define the $su(1, 1)$ algebra

$$\begin{aligned} \{x^0, x^1\} &= x^2, \\ \{x^0, x^2\} &= -x^1, \\ \{x^1, x^2\} &= -x^0. \end{aligned} \quad (2.28)$$

Since the Poisson brackets for solutions (b) and (c) define Lie algebras, their noncommutative analogues are easy to obtain. One simply replaces the Poisson brackets by commutation relations. With the exception of these two cases, obtaining the matrix analogues of classical solutions is nontrivial. We give a procedure for finding ‘rotationally invariant’ matrix solutions in the following subsections.

B. Matrix solutions

Here we search for matrix analogues of the rotationally invariant solutions of Sec. II A above to the commutative equations of motion (2.7). Our aim is to obtain the spectra of the matrices which solve the equations, which then give lattice versions of the commutative solutions depicted in the plots in Fig. 1. After first defining the meaning of rotational invariance for the matrices in Sec. II B 1, we obtain recursion relations for the spectra in Sec. II B 2. Exact

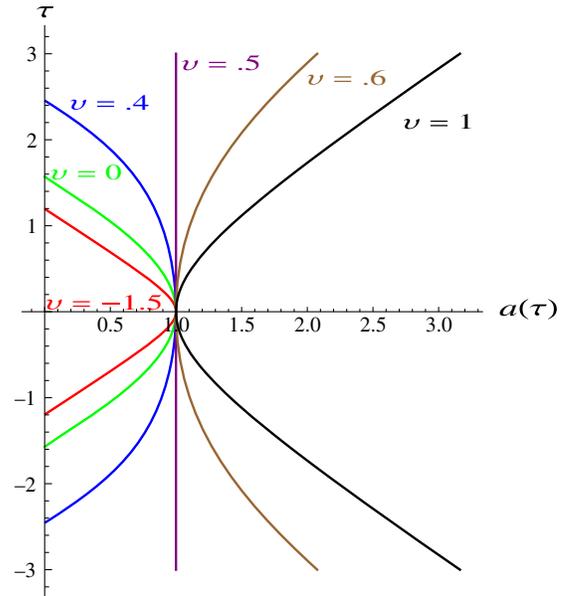


FIG. 1. Numerical solution to (2.16) for $v = -1.5, 0, 0.4, 0.5, 0.6$ and 1 . $v = 0.5$ and 1 correspond to the cylinder and de Sitter solutions, respectively. The boundary values are $a(0) = 1$ and $a'(0) = 0$.

solutions to the recursion relations are discussed in Secs. II B 3 and II B 4, and additional remarks concerning finite-dimensional solutions and stability are made in Secs. II B 5 and II B 6.

As a preliminary step it is convenient to write down an alternative expression for the commutative solutions of Sec. II A. For this we utilize a different parametrization of the two-dimensional manifolds. We replace τ by some other time coordinate t , which along with σ , satisfies the fundamental Poisson bracket

$$\{\sigma, t\} = 1, \quad (2.29)$$

which has a simple noncommutative extension. The previous commutative solutions can now be written as $y^\mu = x^\mu(t, e^{i\sigma})$. We then regard x^0 and the scale factor, which we now denote by \tilde{a} , as functions of t , thereby replacing (2.12) with

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} x^0(t) \\ \tilde{a}(t) \cos \sigma \\ \tilde{a}(t) \sin \sigma \end{pmatrix}. \quad (2.30)$$

Then the equations of motion (2.7) give

$$\begin{aligned} 2\partial_t \tilde{a}(\partial_t x^0 - v) + \tilde{a}\partial_t^2 x^0 &= 0, \\ (\partial_t x^0 - v)^2 - v^2 + \partial_t(\tilde{a}\partial_t \tilde{a}) &= 0. \end{aligned} \quad (2.31)$$

The first equation implies that

$$k = \tilde{a}^2(\partial_t x^0 - v) \quad (2.32)$$

is independent of t . The second equation then says that the dynamics of \tilde{a}^2 is determined by a simple force equation:

$$\frac{1}{2}\partial_t^2(\tilde{a}^2) = -\frac{k^2}{\tilde{a}^4} + v^2. \quad (2.33)$$

The solutions are characterized by k and the conserved ‘‘energy’’ $\frac{1}{4}(\partial_t(\tilde{a}^2))^2 - \frac{k^2}{\tilde{a}^2} - v^2\tilde{a}^2$. They are, of course, equivalent to those found previously in Sec. II A. To see this one only needs to apply the reparametrization $t \rightarrow \tau = x^0(t)$.

1. A rotationally invariant ansatz

We now return to the matrix model described by the action (2.1). Upon defining Y_\pm as in (1.2) the equations (2.2) can be written according to

$$\begin{aligned} [Y_+, [Y_-, Y^0]] + \frac{1}{2}[Y^0, [Y_+, Y_-]] - \alpha[Y_+, Y_-] &= 0, \\ [Y^0, [Y^0, Y_-]] + \frac{1}{2}[Y_-, [Y_-, Y_+]] + 2\alpha[Y^0, Y_-] &= 0. \end{aligned} \quad (2.34)$$

We wish to write down a rotationally invariant ansatz for the matrices Y^0 and Y_\pm which reduces to (2.30) in the

commutative limit. Different definitions are possible. We require that our choice satisfies (1.1). Our ansatz shall be expressed in terms of functions of two infinite-dimensional matrices \hat{t} and $e^{i\hat{\sigma}}$, and are the matrix analogues of t and $e^{i\sigma}$, respectively. The former is Hermitean and the latter is unitary. The matrix analogue of the Poisson bracket (2.29) is the commutation relation

$$[e^{i\hat{\sigma}}, \hat{t}] = -\Delta e^{i\hat{\sigma}}, \quad (2.35)$$

where Δ is a central element with units of time which is assumed to be linear in the noncommutative parameter. $e^{i\hat{\sigma}}$ generates time translations $\hat{t} \rightarrow \hat{t} + \Delta$. Together $e^{i\hat{\sigma}}$ and \hat{t} generate the algebra of the noncommutative cylinder [17,38,39].

For solutions to (2.34), which we denote by $Y^\mu = X^\mu$, we take

$$X_+ = X^1 + iX^2 = A(\hat{t})e^{i\hat{\sigma}}, \quad X^0 = X^0(\hat{t}). \quad (2.36)$$

This is consistent with our definition (1.1) of rotation invariance. Here we restrict X^0 and A to being real polynomial functions of \hat{t} . Then $A(\hat{t})$ and $X^0(\hat{t})$ are infinite-dimensional Hermitean matrices. In the commutative limit, the ansatz (2.36) agrees with the expression (2.30). After substituting the ansatz into (2.34) one gets

$$\begin{aligned} (X^0(\hat{t}) - X^0(\hat{t} - \Delta) - \alpha)A(\hat{t})^2 - (X^0(\hat{t} + \Delta) \\ - X^0(\hat{t}) - \alpha)A(\hat{t} + \Delta)^2 &= 0, \\ \frac{1}{2}(A(\hat{t} - \Delta)^2 + A(\hat{t} + \Delta)^2 - 2A(\hat{t})^2) + (X^0(\hat{t}) \\ - X^0(\hat{t} - \Delta) - \alpha)^2 - \alpha^2 &= 0. \end{aligned} \quad (2.37)$$

The first equation states that $(X^0(\hat{t}) - X^0(\hat{t} - \Delta) - \alpha)A(\hat{t})^2$ is invariant under discrete translations $\hat{t} \rightarrow \hat{t} + n\Delta$, $n = \text{integer}$, and is the matrix analogue of (2.32).

2. Recursion relations

We next write down recursion relations for the eigenvalues of the matrices $X^0(\hat{t})$ and $A(\hat{t})$. The spectrum for the operator \hat{t} is discrete, with equally spaced eigenvalues

$$t_n = t_0 - n\Delta, \quad n \in \mathbb{Z}, \quad (2.38)$$

where t_0 is real. This follows since from the commutation relations (2.35), $e^{2\pi i \hat{t}/\Delta}$ is a central element. It is a constant phase $e^{2\pi i t_0/\Delta} \mathbf{1}$ in any irreducible representation of the algebra, from which (2.38) results.

The eigenvalues of $X^0(\hat{t})$ and $A(\hat{t})$ are real and we denote them by

$$x_n^0 = X^0(t_n), \quad a_n = A(t_n). \quad (2.39)$$

From (2.36), X_+ and X_- act as lowering and raising operators, respectively, on the corresponding eigenvectors. Since the eigenvalues of $X_1^2 + X_2^2 = \frac{1}{2}(X_+X_- + X_-X_+)$ are positive definite, we get that $a_n^2 + a_{n-1}^2 \geq 0$, for all n . However, since we want $A(\hat{t})$ to be Hermitean, we get the stronger condition that

$$a_n^2 \geq 0, \quad (2.40)$$

for all n . From the equations of motion (2.37) we get the following recursion relations for the eigenvalues:

$$\begin{aligned} (x_n^0 - x_{n+1}^0 - \alpha)a_n^2 - (x_{n-1}^0 - x_n^0 - \alpha)a_{n-1}^2 &= 0, \\ \frac{1}{2}(a_{n+1}^2 + a_{n-1}^2 - 2a_n^2) + (x_n^0 - x_{n+1}^0 - \alpha)^2 - \alpha^2 &= 0. \end{aligned} \quad (2.41)$$

From the first equation, $k = (x_n^0 - x_{n+1}^0 - \alpha)a_n^2$ is independent of n , and then from the second equation we get a recursion relation for just a_n

$$\frac{1}{2}(a_{n+1}^2 + a_{n-1}^2 - 2a_n^2) + \frac{k^2}{a_n^4} - \alpha^2 = 0, \quad (2.42)$$

which is valid provided a_n does not vanish. Equation (2.42) is the lattice version of (2.33). Given the values for any neighboring pair of eigenvalues for A , we can determine the entire series $\{a_n\}$. Then starting with one time eigenvalue, we can determine all of $\{x_n^0\}$ using $x_n^0 - x_{n+1}^0 = \alpha + k/a_n^2$.

Solutions are plotted in Fig. 2 for $\alpha = 0.5, 0.51$ and 0.6 with $k = 0.5$ and boundary values $a_0 = a_1 = 1, x_0^0 = 0$. $\alpha = 0.5$ corresponds to the noncommutative cylinder

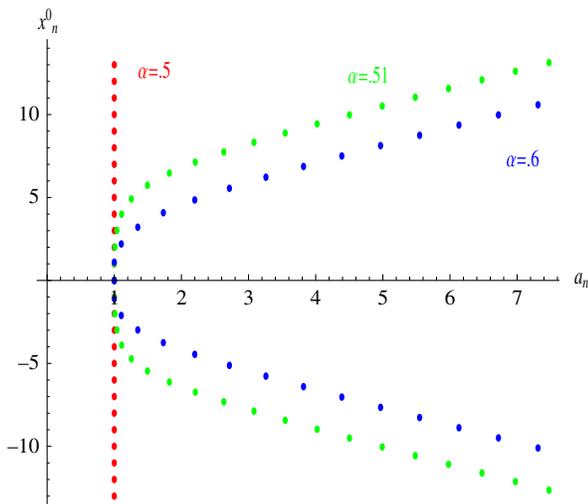


FIG. 2. Solutions to the recursion relation (2.42) for $\alpha = 0.5, 0.51$ and 0.6 with $k = (x_n^0 - x_{n+1}^0 - \alpha)a_n^2$ fixed to be 0.5 . $\alpha = 0.5$ corresponds to the noncommutative cylinder solution, while $\alpha = 0.51$ and 0.6 are examples of noncommutative analogues of open universe solutions. The initial conditions are $a_0 = a_1 = 1$ and $x_0^0 = 0$.

solution which we discuss in Sec. II B 3. $\alpha = 0.51$ and 0.6 are examples of discrete versions of open universe solutions. Another example of a discrete open universe is the noncommutative de Sitter solution which corresponds to $k = 0$. We discuss this case in Sec. II B 4. While Fig. 2 shows matrix analogues of the cylindrical and open space-time solutions, here we are unable to obtain matrix analogues of closed space-times. Related to this issue is the absence of solutions having $\alpha < 0.5$ (or more generally, $|\alpha| < |k|$, along with initial conditions $a_0 = a_1 = 1$). For these cases, a_n^2 decreases to zero as one goes away from the initial values. ($a_n^2 = 0$ is analogous to zero radius in the continuous case; i.e., a cosmological singularity.) As a_n^2 decreases to zero, the k^2/a_n^4 term dominates in the recursion relation (2.42), i.e., the leading term in the expression for a_{n+1}^2 goes like $-k^2/a_n^4 < 0$. Then for some n , a_n^2 becomes negative, which is inconsistent with Hermiticity. Thus, either such solutions do not exist or there must be raising or lowering operators that kill all states with $a_n^2 < 0$. An example of the latter is the discrete series representation of the de Sitter solutions, which is discussed in Sec. II B 4.

We note that the analysis leading to recursion relations (2.41) and (2.42) is only valid for infinite-dimensional solutions to the matrix equations, and moreover when the index n spans all positive and negative integers. Equation (2.36) is not valid if this is not the case. Alternatively, if n does not span all positive and negative integers it still may be possible to write

$$X_+ = AU, \quad (2.43)$$

where A and U are diagonal and unitary matrices, respectively, and X^0 is a diagonal matrix. This is a generalization of the ansatz (2.36). Both (2.36) and (2.43) imply that X_+X_- commutes with X^0 , and so they are consistent with the definition (1.1) of rotational invariance. Then X_+X_- and X^0 have common eigenvalues. In the case where (2.36) holds they are, respectively, a_n^2 and x_n^0 . Even if (2.36) does not hold, it may still be possible that the recursion relations (2.41) for the eigenvalues x_n^0 and a_n^2 of X^0 and A^2 , respectively, are valid after restricting the values of the label n in some fashion. For example, we find this to be the case for the discrete series of the de Sitter solutions, as is discussed in Sec. II B 4.

We next review well-known examples of rotationally invariant matrix model solutions, which are exact solutions of the recursion relations (2.42).

3. Noncommutative cylinder [17,38,39]

A trivial solution of the recursion relation (2.42) is

$$x_n^0 = -2\alpha n + x_0^0, \quad a_n = a_0, \quad (2.44)$$

where $k = \alpha a_0^2$ and $n \in \mathbb{Z}$. x_n^0 and a_0 are real and here are identified with eigenvalues of X^0 and A for the noncommutative cylinder. The solution represents the discrete version of the constant solution for a (2.22). The noncommutative cylinder solution $Y^\mu = X^\mu$ is defined by the commutation relations

$$[X^0, X_+] = 2\alpha X_+, \quad [X_+, X_-] = 0, \quad (2.45)$$

from which one recovers Poisson brackets (2.24) in the limit $\alpha \rightarrow 0$. x_n^0 and a_n in (2.44) are the eigenvalues, respectively, of X^0 and the square root of X_+X_- , which is central in the algebra. The latter is constant in any irreducible representation of the algebra and is the radius squared of the noncommutative cylinder. X_+ and X_- are raising and lowering operators, respectively, for the eigenvectors of X^0 .

4. Noncommutative dS^2 [16]

Another solution of the recursion relations is

$$\begin{aligned} x_n^0 &= -\alpha(n + \epsilon_0), \\ a_n^2 &= \alpha^2 n(n + 2\epsilon_0 + 1) + a_0^2, \end{aligned} \quad (2.46)$$

where ϵ_0 and a_0 are real. Now $k = 0$. This solution can be identified with noncommutative (or fuzzy) dS^2 and it corresponds to the matrix analogue of the solution (2.25). The relevant commutation relations for the matrix solution $Y^\mu = X^\mu$ now define the $su(1, 1)$ Lie algebra

$$[X^0, X_+] = \alpha X_+, \quad [X_+, X_-] = -2\alpha X^0, \quad (2.47)$$

and they yield the Poisson brackets (2.28) in the $\alpha \rightarrow 0$ limit.

Irreducible representations of the $su(1, 1)$ Lie algebra are well known and classified by eigenvalues of the central operator

$$R^2 = \frac{1}{2}(X_+X_- + X_-X_+) - (X^0)^2, \quad (2.48)$$

which we denote by $-\alpha^2 j(j + 1)$, along with ϵ_0 . R is the length scale of the noncommutative de Sitter space. States $|j, \epsilon_0, n\rangle$ in any irreducible representation can be taken to eigenvectors of X^0 , with X_+ and X_- behaving as lowering and raising operators, respectively,

$$\begin{aligned} X^0|j, \epsilon_0, n\rangle &= -\alpha(\epsilon_0 + n)|j, \epsilon_0, n\rangle, \\ X_+|j, \epsilon_0, n\rangle &= i\alpha(j + \epsilon_0 + n)|j, \epsilon_0, n - 1\rangle, \\ X_-|j, \epsilon_0, n\rangle &= i\alpha(j - \epsilon_0 - n)|j, \epsilon_0, n + 1\rangle. \end{aligned} \quad (2.49)$$

It follows that

$$\begin{aligned} X_+X_-|j, \epsilon_0, n\rangle &= \alpha^2((\epsilon_0 + n)(\epsilon_0 + n + 1) \\ &\quad - j(j + 1))|j, \epsilon_0, n\rangle. \end{aligned} \quad (2.50)$$

If we assume (2.36), or more generally (2.43), then we can identify X_+X_- with A^2 , with eigenvalues $a_n^2 \geq 0$. Then comparing (2.46) with (2.50) gives

$$a_0^2 = \alpha^2(\epsilon_0(\epsilon_0 + 1) - j(j + 1)). \quad (2.51)$$

The inequality (2.40) in this case leads to

$$\left(\epsilon_0 + n + \frac{1}{2}\right)^2 \geq \left(j + \frac{1}{2}\right)^2, \quad (2.52)$$

for all n .

Nontrivial representations are known to fall into three categories: principal, supplementary, and discrete series. For the principal and supplementary series, neither $j + \epsilon_0$ nor $j - \epsilon_0$ are integers, so that no states $|j, \epsilon_0, n\rangle$ are killed by either X_+ or X_- . There are then no restrictions on the integers n labeling the states. One takes $j = -\frac{1}{2} + i\rho$, with ρ real, for the principal series, which identically satisfies (2.52). j is assumed to be real for the supplementary series. Then if we choose $-\frac{\pi}{2} \leq \epsilon_0 < \frac{\pi}{2}$, we need that $|j + \frac{1}{2}| \leq |\epsilon_0 + \frac{1}{2}|$.

Finally, for the discrete series one has that either $j + \epsilon_0$ or $j - \epsilon_0$ are integers. For the former, we can choose $j + \epsilon_0 = 0$. Then from (2.49), X_+ kills $|j, -j, 0\rangle$, which then serves the role as the bottom state for the irreducible representation $D^+(j)$. In this case n is restricted to positive integers, including 0. The inequality (2.52) is satisfied for $j \leq 0$. The resulting spectra for X^0 and X_+X_- is given by

$$\begin{aligned} x_n^0 &= \alpha(j - n), \\ a_n^2 &= \alpha^2(n + 1)(n - 2j), \\ n &= 0, 1, 2, \dots \end{aligned} \quad (2.53)$$

The time takes on only negative eigenvalues, assuming $\alpha > 0$. Similarly, if one chooses $j - \epsilon_0 = 0$, then from (2.49), X_- kills $|j, j, 0\rangle$. The latter serves the role as the top state for the irreducible representation $D^-(j)$ and in this case n is restricted to negative integers, including 0. The inequality (2.52) is again satisfied for $j \leq 0$. The spectrum for X_+X_- is the same as in the previous case (2.53), while there is a sign flip for x_n^0 , i.e., the signs are now all positive. We note that (2.36) is not valid for the discrete solutions since the n does not span all integers. [The recursion relations (2.41) are still valid, however, provided that we now restrict the integer n in these relations to $n \geq 1$ for $D^+(j)$, and $n \leq -1$ for $D^-(j)$.] Furthermore, its generalization (2.43) does not hold in general either. Only for $j = -\frac{1}{2}$ can we write X_+ in the form (2.43). For $D^+(-\frac{1}{2})$ the matrices A and U are given simply by

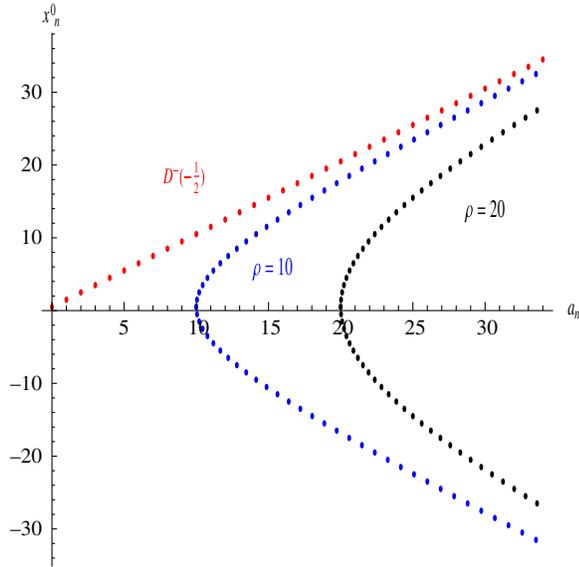


FIG. 3. Noncommutative de Sitter solutions. Plots of the eigenvalues x_n^0 and a_n for the discrete series $D^-(\frac{1}{2})$ and for the principal series with $\rho = 10$ and 20 . $\alpha = 1$ and $\epsilon_0 = -\frac{1}{2}$ are chosen in all cases.

$$A_{nm} = \alpha(n+1)\delta_{n,m}, \quad U_{nm} = i\delta_{n+1,m}. \quad (2.54)$$

We remark that the existence of a bottom (top) state for the discrete series, where there is a corresponding minimum value for the radial eigenvalues a_n , means that there is an initial (final) state. It is thus a discrete analogue of a cosmological singularity.

Plots of the eigenvalues x_n^0 and a_n of the matrices X^0 and A , respectively, are given in Fig. 3 for the discrete series $D^-(\frac{1}{2})$, as well as for the principal series with $\rho = 10$ and 20 .

5. The question of finite-dimensional solutions

In the above example, it is well known that there are no finite-dimensional solutions of noncommutative de Sitter space since there are no finite-dimensional unitary representations of the $SU(1,1)$ group. More generally, one can ask whether or not there exist nontrivial finite-dimensional matrix solutions of the equations of motion (2.2). This question is relevant for knowing whether or not there are matrix solution analogues of the closed space-time cosmologies. The latter are expected to emerge upon taking the $N \rightarrow \infty$ limit of the $N \times N$ matrix solutions, along with initial and final singularities on the resulting space-time manifold. So for example, one can ask if there is a matrix analogue of the closed space-time solution (2.19) of the commutative equations of motion (2.7).

As stated above, if a_n^2 tends to zero, it becomes necessary to terminate the series generated by the recursion relations (2.42) in order to prevent a_n^2 from becoming negative. There must then exist a bottom or top state, which would

correspond, respectively, to an initial or final singularity in the continuum limit. Any matrix analogue of a closed space-time solution must have *both* a bottom and top state, and thus the matrix solution should be finite dimensional. In this regard, we have not been able to find any nontrivial finite-dimensional matrix solutions to (2.2), and thus here we do not have matrix model analogues of the closed space-time solutions of Sec. II A; i.e., all the solutions of (2.16) with $v < \frac{1}{2}$.

Although we do not have a proof that there are no nontrivial $N \times N$ solutions, for arbitrary finite N , to the matrix equations (2.2), it is easy to show that no nontrivial solutions exist for the simplest case of $N = 2$. In that case we can set X^μ equal to a linear combination of Pauli matrices, one of which, say X^0 , we can take up to a factor to be σ_3 . (Terms in X^μ which are proportional to the identity matrix trivially solve the equations of motion.)

$$X^0 = \sigma_3 \quad X^1 = u_i \sigma_i \quad X^2 = v_i \sigma_i,$$

where u_i and v_i are real. Here we are not making any additional restrictions such as rotational invariance. Upon substituting into the equations of motion (2.2) one gets

$$\begin{aligned} u_3 u_i + v_3 v_i - (\vec{u}^2 + \vec{v}^2) \delta_{i3} - \alpha \epsilon_{ijk} u_j v_k &= 0, \\ \vec{u} \cdot \vec{v} u_i - v_3 \delta_{i3} - (\vec{u}^2 - 1) v_i - \alpha \epsilon_{ij3} u_j &= 0, \\ \vec{u} \cdot \vec{v} v_i - u_3 \delta_{i3} - (\vec{v}^2 - 1) u_i + \alpha \epsilon_{ij3} v_j &= 0. \end{aligned}$$

The only real solutions are $u_i = u_3 \delta_{i3}$, $v_i = v_3 \delta_{i3}$, but these are trivial solutions since then all X^μ are proportional to σ_3 . Thus there are no nontrivial 2×2 matrix solutions of the equations of motion (2.2).

6. The question of stability

Issues related to the stability of the rotationally invariant solutions to (2.2) were examined in [18]. More specifically, [18] was concerned with small perturbations about the solutions to (2.18). Leading order effects were examined upon perturbing in the noncommutative parameter Θ , or equivalently α . The perturbations about the solutions were expressed in terms of an Abelian gauge field and scalar field (or non-Abelian gauge fields and N scalar fields if one expands about a stack of N coinciding branes). This could be done in general with the use of an appropriate Seiberg-Witten map [40] on the noncommutative space-time associated with the solution. Gauge transformations correspond to area preserving coordinate transformations on the two-dimensional surface, while the scalar field is associated with perturbations normal to the surface. At leading order in Θ , the resulting perturbed action yielded the usual dynamics for a scalar field coupled to the gauge field on the two-dimensional commutative manifold. Since gauge fields are nondynamical in two-space-time dimensions they can be eliminated leaving only the scalar field degree of

freedom. For all values of the parameter ν appearing in the commutative theory, the remaining scalar field was found to be tachyonic.

The persistence of tachyonic modes and the absence of any finite-dimensional solutions appear to be generic features of the Lorentzian IKKT-type matrix model whose dynamics follows from the action (2.1). On the other hand, they no longer are the case when additional terms are included in the action. For example it was recently found in [19] that finite-dimensional matrix solutions exist when a quadratic, or mass, term is added to the matrix model action. With the same quadratic term the scalar field resulting from perturbations about the rotationally invariant solutions can have a positive mass squared, thus ensuring stability of the commutative field theory. We more generally explore the consequences of including the quadratic term in the following section.

III. INCLUSION OF A QUADRATIC TERM

We now add a quadratic, or mass, term to (2.1). The total matrix model action is then

$$\begin{aligned} S_{\text{total}}(Y) &= S(Y) + \frac{\beta}{2g^2} \text{Tr} Y_\mu Y^\mu \\ &= \frac{1}{g^2} \text{Tr} \left(-\frac{1}{4} [Y_\mu, Y_\nu] [Y^\mu, Y^\nu] - \frac{2}{3} i\tilde{\alpha} \epsilon_{\mu\nu\lambda} Y^\mu Y^\nu Y^\lambda \right. \\ &\quad \left. + \frac{\beta}{2} Y_\mu Y^\mu \right), \end{aligned} \quad (3.1)$$

where β is a real constant and we now denote the coefficient of the cubic term by $\tilde{\alpha}$ in order to distinguish it from the noncommutative parameter α appearing in the rotationally invariant classical solutions of Sec. II B. The matrix equations of motion now read

$$[[Y_\mu, Y_\nu], Y^\nu] - i\tilde{\alpha} \epsilon_{\mu\nu\lambda} [Y^\nu, Y^\lambda] = -\beta Y_\mu. \quad (3.2)$$

Upon comparing with the previous section, the (i) 2 + 1 Lorentz symmetry of the background space, as well as (iii) the unitary gauge symmetry, is preserved by the last term, but (ii) translation symmetry is broken when $\beta \neq 0$.

This system contains new solutions, as well as some of the previous solutions (even when $\beta \neq 0$). Before discussing the matrix solutions, we once again find it convenient to first examine solutions in the commutative limit of the matrix model.

A. Commutative limit

We first write down the modification of the commutative equations of Sec. II A. Now a much larger family of solutions exist. We shall give a (mostly) qualitative discussion of these solutions.

The commutative limit of the matrix model action can once again be expressed using the Poisson bracket (2.3) on

some two-dimensional manifold \mathcal{M}_2 . In order for the cubic term in the action to survive in the limit, we again need for its coefficient to be linear in the noncommutativity parameter Θ , i.e., $\tilde{\alpha} \rightarrow +\nu\Theta$. The quadratic term in the action will survive in the limit provided that β goes like Θ^2 , i.e., $\beta \rightarrow \omega\Theta^2$, with ω finite. Then (2.4) is replaced by

$$\begin{aligned} S_c(y) &= \frac{1}{g_c^2} \int_{\mathcal{M}_2} d\mu(\tau, \sigma) \left(\frac{1}{4} \{y_\mu, y_\nu\} \{y^\mu, y^\nu\} \right. \\ &\quad \left. + \frac{\nu}{3} \epsilon_{\mu\nu\lambda} y^\mu \{y^\nu, y^\lambda\} + \frac{\omega}{2} y_\mu y^\mu \right), \end{aligned} \quad (3.3)$$

where $d\mu(\tau, \sigma) = d\tau d\sigma/h$ is once again the invariant integration measure on \mathcal{M}_2 . Not surprisingly, translational invariance is broken when $\omega \neq 0$. The resulting equations of motion are now

$$\{\{y_\mu, y_\nu\}, y^\nu\} - \nu \epsilon_{\mu\nu\rho} \{y^\nu, y^\rho\} = \omega y_\mu. \quad (3.4)$$

Upon substituting the rotationally invariant expression (2.12) into (3.4), we get the following equations for a and h , both of which are assumed to be functions of only τ :

$$\begin{aligned} (ad'h)'h + h^2 - 2\nu h &= \omega, \\ 2h(h - \nu)a'a + a^2 h'h &= \omega\tau, \end{aligned} \quad (3.5)$$

the prime again denoting a derivative in τ . It follows that $h^2 g - \omega(a^2 - \tau^2)$ is a constant of integration, where once again g is the determinant of the induced metric. We then get an explicit formula for h , and hence the integration measure

$$h = \sqrt{\frac{c_1 - \omega(a^2 - \tau^2)}{(1 - a'^2)a^2}}, \quad (3.6)$$

c_1 being the constant of integration. Here we see that the measure is not simply expressed in terms of metric, except for the case $\omega = 0$ where we recover the result (2.11) of the previous section. For all ω and ν , h can be eliminated from the differential equation for a , which can be written

$$a'' - \frac{a'^2}{a} + \frac{1}{a} - \frac{2\nu}{ah} (1 - a'^2) - \frac{\omega}{h^2} \left(\frac{\tau}{a}\right)' = 0, \quad (3.7)$$

generalizing (2.16). The breaking of time translation symmetry when $\omega \neq 0$ implies the absence of a conserved energy, and consistent with that, we have not found a generalization of the quantity (2.17) which is conserved when $\omega = 0$.

There is now a large family of solutions, including those discussed in 2.1 when $\omega = 0$. Among them are some exact solutions, all of which have h equal to a constant value:

- (1) There are two distinct dS^2 solutions to (3.5) of the form (2.25) when $\nu^2 + 2\omega > 0$ and $\omega \neq 0$. (Here we

assume v and ω are finite.) They yield the following constant values for h :

$$h_{\pm} = \frac{1}{2}(v \pm \sqrt{v^2 + 2\omega}). \quad (3.8)$$

The solution is degenerate when $v^2 + 2\omega = 0$, and no de Sitter solution exists for $v^2 + 2\omega < 0$.

- (2) A dS^2 solution exists to the equations of motion in the limit $v, \omega \rightarrow \infty$, with $\frac{\omega}{v}$ finite and nonzero. In this limit, the kinetic energy (or Yang-Mills) term is absent from the action (3.3). In this case

$$h = -\frac{\omega}{2v}. \quad (3.9)$$

If both the kinetic energy term and the quadratic term are absent from the action and only the totally antisymmetric term remains, i.e., $v \rightarrow \infty$ and $\frac{\omega}{v} \rightarrow 0$, then there are only trivial solutions to the equations of motion, $\{x^\mu, x^\nu\} = 0$. In the matrix model, all matrices X^μ commute in this case. This result does not generalize to higher dimensions where one can have non-trivial solutions of the equations of motion when only a totally antisymmetric term appears in the action, as we show in Sec. VD 2.

- (3) Another solution, which exists only when $\omega \neq 0$, is a sphere, $(x^1)^2 + (x^2)^2 + (x^0)^2 = 1$, embedded in three-dimensional Minkowski space-time [19]. The solution is

$$a(\tau)^2 = 1 - \tau^2, \quad h = 2v, \quad -1 \leq \tau \leq 1, \quad (3.10)$$

which is only valid for $\omega = -4v^2$. (More generally, by introducing another real parameter it is deformed to an ellipsoid embedded in three-dimensional Minkowski space-time.) The invariant measure obtained from the induced metric is

$$ds^2 = -\frac{1 - 2\tau^2}{1 - \tau^2} d\tau^2 + (1 - \tau^2) d\sigma^2, \quad (3.11)$$

which differs from that of the Euclidean sphere. The metric tensor does not have definite signature; it is Lorentzian for $-\frac{1}{\sqrt{2}} < \tau < \frac{1}{\sqrt{2}}$, and Euclidean for $\frac{1}{\sqrt{2}} < |\tau| < 1$. The latitudes with $|\tau| = \frac{1}{\sqrt{2}}$ produce a singularity in the Ricci scalar, and unlike with the curvature of a Euclidean sphere, the Ricci scalar is not constant and it is negative

$$R = -\frac{2}{(1 - 2\tau^2)^2}. \quad (3.12)$$

The solution represents a closed space-time cosmology with the initial and final singularity occurring at

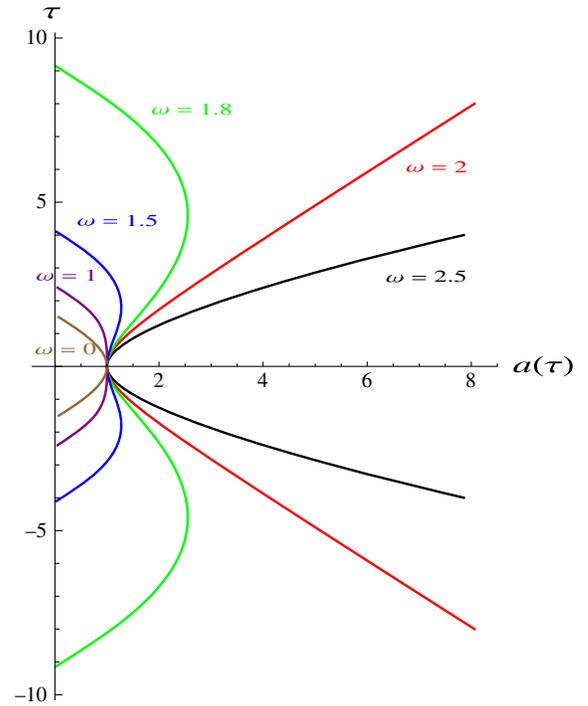


FIG. 4. Numerical solutions to (3.5) for $v = 0$ and $\omega = 1, 1.5, 1.8, 2$ and 2.5 . The boundary conditions are $a(0) = 1$ and $a'(0) = 0$.

the latitudes with $|\tau| = \frac{1}{\sqrt{2}}$, where the spatial radius is not zero. The solution can be expressed in terms of the $su(2)$ Poisson bracket algebra.

The Poisson brackets of the solutions define three-dimensional Lie algebras and it is straightforward to find their matrix analogues. One gets noncommutative de Sitter space-time for both solutions 1 and 2, which we discuss in the next subsection, while for 3 the result is a Lorentzian fuzzy sphere [19]. We note that there are also no cylindrical space-time solutions, $a = \text{constant}$, when $\omega \neq 0$.

In general, solutions to (3.5) are labeled by four independent parameters: v, ω, c_1 [the integration constant in (3.6)] and the value of a' at some $\tau = \tau_0$. (The value of a at a given τ merely determines the overall scale.) The solutions can be obtained numerically and some novel results appear when $\omega \neq 0$. In Fig. 4 we plot τ versus $a(\tau)$ for $v = 0$ and different values of ω . A closed space-time results for $\omega \leq 1$, a dumbbell shaped curve [with two maxima for $a(\tau)$] appears for $1 < \omega < 2$ and an open space-time for $\omega \geq 2$. Another example, depicted in Fig. 5, shows that by starting from initial conditions associated with a rapid inflationary period, one can get a smooth transition to a noninflationary phase. There we take $a = 1$ and $a' = 50$ at some initial time $\tau_0 = 0.01$, along with the following choices for the remaining parameters: $v = 1, \omega = -0.5$ and $c_1 = -2500$. We then find that by starting with large values for the expansion and deceleration rates, $a'(\tau)/a(\tau)$ and $-a''(\tau)/a(\tau)$, respectively, at $\tau = \tau_0$

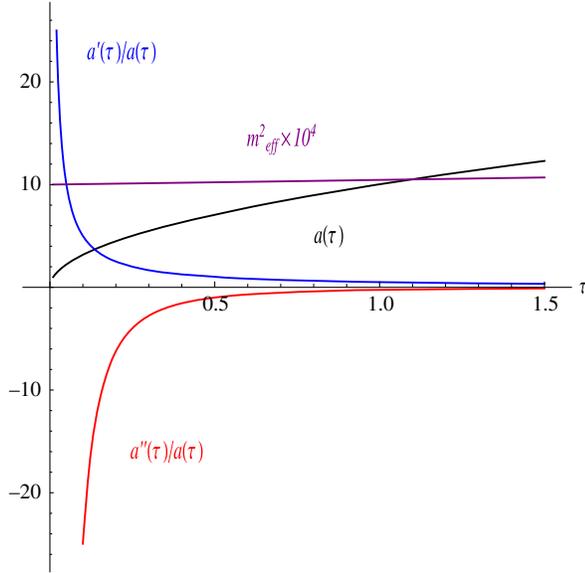


FIG. 5. Numerical solution for $v = 1$, $\omega = -0.5$ and $c_1 = -2500$, with initial conditions $a(0.01) = 1$ and $a'(0.01) = 50$. Plots are given for $a(\tau)$, $a'(\tau)/a(\tau)$ and $a''(\tau)/a(\tau)$, and show that $a'(\tau)/a(\tau)$ and $-a''(\tau)/a(\tau)$ decrease rapidly to zero as τ increases from the initial value $\tau = 0.01$. We also plot the effective mass-squared m_{eff}^2 of the scalar field, which is associated with perturbations normal to the surface (see Sec. IV B). The plot indicates that $m_{\text{eff}}^2 > 0$.

these quantities go rapidly to zero as τ increases from the initial value.

B. Rotationally invariant matrix solutions

1. Recursion relations

When $\beta \neq 0$, the right-hand sides of the matrix equations of motion (2.34) no longer vanish and substitution of the ansatz (2.36) into the equations of motion (3.2) yields

$$\begin{aligned} & (X^0(\hat{t}) - X^0(\hat{t} - \Delta) - \tilde{\alpha})A(\hat{t})^2 - (X^0(\hat{t} + \Delta) \\ & - X^0(\hat{t}) - \tilde{\alpha})A(\hat{t} + \Delta)^2 + \beta X^0(\hat{t}) = 0, \\ & \frac{1}{2}(A(\hat{t} - \Delta)^2 + A(\hat{t} + \Delta)^2 - 2A(\hat{t})^2) + (X^0(\hat{t}) \\ & - X^0(\hat{t} - \Delta) - \tilde{\alpha})^2 - \tilde{\alpha}^2 - \beta = 0. \end{aligned} \quad (3.13)$$

So unlike in Sec. II B 1, $(X^0(\hat{t}) - X^0(\hat{t} - \Delta) - \tilde{\alpha})A(\hat{t})^2$ is not invariant under discrete translations $\hat{t} \rightarrow \hat{t} + n\Delta$, $n = \text{integer}$, except for $\beta = 0$. The recursion relations (2.41) for the eigenvalues (2.39) are now generalized to

$$\begin{aligned} & (x_n^0 - x_{n+1}^0 - \tilde{\alpha})a_n^2 - (x_{n-1}^0 - x_n^0 - \tilde{\alpha})a_{n-1}^2 + \beta x_n^0 = 0, \\ & \frac{1}{2}(a_{n+1}^2 + a_{n-1}^2 - 2a_n^2) + (x_n^0 - x_{n+1}^0 - \tilde{\alpha})^2 - \tilde{\alpha}^2 - \beta = 0. \end{aligned} \quad (3.14)$$

Once again, starting with any two neighboring eigenvalues for A^2 and one time eigenvalue x_n^0 one can use the recursion relations to generate a matrix solution. As before solutions are only valid providing all $a_n^2 \geq 0$. This means that either n spans all positive and negative integers or the series is terminated at some n [and then (2.36) no longer holds]. The latter was the case for the discrete series of the noncommutative de Sitter solution. We discuss noncommutative de Sitter solutions of the equations of motion (3.2) in Sec. III B 2. For finite-dimensional matrix solutions there must be both a largest and smallest value of n . In Sec. III B 3 we show that finite matrix solutions exist for $\beta \neq 0$, and they contain the Lorentzian fuzzy sphere [19].

2. Noncommutative dS^2

When $\beta \neq 0$, and $\tilde{\alpha}^2 + 2\beta > 0$, there exist two distinct solutions to the recursion relations (3.14) of the form (2.46), i.e.,

$$\begin{aligned} x_n^0 &= -\alpha_{\pm}(n + \epsilon_0), \\ a_n^2 &= \alpha_{\pm}^2 n(n + 2\epsilon_0 + 1) + a_0^2, \end{aligned} \quad (3.15)$$

where

$$\alpha_{\pm} = \frac{1}{2} \left(\tilde{\alpha} \pm \sqrt{\tilde{\alpha}^2 + 2\beta} \right). \quad (3.16)$$

They are associated with two distinct noncommutative de Sitter solutions. In the commutative limit ($\tilde{\alpha} \rightarrow v\Theta$, $\beta \rightarrow \omega\Theta^2$) they go to the two de Sitter space-times described in Sec. III A, with $\alpha_{\pm} \rightarrow \Theta h_{\pm}$ and h_{\pm} given in (3.8). In both cases X^{μ} span an $su(1, 1)$ Lie algebra (2.47) with noncommutative parameters $\alpha = \alpha_{\pm}$. The two noncommutative solutions coincide when $\tilde{\alpha}^2 + 2\beta = 0$. In the limit $\beta \rightarrow 0$, one of them ($\alpha_{+} \rightarrow \tilde{\alpha}$) reduces to the solution of Sec. II B 4, while the other ($\alpha_{-} \rightarrow 0$) becomes the vacuum solution. Once again, nontrivial representations fall into the three categories, principal, supplementary, and discrete series, and they can be constructed as in Sec. II B 4 for each of the two solutions.

The noncommutative dS^2 relations (2.47) also solve the matrix equations (3.2) in the limit $\beta, \tilde{\alpha} \rightarrow \infty$, with $\frac{\beta}{\tilde{\alpha}}$ finite. This corresponds to a matrix action where the kinetic energy (or Yang-Mills) term is absent. The noncommutative dS^2 solution in this case has

$$\alpha = \frac{\beta}{2\tilde{\alpha}}, \quad (3.17)$$

and is the matrix analogue of (3.9).

3. Finite-dimensional matrix solutions

When $\beta \neq 0$, there exist finite-dimensional matrix solutions of the equations of motion (3.2), which are

associated with the $su(2)$ algebra. For this we express the matrices X^μ as a linear combination of J_i , which are $N \times N$ Hermitean matrices spanning the $su(2)$ algebra, $[J_i, J_j] = i\epsilon_{ijk}J_k$. Here $i = 1, 2, 3$ are *Euclidean* indices. We consider the following linear combination:

$$X^0 = 2J_3, \quad X^1 = 2v_1J_1, \quad X^2 = 2(v_2J_2 + v_3J_3). \quad (3.18)$$

X^μ are a solution to the equations of motion (3.2) when

$$\beta = -4v_1^2, \quad \tilde{\alpha}^2 = 1 - \frac{v_3^2}{1+v_1^2}, \quad v_2 = -\tilde{\alpha}v_1. \quad (3.19)$$

Hermiticity requires all coefficients v_i to be real and thus $\beta < 0$ and $\tilde{\alpha}^2 \leq 1$. Given that J_i define an irreducible representation of $su(2)$, the time eigenvalues for this solution are

$$2m = -N + 1, -N + 3, \dots, N - 1. \quad (3.20)$$

Like with the fuzzy sphere embedded in three-dimensional Euclidean space [11,31], the matrices X^μ span the $su(2)$ algebra. Here the $su(2)$ generators are given by

$$J_1 = \frac{1}{2v_1}X^1, \quad J_2 = \frac{1}{2v_2}(X^2 - v_3X^0), \quad J_3 = \frac{1}{2}X^0, \quad (3.21)$$

for $v_1, v_2 \neq 0$ and X^μ solve the Lorentzian, rather than Euclidean, matrix equations.

The above solutions are not in general rotationally invariant; i.e., (1.1) may not apply for these solutions, and therefore neither the ansatz (2.36) nor (2.43) in general hold. An exceptional case is $v_3 = 0$. After setting $\beta = -4\tilde{\alpha}^2v_1^2$ and doing a rescaling we can write

$$X^0 = 2\tilde{\alpha}J_3, \quad X^1 = \sqrt{-\beta}J_1, \quad X^2 = -\sqrt{-\beta}J_2. \quad (3.22)$$

For this case, X_+X_- commutes with X^0 . We can then identify the matrix A^2 with $-\beta(\tilde{J}^2 - J_3^2 - J_3)$. The N -dimensional irreducible representations for X^μ satisfying (3.22) define Lorentzian fuzzy spheres and were discussed previously in [19]. The commutative solution is recovered by taking the $N \rightarrow \infty$ limit, along with $\tilde{\alpha}, \beta \rightarrow 0$. For this we need to keep both $\tilde{\alpha}N$ and $\sqrt{-\beta}N$ finite, with $\frac{\sqrt{-\beta}}{2\tilde{\alpha}} \rightarrow a_0$, in the limit.

IV. STABILITY ANALYSIS

Here we examine small perturbations about the rotationally invariant solutions obtained in the previous two sections.

After substituting the perturbations back into the matrix action and taking the commutative limit we obtain a scalar field coupled to a gauge field on the space-time manifold associated with the commutative solution. Upon eliminating the gauge fields one gets an effective mass term for the scalar. As stated earlier, the effective mass squared is negative for the systems studied in Sec. II, as was shown in [18]. The result changes when the quadratic term is included in the matrix model action. We show that the effective mass squared is positive for a range of $\omega \neq 0$ ensuring the stability of the field theory in the commutative limit.

We begin in Sec. IVA with the specific example of noncommutative de Sitter space, which is a solution of the matrix model, with or without the inclusion of the quadratic term (cf. Sec. II B 4 and III B 2). A general analysis which is valid for all rotationally invariant solutions is carried out in Sec. IV B. It relies on finding the appropriate Seiberg-Witten map for the system.

A. Noncommutative dS^2

Here we expand the matrices Y^μ about the noncommutative de Sitter solution X^μ , which satisfy the $su(1,1)$ commutation relations (2.47). Taking the expansion parameter to be the noncommutative parameter α , one can write

$$Y^\mu = X^\mu - \alpha R A^\mu, \quad (4.1)$$

at leading order. R is a distance scale, with R^2 being the value of the central operator in (2.48). As in [11], the infinite-dimensional Hermitean matrices A^μ are functions of X^μ . A^μ can be regarded as noncommutative potentials. This is since infinitesimal gauge transformations (iii) of the form $U = 1 + i\alpha R \Lambda$, where Λ is an infinite-dimensional Hermitean matrix with infinitesimal elements, lead to the gauge variations

$$\delta A^\mu = -i[\Lambda, X^\mu] + i\alpha R[\Lambda, A^\mu]. \quad (4.2)$$

Following [11] we define noncommutative field strengths $F_{\mu\nu}$ according to

$$\alpha^2 R^2 F_{\mu\nu} = [Y_\mu, Y_\nu] + i\alpha \epsilon_{\mu\nu\lambda} Y^\lambda. \quad (4.3)$$

They transform covariantly, $\delta F_{\mu\nu} = +i\alpha R[\Lambda, F_{\mu\nu}]$. $F_{\mu\nu}$ vanishes when it is evaluated on the noncommutative de Sitter solution $Y^\mu = R X^\mu$, where X^μ satisfies (2.47).

We next express the matrix model action in terms of A_μ and $F_{\mu\nu}$. We do this for the action (2.1) in Sec. IVA 1 and then consider the quadratic term in IVA 2.

1. $\beta = 0$

Upon substituting (4.1) and (4.3) into the action (2.1) we get

$$S(Y) = \frac{\alpha^4 R^2}{g^2} \text{Tr} \left\{ -\frac{R^2}{4} F_{\mu\nu} F^{\mu\nu} - \frac{iR}{6} \epsilon_{\mu\nu\lambda} F^{\mu\nu} A^\lambda + \frac{i}{6\alpha} \epsilon_{\mu\nu\lambda} [A^\mu, X^\nu] A^\lambda + \frac{1}{6} A_\mu A^\mu \right\} + S(X). \quad (4.4)$$

The result can be reexpressed on the two-dimensional de Sitter manifold, with embedding coordinates x^μ satisfying $(x^1)^2 + (x^2)^2 - (x^0)^2 = R^2$, upon using the appropriate star product for de Sitter space [16]. The latter can be expanded in the noncommutative parameter α . At lowest order, the star commutator of the symbols of the functions $\mathcal{F}(X)$ and $\mathcal{G}(X)$ goes to $+i\alpha/R$ times the Poisson bracket of $\mathcal{F}(x)$ and $\mathcal{G}(x)$, i.e., $+\frac{i\alpha}{R} \{\mathcal{F}(x), \mathcal{G}(x)\}$. Then the commutative limit of the action (4.4) may be written as an integral of symbols according to

$$S(X) - S(\bar{X}) \rightarrow \frac{\alpha^4 R^2}{g_c^2} \int d\mu \left\{ \frac{1}{4} \left(\{A_\mu, x_\nu\} - \{A_\nu, x_\mu\} \right)^2 - \frac{R}{2} \epsilon_{\mu\nu\lambda} \{A^\mu, x^\nu\} A^\lambda \right\}, \quad (4.5)$$

where $d\mu$ is the invariant measure over de Sitter space which we now specify. Instead of using (2.12), we find it more convenient to use a different parametrization of the surface. Following [16], we take

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix} = R \begin{pmatrix} \tan \eta \\ \sec \eta \cos \sigma \\ \sec \eta \sin \sigma \end{pmatrix}, \quad (4.6)$$

where $\{-\frac{\pi}{2} \leq \eta \leq \frac{\pi}{2}, -\pi \leq \sigma < \pi\}$ and we have included the distance scale. For the integration measure we take $d\mu = d\eta d\sigma R^2 / \cos^2 \eta$. The induced metric tensor resulting from (4.6) is

$$-\mathfrak{g}_{\eta\eta} = \mathfrak{g}_{\sigma\sigma} = \frac{R^2}{\cos^2 \eta}, \quad \mathfrak{g}_{\eta\sigma} = 0. \quad (4.7)$$

The fundamental Poisson brackets are $\{\eta, \sigma\} = \cos^2 \eta$, which is consistent with the $su(1,1)$ Lie algebra, $\{x^\mu, x^\nu\} = R \epsilon^{\mu\nu\lambda} x_\lambda$.

Like in [11], we introduce a pair of tangent vectors K_μ^a , $a = \eta, \sigma$ on the manifold defined by

$$K_\mu^a = \frac{\cos^2 \eta}{R^2} \epsilon^{ab} \epsilon_{\mu\nu\lambda} x^\nu \partial_b x^\lambda. \quad (4.8)$$

K_μ^a along with the normal vector x_μ form the orthogonal basis. Moreover,

$$\begin{aligned} K_\mu^a K^{b\mu} &= R^2 \mathfrak{g}^{ab}, \\ K_\mu^\eta x^\mu &= K_\mu^\sigma x^\mu = 0, \\ \epsilon_{\mu\nu\lambda} K^{\eta\mu} K^{\sigma\nu} x^\lambda &= R \cos^2 \eta. \end{aligned} \quad (4.9)$$

Additional identities are

$$\begin{aligned} \frac{\partial K_\mu^\eta}{\partial \eta} &= \frac{\partial K_\mu^\sigma}{\partial \sigma} = -\tan \eta K_\mu^\eta - \frac{1}{R} x_\mu, \\ \frac{\partial K_\mu^\sigma}{\partial \eta} &= \frac{\partial K_\mu^\eta}{\partial \sigma} = -\tan \eta K_\mu^\sigma, \\ \frac{\partial x_\mu}{\partial \eta} &= -\frac{R}{\cos^2 \eta} K_\mu^\eta, \quad \frac{\partial x_\mu}{\partial \sigma} = \frac{R}{\cos^2 \eta} K_\mu^\sigma. \end{aligned} \quad (4.10)$$

We now expand $A_\mu(\eta, \sigma)$ in the above basis and define $U(1)$ gauge potentials $(\mathcal{A}_\eta, \mathcal{A}_\sigma)$ and a scalar field ϕ on de Sitter space:

$$R A_\mu(\eta, \sigma) = \mathcal{A}_\sigma(\eta, \sigma) K_\mu^\eta + \mathcal{A}_\eta(\eta, \sigma) K_\mu^\sigma + \frac{1}{R} \phi(\eta, \sigma) x_\mu. \quad (4.11)$$

Then using the above identities

$$\begin{aligned} \{A_\mu, x_\nu\} - \{A_\nu, x_\mu\} &= \mathcal{F}_{\eta\sigma} K_\mu^\eta K_\nu^\sigma + \phi \epsilon_{\mu\nu\lambda} \frac{x^\lambda}{R} + (\partial_\eta \phi - \mathcal{A}_\sigma) \frac{x_\mu}{R} K_\nu^\sigma \\ &\quad + (\partial_\sigma \phi - \mathcal{A}_\eta) \frac{x_\mu}{R} K_\nu^\eta - (\mu \rightleftharpoons \nu), \end{aligned} \quad (4.12)$$

where $\mathcal{F}_{\eta\sigma} = \partial_\eta \mathcal{A}_\sigma - \partial_\sigma \mathcal{A}_\eta$ is the $U(1)$ field strength on the de Sitter manifold. Substituting (4.11) and (4.12) into the commutative limit (4.5) of the action gives

$$\begin{aligned} S(X) - S(\bar{X}) &\rightarrow \frac{\alpha^4 R^2}{g_c^2} \int d\eta d\sigma \left\{ -\frac{1}{2} \cos^2 \eta (\mathcal{F}_{\eta\sigma})^2 + \frac{1}{2} (\partial_\eta \phi)^2 \right. \\ &\quad \left. - \frac{1}{2} (\partial_\sigma \phi)^2 + 2\phi \mathcal{F}_{\eta\sigma} - \frac{\phi^2}{\cos^2 \eta} \right\} \\ &\rightarrow \frac{\alpha^4 R^2}{g_c^2} \int d\eta d\sigma \sqrt{-\mathfrak{g}} \left\{ \frac{R^2}{4} \mathcal{F}_{ab} \mathcal{F}^{ab} - \frac{1}{2} \partial_a \phi \partial^a \phi \right. \\ &\quad \left. + \frac{2}{\sqrt{-\mathfrak{g}}} \phi \mathcal{F}_{\eta\sigma} - \frac{1}{R^2} \phi^2 \right\}, \end{aligned} \quad (4.13)$$

where \mathfrak{g} is the determinant of the induced metric (4.7). The mass squared for the scalar field appearing in the action is positive, $m^2 = 2/R^2$. However, the gauge field is non-dynamical on a two-dimensional space-time and can be eliminated, which leads to the effective field Lagrangian

$$\mathcal{L}_{\text{eff}} = -\frac{1}{2} \partial_a \phi \partial^a \phi + \frac{1}{R^2} \phi^2. \quad (4.14)$$

We then get a switch of sign for the mass squared of the scalar field, and so the scalar field is tachyonic. More generally, tachyonic excitations were shown to occur for all spherically symmetric solutions of the matrix model described in Sec. II [18].

We note that the kinetic energies of the gauge and scalar fields in (4.13) have opposite signs. This appears to be a generic feature of the Lorentzian matrix model [18,19], and is not totally unexpected since the matrix model action, specifically the Yang-Mills term, is not positive definite. This situation is harmless in two space-time dimensions since the gauge field can be eliminated. However, the same does not apply in higher dimensions, and this issue is yet to be resolved.

2. $\beta \neq 0$

We now include the quadratic term in the total action (3.1). As stated in Sec. III B 2, there exist two noncommutative de Sitter solutions (2.47) when $\tilde{\alpha}^2 + 2\beta > 0$ (and one when $\tilde{\alpha}^2 + 2\beta = 0$). They correspond to $\alpha = \alpha_{\pm}$ as given in (3.16). There is also one noncommutative de Sitter solution in the limit $\beta, \tilde{\alpha} \rightarrow \infty$, with $\frac{\beta}{\tilde{\alpha}}$ finite. In this case, $\alpha = +\frac{\beta}{2\tilde{\alpha}}$.

An expansion (4.1) in the action (3.1) around one of the de Sitter solutions gives

$$S_{\text{total}}(Y) = \frac{\alpha^4 R^2}{g^2} \text{Tr} \left\{ -\frac{R^2}{4} F_{\mu\nu} F^{\mu\nu} - i\gamma R \epsilon_{\mu\nu\lambda} F^{\mu\nu} A^\lambda + \frac{i\gamma}{\alpha} \epsilon_{\mu\nu\lambda} [A^\mu, \bar{X}^\nu] A^\lambda + \gamma A_\mu A^\mu \right\} + S_{\text{total}}(X), \quad (4.15)$$

where $\gamma = \frac{1}{2} - \frac{\tilde{\alpha}}{3\alpha}$ and $F_{\mu\nu}$ is again defined by (4.3). We now repeat the previous procedure to obtain the commutative limit of the action. $\frac{\tilde{\alpha}}{\alpha}$ and $\frac{\beta}{\tilde{\alpha}^2}$ go to finite values in the commutative limit, respectively, $+\frac{v}{h}$ and $\frac{\omega}{h^2}$, and hence so does γ . One now gets

$$\begin{aligned} S_{\text{total}}(Y) - S_{\text{total}}(X) &\rightarrow \frac{\alpha^4 R^2}{g_c^2} \int \frac{d\eta d\sigma}{\cos^2 \eta} \left\{ \frac{1}{4} (\{A_\mu, x_\nu\} - \{A_\nu, x_\mu\})^2 \right. \\ &\quad \left. + \left(\frac{1}{2} - \frac{v}{h} \right) R \epsilon_{\mu\nu\lambda} \{A^\mu, x^\nu\} A^\lambda + \left(1 - \frac{v}{h} \right) R^2 A_\mu A^\mu \right\} \\ &\rightarrow \frac{\alpha^4 R^2}{g_c^2} \int d\eta d\sigma \sqrt{-g} \left\{ \frac{R^2}{4} \mathcal{F}_{ab} \mathcal{F}^{ab} - \frac{1}{2} \partial_a \phi \partial^a \phi \right. \\ &\quad \left. + \left(2 - \frac{v}{h} \right) \left(\frac{2}{\sqrt{-g}} \phi \mathcal{F}_{\eta\sigma} - \frac{1}{R^2} \phi^2 \right) \right\} \quad (4.16) \end{aligned}$$

in the commutative limit after applying (4.11) and (4.12). The kinetic energy terms are unaffected by the deformation parameter β , and the previous result (4.13) for the remaining terms are recovered when $h \rightarrow +v$ corresponding to

$\omega \rightarrow 0$. Upon eliminating the nondynamical gauge field, one gets the following effective field Lagrangian:

$$\mathcal{L}_{\text{eff}} = -\frac{1}{2} \partial_a \phi \partial^a \phi + \frac{1}{R^2} \left(2 - \frac{v}{h} \right) \left(3 - 2 \frac{v}{h} \right) \phi^2, \quad (4.17)$$

generalizing (4.14). Thus the mass squared now depends on $\frac{v}{h}$, which from (3.8) has two possible values, $\frac{v^2}{\omega} (-1 \pm \sqrt{1 + \frac{2\omega}{v^2}})$. It is positive for

$$\frac{3}{2} < +\frac{v}{h} < 2, \quad (4.18)$$

and so the system is stabilized for this range of parameters. The value $\omega = 0$ corresponds to $v/h \rightarrow +1$ or ∞ and hence it is not included in this range. This is consistent with the result of the previous subsection that the mass squared is negative when the quadratic term is not included in the matrix model. The corresponding values for ω are obtained using $\omega = 2h(h - v)$. The scalar field is massless for the special values $(\frac{v}{h}, \frac{\omega}{2h^2}) = (+\frac{3}{2}, -\frac{1}{2})$ and $(+2, -1)$.

B. General analysis using a Seiberg-Witten map

Here we consider perturbing about arbitrary solutions to the matrix equations (3.1). Since in general we do not have exact solutions, we cannot follow the previous procedure, which requires defining a noncommutative field strength that vanishes for zero perturbations. An alternative procedure is to expand the action about the commutative solution up to second order in the noncommutative parameter Θ . The perturbations can be expressed in terms of commutative gauge fields and a scalar field upon applying the appropriate Seiberg-Witten map on the two-dimensional target manifold \mathcal{M}_2 . The map was found in [18] and is reviewed in Appendix A. The commutative gauge fields correspond to perturbations along the tangent directions of \mathcal{M}_2 , while the scalar field is associated with perturbations normal to the surface. In [18], the procedure was applied to the rotationally invariant manifolds which were solutions to the equations of motion (2.7). These equations followed from the action (2.4) which did not include a quadratic term. There we found that for all rotationally invariant solutions, the effective dynamics for the scalar field gave a tachyonic mass. This indicates an instability with respect to perturbations normal to the surface. Here we include the quadratic term in the matrix model action, i.e., we start with (3.1), and repeat the stability analysis for the rotationally invariant solutions to equations of motion (3.4). Just as in Sec. IV A 2, we find that when $\beta \neq 0$, the effective mass squared for the scalar field can be positive for a certain range of parameters.

As in Secs. IV A 1 and IV A 2, we denote perturbations of the embedding coordinates by A^μ . They are perturbations about the commutative space-time solutions, as well as

functions on the noncommutative manifold. Assuming the existence of a general rotationally invariant star product one can write the perturbations as functions on a smooth manifold parametrized by τ and $e^{i\sigma}$, and so $A^\mu = A^\mu(\tau, e^{i\sigma})$. The perturbation parameter shall again be identified with the noncommutativity parameter Θ , and thus

$$y^\mu = x^\mu + \Theta A^\mu. \quad (4.19)$$

The perturbations (4.19) induce nonvanishing fluctuations in the induced metric tensor at first order in Θ , and thus affect the space-time geometry. As in the previous subsection, A_μ transform as noncommutative gauge potentials. Up to first order in Θ , the infinitesimal gauge variations of A_μ are given by

$$\delta A_\mu = \{\Lambda, x_\mu\} + \Theta \{\Lambda, A_\mu\}. \quad (4.20)$$

Using the Poisson brackets (2.3), gauge variations at zeroth order in Θ are along the tangential directions of \mathcal{M}_2 ,

$$\delta A_\mu = -h(\partial_\tau \Lambda \partial_\sigma x_\mu - \partial_\sigma \Lambda \partial_\tau x_\mu) + \mathcal{O}(\Theta). \quad (4.21)$$

We shall choose $h = h(\tau)$, which is consistent with all the solutions in Secs. II A and III A.

Using a Seiberg-Witten map [40], the noncommutative potentials A_μ can be reexpressed in terms of commutative gauge potentials, denoted by $(\mathcal{A}_\tau, \mathcal{A}_\sigma)$, on \mathcal{M}_2 , along with their derivatives. Since the noncommutative potentials A_μ have three components and the commutative potentials have only two, an additional degree of freedom, associated with a scalar field ϕ should be included in the map: $A_\mu = A_\mu[\mathcal{A}_\tau, \mathcal{A}_\sigma, \phi]$. The Seiberg-Witten map is defined so that commutative gauge transformation, $(\mathcal{A}_\tau, \mathcal{A}_\sigma) \rightarrow (\mathcal{A}_\tau + \partial_\tau \lambda, \mathcal{A}_\sigma + \partial_\sigma \lambda)$, for arbitrary functions λ on \mathcal{M}_2 , induce noncommutative gauge transformations for A_μ , which are given by (4.20) for infinitesimal gauge transformations. Λ in this case is a function of λ , along with commutative potentials and their derivatives, $\Lambda = \Lambda[\lambda, \mathcal{A}_\tau, \mathcal{A}_\sigma]$.

The Seiberg-Witten map consistent with (2.3) and (2.12) was obtained in [18] up to first order in Θ and is written down explicitly in Appendix A, cf., (A1)–(A3). In this regard the first order map is sufficient for our purposes since we wish to expand the action S_c , and hence also y^μ , up to second order in Θ . The task is to thus substitute (4.19), along with the map (A1)–(A3) into the action (3.3). After some work we find

$$S_c(y) = -\frac{\Theta^2}{g_c^2} \int d\tau d\sigma h(\tau)^3 g^2 \left(\frac{1}{2} \mathcal{F}_{\tau\sigma} \mathcal{F}^{\tau\sigma} - \frac{1}{2} \partial_a \phi \partial^a \phi + \gamma(\tau) \mathcal{F}_{\tau\sigma} \phi - \frac{1}{2} m^2(\tau) \phi^2 \right) + S_c(x), \quad (4.22)$$

where indices $a, b, \dots = \tau, \sigma$ are raised and lowered with the induced metric associated with the invariant interval (5.7). The time-dependent coupling coefficient $\gamma(\tau)$ and mass $m(\tau)$ are given by

$$\begin{aligned} \gamma(\tau) &= \frac{2a(\tau)^2}{h(\tau)^2 g^2} \left(-v h(\tau) g_{\tau\tau} + \omega a(\tau) \left(\frac{\tau}{a(\tau)} \right)' \right), \\ m(\tau)^2 &= \frac{a(\tau)^2}{h(\tau)^2 g^2} \left(g_{\tau\tau} \left(2h(\tau)^2 - 4v h(\tau) - \omega \right) + 2\omega a(\tau) \left(\frac{\tau}{a(\tau)} \right)' \right). \end{aligned} \quad (4.23)$$

Here we see a common feature for these systems, which is that the kinetic energies of the gauge and scalar fields have opposite signs. Upon eliminating the gauge field using its equation of motion, $\mathcal{F}^{\tau\sigma} + \gamma(\tau)\phi = \text{constant}/(h(\tau)^3 g^2)$, and substituting back into (4.22), we now get the effective action

$$S_c^{\text{eff}}(y) = -\frac{\Theta^2}{g_c^2} \int d\tau d\sigma h(\tau)^3 g^2 \left(-\frac{1}{2} \partial_a \phi \partial^a \phi - \frac{1}{2} m_{\text{eff}}^2(\tau) \phi^2 \right) + S_c^{\text{eff}}(x), \quad (4.24)$$

where the mass squared for the scalar field gets modified to

$$\begin{aligned} m_{\text{eff}}^2(\tau) &= m(\tau)^2 + g\gamma(\tau)^2 \\ &= \frac{2}{h(\tau)^2 g} \left(v^2 + (h(\tau) - v)^2 - \frac{\omega}{2} + \frac{\omega a(\tau)}{g_{\tau\tau}^2 h(\tau)^2} \right. \\ &\quad \times \left. \left(\frac{\tau}{a(\tau)} \right)' \left(g_{\tau\tau} h(\tau) (h(\tau) - 4v) + 2\omega a(\tau) \left(\frac{\tau}{a(\tau)} \right)' \right) \right). \end{aligned} \quad (4.25)$$

The result for m_{eff}^2 is in general a function of the time parameter τ and it can be positive, negative or zero. It is negative when $\omega = 0$ for all spherically symmetric solutions. This follows from the Lorentzian signature of the metric, $g < 0$. The result agrees with what was found in [18].

We get the following results for the three exact solutions found in Sec. III A:

- (1) γ , m^2 and m_{eff}^2 are constants for the de Sitter solution (2.25) and (3.8) with v and ω finite. In this case (4.23) and (4.25) yield

$$\begin{aligned}\gamma &= m^2 = 2\left(2 - \frac{v}{h}\right), \\ m_{\text{eff}}^2 &= 2\left(2 - \frac{v}{h}\right)\left(-3 + 2\frac{v}{h}\right).\end{aligned}\quad (4.26)$$

Of course this case has already been discussed in Sec. IV A. The result agrees with the mass squared appearing in (4.17), which as we saw can be positive, negative or zero. (Here we set $R = 1$.)

- (2) For the dS^2 solution (3.9) obtained in the limit $v, \omega \rightarrow \infty$, with $\frac{\omega}{v}$ finite, the result of the perturbations is a BF Lagrangian with no kinetic energy terms,

$$\mathcal{L}_{\text{eff}} = \mathcal{F}_{\tau\sigma}\phi - \frac{1}{2}\phi^2. \quad (4.27)$$

Thus both the gauge field and scalar field are nondynamical, with the equations of motion giving $\mathcal{F}_{\tau\sigma} = \phi = \text{constant}$. The result for the field theory action can also be seen from (4.16), since the kinetic terms do not contribute in the limiting case. Of course the result is not surprising since there are no Yang-Mills terms in the matrix action.

- (3) For the case of the sphere embedded in Minkowski space-time (3.10), we get

$$\begin{aligned}\gamma &= -\frac{1 + 2\tau^2}{(1 - 2\tau^2)^2}, \\ m^2 &= -\frac{3 - 2\tau^2}{(1 - 2\tau^2)^2}, \\ m_{\text{eff}}^2 &= 4\frac{1 - \tau^2 + 2\tau^4}{(1 - 2\tau^2)^3}.\end{aligned}\quad (4.28)$$

The latter is negative in the region with Lorentzian signature and hence the scalar field is tachyonic. As alluded to in Sec. III A, this solution is a special case of a one-parameter family of solutions associated with ellipsoids in Minkowski space-time. m_{eff}^2 is positive for some range of the parameter, corresponding to stable dynamical systems [19].

For all the remaining solutions m_{eff}^2 can be computed numerically. In the case of the numerical solution shown in Fig. 5 depicting the transition from a rapid inflationary to a noninflationary phase, we find that $m_{\text{eff}}^2 > 0$ for all τ . The result for m_{eff}^2 is also plotted in Fig. 5. We can then conclude that the solution is stable for leading order perturbations.

V. THE FIVE-DIMENSIONAL MATRIX MODEL

We now consider five Hermitean matrices Y^μ , with indices $\mu, \nu, \dots = 0, 1, 2, 3, 4$, which are raised and lowered with the five-dimensional Minkowski space-time metric $\eta = \text{diag}(-1, 1, 1, 1, 1)$. In analogy with Sec. III, we demand that the dynamics for Y^μ is invariant under $4 + 1$

Lorentz transformations and unitary gauge transformations, but not necessarily translations. The Yang-Mills and quadratic terms in the matrix action (3.1) generalize straightforwardly to the five-dimensional case. The totally antisymmetric cubic term Chern-Simons type in (3.1) can be replaced by a fifth order term. This term has been introduced previously in Euclidean matrix models [12,41,42]. Then we can write the five-dimensional matrix action according to

$$\begin{aligned}S_{\text{total}}(Y) &= \frac{1}{g^2}\text{Tr}\left(-\frac{1}{4}[Y_\mu, Y_\nu][Y^\mu, Y^\nu] \right. \\ &\quad \left. + \frac{4}{5}\alpha_5\epsilon_{\mu\nu\lambda\rho\sigma}Y^\mu Y^\nu Y^\lambda Y^\rho Y^\sigma + \frac{\beta}{2}Y_\mu Y^\mu\right),\end{aligned}\quad (5.1)$$

where $\epsilon_{01234} = 1$ and g, α_5 and β are real constants. The resulting equations of motion are

$$[[Y_\mu, Y_\nu], Y^\nu] + 4\tilde{\alpha}_5\epsilon_{\mu\nu\lambda\rho\sigma}Y^\nu Y^\lambda Y^\rho Y^\sigma = -\beta Y_\mu. \quad (5.2)$$

An exact solution to these equations was found for the Euclidean metric and $\beta = 0$ and it was called the fuzzy four sphere [12,41,42]. An analogous construction should be possible in Minkowski space-time to obtain a noncommutative four-dimensional de Sitter space. Other nontrivial solutions to this model are not known; however, we can show that large families of solutions exist after taking the commutative limit of this matrix model. This is done in Sec. V A. One of the solutions is four-dimensional de Sitter space, but it differs from the commutative limit of the Lorentzian counterpart of the four-sphere. We consider perturbations about the de Sitter solution in Sec. V B. Finally in Sec. V C, we propose an ansatz for rotationally invariant solutions to the five-dimensional matrix model.

A. Solutions in the commutative limit

In order to take the commutative limit, it is convenient to write the fifth order term in the trace (5.1) using the commutator, $\frac{1}{5}\alpha_5\epsilon_{\mu\nu\lambda\rho\sigma}Y^\mu[Y^\nu, Y^\lambda][Y^\rho, Y^\sigma]$. Then we can easily apply the usual procedure to get the commutative theory. We replace matrices Y^μ by coordinate functions y^μ , now defined on a four-dimensional manifold \mathcal{M}_4 , and matrix commutators by $i\Theta$ times the corresponding Poisson bracket on \mathcal{M}_4 . Θ once again denotes the noncommutativity parameter. If all three terms in the action are to survive in the limit we need that α_5 goes to a finite value, $\alpha_5 \rightarrow v_5$ and, as before, $\beta \rightarrow \omega\Theta^2$, with v_5 and ω finite. The limiting action in that case is

$$\begin{aligned}S_c(y) &= \frac{1}{g_c^2}\int_{\mathcal{M}_4} d\mu_4\left(\frac{1}{4}\{y_\mu, y_\nu\}\{y^\mu, y^\nu\} \right. \\ &\quad \left. - \frac{v_5}{5}\epsilon_{\mu\nu\lambda\rho\sigma}y^\mu\{y^\nu, y^\lambda\}\{y^\rho, y^\sigma\} + \frac{\omega}{2}y_\mu y^\mu\right),\end{aligned}\quad (5.3)$$

where $d\mu_4$ denotes the invariant integration measure, on \mathcal{M}_4 . The equations of motion resulting from variations of y^μ are

$$\{\{y_\mu, y_\nu\}, y^\nu\} + v_5 \epsilon_{\mu\nu\lambda\rho\sigma} \{y^\nu, y^\lambda\} \{y^\rho, y^\sigma\} = \omega y_\mu. \quad (5.4)$$

Infinitesimal gauge variations again have the form $\delta y^\mu = \Theta \{\Lambda, y^\mu\}$, where Λ is an infinitesimal function on \mathcal{M}_4 .

We denote coordinates on a local patch of \mathcal{M}_4 by τ, σ and ξ^i , $i, j, \dots = 1, 2, 3$. τ is timelike, while σ and ξ^i are spacelike, the latter spanning a unit two-sphere $(\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 = 1$. A rotationally invariant ansatz for solutions $y^\mu = x^\mu(\tau, \sigma, \xi^i)$ to the equations of motion (5.3) is

$$\begin{pmatrix} x^0 \\ x^i \\ x^4 \end{pmatrix} = \begin{pmatrix} \tau \\ a(\tau) \xi^i \sin \sigma \\ a(\tau) \cos \sigma \end{pmatrix}. \quad (5.5)$$

This is an obvious generalization of (2.12). The spatial coordinates x^1, x^2, x^3, x^4 span a three-sphere of radius $a(\tau)$ at time slice,

$$\vec{x}^2 + (x^4)^2 = a^2(\tau), \quad (5.6)$$

where $\vec{x}^2 = x^i x^i$, and the isometry group is $SO(4)$. For this one assumes $0 \leq \sigma \leq \pi$, with $\sigma = 0, \pi$ corresponding to the poles. The invariant interval on the surface is

$$ds^2 = -(1 - a'(\tau)^2) d\tau^2 + a(\tau)^2 (d\sigma^2 + \sin^2 \sigma ds_{S^2}^2), \quad (5.7)$$

where $ds_{S^2}^2$ is the invariant interval on the two-sphere and $d\sigma^2 + \sin^2 \sigma ds_{S^2}^2$ is the invariant interval on the three-sphere. The resulting Ricci scalar and Einstein tensor are nonvanishing, indicating the presence of a gravitational source. They are, respectively,

$$R = \frac{6(1 - a'(\tau)^2 + a(\tau)a''(\tau))}{a(\tau)^2(1 - a'(\tau)^2)^2} \quad (5.8)$$

and

$$\begin{aligned} G_{\tau\tau} &= \frac{3}{a(\tau)^2}, & G_{\sigma\sigma} &= -\frac{(1 - a'(\tau)^2 + 2a(\tau)a''(\tau))}{(1 - a'(\tau)^2)^2} \\ G_{\theta\theta} &= \sin^2 \sigma G_{\sigma\sigma}, & G_{\phi\phi} &= \sin^2 \sigma \sin^2 \theta G_{\sigma\sigma}, \end{aligned} \quad (5.9)$$

θ and ϕ being the usual spherical coordinates on the two-sphere.

It remains to define a symplectic structure on the four-dimensional space-time manifold. Although there is no nonsingular Poisson structure on the three sphere, we can write Poisson brackets which are consistent with three-dimensional rotation symmetry, i.e., corresponding to

rotations among the three spatial coordinates x^i . The fundamental Poisson brackets are

$$\{\sigma, \tau\} = h(\tau, \sigma) \quad \{\xi^i, \xi^j\} = \kappa \epsilon_{ijk} \xi^k, \quad (5.10)$$

where κ is constant. The Jacobi identity is trivially satisfied. Here, unlike for the two-dimensional case discussed in Secs. II A and III A, we allow for h to be a function of σ as well as τ . This ansatz along with (5.5) will allow for nontrivial solutions to the equations of motion.

Below we examine three different families of solutions to the equations of motion (5.4). In each case only two out of the three terms in the action (5.3) contribute:

- (1) We first consider the limiting case where both $\omega, v_5 \rightarrow \infty$, with $\frac{\omega}{v_5}$ and κ finite and nonvanishing.

In this limit the first term in the action (5.3) (i.e., the Yang-Mills term) does not contribute. We examined the analogous example in the three-dimensional matrix model (case 2 in Sec. III A) where the Yang-Mills term does not contribute. The equations of motion are then

$$v_5 \epsilon_{\mu\nu\lambda\rho\sigma} \{y^\nu, y^\lambda\} \{y^\rho, y^\sigma\} = \omega y_\mu. \quad (5.11)$$

They are solved by $a(\tau)^2 = \tau^2 + 1$ which defines the four-dimensional de Sitter space dS^4

$$-(x^0)^2 + \vec{x}^2 + (x^4)^2 = 1. \quad (5.12)$$

The Poisson structure on this space is determined by the two parameters $\frac{\omega}{v_5}$ and κ . The solution for $h(\tau, \sigma)$ is

$$h(\tau, \sigma) = -\frac{\omega \csc^2 \sigma}{8\kappa v_5 a(\tau)^2}. \quad (5.13)$$

From (5.13), the Poisson brackets of the embedding coordinates x^μ are

$$\begin{aligned} \{x^0, x^i\} &= \frac{\omega}{8\kappa v_5} \frac{x^i x^4}{(\vec{x}^2)^{\frac{3}{2}}}, & \{x^4, x^i\} &= \frac{\omega}{8\kappa v_5} \frac{x^i x^0}{(\vec{x}^2)^{\frac{3}{2}}}, \\ \{x^0, x^4\} &= -\frac{\omega}{8\kappa v_5} \frac{1}{\sqrt{\vec{x}^2}}, & \{x^i, x^j\} &= \kappa \sqrt{\vec{x}^2} \epsilon_{ijk} x^k. \end{aligned} \quad (5.14)$$

It can be checked that the Poisson bracket relations are consistent with the de Sitter space condition (5.12). The Poisson brackets are invariant under the action of the three-dimensional rotation group, although they do not preserve all the isometries of de Sitter space. More specifically, $SO(4, 1)$ is broken to $SO(3) \times \mathcal{L}_2$, where \mathcal{L}_2 is the two-dimensional Lorentz group.

This solution is the four-dimensional analogue of the previous dS^2 solution (3.9) to the equations of motion (3.4) in the limit $v, \omega \rightarrow \infty$, with $\frac{\omega}{v}$ finite. Unlike the case with dS^2 , the Poisson brackets of the coordinate functions x^μ on dS^4 are not associated with a finite-dimensional Lie algebra, and so its matrix analogue of the commutative solution is nontrivial. By changing the background metric to $\eta = \text{diag}(-1, 1, 1, 1, -1)$ we can obtain a four-dimensional anti-de Sitter space and by switching the background metric to $\eta = \text{diag}(1, 1, 1, 1, 1)$ we recover a four sphere. The solutions are given explicitly in Appendix B.

- (2) Here we set $\kappa = v_5 = 0$ and take $h = h(\tau)$. Now the second term in the action (5.3) does not contribute to the dynamics. The Poisson brackets are noninvertible in this case, and the equations of motion trivially reduce to the two-dimensional system (2.15) with $v = 0$; i.e. $h(\tau)$ and $a(\tau)$ should satisfy

$$\begin{aligned} (aa'h)'h + h^2 &= \omega, \\ 2h^2a'a + a^2h'h &= \omega\tau. \end{aligned} \quad (5.15)$$

There is a one-parameter (not including integration constants) family of solutions which can be obtained numerically. Solutions for different values of ω were already plotted in Fig. 4. We recall that σ , $0 \leq \sigma < 2\pi$, was periodic in Secs. II–IV. Here σ parametrizes the longitudes on the three-sphere and ranges from 0 to π , where $\sigma = 0$ and π denote the poles and correspond to coordinate singularities.

- (3) Finally we consider $\omega = 0, \kappa \neq 0$. Now the third term in the action (5.3) does not contribute. If we set

$$h(\tau, \sigma) = f(\tau)\sin^2\sigma, \quad (5.16)$$

then the solution to (5.4) with space-time index $\mu = 0$ is

$$f(\tau) = 2\kappa v_5 a(\tau)^2 + \frac{c_1}{a(\tau)^2}, \quad (5.17)$$

where c_1 is an integration constant. The remaining equations of motion, $\mu = i, 4$, are solved when $a(\tau)$ satisfies

$$a'(\tau)^2 = \left(\frac{\kappa a(\tau)}{f(\tau)} \right)^2 + 1. \quad (5.18)$$

This implies that $|a'(\tau)|$ cannot be less than one. So, for instance, $a(\tau)$ cannot have turning points and there can be no closed space time solutions. Moreover, from (5.7) the induced metric has a Euclidean signature, even though the background metric is Lorentzian. Solutions of this form have no

two-dimensional analogues. They simplify in some limiting cases:

- (i) In the case $v_5 \rightarrow 0$, one gets

$$\begin{aligned} a'(\tau)^2 &= \frac{\kappa^2}{c_1^2} a(\tau)^6 + 1, \\ f(\tau) &= \frac{c_1}{a(\tau)^2}. \end{aligned} \quad (5.19)$$

$a(\tau)$ is then expressible in terms of inverse elliptic integrals.

- (ii) The limit $\kappa \rightarrow 0, c_1 \neq 0$ gives a linearly expanding (or contracting) universe

$$a(\tau) = \pm\tau, \quad f(\tau) = \frac{c_1}{\tau^2}. \quad (5.20)$$

- (iii) Another simplifying limit is $c_1 \rightarrow 0$, leading to

$$\begin{aligned} a'(\tau)^2 &= \frac{1}{4v_5 a(\tau)^2} + 1, \\ f(\tau) &= 2\kappa v_5 a(\tau)^2, \end{aligned} \quad (5.21)$$

which can be easily integrated

$$a(\tau) = \frac{1}{2} \sqrt{(2\tau + c_2)^2 - \frac{1}{v_5^2}}, \quad (5.22)$$

where c_2 is an integration constant. This solution describes an open space-time. Here we must restrict the time domain to $|2\tau + c_2| \geq 1/|v_5|$. The solution for $a(\tau)$ goes asymptotically to τ and it is singular in the limit $v_5 \rightarrow 0$, as well as $\kappa \rightarrow 0$.

In general, solutions of (5.17) and (5.18) are parametrized by κ, v_5 and the integration constant c_1 . (The initial value for a just determines the overall scale.) Some examples of numerical solutions for $a(\tau)$ are plotted in Fig. 6 for different values for v_5 and c_1 and fixed $\kappa = 1$. In one example, $c_1 = 0$ and $v_5 = 5$, there is an initial rapid inflation followed by a linear expansion, which is very similar to the solution plotted in Fig. 5. In contrast, the example, $c_1 = 1$ and $v_5 = 0$, a linear expansion is followed by a rapid inflation.

B. Perturbations about dS^4

Here we consider the four-dimensional de Sitter solutions (1 in Sec. VA) which resulted upon taking the limit $\omega, v_5 \rightarrow \infty$, with $\frac{\omega}{v_5}$ finite. The solutions have a Poisson structure which is determined by two finite nonvanishing parameters $\frac{\omega}{v_5}$ and κ . In this subsection we expand about the commutative dS^4 solutions, expressing perturbations in terms of commutative gauge and scalar fields. This requires

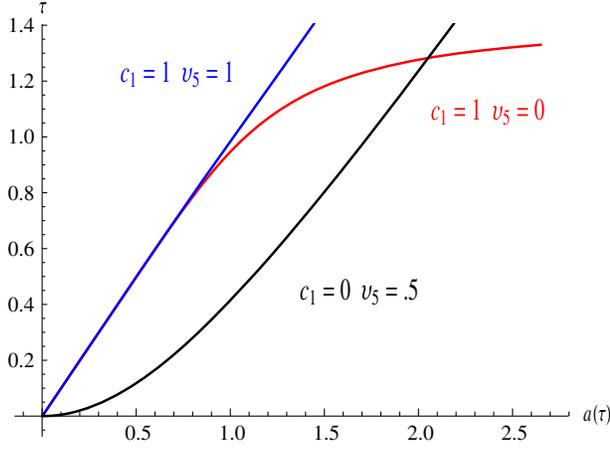


FIG. 6. Numerical solution for (5.17) and (5.18) for $\kappa = 1$ and three different sets of values for v_5 and c_1 . τ is plotted versus $a(\tau)$, with initial condition $a(0) \approx 0$ in all cases. For the choice of $c_1 = 1$ and $v_5 = 1$, one gets an approximate linear expansion. The choice of $c_1 = 0$ and $v_5 = 5$ has an initial linear inflation followed by a linear expansion, while the choice $c_1 = 1$ and $v_5 = 0$ has an initial linear expansion followed by a rapid inflation.

finding the appropriate Seiberg-Witten map, which is given explicitly in Appendix C.

We again write the embedding coordinates y^μ according to (4.19) where Θ is the perturbation parameter and the

perturbations A^μ are now functions on dS^4 . Locally then A^μ are functions of τ, σ and coordinates on S^2 , which we take to be the usual spherical coordinates θ and ϕ , $0 < \theta < \pi$ and $0 \leq \phi < 2\pi$. The action (5.1) is gauge invariant, at least up to first order in Θ . So A_μ can be regarded as non-commutative gauge potentials up to first order in Θ . A Seiberg-Witten map can be constructed on dS^4 , so that A_μ can be reexpressed in terms of commutative gauge potentials \mathcal{A}_a , $a = \tau, \sigma, \theta, \phi$, and a scalar field Φ . As in Sec. IV B, in order to produce the leading order correction to the action we need to obtain the Seiberg-Witten map up to first order. The result is given in (B2) and (B3) of Appendix C.

We next substitute the expression for y^μ , (4.19), along with (B2) and (B3), into the action $S_c(y)$. For the integration measure we take

$$\begin{aligned} d\mu_4 &= \frac{d\mu_{S^2}}{h(\tau, \sigma)} = \frac{\sin \theta d\tau d\sigma d\theta d\phi}{h(\tau, \sigma)} \\ &= -\frac{8\kappa v_5}{\omega} \sqrt{-g} d\tau d\sigma d\theta d\phi, \end{aligned} \quad (5.23)$$

where $d\mu_{S^2}$ is the invariant measure on the sphere, $h(\tau, \sigma)$ is given in (5.13) and g is the determinant of the metric on dS^4 . Thus

$$S_c(y) \propto \int \sqrt{-g} d\tau d\sigma d\theta d\phi \left(-\frac{v_5}{5} \epsilon_{\mu\nu\lambda\rho} y^\mu \{y^\nu, y^\lambda\} \{y^\rho, y^\sigma\} + \frac{\omega}{2} y_\mu y^\mu \right). \quad (5.24)$$

After removing all of the total derivatives we arrive at the following simple expression for the leading order (i.e., quadratic) terms in the action $S_c(y)$ (up to a proportionality constant)

$$\begin{aligned} \Theta^2 \int \sqrt{-g} d\tau d\sigma d\theta d\phi &\left(\frac{\omega}{v_5} \frac{\mathcal{F}_{\tau\sigma} \Phi}{a(\tau)^2 \sin^2 \sigma} \right. \\ &\left. + 8\kappa^2 \frac{\mathcal{F}_{\theta\phi} \Phi}{\sin \theta} - 12\kappa \Phi^2 \right), \end{aligned} \quad (5.25)$$

where $\mathcal{F}_{ab} = \partial_a \mathcal{A}_b - \partial_b \mathcal{A}_a$ are the commutative field strengths. This generalizes the effective Lagrangian in (4.27). Just as in that example, no kinetic energy terms appear for the scalar and gauge fields, which is not surprising since there were no kinetic energy term for y^μ in (5.24). (We therefore do not have to be concerned with the issue of the kinetic energy terms having opposing signs in this case.) The scalar field couples to the radial component of the electric and magnetic fields. The field equations imply that Φ is frozen to a constant value, while the field strengths are constrained by

$$\frac{\omega}{v_5} \frac{\mathcal{F}_{\tau\sigma}}{a(\tau)^2 \sin^2 \sigma} + 8\kappa^2 \frac{\mathcal{F}_{\theta\phi}}{\sin \theta} = 24\kappa \Phi. \quad (5.26)$$

So for example, in the absence of an electric field, the perturbations give rise to a magnetic monopole source with charge equal to $\frac{1}{4\pi} \int \mathcal{F}_{\theta\phi} d\theta \wedge d\phi = \frac{3\Phi}{4\pi\kappa} \int \sin \theta d\theta \wedge d\phi = \frac{3\Phi}{\kappa}$. The monopoles spontaneously break the de Sitter group symmetry down to the three-dimensional rotation group, due to the same symmetry breaking that is present in the Poisson brackets on the surface (5.14).

C. The question of rotationally invariant matrix solutions

In Sec. II B 1 we wrote down an ansatz for rotationally invariant matrices for the three-dimensional matrix model and their resulting equations of motion. Here we propose to do the same in the five-dimensional case. Rotational symmetry in this case is applied to the three matrices X^i , $i = 1, 2, 3$ (and not also X_4). This is consistent with the $SO(3)$ symmetry of the solutions in Sec. VA of the commutative equations of motion. We show that the five

matrix equations (5.2) reduce to three upon taking into account rotational symmetry, and in a special case reduce to the matrix equations of Sec. III.

A natural choice for the matrix analogues of the ansätze (5.5) and (5.10) is

$$\begin{pmatrix} Y^0 \\ Y^i \\ Y^4 \end{pmatrix} = \begin{pmatrix} t \otimes 11 \\ u \otimes j_i \\ v \otimes 11 \end{pmatrix} \quad (5.27)$$

where t, u, v are Hermitian matrices and j_i generate the fuzzy sphere

$$[j_i, j_j] = i\alpha\kappa\epsilon_{ijk}j_k. \quad (5.28)$$

Without loss of generality, we can choose its radius to be one, $j_i j_i = 11$. Upon substituting into the five matrix equations (5.2) one gets that t, u, v must satisfy

$$\begin{aligned} -[[t, u], u] - [[t, v], v] + 4i\tilde{\alpha}_5\alpha\kappa[u^2, [u, v]]_+ &= -\beta t, \\ -[[u, t], t] + [[u, v], v] - 4i\tilde{\alpha}_5\alpha\kappa[u^2, [t, v]]_+ \\ + 4i\tilde{\alpha}_5\alpha\kappa[[u, v], [t, u]] + 2\alpha^2\kappa^2 u^3 &= -\beta u, \\ -[[v, t], t] + [[v, u], u] + 4i\tilde{\alpha}_5\alpha\kappa[u^2, [t, u]]_+ &= -\beta v, \end{aligned} \quad (5.29)$$

where $[\cdot, \cdot]_+$ denotes the anticommutator.

While we have not found general solutions to these matrix equations, the system simplifies when $\kappa = 0$, or equivalently $\alpha = 0$. In that case the equations reduce to those of the three-dimensional system (3.2) with $(t, u, v) = (Y^0, Y^1, Y^2)$ and $\tilde{\alpha} = 0$. Therefore the recursion relations (3.14) can be applied in this case to numerically generate the spectra of matrices satisfying (2.36). The commutative limit of such solutions corresponds to solutions 2 of Sec. VA.

D. Alternative background metrics

We remark that all of the previous solutions to the commutative equations have analogs when one changes the signature of the background metric η . As an example, here we consider the four-dimensional de Sitter solution dS^4 to the equations of motion (5.11). These equations were associated with the limit where both $\omega, v_5 \rightarrow \infty$, with $\frac{\omega}{v_5}$ and κ finite and nonvanishing. If we now switch to the Euclidean background metric $\eta = \text{diag}(1, 1, 1, 1)$ we recover an S^4 solution to the equations of motion. If instead we replace the background metric by $\eta = \text{diag}(-1, 1, 1, 1, -1)$ we recover AdS^4 . A fuzzy four-sphere was obtained previously to a matrix model [12,41,42]. However, its commutative limit differs from that of our solution. In fact, Poisson brackets for the former do not close among the coordinates. The AdS^4 solution we obtain has a nontrivial Poisson structure. The Poisson brackets are

also nontrivial at the AdS^4 boundary for generic cases of the constants. There is an exceptional case, however, where the lowest order noncommutativity vanishes at the boundary, but not in the interior. It corresponds to the limit where both κ and ω vanish.

We first discuss the S^4 solution and then AdS^4 .

1. S^4

For the case of the Euclidean metric $\eta = \text{diag}(1, 1, 1, 1)$ we can keep the ansatz (5.5), only now τ is a spacelike coordinate. The equations of motion have the solution $a(\tau)^2 = 1 - \tau^2$ for $-1 \leq \tau \leq 1$ with $h(\tau, \sigma)$ given by (5.13). This is the solution for S^4

$$(x^0)^2 + \vec{x}^2 + (x^4)^2 = 1, \quad (5.30)$$

with Poisson brackets

$$\begin{aligned} \{x^0, x^i\} &= \frac{\omega}{8\kappa v_5} \frac{x^i x^4}{(\vec{x}^2)^{\frac{3}{2}}}, & \{x^4, x^i\} &= -\frac{\omega}{8\kappa v_5} \frac{x^i x^0}{(\vec{x}^2)^{\frac{3}{2}}}, \\ \{x^0, x^4\} &= -\frac{\omega}{8\kappa v_5} \frac{1}{\sqrt{\vec{x}^2}}, & \{x^i, x^j\} &= \kappa \sqrt{\vec{x}^2} \epsilon_{ijk} x^k. \end{aligned} \quad (5.31)$$

They break the $SO(5)$ rotational symmetry to $SO(3) \times SO(2)$. This result differs from the commutative limit of the fuzzy four-sphere [12,41,42]. The algebra for the latter does not close among the embedding coordinates. Conversely, the matrix analogue of the system (5.30) and (5.31), if it exists, is not the fuzzy four-sphere.

2. AdS^4

A global parametrization of AdS^4 is

$$\begin{pmatrix} x^0 \\ x^i \\ x^4 \end{pmatrix} = \begin{pmatrix} \sin \tau \cosh \sigma \\ \sinh \sigma \xi^i \\ \cos \tau \cosh \sigma \end{pmatrix}, \quad (5.32)$$

where once again $\xi^i, i = 1, 2, 3$, span the unit two-sphere $(\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 = 1$. The background metric is now $\eta = \text{diag}(-1, 1, 1, 1, -1)$ and so

$$-(x^0)^2 + \vec{x}^2 - (x^4)^2 = -1. \quad (5.33)$$

The induced metric on the surface is given by

$$ds^2 = -\cosh^2 \sigma d\tau^2 + d\sigma^2 + \sinh^2 \sigma ds_{S^2}^2, \quad (5.34)$$

$ds_{S^2}^2$ being associated with the unit two-sphere. The time-like parameter τ is periodic and closed timelike curves exist on this space. Equation (5.32) solves the equations of motion (5.11) upon taking the fundamental Poisson brackets to be (5.10), with $h(\tau, \sigma)$ given by

$$h(\tau, \sigma) = \frac{\omega}{8\kappa v_5} \frac{1}{\sinh^2 \sigma \cosh \sigma}. \quad (5.35)$$

The resulting Poisson brackets of the coordinates x^μ of the embedding space are

$$\begin{aligned} \{x^0, x^i\} &= -\frac{\omega}{8\kappa v_5} \frac{x^i x^4}{(\vec{x}^2)^{\frac{3}{2}}}, & \{x^4, x^i\} &= \frac{\omega}{8\kappa v_5} \frac{x^i x^0}{(\vec{x}^2)^{\frac{3}{2}}}, \\ \{x^0, x^4\} &= -\frac{\omega}{8\kappa v_5} \frac{1}{\sqrt{\vec{x}^2}}, & \{x^i, x^j\} &= \kappa \sqrt{\vec{x}^2} \epsilon_{ijk} x^k. \end{aligned} \quad (5.36)$$

They break the $SO(3, 2)$ space-time symmetry to $SO(3) \times SO(2)$.

The result can be reexpressed in terms of Fefferman-Graham coordinates $(z, \zeta^0, \zeta^1, \zeta^2)$, which only cover a local patch of AdS^4 [43]. The map from (τ, σ, ξ^i) to $(z, \zeta^0, \zeta^1, \zeta^2)$ is given by

$$\begin{aligned} \zeta^0 &= z \sin \tau \cosh \sigma, & \zeta^1 &= z \sinh \sigma \xi^1, & \zeta^2 &= z \sinh \sigma \xi^2, \\ \frac{1}{z} &= \cos \tau \cosh \sigma - \xi^3 \sinh \sigma. \end{aligned} \quad (5.37)$$

In terms of these coordinates the induced metric is given by

$$ds^2 = \frac{dz^2 - (d\zeta^0)^2 + (d\zeta^1)^2 + (d\zeta^2)^2}{z^2}, \quad (5.38)$$

while the Poisson brackets are mapped to

$$\begin{aligned} \{z, \zeta^0\} &= \frac{\omega}{8\kappa v_5} z^3 \text{csch} \sigma (\xi^3 \text{ctnh} \sigma \cos \tau - 1), \\ \{z, \zeta^a\} &= \kappa z^2 \sinh \sigma \epsilon^{ab} \zeta^b - \frac{\omega}{8\kappa v_5} z \text{csch}^3 \sigma \zeta^0 \zeta^a, \\ \{\zeta^0, \zeta^a\} &= \kappa z \sinh \sigma \epsilon^{ab} \zeta^0 \zeta^b - \frac{\omega}{8\kappa v_5} z^2 \text{csch}^3 \sigma \zeta^a, \\ \{\zeta^1, \zeta^2\} &= -\kappa z^3 \sinh^2 \sigma (\sinh \sigma - \xi^3 \cosh \sigma \cos \tau), \end{aligned} \quad (5.39)$$

where $a, b = 1, 2$.

The boundary at infinity corresponds to $z \rightarrow 0$, or equivalently $\sigma \rightarrow \infty$ with $z e^\sigma$ finite. In that limit, (5.37) reduces to

$$\begin{aligned} \frac{2}{z} e^{-\sigma} \zeta^0 &\rightarrow \sin \tau, & \frac{2}{z} e^{-\sigma} \zeta^a &\rightarrow \xi^a, \\ \frac{2}{z} e^{-\sigma} &\rightarrow \cos \tau - \xi^3. \end{aligned} \quad (5.40)$$

These equations are solved by

$$\begin{aligned} z^2 e^{2\sigma} &\rightarrow (\zeta^0)^4 + 2(\zeta^0)^2(1 - \zeta^a \zeta^a) + (1 + \zeta^a \zeta^a)^2, \\ \cos \tau &\rightarrow \frac{1}{z} e^{-\sigma} (1 - (\zeta^0)^2 + \zeta^a \zeta^a), \\ \xi^3 &\rightarrow \frac{1}{z} e^{-\sigma} (-1 - (\zeta^0)^2 + \zeta^a \zeta^a). \end{aligned} \quad (5.41)$$

We can consistently take the limit $z \rightarrow 0$ on the Poisson brackets (5.39) since z has zero Poisson brackets with the coordinates on the boundary $(\zeta^0, \zeta^1, \zeta^2)$. The remaining Poisson brackets at $z \rightarrow 0$ are

$$\begin{aligned} \{\zeta^0, \zeta^a\} &= \frac{1}{2} \kappa z e^\sigma \epsilon^{ab} \zeta^0 \zeta^b, \\ \{\zeta^1, \zeta^2\} &= -\frac{1}{4} \kappa z e^\sigma (1 + (\zeta^0)^2 + \zeta^a \zeta^a), \end{aligned} \quad (5.42)$$

with given $z e^\sigma$ in (5.41). A central element in the algebra is $\mathcal{C} = (1 - (\zeta^0)^2 + \zeta^a \zeta^a) / \zeta^0$ and the Poisson bracket is then nonsingular on the two-dimensional surfaces with $\mathcal{C} = \text{constant}$ corresponding to symplectic leaves.

All Poisson brackets vanish at the boundary in the limit $\kappa \rightarrow 0$. Then for $h(\tau, \sigma)$ in (5.35) to be well defined, we would also need to send $\frac{\omega}{v_5} \rightarrow 0$. Only the totally antisymmetric term in the action (5.3) survives in this case, and the equations of motion reduce to

$$\epsilon_{\mu\nu\lambda\rho\sigma} \{y^\nu, y^\lambda\} \{y^\rho, y^\sigma\} = 0. \quad (5.43)$$

A general solution is

$$\begin{aligned} \{x^0, x^i\} &= -\rho x^i x^4, & \{x^4, x^i\} &= \rho x^i x^0, \\ \{x^0, x^4\} &= -\rho \vec{x}^2, & \{x^i, x^j\} &= 0, \end{aligned} \quad (5.44)$$

where ρ can be any function of \vec{x}^2 . These Poisson brackets are consistent with the AdS^4 constraint (5.33) and the Jacobi identity. They agree with (5.36) in the limit $\kappa, \omega \rightarrow 0$ for $\rho \sim 1/(\vec{x}^2)^{\frac{3}{2}}$. If we express x^μ using the parametrization (5.32), then the Poisson brackets (5.44) result from taking $\{\sigma, \tau\} = \rho \tanh \sigma$ and $\{\xi^i, \xi^j\} = 0$. The Poisson brackets (5.44) generalize to any dimension $d > 2$ although they may not in general solve a matrix model equation. Once again, they can be reexpressed in terms of Fefferman-Graham coordinates. They vanish upon being projected to the AdS boundary $z \rightarrow 0$. Therefore in this case the boundary is commutative (at least at lowest order), with space-time symmetry corresponding to the full three-dimensional Poincaré group, while the interior of AdS is noncommutative.

VI. CONCLUDING REMARKS

In the Introduction we wrote down a general definition (1.1) of rotationally invariant matrices embedded in three-dimensional space-time, and in Sec. II we obtained recursion relations for such matrices which solve the Lorentzian matrix model equations of motion. These recursion relations allow one to generate discrete versions of open two-dimensional universes. For a matrix analogue of a closed space-time solution, we need to require the existence of bottom and top states, i.e., there must be both a minimum and maximum time eigenvalue. If the recursion relations are valid for such a solution, the recursion procedure must then terminate at the minimum and maximum time eigenvalues. Matrix solutions in this case would be finite dimensional. Here and in [19], we obtained finite-dimensional matrix solutions corresponding to Lorentzian fuzzy spheres, which resolve cosmological singularities. Here we showed that infinite-dimensional solutions to the toy matrix model corresponding to the discrete series representations of the de Sitter group also resolve cosmological singularities. In both of these examples, singularities in the induced metric emerge after taking the continuum (or commutative) limit. The commutative limit also allowed for other space-times with desirable features, such as a solution which transitions from a rapid initial inflation to a noninflationary phase. The quadratic term in the matrix model action studied in Sec. III played an important role for finding the novel solutions to the Lorentzian matrix model. It was also shown to be useful for stabilizing the leading order field theory which resulted from perturbations about the classical solutions.

Some of the matrix solutions describing two-dimensional space-times in the commutative limit generalize in a straightforward way to higher-dimensional space-time geometries, while others do not. In Sec. V we saw that solutions to the commutative limit with $v = 0$ have an obvious generalization to four dimensions. (This was case 2 in Sec. VA.) Since they do not require a totally antisymmetric term in the action, analogous solutions exist in any dimension d . Another example of a solution which generalizes to $d > 2$ is the de Sitter solution. This is the case where the matrix model has no kinetic energy term. In the commutative limit, the two-dimensional solution is given by (2.26) and (2.28), while its four-dimensional counterpart (case 1 in Sec. VA) is given by (5.12) and (5.14). The corresponding S^4 and AdS^4 solutions along with their attached Poisson structures were given explicitly in Sec. VD. The Poisson brackets of the S^4 solution differed from the commutative limit of the fuzzy four-sphere obtained in [12,41,42]. This is obvious because the commutation algebra of the coordinates for the fuzzy four-sphere does not close. On the other hand, due to the nontrivial nature of our Poisson brackets for the $d = 4$ commutative solutions, the matrix model analogues of these solutions are not straightforward to obtain, unlike

the case with $d = 2$. Concerning our AdS^4 solution, we found that the general Poisson brackets (5.36) are nonzero when projected to the boundary. The exceptional case corresponds to the limit where both κ and ω vanish. In this case the boundary remains commutative, at least at lowest order, where the space-time symmetry is the full three-dimensional Poincaré group. The corresponding matrix action in this case consists only of the totally antisymmetric term, $\epsilon_{\mu\nu\lambda\rho}\text{Tr}Y^\mu Y^\nu Y^\lambda Y^\rho Y^\sigma$. These results could have interesting implications for the AdS/CFT correspondence.

In addition to the solutions which generalize from $d = 2$, there are some solutions to the higher-dimensional theories which have no $d = 2$ analogue. This was true for case 3 in Sec. VA. Concerning the different families of $d = 4$ space-time manifolds obtained in Sec. VA, it may be possible to find other solutions to the commutative equations, and even the matrix equations. For example, we can use the fact that fuzzy coset models are higher-dimensional generalizations of the fuzzy sphere [36]. The latter was shown in [19] to solve the three-dimensional Lorentzian matrix model, and so it is natural to ask if the former solve higher-dimensional Lorentzian matrix models. More specifically, fuzzy CP^2 may solve the five-dimensional model. In the commutative limit, the solutions would yield cosets manifolds embedded in Minkowski space-time.

Another possibility for finding more solutions, at least in the commutative limit, is to modify the ansatz (5.5) in Sec. V, which for any time slice describes S^3 . For example, we can let the spatial coordinates x^1, x^2, x^3, x^4 instead span $S^2 \times S^1$. For this we can introduce a second radial quantity $b(\tau)$ and replace (5.5) by

$$\begin{pmatrix} x^0 \\ x^i \\ x^4 \end{pmatrix} = \begin{pmatrix} \tau \\ (a(\tau) \sin \sigma + b(\tau)) \xi^i \\ a(\tau) \cos \sigma \end{pmatrix}. \quad (6.1)$$

Here σ is a periodic parameter, $0 \leq \sigma < 2\pi$. Its canonical conjugate will have a regularly spaced spectrum in the noncommutative version of the theory, similar to that of the operator \hat{t} in (2.38). This ansatz is a generalization of (5.5), since it reduces to it in the limit $b \rightarrow 0$. $SO(3)$ is an isometry for this system, and this three-dimensional rotation symmetry is preserved if we once again impose the Poisson brackets (5.10).

While the focus of this article has been to search for classical solutions to toy matrix models which give rise to cosmological space-times in the continuum limit, one can have for matrix analogues of other solutions for general relativity, such as black hole solutions [14]. The eigenvalues of such a matrix solution give a lattice description of a black hole. Bounded solutions would necessarily give a resolution of the black hole singularity. It would be of interest to demonstrate how to recover black hole metrics,

along with their singularities, from the induced metric upon taking the continuum limit.

We examined perturbations about the rotationally invariant solutions to the three-dimensional Lorentzian matrix in Sec. IV, and the de Sitter solution of the five-dimensional model in Sec. VB. In the commutative limit the result is a gauge theory coupled to a scalar field theory. The gauge fields are associated with longitudinal perturbations, while the scalar fields denote perturbations normal to the space-time surface. A persistent feature of the emergent field theory is that the kinetic energies of the gauge and scalar fields have opposite signs. This presents no obstacle to the two-dimensional field theories, since the gauge fields are nondynamical and can be eliminated from the action. The result is an effective field theory for the remaining scalar field which for different choices of the parameters can be massive, massless, or tachyonic. It was also not an issue for perturbations about the four-dimensional de Sitter solution, as the kinetic energy vanished in that case. This system led to magnetic monopoles on the surface. For higher dimensional matrix solutions, the difference in signs in the kinetic energy terms remains an issue which requires a creative solution.

ACKNOWLEDGMENTS

We are very grateful to A. Pinzul and C. Uhlemann for valuable discussions.

APPENDIX A: 2D ROTATIONALLY INVARIANT SEIBERT-WITTEN MAP

Here we review the general Seiberg-Witten map up to first order in Θ on a two-dimensional rotationally invariant surface [18]. It is required to be consistent with (2.3) and (2.12). At lowest order in Θ , contributions to the noncommutative potentials A_μ come from the commutative gauge potentials $(\mathcal{A}_\tau, \mathcal{A}_\sigma)$ along the tangent directions to the surface, and the scalar field is associated with perturbations normal to the surface. Also at lowest order, the noncommutative gauge parameter Λ can be identified with the commutative gauge parameter λ . The next order is obtained by demanding consistency with (4.20). The result is

$$\begin{aligned} A_\mu &= A_\mu^{(0)} + \Theta A_\mu^{(1)} + \mathcal{O}(\Theta^2) \\ \Lambda &= \Lambda^{(0)} + \Theta \Lambda^{(1)} + \mathcal{O}(\Theta^2), \end{aligned} \quad (\text{A1})$$

$$\begin{aligned} A^{(0)0} &= -h(\tau)(-\mathcal{A}_\sigma + a'(\tau)a(\tau)\phi) \\ A_\pm^{(0)} &= -h(\tau)e^{\pm i\sigma}(\pm ia(\tau)\mathcal{A}_\tau - a'(\tau)\mathcal{A}_\sigma + a(\tau)\phi) \\ \Lambda^{(0)} &= \lambda \end{aligned} \quad (\text{A2})$$

$$\begin{aligned} A^{(1)0} &= h(\tau) \left(\frac{1}{2} \partial_\tau (h(\tau) \mathcal{A}_\sigma^2) + a'(\tau) h(\tau) \mathcal{A}_\tau \partial_\sigma (a(\tau) \phi) \right. \\ &\quad \left. - \mathcal{A}_\sigma \partial_\tau (a'(\tau) h(\tau) a(\tau) \phi) \right) \\ A_\pm^{(1)} &= h(\tau) e^{\pm i\sigma} \left(\mp i \partial_\tau (a(\tau) h(\tau) \mathcal{A}_\tau) \mathcal{A}_\sigma \right. \\ &\quad \mp ia(\tau) h(\tau) \mathcal{A}_\tau \mathcal{F}_{\tau\sigma} \pm ih(\tau) a(\tau) \mathcal{A}_\tau \phi \\ &\quad + h(\tau) \mathcal{A}_\tau \partial_\sigma (a(\tau) \phi) - \mathcal{A}_\sigma \partial_\tau (h(\tau) a(\tau) \phi) \\ &\quad \left. + \frac{1}{2} \partial_\tau (a'(\tau) h(\tau) \mathcal{A}_\sigma^2) - \frac{1}{2} a(\tau) h(\tau) \mathcal{A}_\tau^2 \right) \\ \Lambda^{(1)} &= h(\tau) \mathcal{A}_\sigma \partial_\tau \lambda \end{aligned} \quad (\text{A3})$$

where $A_\pm^{(n)} = A_1^{(n)} \pm A_2^{(n)}$ and $\mathcal{F}_{\tau\sigma} = \partial_\tau \mathcal{A}_\sigma - \partial_\sigma \mathcal{A}_\tau$ is the $U(1)$ gauge field on the surface.

APPENDIX B: SEIBERG-WITTEN MAP ON dS^4

Here we obtain the Seiberg-Witten map up to first order in the noncommutativity parameter for the four-dimensional de Sitter solution of Sec. V.

We first obtain the zeroth order result. This is easy to determine by comparing the gauge transformation properties of the commutative gauge potentials \mathcal{A}_a , $a = \tau, \sigma, \theta, \phi$ with those of the noncommutative potentials A_μ , using the Poisson brackets (5.10). The gauge variations of the former are simply $\delta \mathcal{A}_a = \partial_a \lambda$, λ being an infinitesimal commutative gauge parameter on dS^4 , while the latter is given by (4.20), where Λ is an infinitesimal noncommutative gauge parameter. The result is

$$\begin{aligned} \delta A_\mu &= -h(\partial_\tau \Lambda \partial_\sigma x_\mu - \partial_\sigma \Lambda \partial_\tau x_\mu) \\ &\quad + \frac{\kappa}{\sin \theta} (\partial_\theta \Lambda \partial_\phi x_\mu - \partial_\phi \Lambda \partial_\theta x_\mu) \\ &\quad + \Theta \delta A_\mu^{(1)} + \mathcal{O}(\Theta^2), \end{aligned} \quad (\text{B1})$$

where $h = h(\tau, \sigma)$ is given in (5.13). Then at zeroth order in Θ the commutative gauge potentials are tangent to dS^4 , while an additional degree of freedom Φ is associated with perturbations normal to the surface. Thus the zeroth order result for A_μ and Λ is given in

$$\begin{aligned} A_\mu &= -h(\mathcal{A}_\tau \partial_\sigma x_\mu - \mathcal{A}_\sigma \partial_\tau x_\mu) \\ &\quad + \frac{\kappa}{\sin \theta} (\mathcal{A}_\theta \partial_\phi x_\mu - \mathcal{A}_\phi \partial_\theta x_\mu) + \Phi x_\mu \\ &\quad + \Theta A_\mu^{(1)} + \mathcal{O}(\Theta^2) \\ \Lambda &= \lambda + \Theta \Lambda^{(1)} + \mathcal{O}(\Theta^2). \end{aligned} \quad (\text{B2})$$

For the first order terms, $A_\mu^{(1)}$ and $\Lambda^{(1)}$, we demand consistency with (4.20). A solution is

$$\begin{aligned}
A_\mu^{(1)} = & \partial_\tau x_\mu \left(\frac{a^3 a'}{2} (h^2 \mathcal{A}_\tau^2 + \kappa^2 \sin^2 \sigma \mathcal{A}_\Omega^2) + \frac{ha}{2} \partial_\tau \left(\frac{h}{a} \mathcal{A}_\sigma^2 \right) - \frac{h\kappa}{\sin \theta} (\mathcal{A}_\theta \mathcal{F}_{\sigma\phi} + \mathcal{A}_\phi \partial_\theta \mathcal{A}_\sigma) \right) \\
& + \partial_\sigma x_\mu \left(h^2 \mathcal{A}_\tau \mathcal{F}_{\sigma\tau} - \frac{h}{a} \mathcal{A}_\sigma \partial_\tau (ah \mathcal{A}_\tau) + \frac{1}{4} \partial_\sigma h^2 \mathcal{A}_\tau^2 - \frac{\kappa^2 \sin(2\sigma)}{4} \mathcal{A}_\Omega^2 + \frac{h\kappa}{\sin \theta} (\mathcal{A}_\theta \mathcal{F}_{\tau\phi} + \mathcal{A}_\phi \partial_\theta \mathcal{A}_\tau) \right) \\
& + \frac{\kappa \partial_\theta x_\mu}{\sin \theta} \left(h \left(\mathcal{A}_\tau \mathcal{F}_{\sigma\phi} + \cot \sigma \mathcal{A}_\tau \mathcal{A}_\phi - \frac{1}{a} \mathcal{A}_\sigma \partial_\tau (a \mathcal{A}_\phi) \right) + \frac{\kappa}{2} \left(\frac{\partial_\theta \mathcal{A}_\phi^2}{\sin \theta} - \cos \theta \mathcal{A}_\Omega^2 \right) \right) \\
& - \frac{\kappa \partial_\phi x_\mu}{\sin \theta} \left(h \left(\mathcal{A}_\tau \mathcal{F}_{\sigma\theta} + \cot \sigma \mathcal{A}_\tau \mathcal{A}_\theta - \frac{1}{a} \mathcal{A}_\sigma \partial_\tau (a \mathcal{A}_\theta) \right) + \frac{\kappa}{\sin \theta} (\mathcal{A}_\theta \mathcal{F}_{\theta\phi} + \mathcal{A}_\phi \partial_\theta \mathcal{A}_\theta) \right) \\
& - \frac{x_\mu}{2} \left(h^2 \left(a^2 \mathcal{A}_\tau^2 - \frac{\mathcal{A}_\sigma^2}{a^2} \right) + \kappa^2 a^2 \sin^2 \sigma \mathcal{A}_\Omega^2 \right) \\
& - h (\mathcal{A}_\tau \partial_\sigma (\Phi x_\mu) - \mathcal{A}_\sigma \partial_\tau (\Phi x_\mu)) + \frac{\kappa}{\sin \theta} (\mathcal{A}_\theta \partial_\phi (\Phi x_\mu) - \mathcal{A}_\phi \partial_\theta (\Phi x_\mu)) \\
\Lambda^{(1)} = & h \mathcal{A}_\sigma \partial_\tau \lambda - \frac{\kappa}{\sin \theta} \mathcal{A}_\phi \partial_\theta \lambda,
\end{aligned} \tag{B3}$$

where we define $\mathcal{A}_\Omega^2 = \mathcal{A}_\theta^2 + \mathcal{A}_\phi^2 / \sin^2 \theta$. $\mathcal{F}_{ab} = \partial_a \mathcal{A}_b - \partial_b \mathcal{A}_a$ are the commutative field strengths. In obtaining (B3) we have used the explicit expression for the de Sitter solution, $a^2 = \tau^2 + 1$.

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