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ABSTRACT: Most holographic models of superconducting systems neglect the effects of dynamical boundary gauge fields during the process of spontaneous symmetry-breaking. Usually a global symmetry gets broken. This yields a superfluid, which then is gauged “weakly” afterwards. In this work we build and probe the dynamics of a holographic model in which a local boundary symmetry is spontaneously broken instead. We compute two-point functions of dynamical non-Abelian gauge fields in the normal and in the broken phase, and find non-trivial gapless modes. Our AdS₃ gravity dual realizes a p-wave superconductor in (1+1) dimensions. The ground state of this model also breaks (1 + 1)-dimensional parity spontaneously, while the Hamiltonian is parity-invariant. We discuss possible implications of our results for a wider class of holographic liquids.

KEYWORDS: Gauge-gravity correspondence, Holography and condensed matter physics (AdS/CMT), Field Theories in Lower Dimensions, Spontaneous Symmetry Breaking

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1 Introduction

The gauge/gravity correspondence [1], also known as AdS/CFT correspondence or “holography”, has recently provided the tools to study models of strongly correlated systems exhibiting superconducting phases [2–4]. All of these constructions have in common that first a superfluid emerges from a process which spontaneously breaks a global symmetry. Then the resulting field theory is “weakly gauged” afterwards. In other words there are no dynamical gauge fields involved in the symmetry-breaking process, which means that

these models neglect effects of order $\mathcal{O}(1/g_{\text{YM}}^2)$ due to loop corrections which would involve photons in a real superconductor (g_{YM} is the gauge coupling). This may be seen as a shortcoming when trying to model “honest” superconductors in which dynamical gauge fields are present during the symmetry-breaking. In this work we address this issue.

Our first main goal in this paper is to study spontaneous symmetry-breaking in presence of dynamical gauge fields. In this sense we are building an “honest” holographic superconductor. S-wave superconducting ground states in presence of dynamical gauge fields have been studied in higher dimensions [5–8]. In the present work we extend these studies in three ways: i) we examine the dynamics of such a setup, most prominently the two-point functions of dynamical gauge-fields, ii) we consider a vector order parameter, i.e. a p-wave solution, and iii) we work in $(1+1)$ dimensions. Note that the Mermin-Wagner theorem is evaded in the large N limit [9, 10], allowing spontaneous symmetry-breaking in two spacetime dimensions. The normal phase of $(1+1)$ -dimensional fluids in presence of an Abelian field was studied holographically in [11], but using an interpretation distinct from ours.¹

Our second main goal in this paper is to study the two-point functions of our dual field theory as mentioned in point i) above. These are in particular interesting since our setup forces us to work in the so-called “alternative quantization”. We are exploring this alternative formulation and its low-energy effective field theory, which we call “non-equilibrium field dynamics”. Gauge fields in AdS_3 can in general have Neumann or Dirichlet boundary conditions. But in this paper we restrict ourselves to a special case where only Neumann conditions are possible. We consider a Yang-Mills action without any other terms, in particular we have no Chern-Simons term. In this special case gauge fields in AdS_3 can only have Neumann boundary conditions, so the dual field theory is always written in the alternative quantization. This is discussed in greater detail below towards the end of this introduction. In this context we also note that we can only study the superconductor (dynamical boundary gauge field), not a superfluid (external non-dynamical boundary gauge field) in our setup. Therefore we can not directly compare the two within our setup. But we find that in presence of our dynamical gauge field a spontaneous symmetry breaking occurs in our model. And formally this symmetry breaking in our holographic superconductor is very similar to the symmetry breaking known from higher-dimensional holographic superfluids.

We will focus here on the p-wave solution to a Yang-Mills action in AdS_3 -space

$$S = \int d^3x \sqrt{-g} \mathcal{L} = - \int d^3x \sqrt{-g} g^{\mu\mu'} g^{\nu\nu'} F_{\mu\nu}^a F_{\mu'\nu'}^a, \tag{1.1}$$

where g is the metric of AdS_3 , and F is an $\text{SU}(2)$ field strength. Note that we allow no Chern-Simons term here. This corresponds to forcing the chiral anomaly in the dual gauge theory to zero [12, 13]. An action such as (1.1) arises in a particular string theory setup, see section 3.1 for details. It can be thought of as a small-field expansion of the non-Abelian Dirac-Born-Infeld action of two coincident probe D3-branes in a background generated by

¹We will return to this work for comparison below.

N D3 branes with $N \rightarrow \infty$ [16]. For earlier discussions of this brane setup see [11, 17, 18].² The Yang-Mills-theory with action (1.1) is interesting to us for three main reasons:

First, the expectation is that we introduce gapless modes into an otherwise simple setup. For example, in [18] it was found that the duals to AdS₃ Maxwell-Chern-Simons theory do not show any hydrodynamic behavior, all correlators are fixed by symmetries. Now let us naively think of our bulk Yang-Mills-theory defined by (1.1) as such a Maxwell-Chern-Simons theory but with vanishing Chern-Simons coupling. When we introduce the p-wave condensate, we introduce gapless modes (emerging from the Goldstone modes) into an otherwise empty system. So the hope is to study separately these gapless modes, possibly analytically.

Second, our p-wave solution breaks parity in (1+1) dimensions. That means we have a parity-breaking ground state from a parity-invariant action. And the effective field theory description of the field theory dual to this setup needs to incorporate parity-violating terms. In contrast to the Chern-Simons setups in AdS₅ [24–26], here the parity-violating transport is not related to an anomaly, but to spontaneous symmetry-breaking.

Third, Yang-Mills-theory in AdS₃ is different in that it allows only one particular quantization: the bulk gauge fields $\mathcal{A}(t, x, z) = A(t, x, z) + a(t, x, z)$ (background and fluctuations) correspond to dynamical gauge fields in the boundary field theory. This boundary theory also contains currents, but they are non-dynamical, as they act as sources. In other words, the currents and gauge fields have switched their roles compared to the ordinary quantization. This claim is based on gauge-invariance and the presence of leading logarithmic terms in the boundary expansion of gravity fields. This allows only Neumann boundary conditions for the bulk gauge fields. For a careful analysis see [18, 27], (see also comments in [28]). One has to carefully repeat the analysis of [29] but for vector fields instead of scalar fields.

For scalars [29] showed that the two possible quantizations are related by a Legendre transformation and change of boundary conditions. Also the scalar operator two-point function in one quantization is simply the inverse of the scalar operator two-point function in the other quantization.³ A generalization to vector fields in (2+1)-dimensional conformal gauge theories (with AdS₄ gravity duals) was carried out in [30]: the two quantizations are related by an $SL(2, \mathbb{Z})$ transformation in the bulk. On the gravity side, the two bulk theories are dual to each other in the sense that $SL(2, \mathbb{Z})$ is an extension of the traditional electric-magnetic duality symmetry. On the field theory side, there is no obvious symmetry relation between the two (respectively corresponding) boundary theories. They are in general physically inequivalent,⁴ but related to each other: the application of $SL(2, \mathbb{Z})$ to one boundary theory takes us to the other boundary theory. So we can compute observables in one quantization if we know the other. Now in stark contrast to this, we are interested in a non-conformal theory with gauge fields in (1+1) dimensions where only one quantization

²Closely related to this are the constructions [19–22]. For a review of the relevant methods see [23].

³Note that changing quantizations amounts to changing the boundary conditions *and* performing a Legendre transformation.

⁴It has been shown that in the conformally-symmetric case there exists a group-theoretic equivalence between the two different quantizations [31].

exists if we do not allow a Chern-Simons term in our action (1.1). So a possible relation of conserved current Green's functions to gauge field Green's functions is less clear here, if it exists at all.⁵

In our alternative quantization one has to carefully question some of the concepts which are by now well established in the normal quantized versions of gauge/gravity. What we do not doubt is that gauge/gravity is a correspondence between bulk fields and operators on the boundary. In this generality we are going to use gauge/gravity in this work and consequently analyze one-point functions and two-point functions of our operators, by which we mean our dynamical boundary gauge fields, denoted by $\mathfrak{A}(t, x)$ for background fields, and $\mathfrak{a}(t, x)$ for fluctuations around the background. So in other words $A(t, x, z) + a(t, x, z)$ in the bulk corresponds to $\mathfrak{A}(t, x, z) + \mathfrak{a}(t, x, z)$ on the boundary. In this context we are going to make two further assumptions (which are standard in AdS/CFT):

1. Gauge-invariant bulk field combinations correspond to physical operators in the boundary field theory.
2. Quasinormal modes of gauge-invariant bulk field combinations coincide with the poles of two-point functions of physical operators in the boundary field theory.

Our setup with dynamical gauge fields and fixed currents also obscures the standard interpretation in terms of hydrodynamics. Hydrodynamics is a low-energy effective field theory which is always expressed in terms of dynamical conserved currents (organized in constitutive equations and conservation equations, see e.g. [32] for a review). In contrast to this, we have to consider a theory with dynamical gauge fields in the present work. We call the low-energy effective field theory of our field theory “non-equilibrium field dynamics”, or shorter “field dynamics” (in order to distinguish it from hydrodynamics). When doing linear response in our setup, we have only access to two-point-functions of gauge fields (gauge field propagators), but not to current-current correlators. We note that this formulation is very similar to the language used in thermal quantum field theory, see e.g. [33], and when discussing non-equilibrium thermal dynamics. Therefore one may hope that the gravity setting with Neumann boundary conditions provides results which compare more directly with field theory approaches. We stress again that the main point of our work is not a comparison of superfluids with superconductors. Instead we want to explore the alternative setup. In particular, we explore what we call field dynamics, i.e. the low-energy limit of our alternative description. This is analogous to hydrodynamics in the ordinary description.

We start in section 2 with a few thoughts on possible field theoretic descriptions of systems with dynamical gauge fields in two spacetime dimensions. In section 3 we introduce a concrete holographic setup, the Yang-Mills action in AdS₃ and our symmetry-breaking solution with vector hair. Then, in section 4 we present our analysis of fluctuations, explaining in detail how standard methods need to be adjusted to the situation in AdS₃.

⁵According to [31], for any tensor field (including scalars and vectors) in a conformal theory in any dimension the two quantizations are related by a Weyl reflection, and two-point functions are related to each other. But our model at hand breaks conformal symmetry.

Finally we close with a discussion of open tasks in section 5. Details on gauge-covariant and gauge-invariant fields are collected in appendix A. In appendix B we cross-check some of our results when using the ordinary interpretation with gauge fields being the sources.

2 Dynamical gauge fields in (1 + 1) dimensions

Although we are going to study a simple system on the gravity side, there will be several complications on the field theory side. We know of no suitable formulation of superfluids or superconductors in (1 + 1) dimensions expressed in a way which would be directly comparable to our setup. Hence, in this section we will outline our best estimate for such a suitable description. In order to disentangle the relevant topics, we separate and discuss them here from the field theory perspective (as opposed to the gravity perspective). We will be rather schematic and make qualitative observations here, before we introduce a concrete model in section 3.

2.1 Thermodynamics

As mentioned above, we are going to study gauge fields and their dynamics in this work. Finding a version of hydrodynamics with dynamical gauge fields (instead of the usual dynamical currents) turns out to be a bit tricky (as discussed below). However, our equilibrium states will have a fairly simple standard interpretation. For later convenience we here review the canonical and the grand-canonical ensemble. The grand-canonical ensemble is defined to be in contact with a particle reservoir. Therefore it has a fixed chemical potential μ , and the grand-canonical potential density is $\Omega(T, \mu)$. The canonical ensemble has a fixed charge density ρ and the relevant potential is the free energy density $\mathcal{F}(T, \rho)$. A Legendre transformation takes us from one ensemble into the other

$$\Omega = \mathcal{F} - \mu\rho. \tag{2.1}$$

So here we have two distinct setups: first, the grand-canonical ensemble (at fixed chemical potential μ) which we will associate with the “ordinary” setup later. Second, there is the canonical ensemble (at fixed charge density ρ), which we will associate with the “alternative” setup later. In this context these two setups have been used many times before, and usually one does not talk about this as “two distinct quantizations”. But the two setups are indeed related by a Legendre transform on the field theory side (plus appropriate choice of boundary conditions fixing either μ or ρ). Also the gravity duals to both setups are related by a bulk Legendre transformation (plus a change of boundary conditions), which takes us from one quantization of the gauge field to the other. This has been first discussed for a scalar in [29]. In the stationary state which captures thermodynamics, it is thus clear how to interpret the two distinct setups, the “ordinary” and the “alternative” one.

2.2 Hydrodynamics vs. field dynamics

Let us consider the linear response of a normal fluid at low momentum and frequency in $1 + d$ spacetime dimensions

$$\langle j \rangle = G_{ij}^R \cdot \mathbf{a}, \tag{2.2}$$

where the angular brackets indicate that $\langle j \rangle$ is a one-point function of the conserved current $j = (j^t, j^\alpha)^T$, and $\mathbf{a} = (\mathbf{a}_t, \mathbf{a}_\alpha)^T$ is the gauge field, which acts as a source here. Greek indices run over the d spatial dimensions. By G_{jj}^R we mean the $(1+d) \times (1+d)$ matrix of two-point functions of the current j with itself. We define this to be the “ordinary” setup. This is the setup which has been realized in most holographic models (including those of holographic superconductors) to date. The low-frequency and low-momentum limit of this setup is simply ordinary “hydrodynamics”.

Alternatively, we can define a different system by switching the roles of sources and one-point functions, so that we get

$$\langle \mathbf{a} \rangle = G_{\mathbf{a}\mathbf{a}}^R \cdot j, \tag{2.3}$$

where now the angular brackets indicate that $\langle \mathbf{a} \rangle$ is a one-point function of the gauge field \mathbf{a} , and j is a conserved current, which now acts as source. By $G_{\mathbf{a}\mathbf{a}}^R$ we mean the $(1+d) \times (1+d)$ matrix of two-point functions of the gauge field \mathbf{a} with itself. We will refer to this system as the “alternative” setup. This is the setup which our gravitational setup (with Neumann boundary conditions on the gravity fields) will correspond to through the gauge/gravity correspondence. We refer to the low-frequency, low-momentum limit of this setup as “field dynamics” (in order to distinguish it from the hydrodynamic description above).

It is not obvious, how the ordinary setup is related to the alternative one, i.e. how “hydrodynamics” is related to “field dynamics”. It is also not *a priori* obvious what physics is captured in the alternative setup at low frequencies and momenta (i.e. in “field dynamics”). So one of our goals will be to study the low-frequency, low-momentum fluctuations of our gravity model in the alternative setup. As noted before, in our AdS₃ gravity model we have no access to the ordinary setup. Therefore a direct comparison between the (well-known) ordinary “hydrodynamics” and the (less conventional) alternative “field dynamics” setup is not possible in our case.

Note also that it is not possible to naively invert ordinary hydrodynamics in order to get an alternative version of hydrodynamics in terms of gauge fields. In order to see this, let us consider a simple ordinary hydrodynamics setup with a constitutive equation for the dynamical current

$$\langle j^\mu \rangle = \rho u^\mu + \sigma \left(e^\mu - \nabla^\mu \left(\frac{\mu}{T} \right) \right), \tag{2.4}$$

where $e_\mu = u^\nu (\partial_\mu \mathbf{a}_\nu - \partial_\nu \mathbf{a}_\mu)$ is the electric field, μ is the chemical potential which couples to the charge density ρ , and $u^\mu = (1, 0)$ is the 2-velocity (all to first order in derivatives). Let us assume small field fluctuations \mathbf{a}^t and \mathbf{a}^x , to which the system responds by charge fluctuations in ρ . We keep the temperature T and the 2-velocity fixed. Now our goal would be to solve this equation for \mathbf{a}^μ (such that we could write $\mathbf{a}^\mu = \dots$, where \dots are terms which do not explicitly depend on \mathbf{a}^μ). Using the conservation equation $\nabla_\mu \langle j^\mu \rangle = 0$, and Fourier-transforming by $\mathbf{a}^\mu \rightarrow \mathbf{a}^\mu \exp(-i\omega t + ikx)$, we may rewrite (2.4) as a matrix equation

$$\begin{pmatrix} \langle j^t \rangle \\ \langle j^x \rangle \end{pmatrix} = -\frac{i\sigma}{\omega + iDk^2} \begin{pmatrix} k^2 & \omega k \\ \omega k & \omega^2 \end{pmatrix} \begin{pmatrix} \mathbf{a}^t \\ \mathbf{a}^x \end{pmatrix} = \mathcal{W} \cdot \mathbf{a}, \tag{2.5}$$

with the diffusion constant $D = \sigma/\chi$, where χ is the electric susceptibility. For this system to be invertible, we would need the matrix \mathcal{W} to have an inverse. But as we easily see, this is not the case since $\det \mathcal{W} = 0$.

2.3 Superfluid field dynamics

In ordinary superfluid hydrodynamics we have to add the phase of the order parameter as a new degree of freedom compared to the normal-fluid hydrodynamics. This is due to the Goldstone theorem which tells us that for every spontaneously broken continuous symmetry a massless bosonic field appears in the spectrum of our theory. Superfluids just below the critical temperature can be thought of as a mixture of two phases, a superfluid, and a normal-fluid one. Once the symmetry-breaking has occurred, superconductors can be described as charged superfluids for the purpose of studying low-energy excitations and transport phenomena. Therefore in the literature superconductors and superfluids are often described in the formalism of superfluid hydrodynamics. This is the ordinary setup which we will have no access to in our AdS₃ model. Hence we are not able to formulate our results in this common way.

In our alternative setup we still expect new degrees of freedom since we are still breaking a continuous symmetry spontaneously. To be more precise, our gravity solution will develop vector hair below a critical temperature, i.e. we have a non-trivial background gauge field $A_x^1(z)$. This breaks a residual U(1) gauge symmetry spontaneously. Thus we expect to see the Goldstone-modes from this symmetry-breaking in the two-point functions of our dynamical gauge fields. But it is not clear what the behavior of these modes is going to be. For example one could study their dispersion relations $\omega(k)$. In the broken phase however, we restrict ourselves to the case of vanishing spatial momentum, and compute selected Green's functions. Only for the normal phase we consider non-zero momentum and find a linear dispersion, as discussed below.

2.4 Parity-violating field dynamics

Our gravity solution leads to spontaneous parity-breaking in the dual (1 + 1) dimensional field theory. Therefore we also expect to see effects of parity-violation in our low-frequency, low-momentum modes. Note that parity-violating hydrodynamics (in the ordinary formulation) for superfluids with a scalar order parameter has been studied in 3+1 dimensions [34]. There, parity-breaking leads to a large number of additional transport coefficients. In principle it should be possible to carry out a similar analysis in the alternative setup in order to derive a formulation of parity-violating field dynamics. For example one could use the fluid/gravity framework [35] and compute the constitutive relations (presumably for dynamical gauge fields) and corresponding coefficients explicitly. It is not clear what to expect from such an analysis. We note that adding a Chern-Simons term to our action (1.1) would lead to a chiral anomaly on the field theory side. This would break parity as well and such setups have been examined recently in [18, 28, 36]. However, this would also change the boundary behavior of the gauge fields.

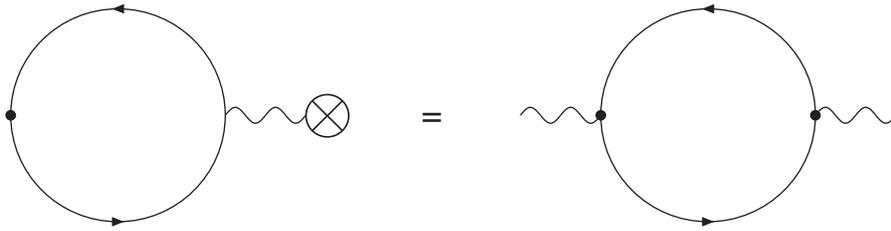


Figure 1. In $(1 + 1)$ dimensions the anomaly diagram on the left is identical to the self-energy diagram for the gauge field on the right.

2.5 Massive gauge bosons and the Schwinger model

One may speculate that the physics of dynamical gauge bosons should be best captured by thermal quantum field theory, at least at weak coupling. In this case the correlators G_{aa}^R between gauge bosons \mathbf{a} discussed around (2.3) should be interpreted as gauge boson propagators. Due to the finite temperature we naively expect our gauge bosons to have a thermal mass. Furthermore, they should acquire a mass through a Higgs mechanism due to the spontaneous symmetry-breaking. The latter mass contribution should depend on the condensate which breaks the symmetry. So in particular this mass contribution should vanish as we approach the phase transition from the broken phase. But we will see below that neither the thermal mass nor the mass from symmetry-breaking can appear in our setup.

Already at zero temperature gauge bosons in $(1 + 1)$ dimensions are different from higher-dimensional cases. Schwinger discussed quantum electrodynamics in two spacetime dimensions [37]. Among other things he found that photons in $(1 + 1)$ dimensions can have a mass term which does not break gauge invariance. Furthermore the self-energy diagram of gauge fields in $(1 + 1)$ dimensions is identical to the diagram for the chiral anomaly in $(1 + 1)$ dimensions, see figure 1. The mathematical reason for this is that there are two γ -matrices in $(1 + 1)$ dimensions, γ^0, γ^1 , plus the $\gamma^5 = \gamma^0\gamma^1$, and they obey the relation $\gamma^\mu\gamma^5 = -\epsilon^{\mu\nu}\gamma_\nu$, with the totally antisymmetric symbol in two spacetime dimensions $\epsilon^{\mu\nu}$. Thus in the fermion loop in figure 1 on the left we can replace the coupling to an external axial current $i\gamma^\mu\gamma_5$ by the coupling to an external gauge field $-i\gamma_\nu$. This implies a strong constraint: if a gauge field in $(1 + 1)$ dimensions has a mass, then the theory has a chiral anomaly, and vice versa.

We will use this result below in order to interpret the $(1 + 1)$ -dimensional field theory which is dual to our AdS_3 setup. Note that for a massive photon in $(1 + 1)$ dimensions the Ward identity requires a pole in the gauge boson propagator at two-momentum $k^2 = 0$ [37].

3 Holographic setup and symmetry-breaking solution

While the previous section was very speculative, in the present section we are going to work with a concrete gravity model. On the dual field theory side, this will realize a p-

	t	x	y	z	4	5	6	7	8	9
<i>D3</i>	X	X	X	X						
<i>D3'</i>	X	X			X	X				

Table 1. *D3 – D3'* brane configuration which provides only a Maxwell but no Chern-Simons term. We consider $N \rightarrow \infty$ *D3* branes, and $N_f = 2$ *D3'* probe branes.

wave superconductor in $(1 + 1)$ dimensions with dynamical electromagnetic fields, i.e. the alternative setup described above.

3.1 Dirac Born Infeld action for coincident *D3* probe branes

In the context of probe brane constructions [16] — apart from the DBI-action — the effective brane world volume action also contains a Wess-Zumino term. This leads to the Chern-Simons coupling of the gauge field. However, for particular brane intersections such a Chern-Simons term can be absent and only the Maxwell term remains. We consider a defect brane configuration which realizes this very situation. This is the *D3 – D3'* defect configuration illustrated in table 1. The metric generated by the N background *D3* branes is the standard $\text{AdS}_5 \times S_5$ (black brane) metric

$$ds_{D3}^2 = H^{-1/2}(-f(r)dt^2 + dx^2 + dy^2 + dz^2) + H^{1/2}\left(\frac{dr^2}{f(r)} + r^2 d\Omega_5^2\right), \quad (3.1)$$

where $f(r) = 1 - \frac{r_0^4}{r^4}$, $H = (L/4)^4$ and $d\Omega_5^2 = d\theta^2 + \cos^2\theta d\xi^2 + \sin^2\theta dS_3^2$. Transforming the radial coordinate to $u = r_0/r$, the metric becomes

$$ds_{D3}^2 = \left(\frac{r_0}{L}\right)^2 \frac{-f(u)dt^2 + dx^2 + dy^2 + dz^2}{u^2} + \frac{L^2 du^2}{u^2 f(u)} + L^2 d\Omega_5^2. \quad (3.2)$$

We consider N_f coincident probe *D3'* branes with a trivial embedding, i.e. $\theta(r) = 0$. Then the metric induced on the world volume of the *D3'* branes is

$$ds_{D3'}^2 = \left(\frac{r_0}{L}\right)^2 \frac{-f(u)dt^2 + dx^2}{u^2} + \frac{L^2 du^2}{u^2 f(u)} + L^2 d\xi^2. \quad (3.3)$$

From this metric we can see that the probe *D3'* branes cover AdS_3 as well as an S_1 cycle inside the five-sphere. The existence of such $\text{AdS}_3 \times S^1$ embeddings for the *D3'* branes has been demonstrated in [17]. Furthermore we switch on a $U(N_f)$ gauge field living on our $(1 + 1)$ -dimensional defect. In fact our gauge field lives on the world volume of the *D3'* branes. These probe branes do not wrap any cycle with Ramond-Ramond flux and so the Wess-Zumino part of the brane action does not give rise to a Chern-Simons coupling for our $U(N_f)$ gauge field. This argument was first made in [18] (see also [11]) and works for Abelian as well as non-Abelian gauge fields. Then up to some constant the action for the probe *D3'*-branes will be the non-Abelian DBI-action [38], which after exploiting our symmetries⁶ amounts to

$$S_{D3'} = -T_{D3} N_f \text{Str} \int dt dx dz d\xi \sqrt{-\det(g_{ab} + (2\pi\alpha')F_{ab})}, \quad (3.4)$$

⁶See section 3 of [21] for details of the analogous computation for probe *D7* branes.

with $F = dA + A \wedge A$, and the symmetrized trace Str is taken over $U(N_f)$ representation matrices (we have renamed the radial coordinate $u \rightarrow z$ for convenience). Here we choose the metric and gauge field to be independent of the S^1 -coordinate ξ . After expanding the square root in (3.4) in small field strengths F the leading contribution is extremized by the AdS_3 black hole background. The subleading contribution merely has the form of the Yang-Mills action which we already advertised in equation (1.1). It should be noted here that the non-Abelian DBI action is only valid up to fourth order in field strengths [14, 15], and it has been shown to disagree with corresponding string scattering amplitude computations beyond this order. However, in the present paper we are only interested in the leading (quadratic in field strengths) order.

The field theory which is holographically dual to this gravity setup is known and has originally been studied in [17], see also [11, 18]. This dual field theory is $U(N_f) \times U(N) \mathcal{N} = 4$ SYM theory coupled to a bifundamental hypermultiplet along the $(1 + 1)$ -dimensional defect. In particular there is a dynamical gauge field \mathfrak{A} living on the $(1 + 1)$ -dimensional defect. This dynamical gauge field is dual to the dynamical bulk $U(N_f)$ gauge field A on the stack of N_f probe branes. Therefore the setup presented in this subsection provides an example of the dynamical boundary gauge fields which we want to study in this work. It should be noted that our dynamical boundary gauge field \mathfrak{A} (dual to the bulk gauge field A) is still a gauge-independent operator under the original ‘‘color’’ group $U(N \rightarrow \infty)$. It is only dynamical and hence gauge-dependent under the additional ‘‘flavor’’ $U(N_f)$ which we have introduced.

3.2 Yang-Mills-action in AdS_3

In the previous subsection we have illustrated how the Yang-Mills action (1.1) may arise from a certain string construction. In this subsection we focus on this Yang-Mills action for a non-Abelian gauge field A in AdS_3 . The neutral AdS_3 black hole background in Poincaré coordinates is given by

$$ds^2 = \frac{L^2}{z^2} \left(-f(z)dt^2 + dx^2 + \frac{dz^2}{f(z)} \right), \tag{3.5}$$

in which $f(z) = 1 - z^2$, $z = r_+/r$, and r_+ is the horizon of the black hole. The black hole temperature is given by

$$T = \frac{r_+}{2\pi}. \tag{3.6}$$

We are free to set $L = 1$ and $r_+ = 1$. To this background, we add $SU(2)$ gauge fields \mathcal{A}_μ^a , with $a = 1, 2, 3$. We split this bulk gauge field \mathcal{A}_μ^a in background fields A_μ^a , and fluctuations a_μ^a around this background. Our Ansatz for the background fields is

$$A = \Phi(z)dt\tau^3 + W(z)dx\tau^1, \tag{3.7}$$

where τ^a are the generators of the $SU(2)$. For our fluctuations we choose the gauge $a_z^b \equiv 0$ but leave them otherwise unrestricted for now. The field strength is $\mathcal{F}_{\mu\nu}^a = \partial_\mu \mathcal{A}_\nu^a - \partial_\nu \mathcal{A}_\mu^a + g_{\text{YM}} \epsilon^{abc} \mathcal{A}_\mu^b \mathcal{A}_\nu^c$. For simplicity we set $g_{\text{YM}} \equiv 1$ in the remainder of this work. Note that

the value of g_{YM} would be important if we were to relax the probe limit, comparing the gravitational coupling to the gauge coupling.

The equations of motion for the p-wave background fields Φ and W are

$$0 = \partial_z(zf\partial_z W) + \frac{z}{f}\Phi^2 W, \tag{3.8}$$

$$0 = \partial_z(z\partial_z\Phi) - \frac{z}{f}\Phi W^2. \tag{3.9}$$

3.3 Alternative boundary conditions

The boundary condition at the horizon is $\Phi(z = 1) = 0$, which is to make Φdt well-defined at $z = 1$, while W should be regular at the horizon. The horizon expansion is

$$\Phi = -\Phi_1(z - 1) + \frac{1}{4}(2\Phi_1 + W_0^2\Phi_1)(z - 1)^2 + \mathcal{O}((z - 1)^3), \tag{3.10}$$

$$W = W_0 + \mathcal{O}((z - 1)^3). \tag{3.11}$$

This expansion has the same structure as in the $\text{AdS}_5/\text{CFT}_4$ and $\text{AdS}_4/\text{CFT}_3$ cases. In contrast to that, at the boundary, the story is different. The asymptotic behavior of W and Φ at the boundary $z = 0$ is

$$\begin{aligned} W &= -\mathfrak{J}_x \ln z + \langle \mathfrak{A}_x \rangle, \\ \Phi &= -\mathfrak{J}_t \ln z + \langle \mathfrak{A}_t \rangle, \end{aligned} \tag{3.12}$$

where $\mathfrak{J}_t =: \rho$ and \mathfrak{J}_x are the sources, and $\langle \mathfrak{A}_t \rangle =: \mu$, as well as $\langle \mathfrak{A}_x \rangle$ are the vacuum expectation values⁷ of the operator dual to the bulk background field A_μ^a . Note that we would get a different boundary behavior for the gauge fields had we allowed a Chern-Simons term in our action (1.1). In that case the asymptotics depend on the Chern-Simons level, see for example [18].

In (3.12) we now encounter the alternative quantization mentioned above, which corresponds to the choice of Neumann boundary conditions [5–8]. In AdS_4 for example, we would be free to interpret the leading term of Φ as the source (chemical potential), and the subleading term as the expectation value of the dual current operator (charge density), or the other way around. However, this is not possible in AdS_3 [18, 27]. We have to choose the Neumann boundary condition. One may be concerned that [18, 27] consider Abelian gauge fields for the most part. But it has been noted in [27] and earlier in [30] that the asymptotics and hence also the boundary conditions are determined by the linear theory. Our non-Abelian terms contribute to non-linear (interaction) terms.⁸ Therefore our non-Abelian gauge fields in AdS_3 should only satisfy the Neumann boundary condition, just like their Abelian counterparts.

So compared to the higher dimensional cases, we observe two changes: first, the constant term is not the leading contribution anymore. A logarithmic term appears and contributes to leading order. Second, as explained in [18, 27], the leading (now logarithmic)

⁷Note that we have suppressed gauge indices a, b, \dots since according to our Ansatz (3.7) in the background the $a = 3$ index always appears with the spacetime index $\mu = t$ and $a = 1$ appears always with $\mu = x$.

⁸As a concrete example, boundary conditions have been discussed in [27] for $\text{SO}(8)$ non-Abelian gauge fields of AdS_4 supergravity.

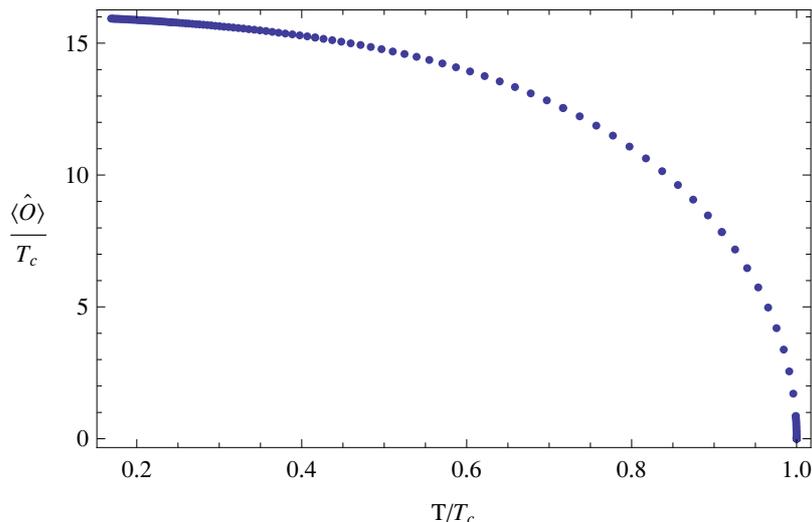


Figure 2. Condensate $\langle \hat{O} \rangle / T_c = \langle \mathfrak{A}_x \rangle / (\rho 2\pi \rho_c)$ versus temperature $T/T_c = \rho_c / \rho$; the critical temperature is given by $T_c = 0.046\rho$.

term has to be interpreted as a source and corresponds to a (non-dynamical) boundary current. Therefore, the source in this system is the boundary current \mathfrak{J} . The subleading (constant) term is the expectation value of the operator dual to the field A . We denote this operator by \mathfrak{A} . According to the analysis in [18, 27], this operator \mathfrak{A} has to be interpreted as a gauge field living on the AdS_3 -boundary. So in summary, our bulk gauge field A is dual to a boundary gauge field \mathfrak{A} . This boundary gauge field \mathfrak{A} is sourced by the current \mathfrak{J} .

Thus our fixed charge density $\rho = \mathfrak{J}_t$ sources the dynamical gauge field $\mu = \langle \mathfrak{A}_t \rangle$. This particular component $\langle \mathfrak{A}_t \rangle$ can be interpreted as a chemical potential (which does not act as a source, but is determined by the equations of motion). Meanwhile $\langle \mathfrak{A}_x \rangle$ is the order parameter of the symmetry-breaking, while \mathfrak{J}_x is the source term which we demand to vanish, for the breaking to be spontaneous. Our operator \mathfrak{A}_x is the spatial component of a dynamical boundary gauge field, which acquires an expectation value. In the context of thermodynamics our alternative setup has a simple interpretation: since we are working at fixed charge density, this is the canonical ensemble (provided we add an appropriate boundary term to the action (1.1) as discussed below).

3.4 Symmetry-breaking solution and free energy

For simplicity we choose to work in the probe limit, i.e. the gauge field does not backreact on the AdS_3 metric. After solving the equations of motion (3.8) with boundary conditions (3.10), the order parameter $\langle \mathfrak{A}_x \rangle$ is plotted versus the temperature in figure 2, both in units of T_c . Note that in the canonical ensemble T_c is determined by the only other dimensional quantity in the problem, namely ρ , see [3, 4]. So we actually plot $\langle \mathfrak{A}_x \rangle$ rescaled by $(\rho 2\pi \rho_c)$ versus the temperature as measured by $T/T_c = \rho_c / \rho$.

Near the critical temperature T_c , the condensate $\langle \mathfrak{A}_x \rangle$ behaves like

$$\langle \mathfrak{A}_x \rangle \approx 25.5515 T_c \sqrt{1 - T/T_c}. \tag{3.13}$$

The critical exponent is $1/2$ which is identical to the one in mean field theory. Note that this solution also breaks parity in $(1+1)$ dimensions. In any odd number of spatial dimensions parity is defined as inversion of only one of the spatial coordinates. In our setup this would be inversion of the one spatial coordinate x . Our order parameter $\langle \mathcal{A}_x \rangle$ is a vector pointing along this x -direction (or against it depending on the sign conventions). Therefore parity symmetry, the invariance of our system under inversion of the x -coordinate, is broken spontaneously.

In the following we will compute the free energy difference between the normal state and the broken state in order to verify that the broken phase is preferred at low temperatures. With the Ansatz (3.7), we can rewrite

$$\frac{1}{2}\sqrt{-g}\mathcal{L} = -\frac{1}{2}\sqrt{-g}F_{\mu\nu}^a F^{a\mu\nu} = -zf(\partial_z W)^2 + \frac{z}{f}\Phi^2 W^2 + z(\partial_z W)^2. \quad (3.14)$$

Then the on-shell action

$$\begin{aligned} S_{\text{os}} &= \int \sqrt{-g}\mathcal{L}d^3x = 2 \lim_{\epsilon \rightarrow 0} \int dt dx (-zfW\partial_z W + z\Phi\partial_z\Phi)_{z=\epsilon} - 2 \int dz dt dx \frac{z}{f}\Phi^2 W^2, \\ &= 2 \int dt dx (-\mu\rho + \rho^2 \log(z)) - 2 \int dz dt dx \frac{z}{f}\Phi^2 W^2, \end{aligned} \quad (3.15)$$

can be obtained by integrating our action (1.1) by parts, and using the equations of motion (3.8) in the form

$$- \int dz [zf(\partial_z W)^2] = -zfW\partial_z W|_{z=\epsilon} - \int dz \frac{z}{f}\Phi^2 W^2, \quad (3.16)$$

and

$$\int dz [z(\partial_z \Phi)^2] = z\Phi\partial_z\Phi|_{z=\epsilon} - \int dz \frac{z}{f}\Phi^2 W^2. \quad (3.17)$$

In order to place the dual field theory in the canonical ensemble we need to add a boundary term [18, 27]

$$S_{\text{bdy}} = -2 \int dt dx (\sqrt{-g}A_\mu F^{z\mu})_{z=0} = 2 \int dt dx (-zfW'W + z\Phi'\Phi)_{z=0}, \quad (3.18)$$

where the prime denotes a derivative with respect to the radial coordinate z . Note that this contribution (3.18) is identical to a part of the on-shell action (3.15). Now we have to add a counterterm

$$S_{\text{counter}} = 4 \int dt dx \sqrt{-h}F_{z\mu}F^{z\mu} \log(z) = -4 \int dt dx \rho^2 \log(z), \quad (3.19)$$

in order to cancel the logarithmic divergences in (3.15) and (3.18). Here h is the metric induced on the boundary. Hence we obtain the renormalized on-shell action

$$S_{\text{ren}} = S_{\text{os}} + S_{\text{bdy}} + S_{\text{counter}} = -4 \int dt \int dx \rho\mu - 2 \int dx^3 \frac{z}{f}\Phi^2 W^2, \quad (3.20)$$

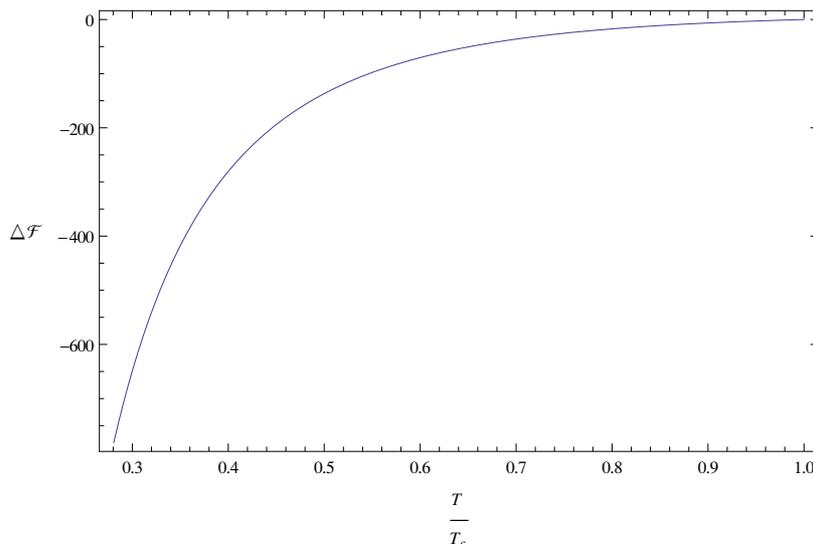


Figure 3. The free energy difference between the normal and broken phase.

where we have used that the source has to vanish, i.e. $\mathfrak{J}_x = 0$ is required, for spontaneous symmetry-breaking. Finally the free energy density \mathcal{F} is given by the Euclidean continuation of S_{ren} as

$$\mathcal{F} = -TS_{\text{ren}} = 4\mu\rho + 2 \int dz \frac{z\Phi^2 W^2}{f}. \quad (3.21)$$

Note that the free energy of the normal state in our conventions is always zero, because in that phase the general gauge field solution is $A_t^3 = -\rho \log(z) + C_0$ and the constant C_0 has to vanish for $A_t^3(z)$ to be well-defined at the horizon. This situation would change if we were to consider backreaction; then we would have finite ρ and μ .

The free energy difference between the broken state and the normal state is plotted in figure 3, from which we can see that the state with a condensation is thermodynamically preferred when $T < T_c$. The condensate vanishes with a square root behavior.

Thus our first result is that there is a second order phase transition for the Yang-Mills theory (1.1) in the AdS₃ black hole background at the critical temperature T_c . Our broken phase develops a (parity-breaking) vector condensate, which corresponds to vector hair of the AdS₃ black hole.

The p-wave solution we have presented here is different from the one discussed in [39]. Our lowest energy solution for the field $W(z)$ has one node at some $z = z^*$, while the p-wave in AdS₄ has no nodes. One may argue that more nodes should correspond to a higher energy for the solution [39]. According to that reasoning the zero node solution should always be lowest in energy. However, in our AdS₃ case the zero node solution coincides with the solution which has $W(z) \equiv 0$. Or in other words: there is no symmetry-breaking solution with zero nodes in our setup.

4 Fluctuations

In this work we will not be able to address all the field theory features presented in section 2 at once. Instead we are going to pick selected modes in order to investigate the dynamics of the full system in the normal phase and in the phase with spontaneously broken symmetry, i.e. the broken phase. Our goal is to show that there are non-trivial dynamical low-energy modes near the phase transition in the broken phase of this system.

4.1 Fluctuation equations of motion

We start by writing down the full set of equations of motion after the gauge choice $a_z \equiv 0$ is made

$$\begin{aligned}
 0 &= a_t^{1''} + \frac{1}{z}a_t^{1'} - \frac{k^2}{f}a_t^1 - \frac{k\omega a_x^1 - (ika_x^2 + Wa_x^3)\Phi}{f}, \\
 0 &= a_x^{1''} + \left(\frac{1}{z} + \frac{f'}{f}\right)a_x^{1'} + \frac{\omega^2 + \Phi^2}{f^2}a_x^1 + \frac{\omega ka_t^1 + (-i(ka_t^2 + 2\omega a_x^2) + 2a_t^3W)\Phi}{f^2}, \\
 0 &= a_t^{2''} + \frac{1}{z}a_t^{2'} - \frac{k^2 + W^2}{f}a_t^2 - \frac{k\omega a_x^2 + i(2ka_t^3W + \omega a_x^3W + ka_x^1\Phi)}{f}, \\
 0 &= a_x^{2''} + \left(\frac{1}{z} + \frac{f'}{f}\right)a_x^{2'} + \frac{\omega^2 + \Phi^2}{f^2}a_x^2 + \frac{k\omega a_t^2 + i(\omega a_t^3W + (ka_t^1 + 2\omega a_x^1)\Phi)}{f^2}, \\
 0 &= a_t^{3''} + \frac{1}{z}a_t^{3'} - \frac{k^2 + W^2}{f}a_t^3 - \frac{k\omega a_x^3 + W(-i(2ka_t^2 + \omega a_x^2) + 2a_x^1\Phi)}{f}, \\
 0 &= a_x^{3''} + \left(\frac{1}{z} + \frac{f'}{f}\right)a_x^{3'} + \frac{\omega^2}{f^2}a_x^3 + \frac{k\omega a_t^3 - i\omega a_t^2W - a_t^1W\Phi}{f^2}. \tag{4.1}
 \end{aligned}$$

Variation of the action with respect to the fields a_z yields the following constraints

$$\begin{aligned}
 0 &= -i\omega a_t^{1'} - \Phi a_t^{2'} - ikfa_x^{1'} + a_t^2\Phi', \\
 0 &= \Phi a_t^{1'} - i\omega a_t^{2'} - f(ika_x^{2'} - Wa_x^{3'} + a_x^3W') - a_t^1\Phi', \\
 0 &= -i\omega a_t^{3'} - f(Wa_x^{2'} + ik a_x^{3'} - a_x^2W'). \tag{4.2}
 \end{aligned}$$

Expanding the fluctuations (4.1) around the AdS-boundary, we get

$$a_t^a = a_t^{a(\text{source})} \log(1/z) + a_t^{a(\text{vev})} + \dots, \tag{4.3}$$

$$a_x^a = a_x^{a(\text{source})} \log(1/z) + a_x^{a(\text{vev})} + \dots, \tag{4.4}$$

where $a = 1, 2, 3$ is the gauge index, (source) denotes the source, and (vev) denotes the vacuum expectation value. We follow the same logic which was applied for our background fields A below (3.12). So the coefficient of the logarithmic term in (4.3) is interpreted as the boundary theory current $j_\mu^b = a_\mu^{b(\text{source})}$ which acts as a source. The constant term is interpreted as the vacuum expectation value of a boundary theory gauge field $\langle \mathbf{a}_\mu^b \rangle = a_\mu^{b(\text{vev})}$.

Near the horizon these fluctuations behave as

$$a_t^a = (1-z)^{-i\frac{\omega}{2}} (\mathcal{O}(1-z)), \tag{4.5}$$

$$a_x^a = (1-z)^{-i\frac{\omega}{2}} (a_x^{a(\text{H})} + \mathcal{O}(1-z)). \tag{4.6}$$

All higher coefficients depend recursively on the three initial horizon values $a_x^{1(H)}$, $a_x^{2(H)}$ and $a_x^{3(H)}$ of the spatial components.

4.2 Analytical results

Gauge-invariant field combinations. Above we have constructed a ground state which breaks the SU(2) gauge symmetry explicitly to U(1)₃, which is broken spontaneously by the same ground state. Therefore our ground state does not have any gauge symmetry left. However, the Lagrangian of our theory still enjoys the full SU(2) gauge symmetry. Thus, while the SU(2)-symmetry is completely broken for our background fields which describe the ground state, the fluctuations still transform under a full SU(2) gauge symmetry in the following way

$$\delta a_\mu^a = \partial \lambda^a + \epsilon^{abc} A_\mu^b \lambda^c, \tag{4.7}$$

with the transformation parameters $\lambda^a(t, x)$ where $a = 1, 2, 3$. These λ^a do not depend on the radial coordinate z because we have chosen to work in the axial gauge $\mathcal{A}_z \equiv 0$ earlier.

First, we note that there are three pure gauge solutions for the fields $(a_t^1, a_x^1, a_t^2, a_x^2, a_t^3, a_x^3)$, which in this notation read

$$\begin{aligned} \text{(I)} & : (-i\omega, ik, \Phi, 0, 0, 0), \\ \text{(II)} & : (-\Phi, 0, -i\omega, ik, 0, W), \\ \text{(III)} & : (0, 0, 0, -W, -i\omega, ik). \end{aligned} \tag{4.8}$$

We have labeled these solutions (I), (II), (III) for later convenience.

We find the following gauge-invariant bulk field combination:

$$\hat{a}_x^3 = a_x^3 + \frac{k}{\omega} a_t^3 + W \frac{\Phi a_t^1 + i\omega a_t^2}{\Phi^2 - \omega^2}. \tag{4.9}$$

At vanishing momentum this expression reduces to the gauge-invariant combination considered by Gubser & Pufu in [39].

Of course we can build more gauge-invariant combinations such as

$$\hat{a}_x^2 = a_x^2 + \frac{i}{\omega W} (W^2 - k^2) a_t^3 - \frac{ik}{W} a_x^3.$$

These are discussed in appendix A.

In the most general case below T_c , all six fluctuations are coupled through the condensates, momenta and frequency terms in the equations of motion (4.1). Therefore all the two-point functions will have the same poles. For a more detailed discussion see [40, 41]. We will be interested in finding these common poles in order to identify low-energy modes. These are the poles at low frequency and small momentum which govern the long-time behavior of our system. By one of our initial assumptions we identify the poles in our two-point functions with the quasinormal modes of the gauge-invariant bulk field combinations \hat{a}_μ^b defined above. Our quasinormal modes can in turn be found by a method which we outline below. In the same way in which all correlators contain the same poles, the bulk fields will all have identical quasinormal modes in the fully coupled system of six equations of motion (4.1).

Gauge-covariant field combinations. In the normal phase the symmetry is only broken partly by the ground state. There is a $U(1)_3$ remaining intact. Therefore the gauge transformation for the fluctuations in this case amounts to

$$\delta a_\mu^a = \partial \lambda^a + \epsilon^{abc} (A_\mu^b \lambda^c + a_\mu^b \Lambda^3 \delta_3^c), \quad (4.10)$$

where Λ^3 is the gauge transformation parameter of the remaining $U(1)_3$ in the background.

It is not possible to form gauge-invariant combinations of the fluctuations a_μ^b as we did before in the broken phase. However, we can form gauge-covariant combinations, i.e. combinations which will have a well-defined homogeneous transformation under the gauge-symmetries.

We consider the following gauge-covariant field combinations

$$e_L^+ = (\Phi(z) - \omega) e_x^+ - k e_t^+, \quad (4.11)$$

$$e_L^- = (\Phi(z) + \omega) e_x^- + k e_t^-, \quad (4.12)$$

$$e_3 = k a_t^3 + \omega a_x^3, \quad (4.13)$$

where $e_t^\pm = a_t^1 \pm i a_t^2$, $e_x^\pm = a_x^1 \pm i a_x^2$. Note that e_3 is gauge-invariant and thus corresponds to a physically meaningful operator on the boundary by our initial assumptions. Details are discussed in appendix A, see also [42].

Determinant method & quasinormal modes. Quasinormal modes are those bulk field solutions at a particular $\omega = \omega_{n,\text{QNM}}$ which are infalling at the horizon and satisfy a Dirichlet condition at the AdS-boundary; we choose them to vanish at the AdS-boundary. In a coupled system it is not a trivial task to identify the relevant field combinations which have to vanish for a quasinormal mode to be present. A systematic method has been developed in [40, 41]. However, we will see that this method needs to be extended in our case at hand, because there are logarithmic sources.

Let us solve the equations of motion numerically, for example with the horizon values $a_x^{1(\text{H})} = -1$, $a_x^{2(\text{H})} = 1$, and $a_x^{3(\text{H})} = 1$, see (4.5). We label this solution (IV). Then we solve the equations of motion again using different horizon values $a_x^{1(\text{H})} = 1$, $a_x^{2(\text{H})} = -1$, and $a_x^{3(\text{H})} = 1$. This solution is linearly independent from the first one and we label it (V). There is a third linearly independent solution which we find using the values $a_x^{1(\text{H})} = 1$, $a_x^{2(\text{H})} = 1$, and $a_x^{3(\text{H})} = -1$, and we label it (VI). It was shown in [40, 41], that these three solutions can be combined in a matrix together with the three pure gauge solutions given in (4.8). The determinant of this matrix, evaluated at the boundary $z = 0$ was shown to yield locations $\omega_{\text{QNM}}^{(n)}(k)$ of quasinormal modes of the coupled system. Quasinormal modes can be found by considering the following determinant

$$\mathcal{D} = \det \begin{pmatrix} -i\omega & ik & \Phi & 0 & 0 & 0 \\ -\Phi & 0 & -i\omega & ik & 0 & W \\ 0 & 0 & 0 & -W & -i\omega & ik \\ a_t^{1(\text{IV})} & a_x^{1(\text{IV})} & a_t^{2(\text{IV})} & a_x^{2(\text{IV})} & a_t^{3(\text{IV})} & a_x^{3(\text{IV})} \\ a_t^{1(\text{V})} & a_x^{1(\text{V})} & a_t^{2(\text{V})} & a_x^{2(\text{V})} & a_t^{3(\text{V})} & a_x^{3(\text{V})} \\ a_t^{1(\text{VI})} & a_x^{1(\text{VI})} & a_t^{2(\text{VI})} & a_x^{2(\text{VI})} & a_t^{3(\text{VI})} & a_x^{3(\text{VI})} \end{pmatrix}. \quad (4.14)$$

In [40, 41] this kind of determinant was required to vanish in the following way:

$$0 = \lim_{z \rightarrow 0} \mathcal{D}. \tag{4.15}$$

This procedure essentially sets the relevant sources to zero if we work with bulk fields which have a constant as the source term, and no more divergent terms. However, in our AdS₃ case we have logarithms. Therefore the recipe has to be changed in order to extract the sources which now hide in the logarithmic terms. One sensible prescription appears to be

$$0 = \lim_{z \rightarrow 0} z \partial_z \mathcal{D}. \tag{4.16}$$

Whenever we set a fluctuation consistently to zero, we should eliminate the corresponding column and one row of solutions from this determinant. For example this prescription yields exactly the gauge-invariant combination \hat{a}_x^3 defined in (4.9), when we set $a_x^1 = 0 = a_x^2$ at $\Phi, W, \neq 0$, but $k = 0$.

Let us work through this example: we have two pure gauge solutions $a = (-i\omega, \Phi, 0)$ and $a = (-\Phi, -i\omega, W)$, and one non-trivial solution, which depends on one parameter, namely the source of a_x^3 at the horizon. So this gives a solution matrix

$$H = \begin{pmatrix} -i\omega & \Phi & 0 \\ -\Phi & -i\omega & W \\ a_t^1 & a_t^2 & a_x^3 \end{pmatrix}. \tag{4.17}$$

Now we compute the determinant $\mathcal{D} = \det H = a_x^3 - W(\Phi a_t^1 + i\omega a_t^2)/(\omega^2 - \Phi^2)$, and require it to vanish in general. This yields exactly the gauge-invariant combination \hat{a}_x^3 which was found in the analogous AdS₄-system [39]. But there is a crucial difference now. In AdS₄ we simply had to take the prescription (4.15) because the sources were the asymptotic values of the fields, and \hat{a}_x^3 was just a linear combination of these. Luckily the latter is still true in AdS₃, but now we have to extract the logarithm coefficients. This leads to the adjusted prescription (4.16). The reason why this works here is that for $k = 0$ we can reduce (4.1) to three second order equations of motion for a_t^1, a_t^2 , and a_x^3 . These three equations are coupled. But gauge symmetry restricts them, so that they can be reduced to one single equation. In other words: when we work out all of the 9 equations of motion (not choosing a gauge yet), then we see that there are 2 constraint equations from (4.2) which relate a_t^1 to a_t^2 and a_x^3 . Then the counting is: 3 equations of motion with each 2 boundary conditions give 6 boundary conditions. Choosing the infalling solution for each of the 3 fields there are 3 boundary conditions left to be fixed. Now we can use our 2 constraints, which leaves merely one solitary boundary condition. This means that there is only one relevant field \hat{a}_x^3 and our one boundary condition determines the normalization of this field.

As a second example, we consider the normal phase with $W = 0$. Here the adjusted method (4.16) yields accordingly $e_3 = \omega a_x^3 + k a_t^3$ at $a_{t,x}^1 = 0 = a_{t,x}^2$ and $\Phi \neq 0$, and correctly requires the leading logarithmic term in e_3 to vanish. However, for the genuinely coupled cases the prescription (4.16) may have to be modified.⁹ In particular the gauge-covariant

⁹Equivalently, at least in the normal phase, one can carry out an analysis formally similar to [11].

gauge fields will not necessarily be linear combinations of the individual fields a_μ^a anymore. We postpone their consideration to future work.

Numerically we can not take the limit $z \rightarrow 0$ exactly. Therefore we have to choose the same cut-off $\Lambda = 10^{-9}$ which we had chosen when computing our background data Φ and W . See [40, 41] for a more detailed discussion.

4.3 Numerical results

Here we will simplify our lives and consider only particular gauge-invariant field combinations which decouple from all others. Such field combinations are completely determined by requiring the infalling boundary condition at the horizon, and by giving a single second boundary condition fixing the normalization of the solution. For a discussion of how to find these decoupled solutions see section 4.2 and the appendix A. Note that the frequency ω in the following plots should be understood as the dimensionless quantity $\omega \rightarrow \omega/(2\pi T)$, and similarly $k \rightarrow k/(2\pi T)$.

Normal phase at vanishing spatial momentum $k = 0$. We solve the equations of motion (4.1) in the normal phase at finite chemical potential induced by $\Phi \neq 0$ at $\omega \neq 0$, but at $W = 0$ and $k = 0$. In this case the last equation in (4.1) can be consistently decoupled from the others. Therefore we consider that single equation for the fluctuation a_x^3 , setting all other fluctuations to zero

$$0 = a_x^{3''} + \frac{1 - 3z^2}{z(1 - z^2)} a_x^{3'} + \frac{\omega^2}{(1 - z^2)^2} a_x^3. \quad (4.18)$$

Note that this equation does not depend on the background field Φ in other words the dual operator is not charged under $U(1)_3$. We consider the Green's function of the dynamical gauge field combination \mathbf{a}_x^3 in terms of the gauge-invariant electric field $\mathbf{e}_x^3 = \omega \mathbf{a}_x^3$. The straightforward recipe for correlation functions (up to contact terms) can be applied here as shown in [18], to yield

$$G_{\mathbf{e}_x^3 \mathbf{e}_x^3}^R = \omega^2 G_{\mathbf{a}_x^3 \mathbf{a}_x^3}^R = \omega^2 \frac{(\text{vev})}{(\text{source})} = \omega^2 \frac{a_x^3 - \log(z) z \partial_z a_x^3}{z \partial_z a_x^3}. \quad (4.19)$$

Note that we are consequently ignoring numerical factors for the sake of a simple presentation here. Solving the equation of motion (4.18), we see that the imaginary part of the Green's function here asymptotes to a constant at large frequencies and has a pole at $\omega = 0$ independent of charge density.

Normal phase at non-vanishing spatial momentum $k \neq 0$. We solve the equations of motion (4.1) in the normal phase with $W = 0$ at finite momentum $k \neq 0$ and at finite chemical potential induced by $\Phi \neq 0$, while $\omega \neq 0$. Now the last two equations in (4.1) decouple from the others. Therefore we consider only the two coupled equations for the two fluctuations a_t^3 and a_x^3 . These can be combined into e_3 as discussed above. For simplicity we set all other fluctuations to zero and only study e_3 here. There is one pure gauge solution $a = (-i\omega, ik)$, and one non-trivial solution determined by a single parameter, which we

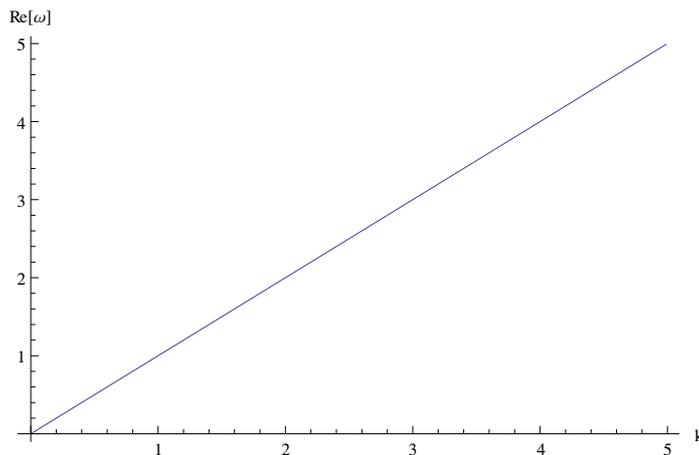


Figure 4. Linear dispersion relation from our numerics: the quasinormal mode with lowest energy of the bulk field E^3 in the normal phase. This corresponds to the lowest-lying pole of the corresponding Green’s function $G_{\epsilon\epsilon}^R$ of the dynamical electric field $\epsilon = \omega\hat{a}_x^3 + k\hat{a}_t^3$.

choose to be $a_x^3(z = z_H)$ at the horizon $z_H = 1$. Similar to the previous case we define the Green’s function

$$G_{\epsilon_3\epsilon_3}^R = \frac{(\text{vev})}{(\text{source})} = \frac{e_3 - \log(z)z\partial_z e_3}{z\partial_z e_3}. \tag{4.20}$$

Note that e_L^\pm satisfy coupled equations and would require a more careful treatment. For e_3 however we numerically find a linear dispersion relation

$$\omega = \pm k, \tag{4.21}$$

as seen in figure 4; in formal agreement with [11]. However, note that the authors of [11] choose to interpret their correlation functions as current-current correlation functions. In contrast to that we interpret our two-point functions according to the alternative setup as gauge field correlators. In this context it is reassuring that the gapless mode propagates with the speed of light $c = 1$ at finite temperature and charge density. It behaves just like a free photon.

Broken phase at vanishing spatial momentum $k = 0$. Here we are going to demonstrate that there is a non-trivial gapless mode appearing in the spectrum of our $(1 + 1)$ -dimensional theory. We work at finite W , Φ and ω , but vanishing momentum $k = 0$. Then we have three coupled fields a_t^1, a_t^2, a_x^3 , but these can be reduced to one gauge-invariant field \hat{a}_x^3 being dual to $\hat{\mathbf{a}}_x^3$, and obeying a single boundary condition (apart from the infalling condition), as discussed in section 4.2 above.

We consider the Green’s function of the dynamical gauge field combination $\hat{\mathbf{a}}_x^3$ in terms of the gauge-invariant electric field $\hat{\epsilon}_x^3 = \omega\hat{a}_x^3$ [18]

$$G_{\hat{\epsilon}_x^3\hat{\epsilon}_x^3}^R = \omega^2 G_{\hat{\mathbf{a}}_x^3\hat{\mathbf{a}}_x^3}^R = \omega^2 \frac{(\text{vev})}{(\text{source})} = \omega^2 \frac{\langle \hat{\mathbf{a}}_x^3 \rangle}{\hat{\mathbf{j}}_x^3} = \omega^2 \frac{\hat{a}_x^3 - \log(z)z\partial_z \hat{a}_x^3}{z\partial_z \hat{a}_x^3}. \tag{4.22}$$

Due to the presence of a Goldstone mode from the spontaneous breaking of the $U(1)_3$, we expect that at $T = T_c$ there has to be a gapless mode at $\omega(k = 0) = 0$. We follow this

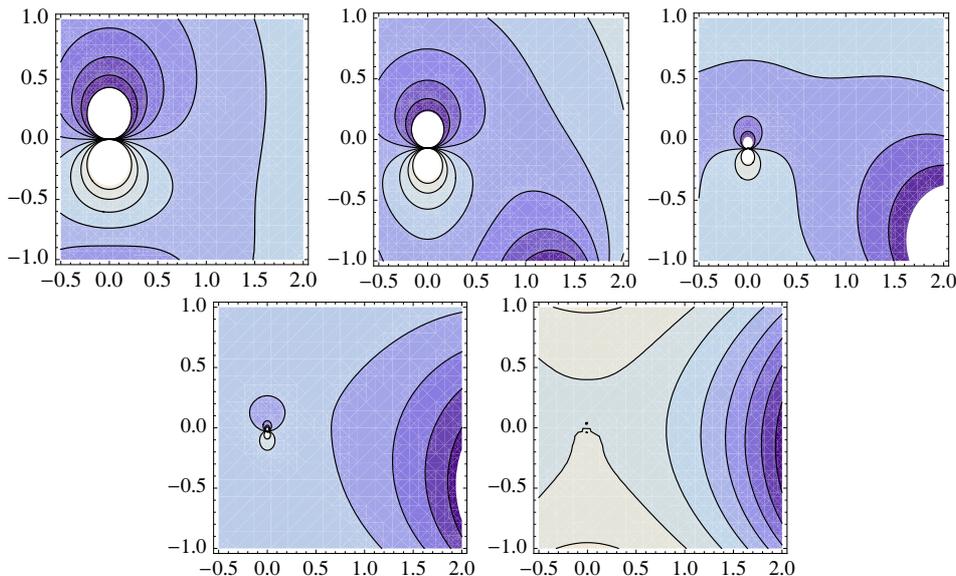


Figure 5. Dynamics in the broken phase: poles of the correlator $G_{\hat{c}_x^3 \hat{c}_x^3}^R$ (derived from the bulk fluctuation \hat{a}_x^3) in the complex frequency plane are visible in these contour plots of the spectral function $\propto \text{Im}G_{\hat{c}_x^3 \hat{c}_x^3}^R$. A gapless mode with non-trivial dynamics appears in the broken phase. The temperature is lowered from $T = T_c$ to $T = 0.91T_c, 0.73T_c, 0.57T_c, 0.44T_c$.

mode to lower temperatures $T < T_c$ keeping the momentum fixed at $k = 0$. The result is shown in figure 5. Indeed the mode appears at $\omega = 0$ and moves down the imaginary frequency axis as the temperature is lowered. So the Goldstone mode becomes a purely dissipative mode at lower temperatures, resembling a diffusive mode.¹⁰ However, it then turns around and approaches $\omega = 0$ again for $T \rightarrow 0$. Figure 6 shows the temperature-dependence of the imaginary gap in this mode. Note that this pole seems to either vanish or to become very steep at low temperatures. In the $T = 0.44T_c$ picture of figure 5 it is barely visible.

It may seem surprising that this mode does not acquire a mass gap, i.e. a gap in real values of ω . This is what one naively expects from spontaneous symmetry-breaking in presence of a dynamical gauge field: the Goldstone mode should be eaten by the gauge field, making the latter massive. But recall that gauge theories in two spacetime dimensions are peculiar as described in section 2.5. Field theory tells us that any non-zero mass for the gauge field would also contribute to the chiral anomaly through the relation given diagrammatically in figure 1. However, holographically the chiral anomaly is dual to a non-vanishing Chern-Simons term of appropriate dimensionality in the bulk action [24–26]. Assuming that there is no other way in which the bulk physics can generate an anomaly, we conclude that our bulk action (1.1) can not give rise to massive gauge fields. This assumption seems safe since the chiral anomaly is a non-perturbative topological quantity in field theory. It necessarily needs to be dual to a topological quantity in the

¹⁰A similar mode was observed in the holographic s-wave superconductor in three spacetime dimensions [40].

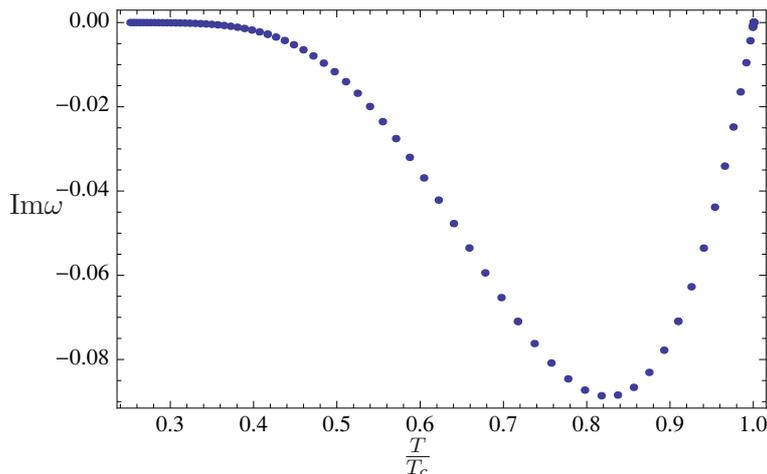


Figure 6. The imaginary gap: imaginary part of the frequency of the mode which was gapless at $T = T_c$. We keep $k = 0$ fixed.

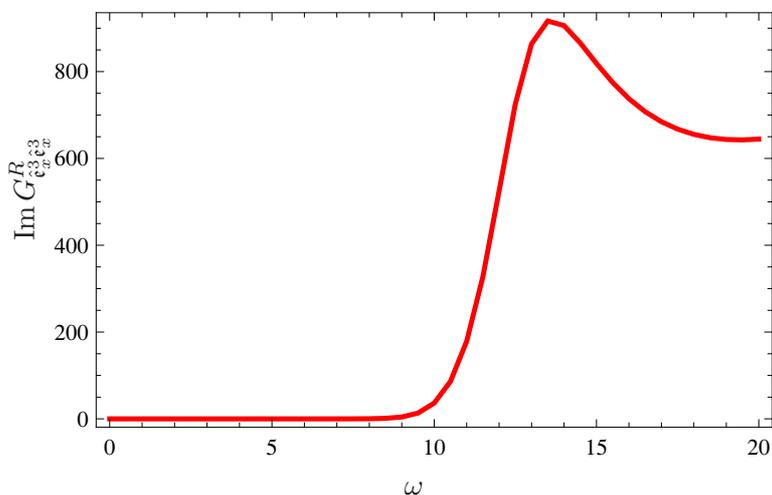


Figure 7. Spectral function in broken phase shows frequency-gap $\omega_{\text{gap}} \approx 10$: imaginary part of the Green's function of the dynamical electric field $\hat{\epsilon}_x^3$ on the boundary as a function of frequency ω at $T = 0.169T_c$. In the gap the gauge field (our analog of a photon) essentially does not propagate. This can be interpreted as the gauge-interaction (our analog of electromagnetism) becoming short-ranged.

dual geometry. A pure Yang-Mills term as in (1.1) can not give rise to topologically distinct solutions.

Figures 7 and 8 show the two-point function of the dynamical field strength $\hat{\epsilon}_x^3$, or in other words the electric field on the boundary. We find an energy gap of size ω_{gap} in the spectral function, which is presumably related to the energy needed for breaking a charge carrier out of the condensate.

In fact we could employ the picture of a photon propagating through a superconductor here. A zero value of that propagator in the frequency-gap means that the photons do not propagate. Thus we indeed see that the U(1)-interactions become short-ranged here although the photons do not become massive.

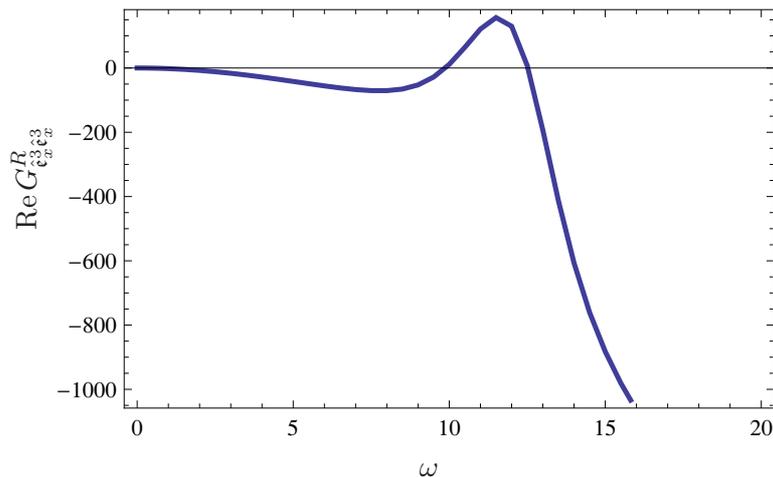


Figure 8. Broken phase: real part of the Green's function of the dynamical electric field \hat{e}_x^3 at $T = 0.169T_c$.

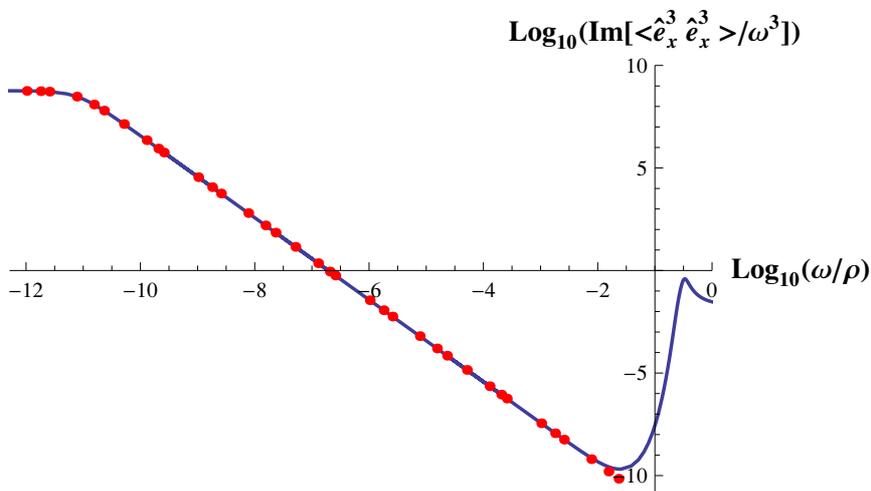


Figure 9. Broken phase: fit of our field strength correlator divided by ω^3 to the Drude model. The relevant parameters are $\sigma_0 \approx 5.8288 \times 10^8$, $\tau \approx 3.2026 \times 10^9$ when $T/T_c \approx 0.169$.

Our currents are not dynamical. However, as suggested in [18] we may formally define a conductivity via Ohm's law as

$$\sigma = i \frac{j_x^3}{\omega \langle \hat{a}_x^3 \rangle}. \quad (4.23)$$

Figure 10 shows this quantity. There is a prominent resonance-like spike near $\omega = 10$ in our conventions. From the numerics it is not clear if this spike has finite height. Note the surprising match between the quantity $\omega^{-3} G_{\hat{e}_x \hat{e}_x}^R$ and the Drude model shown in figure 9, and discussed in appendix B. We have also tried fitting the formal conductivity given in figure 10 using the Drude model but did not succeed. Note that our present setup and field configuration are formally similar to the case of the holographic s-wave superconductor studied in [43]. But note also the distinct interpretation: in [43] the fluctuations of gauge fields have been interpreted using the standard ordinary quantization. In particular this

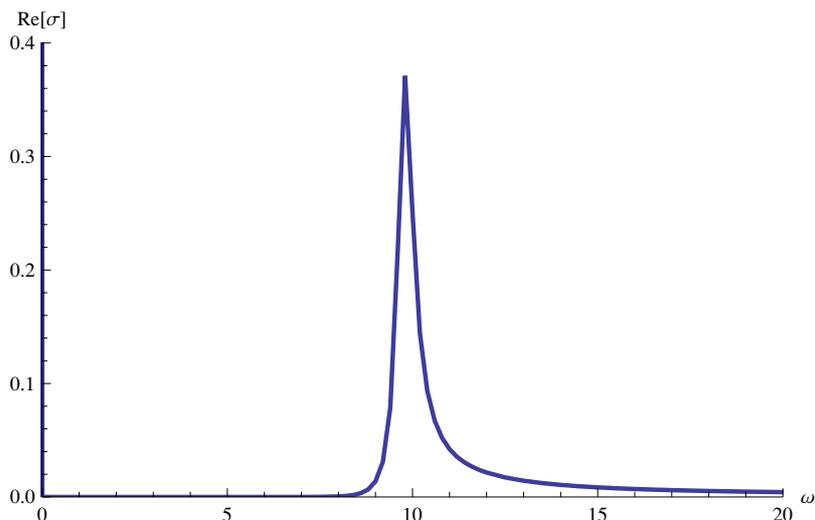


Figure 10. Broken phase: the formally-defined conductivity versus frequency at $T = 0.169T_c$.

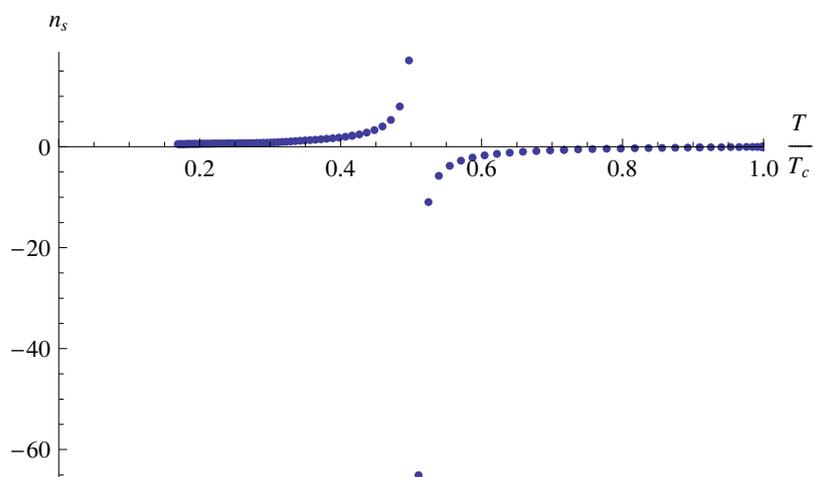


Figure 11. The formally-defined superconducting density diverges at $T^* \approx 0.5T_c$ when the alternative interpretation is employed.

interpretation yielded a superconducting charge density which was negative and divergent near T_c . We find exactly the same divergent behavior of the corresponding superconducting density from our fluctuations if we use that standard interpretation, see figure 12. If we choose the alternative quantization interpretation, the formally-defined superconducting density derived from (4.23) via $n_s = \lim_{\omega \rightarrow 0} \text{Im}(\omega\sigma)$ seems well-behaved at all frequencies, except for one divergence at $T^* \approx 0.5T_c$, see figure 11. This divergence occurs where the graph in figure 12 crossed the real frequency axis. Note that the result displayed in figure 11 is essentially the inverse of figure 12.

This somewhat odd behavior of the formally-defined conductivity suggests that either we should not interpret σ as a conductivity, or our $(1+1)$ -dimensional system shows very peculiar behavior, such as a divergent superconducting density at a particular temperature $T^* = 0.5T_c$.

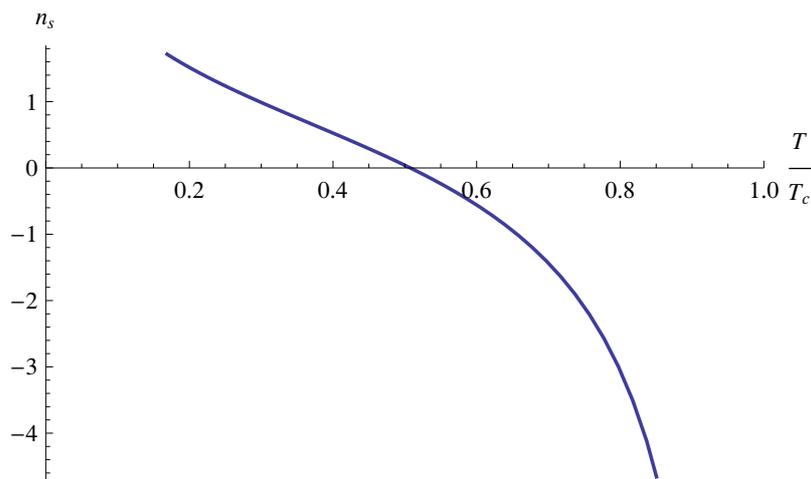


Figure 12. The superconducting density diverges at $T = T_c$ when the ordinary interpretation is employed. It vanishes at a particular $T^* \approx 0.5T_c$.

Broken phase at non-vanishing spatial momentum $k \neq 0$. It is straightforward to find numerical solutions to the full coupled system of equations given in (4.1). However, the interpretation of these solutions is not obvious. All the modes are coupled, which means that in the dual field theory all the operators mix along the renormalization group flow.

In principle we would like to find quasinormal modes and their dispersion relations $\omega_{n,\text{QNM}}(k)$ in this system. However, we can not simply use the prescription given in [41]. This prescription needs to be adjusted to the fact that the sources appear in the logarithmic terms of the bulk field expansions near the AdS-boundary. We leave this point for future work.

5 Discussion

In this work we have shown that a black hole in Yang-Mills theory in AdS₃ can grow vector hair, as seen in figure 2. This solution with a non-trivial profile for the charged gauge field is thermodynamically preferred at low temperatures, as seen from the lower free energy in figure 3. This solution corresponds to a boundary theory which develops a non-zero vacuum expectation value for a dynamical gauge field $\langle \mathfrak{A}_x \rangle \neq 0$. Therefore the local gauge symmetry of this boundary theory is broken spontaneously, leading us to interpret this phase as a p-wave superconductor in (1 + 1) dimensions. Formally the symmetry breaking in presence of our dynamical gauge field occurs just like in the setups with non-dynamical gauge field which yield superfluids. However, we are not able to directly compare the two processes within our gravity model since we are restricted to the alternative quantization, i.e. to the dynamical gauge field case. See section 3.3 for an extended discussion of this point.

Further we have studied the dynamics of this model and have shown that at least one non-trivial gapless mode exists in the broken phase. This may allow a low-energy effective analysis of this setup in what we defined in section 2.2 as “non-equilibrium field dynamics”

(this is analogous to the hydrodynamic description in the standard quantization). The origin of this gapless mode is the Goldstone boson which appears at the transition temperature. Naively we would expect this mode to give mass to our $U(1)_3$ gauge field, just like the photon becomes massive in a traditional superconductor. However, our numerical studies suggest that our gauge field does not acquire a mass, in contrast to our naive expectation. Instead, our Goldstone mode below the transition acquires a larger and larger damping at lower temperatures, as was seen in figure 6.

This fact again confirms our interpretation of these two-point functions as correlators of dynamical gauge fields (i.e. gauge field propagators) in the following way: we know from the Schwinger model that in two spacetime dimensions a $U(1)$ gauge field can only have mass if the theory also has a chiral anomaly, see discussion in section 2.5. However, we have forced our anomaly to vanish by forcing the Chern-Simons coupling to vanish in the action (1.1). Therefore we conclude that our gauge bosons can not acquire a mass in the present setup. See section 4.3 for a detailed discussion. It would be very interesting to investigate the action (1.1) after addition of a Chern-Simons term. We suspect that this will allow a mass generation by a Higgs-like mechanism below the critical temperature.¹¹ In fact, the size of the condensate may be linked to the size of the anomaly, see [36] for a similar setup in AdS_5 . The significance of the formally-defined [18] conductivity (4.23) requires more investigation.

We also see the gauge boson propagator (virtually) vanish in the superconducting phase at low frequencies, see figure 7. This suggests that the gauge interactions mediated by our field theory gauge boson \hat{a}_x^3 (our analog of the photon) become short-ranged. In other words, we see that our analog of the electromagnetic interaction becomes short-ranged. In this context it would be interesting to perform a calculation at finite momentum, which may allow to derive the penetration depth of this superconductor. As a general conclusion this holographic superconductor in the alternative formulation shows signatures which are strikingly similar to the ones that are known in the ordinary formulation. For example the gap in our gauge boson propagator (figure 7) resembles the (conductivity) gap observed in the current-current two-point functions in the ordinary formulation, see e.g. [3]. However, we stress again, that the interpretation of these signatures and quantities is different in the ordinary and the alternative formulations. For example, the gap in the ordinary current-current correlators is interpreted as a gap in the (single electron) conductivity. In our alternative setup the gap in the corresponding gauge-field two-point function is interpreted differently as explained at the beginning of this paragraph.

Although our action (1.1) preserves parity, our ground state solution spontaneously breaks this symmetry. With this property our system may serve as a toy model for unconventional superconductors, which violate parity in their ground state through the order parameter. An experimentally well-studied example for such a system in $2+1$ dimensions is the strontium ruthenate compound Sr_2RuO_4 [44]. In this context it may also be interesting to consider an analog to our setup in AdS_4 . Note that in the ordinary quantization

¹¹In the conclusions of [18] it was noted that for such a theory a perturbative instability is ruled out due to flatness of the gauge connection. However, non-perturbatively there may exist a thermodynamically preferred phase with a (vector) condensate.

the obvious (p+ip)-wave solution which breaks parity in AdS_4 is thermodynamically not favored according to the analysis in [36].

None of our fluctuations indicates an instability of the p-wave phase. However, a study of the full mode spectrum of the coupled system at finite spatial momentum turns out to be involved. In particular this requires an extension of the (determinant) method proposed in [41] in order to account for the logarithmic terms in the AdS_3 -asymptotics. This is postponed to future work.

Our paper leaves some other loose ends which are worthwhile being picked up. In particular one could repeat our analysis in AdS_4 , where both quantizations are possible, and thus can be compared directly. This should allow to make the relation of Green's functions in the alternative quantization to the ones in the ordinary quantization explicit. In AdS_4 we also have a magnetic field, which can destroy the ordered state, and a dynamical response should be observable. So the Meissner-effect could be observed directly. For this purpose the authors of [5–8] provide stationary solutions (in the s-wave superconductor), while we suggest to examine the dynamics of their backgrounds via the bulk field fluctuations along the lines of our present paper.

In analogy to the standard hydrodynamic study of higher-dimensional setups, see for example [35, 45], one could study the alternatively quantized setups at low frequency and momentum. Such a study should yield results resembling those of traditional thermal quantum field theory, and field dynamics out of equilibrium. Due to the simplicity of our setup one may hope for analytic results in the spirit of [46].

In fact our investigations suggest that the alternative interpretation may be interesting for a wide class of holographic fluids: it should be possible to study each of the holographic fluids which have been investigated in AdS_4 , but now with the Neumann boundary condition, i.e. in the alternative quantization. Our approach should be applicable to all holographic fluids, such as holographic quark-gluon plasma and all superfluids, and superconductors (s-wave, p-wave, d-wave), which have been found in AdS_4 .

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A Gauge-invariant and gauge-covariant fields

A.1 Broken phase

In the broken phase (superconducting phase), the gauge transformation of the perturbations are given by

$$\delta(e^{-i\omega t+ikx}a_\mu^a) = \partial_\mu(e^{-i\omega t+ikx}\alpha^a) + e^{-i\omega t+ikx}\epsilon^{abc}A_\mu^b\alpha^c, \quad (\text{A.1})$$

where, α^a can not depend on z because we have chosen an axial gauge. Therefore α^a has constant components.

One way of writing these gauge transformations of a_μ^a from (A.1) explicitly is

$$\delta \begin{pmatrix} a_x^1 \\ a_x^2 \\ a_x^3 \\ a_t^1 \\ a_t^2 \\ a_t^3 \end{pmatrix} = \begin{pmatrix} ik & 0 & 0 \\ 0 & ik & -W(z) \\ 0 & W(z) & ik \\ -i\omega & -\Phi(z) & 0 \\ \Phi(z) & -i\omega & 0 \\ 0 & 0 & -i\omega \end{pmatrix} \times \begin{pmatrix} \alpha^1 \\ \alpha^2 \\ \alpha^3 \end{pmatrix}. \quad (\text{A.2})$$

Actually, there are various gauge invariant linear combinations for one typical perturbation. For example, we can calculate the linear gauge invariant combinations of a_x^3 in general form as follows:

$$\hat{a}_x^3 = a_x^3 + h a_x^1 + j a_x^2 + m a_t^1 + n a_t^2 + p a_t^3, \quad (\text{A.3})$$

where (h, j, m, n, p) are coefficients without a_μ^a . Because of $\delta\hat{a}_x^3 = 0$, we can have 7 types of gauge invariant combinations:

1. $h = j = 0, \quad p = \frac{k}{\omega}, \quad m = \frac{W(z)\Phi(z)}{\Phi(z)^2 - \omega^2}, \quad n = -\frac{i\omega W(z)}{\omega^2 - \Phi(z)^2};$
2. $h = m = n = 0, \quad p = -\frac{W(z)^2 - k^2}{k\omega}, \quad j = \frac{iW(z)}{k};$
3. $h = p = 0, \quad m = -\frac{(W(z)^2 - k^2)\Phi(z)}{W(z)(\omega^2 - \Phi(z)^2)}, \quad n = \frac{i(k^2\omega - \omega W(z)^2)}{W(z)(\omega^2 - \Phi(z)^2)}, \quad j = \frac{ik}{W(z)};$
4. $j = m = 0, \quad h = \frac{W(z)\Phi(z)}{k\omega}, \quad p = \frac{k}{\omega}, \quad n = -\frac{iW(z)}{\omega};$
5. $j = n = 0, \quad h = \frac{\omega W(z)}{k\Phi(z)}, \quad p = \frac{k}{\omega}, \quad m = \frac{W(z)}{\Phi(z)};$
6. $m = p = 0, \quad h = -\frac{(k^2 - W(z)^2)\Phi(z)}{k\omega W(z)}, \quad n = \frac{i(k^2 - W(z)^2)}{\omega W(z)}, \quad j = \frac{ik}{W(z)};$
7. $n = p = 0, \quad h = -\frac{\omega(k^2 - W(z)^2)}{kW(z)\Phi(z)}, \quad m = -\frac{k^2 - W(z)^2}{W(z)\Phi(z)}, \quad j = \frac{ik}{W(z)}.$

In fact, one can also construct the linear gauge-invariant combinations for other perturbations in the same spirit. We will not show them explicitly here.

A.2 Normal phase

In the normal phase, only part of the $SU(2)$ is broken and a $U(1)_3$ remains intact. Then the fluctuations a_μ^a transform as

$$\delta a_\mu^a = \partial_\mu \lambda^a + \epsilon^{abc} (A_\mu^b \lambda^c + a_\mu^b \Lambda^3 \delta_3^c), \quad (\text{A.4})$$

where Λ^3 is the gauge transformation parameter of the $U(1)_3$. From this we obtain

$$\begin{aligned} \delta a_t^1 &= -i\omega \lambda^1 - \Phi(z) \lambda^2 + a_t^2 \Lambda^3, \\ \delta a_t^2 &= -i\omega \lambda^2 + \Phi(z) \lambda^1 - a_t^1 \Lambda^3, \\ \delta a_t^3 &= -i\omega \lambda^3, \\ \delta a_x^1 &= ik \lambda^1 + a_x^2 \Lambda^3, \\ \delta a_x^2 &= ik \lambda^2 - a_x^1 \Lambda^3, \\ \delta a_x^3 &= ik \lambda^3. \end{aligned} \quad (\text{A.5})$$

Define $e_t^\pm = a_t^1 \pm ia_t^2$, $e_x^\pm = a_x^1 \pm ia_x^2$ which are in the fundamental representation of $U(1)_3$, and a_μ^3 is in the adjoint. Using the above transformation, we obtain

$$\begin{aligned} \delta e_t^+ &= \partial_t \lambda^+ + i\Phi(x) \lambda^+ - i\Lambda^3 e_t^+, \\ \delta e_t^- &= \partial_t \lambda^- - i\Phi(x) \lambda^- + i\Lambda^3 e_t^-, \\ \delta e_x^+ &= \partial_x \lambda^+ - i\Lambda^3 e_x^+, \\ \delta e_x^- &= \partial_x \lambda^- + i\Lambda^3 e_x^-, \\ \delta a_\mu^3 &= \partial_\mu \lambda^3. \end{aligned} \quad (\text{A.6})$$

From the a_t^3 and a_x^3 , we can construct a $U(1)_3$ -invariant combination with the help of Fourier transformation ($e^{-i\omega t + ikx} f(t, x) \rightarrow f(\omega, k)$ as before), that is

$$e_3 = k a_t^3 + \omega a_x^3, \quad (\text{A.7})$$

which leads to $\delta e_3 = -ik\omega \lambda^3 + ik\omega \lambda^3 = 0$.

On the other hand, we can build $U(1)_3$ gauge-covariant combinations for e_t^\pm and e_x^\pm ,

$$e_L^+ = (\Phi(z) - \omega) e_x^+ - k e_t^+, \quad (\text{A.8})$$

$$e_L^- = (\Phi(z) + \omega) e_x^- + k e_t^-. \quad (\text{A.9})$$

This leads to

$$\delta e_L^+ = (\Phi(z) - \omega) \delta e_x^+ - k \delta e_t^+ = -i\Lambda^3 e_L^+, \quad (\text{A.10})$$

and

$$\delta e_L^- = (\Phi(z) + \omega) \delta e_x^- + k \delta e_t^- = i\Lambda^3 e_L^-. \quad (\text{A.11})$$

With these gauge-covariant fields e_L^+ and e_L^- and the gauge-invariant field e_3 in the normal phase, we can rewrite the on-shell action. From the on-shell action we can see that e_3 decouples from all other fields while e_L^+ and e_L^- are coupled to each other.

B Ordinary interpretation

B.1 Drude model

The quantity $\omega^{-3}G_{\hat{\epsilon}_x\hat{\epsilon}_x}^R$ is identical to the conductivity one would define in the ordinary interpretation of our setup. Therefore, we tried fitting this using the Drude model

$$\text{Re}(\sigma_{\text{Drude}}) = \frac{\sigma_0}{1 + \omega^2\tau^2}, \quad (\text{B.1})$$

where, $\sigma_0 = nq^2\tau/m$ is the DC conductivity of σ_{xx} , while n , q , m , τ are respectively the electron's number density, charge, mass and the mean free time between collisions. As seen from figure 9 this works surprisingly well at low frequencies. We have no good explanation for this agreement.

B.2 Superconducting density

Here we assume that the ordinary interpretation of sources as fixed boundary gauge fields is correct. In this case our two-point functions would be correlators of dynamical currents. In particular this would allow us to compute the superconducting density using the formula

$$n_s = \lim_{\omega \rightarrow 0} \text{Re} G_{jj}^R(\omega; k = 0). \quad (\text{B.2})$$

Here the Green's functions is obtained from the infalling solution of the bulk field \hat{a}_x^3 given in (4.9). The result is shown in figure 12. We observe the same negative divergent behavior as seen in [43]. We take this as a numerical evidence for the fact that the alternative interpretation should be applied to this kind of setup. See section 4 for a detailed discussion of the alternative interpretation.

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