NONCOMMUTATIVE GEOMETRY

AND

MATRIX MODELS

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ABSTRACT

Noncommutative geometry is a proposed description of spacetime at energies near or beyond the Planck scale. A particularly intriguing representation of a noncommutative algebra is the matrix representation. Matrix models have been shown to include geometry, gravity and nonperturbative aspects of string theory.

We wish to use matrix models to study noncommutative aspects of cosmological models. We begin by studying a matrix model analog of the BTZ black hole, which is a solution of 2+1 general relativity. We propose a Chern-Simons type theory constructed from finite dimensional matrices. After introducing the notion of a rotationally invariant boundary, we count the degeneracy of physical degrees of freedom associated with the boundary. This number coincides with the number of degrees of freedom needed to reproduce the Bekenstein-Hawking entropy relation. Next we study rotationally invariant solutions to d dimensional matrix models. We find d-1 dimensional solutions which have desirable cosmological features, in particular, we find matrix models that resolve cosmological singularities. In the last section, we restrict our attention to a Lorentzian analog of the complex projective plane, which is a four-dimensional solution of an eight-dimensional matrix action.

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1 INTRODUCTION

Motivation

One major challenge of theoretical physics is to formulate a consistent theory of quantum gravity. From a simple powercounting argument, it can be seen that due to the dimensionful coupling constant of general relativity, Newton's constant, the theory is perturbatively non-renormalizable[1]. The higher one goes in a perturbative expansion about a flat background, the more counterterms one must add and with no clear mechanism for a sensible implementation of renormalization to occur. While it may seem unnecessary in most physical processes to fully quantize gravity, there are certain questions that require such a treatment. General relativity predicts singularities in spacetime. These are places in which the theory breaks down and can no longer be trusted. It is believed that a quantum theory of gravity would resolve these singularities. Additionally, Hawking showed that black holes can radiate when aspects of quantum theory and general relativity are considered together [2]. Classically, this process does not happen and there has been much debate about whether this process conserves information and if it violates unitarity, a fundamental principle in quantum theories. A complete treatment of quantum gravity should address these questions.

Different proposals have been set forth to understand the ultraviolet completion of general relativity, such as string theory [3],[4] and loop quantum gravity [5]. One common

[1] The theory may, however, be nonperturbatively renormalizable[1].

theme among these proposals is that the structure of spacetime itself must change at distance scales comparable to the Planck scale. DeWitt made an argument that the energy required to probe spacetime at this scale would be so great that the structure of spacetime itself would be modified; this leads to fundamental uncertainty relations among the coordinates of spacetime points [6],[7]. Similar to the familiar relations among conjugate variables in quantum mechanics, this uncertainty can be captured by a noncommutative feature of the spacetime coordinates. The study of such configurations is called noncommutative geometry and has gained much interest in theoretical physics. It seems possible that noncommutative geometry will play a role in understanding spacetime at or near the Planck scale. Aside from motivation to study noncommutative geometry as a self-contained topic, it naturally arises independently in the context of string theory and emergent geometry, making it a relevant subject in successful approaches to quantum gravity.

Noncommutativity

The procedure to construct a noncommutative spacetime is analogous to that of passing from a classical system to a quantum one; the spacetime coordinates are promoted to operators which do not all commute. An example of such a spacetime is described by a commutation relation such as

$$[\hat{x}_\mu, \hat{x}_\nu] = i\hat{\theta}_{\mu\nu} \tag{1.1}$$

where $\hat{\theta}_{\mu\nu}$ is an antisymmetric tensor operator with dimension length^2. The simplest and canonical example of a noncommutative algebra is given when $\hat{\theta}_{\mu\nu}$ is independent of $\hat{x}_\mu$. $\hat{x}_\mu$ are Hermitian generators of an associative but noncommutative $\mathcal{C}^*$-algebra acting on a Hilbert space and are an essential ingredient in defining non-commuting spacetime. In the *commutative limit*, the right hand side of (1.1) should tend to zero in which one arrives at the ordinary commuting spacetime. Noncommuting spacetime has been referred to as a 'point-less' geometry, inspired by the analogous idea in mechanics that upon quantization, the phase space gets smeared out with points being replaced by Planck cells. The commutation relation (1.1) induces a spacetime uncertainty of

$$\Delta x^\mu \Delta x^\nu \geq \frac{1}{2}|\theta^{\mu\nu}|. \tag{1.2}$$

Because the $\hat{x}_\mu$ do not commute, they cannot be simultaneously diagonalized.

A noncommutative field theory may be constructed by considering an ordinary Lagrangian and proposing that the spacetime coordinates satisfy (1.1). The fields in this case are functions of $\hat{x}^\mu$. Large classes of noncommutative field theories have been studied in this manner, initially with the hope to provide a natural UV regularization mechanism [10]. In another equivalent construction, noncommutative field theory may be formulated by using a $\star$-product approach which generalizes the notion of multiplication among ordinary spacetime variables.

Noncommutative analogs of Yang-Mills theories have also been studied, leading to noncommutative gauge theories which share similarities to the large N Yang Mills of t' Hooft [19]. Noncommutative gauge theories have also been shown to arise in certain limits of string theory and even M-theory [29, 30].

There are also applications with no connection to quantum gravity. Noncommutative geometry arises in the contexts of condensed matter systems. In particular, this spacetime deformation is useful in studying the quantum Hall effect [20].

In many cases the fundamental degrees of freedom of a noncommutative theory may also be expressed in terms of matrices which provide a natural realization of a noncommutative algebra. In addition to being a simple way to express noncommutative geometry, these models are believed to capture nonperturbative aspects of string theory and therefore contain aspects of a dynamical quantum gravity. Moreover, matrix models have been used to provide a mechanism of emergent geometry and gravity, even outside the context of string theory [12]. In such models, spacetime is obtained as a noncommutative brane solution to a Yang-Mills type matrix model. Classical Einstein-Hilbert gravity arises on this 3+1 dimensional brane solution in a semi-classical limit. The mechanism for describing emergent gravity in this scenario is fundamentally different from how gravity arises in string theory matrix models where it lives in the 10 dimensional bulk [18].

The aim of this thesis is to explore the use of matrix models as representations of noncommutative algebras in relation to topics of black hole physics and cosmology. After giving an introduction to the construction of noncommutative geometry, we will propose a matrix model inspired by the 2+1 dimensional Chern-Simons theory in order to describe a two dimensional black hole solution in chapter III. The model allows us to count the density of states associated with the black hole from which we obtain the expected entropy. Chapter IV will be used to obtain solutions resembling open, closed and static universes in a commutative limit from Yang-Mills type matrix models. There we begin by constructing solutions that arise in the commutative limit containing two spacetime dimensions. We then construct solutions containing four spacetime dimensions. In both cases we are able to resolve cosmological singularities. Concluding remarks will be given in chapter V.

2 NONCOMMUTATIVE GEOMETRY

We begin by giving a very brief summary of the axiomatic formulation of noncommutative geometry by Alaine Connes. For more details see [8].

To understand what a noncommutative space is, it is instructive to first consider a commutative space. Let X be a Hausdorff space with continuous complex valued functions, f. These functions form a commutative associative algebra, $\mathcal{A}= C_0(X)$. This is a C^* algebra where addition and multiplication are defined pointwise, involution is given by complex conjugation, and the norm is given by the norm of functions. For each algebra element there exists a homomorphism mapping the algebra to the complex numbers, $\mathbb{C}$: $\phi_x = f(x)$, $x \in X$. The set of all such homomorphisms defines the spectrum of $\mathcal{A}$, $\Phi_{\mathcal{A}}$, which is also the space of irreducible representations of $\mathcal{A}$. Moreover, for each algebra element there is a dual element that maps the algebra spectrum to $\mathbb{C}$ given by $\hat{f}(\phi) = \phi(f)$. This means that there is a mapping $\gamma : C_0(X) \to C_0(\Phi_{\mathcal{A}})$. The Gelfand theorem states that for any C^* algebra, this map is an isometric $*$-isomorphism, and we can identify the topological space, X, with the spectrum of the algebra, $\Phi_{\mathcal{A}}$. Specifically, this means that the irreducible representations of $\mathcal{A}$ can be used to identify points x of the manifold. According to this result from Gelfand, Naimark and Segal, the topological knowledge of the space X is equivalent to the knowledge of the algebra $\mathcal{A}$ [9].

In order to do physics one typically studies a manifold with a metric and uses differential geometry to study objects such as one-forms on the manifold. The idea of Connes was to include all of these geometric features in a purely algebraic way. For this, he proposed that a compact Riemaniann spin manifold can be reconstructed completely by a spectral triple, $(\mathcal{A}, \mathcal{H}, \mathrm{D})$ [8]. This triple is composed of an algebra $\mathcal{A}$ of functions, a Hilbert space $\mathcal{H}$ on which this algebra acts and a self-adjoint Dirac operator D acting on $\mathcal{H}$. The spectrum of $\mathcal{A}$ carries the information about the manifold as a topological space, while D carries the information of the metric. This spectral triple can be generalized by considering a noncommutative but associative algebra of functions instead. This development laid the axiomatic groundwork for noncommutative geometry [8]. While we lose the notion of vector fields, derivatives, and even coordinates themselves in a noncommutative geometry, we are able to characterize these objects in a completely algebraic way.

Before this formal algebraic approach was developed, Weyl introduced a systematic way to describe noncommutativity on classical phase space by introducing a map between phase space functions and Hilbert space operators in the context of quantum mechanics [11]. In the phase space formalism, one can deform the product of functions on phase space using a $\star$-product which results in an algebra isomorphic to the operator algebra in the Hilbert space. The procedure for using this product is known as deformation quantization. In the case of the Heisenberg algebra generated by position and momentum operators used in quantum mechanics and satisfying $[q, p] = i\hbar$, we wish to deform the commutative algebra of $C^{\infty}(\mathbb{R}^2)$. We then write the product of two functions as

$$f \star g = \Sigma_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i\hbar}{2} \right)^n \Pi^n (f, g) \tag{2.1}$$

where the $\star$ denotes an associative but not necessarily commutative product and Π is a Poisson bivector: $\Pi = \Pi^{ij} \partial_i \wedge \partial_j$. This bivector is defined as a map $C^{\infty}(\mathbb{R}^2) \times C^{\infty}(\mathbb{R}^2) \rightarrow$

$C^\infty(\mathbb{R}^2)$ which must be skew symmetric, satisfy the Leibniz rule and Jacobi identity:

$$\Pi(f,g) = -\Pi(g,f) \tag{2.2}$$

$$\Pi(fg,h) = f\Pi(g,h) + g\Pi(f,h) \tag{2.3}$$

$$\Pi(f,\Pi(g,h)) + \Pi(g,\Pi(h,f)) + \Pi(h,\Pi(f,g)) = 0 \tag{2.4}$$

Expansion in powers of n is given by:

$$\Pi^0\,(f,g) = f\,g \tag{2.5}$$

$$\Pi^n\,(f,g) = \Sigma_{k=0}^{n}(-1)^k \binom{n}{k} \frac{\partial^k}{\partial p^k}\frac{\partial^{n-k}}{\partial q^{n-k}} f \;\times\; \frac{\partial^{n-k}}{\partial p^{n-k}}\frac{\partial^k}{\partial q^k} g. \tag{2.6}$$

More generally, the choice of product is not unique. In this procedure of deformation quantization, the $\star$-product is expanded in terms of a small parameter, $\hbar$, such that when $\hbar$ approaches zero, the usual commutative product among functions, the commutative limit, is retained[1]. The n = 1 term in (2.6) gives the ordinary Poisson bracket. Deformation quantization thus develops quantum mechanics by deforming the algebra of functions on a symplectic manifold. In fact, any symplectic manifold may be quantized using a non-unique $\star$-product defined such that

$$f \star g = f\,g + \frac{i\hbar}{2}\,\{f,g\} + \mathcal{O}(\hbar^2) \tag{2.7}$$

In some sense, one may think of the symplectic structure of a manifold as simply an indication that there is a deformation of the algebra of functions into a noncommutative algebra [8]. The above formulas are valid for manifolds with a constant Poisson structure.

[1] $\hbar$ is a fixed dimensionful number in quantum mechanics. As the expansion is also a derivative expansion, 'small' $\hbar$, effectively means that the functions f and g vary slowly.

According to the Darboux theorem, any symplectic manifold can *locally* be written such that the Poisson structure is constant. For a global description, one must construct a Fedosov manifold by introducing a torison-free symplectic connection to the manifold. Finally, for arbitrary Poisson manifolds[2], Kontsevich quantization generalizes (2.1) [14].

This formulation of quantum mechanics can be used for describing noncommutative geometry as well. Construction of such spaces amounts to quantization of even-dimensional symplectic manifolds. While this method of quantization works for general commutation relations, it will be described below in detail assuming a constant non-commutativity parameter and the Moyal $\star$-product. In this case, θ^{ij} clearly satisfies (2.2-2.4).

2.0.1 Weyl Symbols and Star Products

Consider a commutative algebra of functions on $\mathcal{R}^4$ with the usual pointwise product. Assuming that the derivatives of the functions fall off sufficiently rapidly at infinity, we may describe any function, f, by its Fourier transform:

$$\tilde{f}(k) = \int d^4x \;\; e^{ik_\mu x^\mu} f(x), \qquad f(x) = \frac{1}{(2\pi)^4} \int d^4k \;\; e^{-ik_\mu x^\mu} \tilde{f}(k). \tag{2.8}$$

We would like to replace the coordinates x^i of $\mathcal{R}^4$ by operators $\hat{x}^i \in \mathcal{R}^4_\theta$ which will satisfy (1.1) in the case of constant θ. The *Weyl symbol* of f is defined as

$$\hat{W}(f) = \frac{1}{(2\pi)^4} \int d^4k \;\; e^{-ik_\mu \hat{x}^\mu} \tilde{f}(k), \tag{2.9}$$

such that $\hat{W}(e^{iq_\mu x^\mu}) = e^{iq_\mu \hat{x}^\mu}$, i.e., W maps functions to operators [16]. The *Wigner map* is the inverse, mapping operators to functions: $\hat{W}^{-1}(e^{iq_\mu \hat{x}^\mu}) = e^{iq_\mu x^\mu}$.

[2]Recall that symplectic manifolds are special cases of a Poisson manifold in which the symplectic form is non-degenerate.

In general, to define a $\star$-product, we must pick an ordering prescription which maps monomials in the coordinate, x_i to polynomials of the operators, $\hat{x}_i$. One choice is a normal ordering which places all operators $\hat{x}$ to the right and all operators $\hat{y}$ to the left. Here we have chosen the so-called Weyl ordering, which is a totally symmetric ordering prescription. When acting on monomials of x_i, the Weyl symbol gives

$$\hat{W}(x^{i_1}, ..., x^{i_m}) = \frac{1}{m!}\partial_{k_{i_1}}, ..., \partial_{k_{i_m}}(k_i\hat{x}^i)^m \tag{2.10}$$

which in the example of two operators gives,

$$\hat{W}(x^i x^j) = \frac{1}{2!}(\hat{x}^i\hat{x}^j + \hat{x}^j\hat{x}^i) \tag{2.11}$$

One may also combine operators to form new operators. Consider:

$$\hat{W}(f \star g) = \hat{W}(f)\hat{W}(g). \tag{2.12}$$

Using (2.9), the product of two operators $\hat{W}(f)$ and $\hat{W}(g)$ can be written as

$$\hat{W}(f)\,\hat{W}(g) = \frac{1}{(2\pi)^n}\int d^n k\, d^n p\, e^{ik_i\hat{x}^i} e^{ip_j\hat{x}^j}\,\tilde{f}(k)\tilde{g(p)}. \tag{2.13}$$

Since we are assuming the canonical structure for (1.1), we may use the Baker-Hausdorff-Campbell formula on the operator product to give the form:

$$\begin{aligned} e^{ik_i\hat{x}^i} e^{ip_j\hat{x}^j} &= e^{i(k_j+p_j)\hat{x}^j - \frac{1}{2}k_i p_j[\hat{x}^i,\hat{x}^j]} \\ &= e^{-i(k_j+p_j)\hat{x}^j - \frac{i}{2}\theta^{ij}k_i p_j}. \end{aligned} \tag{2.14}$$

Then from (2.9) we have

$$
\begin{aligned}
f \star g &= \hat{W}^{-1}[\hat{W}(f)\hat{W}(g)] \\
&= \hat{W}^{-1}[\frac{1}{(2\pi)^4}\int d^4k d^4p\,(e^{i(k_j+p_j)\hat{x}^j - \frac{i}{2}k_i\theta^{ij}p_j})\tilde{f}(k)\tilde{g}(p)] \\
&= e^{\frac{i}{2}\theta^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}} f(x)g(y)|_{y\to x}.
\end{aligned}
\tag{2.15}
$$

The limit $\theta \to 0$ gives the desired ordinary multiplication among functions. This is known as the Moyal star product (also referred to as the Weyl-Groenewold product) and is defined for constant θ [13]. (2.15) is a representative from an equivalence class of $\star$-products. Two products, $\star$ and $\star'$ are equivalent if and only there exists a differential operator

$$
Df := f + \Sigma_{i=1}^{\infty} D_i(f)\epsilon^i \tag{2.16}
$$

such that

$$
f \star' g = D^{-1}(Df \star Dg) \tag{2.17}
$$

where ϵ is a small deformation parameter.

While in the above we considered θ^{ij} independent of x, every correspondence between spacetime and a noncommutative algebra will require its own class of $\star$-products.

We may also define the operators of derivatives and integrals, detailed in [15]. Formally the derivative should satisfy the Leibniz property for $A, B \in \mathcal{R}^4_\theta$:

$$
\partial_i(AB) = \partial_i(A)B + A\partial_i(B). \tag{2.18}
$$

By writing the Moyal-Weyl star product explicitly, we notice a natural way to express the derivative in terms of θ:

$$[x^i, f(x)]_\star = x^i \star f(x) - f(x) \star x^i = i\theta^{ij}\partial_j f(x). \tag{2.19}$$

This kind of derivative is called an inner derivative. In terms of operators, this derivative is given by the adjoint action of the algebra of operators on itself [16]:

$$[\hat{x}^i, f(\hat{x})] = i\theta^{ij}\partial_j f(\hat{x}) \tag{2.20}$$

$$[\hat{\partial}_i, \hat{x^j}] = \delta_i^j, \quad [\hat{\partial}_i, \hat{\partial}_j] = 0. \tag{2.21}$$

The trace of a Weyl operator is related to integration over spacetime. In the case of $\mathcal{R}^4_\theta$, where $d^4\mu(x)$ is an integration measure on $\mathcal{R}^4$

$$\mathrm{Tr}\,\hat{\mathcal{W}}[f] = \int d^4\mu(x) f(x), \tag{2.22}$$

where an appropriate normalization has been assumed. Generally, integration can be defined as an operation satisfying:

$$\int \mathrm{Tr}\,\partial_i A = 0, \quad \int \mathrm{Tr}\,[A, B] = 0 \tag{2.23}$$

with $A, B \in \mathcal{R}^4_\theta$. Integration is symmetric under cyclic permutations of the functions which is a property inherited from the trace.

Indeed, noncommutative geometry can be approached in two general ways. Either one can work with an ordinary product in the noncommutative algebra of Weyl operators, or we can work with a deformed noncommutative star product, if it exists, among the commutative algebra of functions on spacetime. The aim of this section is to highlight this correspondence in the context of deformation quantization in order to give a physical intuition behind the ideas of noncommutative geometry. On the other hand, the central theme of this thesis is to use matrix representations of noncommutative algebras to study gravity related topics and we will not be utilizing the $\star$-product approach. Next we review the matrix representation as well as how matrix models naturally arise in different areas of physics.

2.0.2 Matrix Representation

An elementary example of a noncommutative algebra is, Mat_N, the algebra of N $\times$ N matrices with entries in the complex numbers. The noncommutative analog of compact spaces is constructed by promoting the algebra of functions on the manifold to operators written as N $\times$ N matrix representations acting on a suitable Hilbert space. Spaces corresponding to these types of noncommutative algebras are then called *fuzzy spaces*. Fuzzy spaces are important in that they preserve the group of isometries of the original space although the algebra of functions is truncated and finite [17]. We will see this explicitly in the case of the fuzzy sphere in the next section. Formally, these spaces may also be described in terms of a spectral triple, $(Mat_N, \mathcal{H}_N, \Delta_N)$ where Δ_N is the matrix analog of the Laplacian and the algebra of N $\times$ N matrices, Mat_N, act on the N dimensional Hilbert space, $\mathcal{H}_N$[22].[3] For manifolds which are not compact, matrix bases are also possible if the Hilbert space is countable, or only analytic functions are considered [21].

Fuzzy Sphere

To illustrate the above ideas, we will review the construction of the fuzzy sphere, a generalization of the ordinary 2-sphere, which is one of the most studied noncommutative spaces [85],[86],[88],[87]. The sphere S^2 can be defined by considering Cartesian

[3]The Laplacian is given by a double commutator just as in quantum mechanics e.g., $\Delta f(\hat{x}) = -[\hat{p},[\hat{p},f(\hat{x})]]$ where $\hat{p}\psi = i\frac{\partial}{\partial x}\psi$ in the position representation for wavefunction ψ.

coordinates x_i with $i = 1, 2, 3$ in $\mathbb{R}^3$ satisfying the relation

$$x_1^2 + x_2^2 + x_3^2 = R^2 \tag{2.24}$$

with $R \in \mathbb{R}$. We will be concerned with properties of the algebra of smooth functions defined on the sphere. At first we may consider discretizing the sphere by replacing it with a lattice. For that we can consider two points of the sphere, the north and south poles, denoted (a,b). The functions at these points can be written as the algebra of 2 $\times$ 2 diagonal matrices:

$$\begin{pmatrix} f(a) & 0 \\ 0 & f(b) \end{pmatrix}.$$

There is no nontrivial action of the group of rotations, SO(3), on this algebra of 2 $\times$ 2 diagonal matrices [84]. To restore rotational invariance, we can replace the algebra of functions with a noncommutative algebra. Consider coordinates replaced with su(2) generators, J_i, in a finite dimensional irreducible representation[4]:

$$x_i \to \tilde{x}_i := \kappa J_i. \tag{2.25}$$

where $[J_i, J_j] = i\epsilon_{ijk}J_k$ and $\kappa \in \mathbb{R}$. The Casimir in an n dimensional irreducible representation is given by $J_1^2 + J_2^2 + J_3^2 = \frac{n^2-1}{4}\mathbb{1}$, where $\mathbb{1}$ is the n dimensional identity matrix. The algebra of functions is now expressed as a noncommutative matrix algebra. The

[4]Alternatively, the algebra of functions satisfying (2.26) form an $\mathcal{U}(su(2))$ module $\star$-algebra with irreducible representations spanned by spherical harmonics, Y_{lm}. The matrix representation is constructed by introducing a cutoff in the spectrum in such a way that matrix alegbra is also a $\mathcal{U}(su(2))$ module $\star$-algebra. Coherent state quantization can be used to extend the map (2.25) into an isometry between between spherical harmonic representations and matrix representations [26].

generators must satisfy (2.24):

$$\hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2 = \kappa^2(J_1^2 + J_2^2 + J_3^2) = R^2. \tag{2.26}$$

Specifically, we have that

$$\kappa^2 = \frac{4R^2}{n^2 - 1}. \tag{2.27}$$

The upshot of this construction is that this matrix algebra respects the rotational symmetry of the sphere. Let $h \in su(2)$ and $U \in SU(2)$. There is a natural definition of this group action on our matrix algebra which leaves the algebra invariant

$$h \to h' := UhU^\dagger \tag{2.28}$$

where $\dagger$ denotes the hermitian conjugate. The generators $\tilde{x}_i$ do not commute, but satisfy

$$[\tilde{x}_i, \tilde{x}_j] = i\kappa\epsilon_{ijk}\tilde{x}_k. \tag{2.29}$$

As the dimension of the representation we choose increases, the more eigenvalues these matrices will have. This corresponds to better and better resolution of the sphere, hence the name *fuzzy sphere*.

From (2.27), if we hold R fixed and let $n \to \infty$, we have that $\kappa \to 0$ and the coordinates become the regular commuting coordinates of S^2.

2.1 Noncommutativity from strings

Aside from constructing fuzzy and general noncommutative spaces from a matrix representation, there are motivations from string theory to study noncommutative geometry and matrix models representations in particular.

2.1.1 Strong fields

One way that noncommutativity arises in string theory is from a perspective similar to that of the Landau problem in condensed matter physics, i.e., from application of a strong magnetic field. Below we review how noncommutativity arises in the Landau problem which is the simplest case[16].

Consider the Lagrangian

$$\mathcal{L} = \frac{m}{2}\dot{\vec{x}}^2 - \dot{\vec{x}} \cdot \vec{A} \tag{2.30}$$

with the vector potential $A_i = -\frac{B}{2}\epsilon_{ij}x^j$ and the electric charge and c set to one. This describes a charged particle moving in the plane $\vec{x} = (x^1, x^2)$ with an applied uniform magnetic field, B, perpendicular to the plane.

In the limit $m \to 0$, (or equivalently in the limit of very strong B field with fixed m) the Lagrangian becomes

$$\mathcal{L}_0 = -\frac{B}{2}\dot{x}^i\epsilon_{ij}x^j \tag{2.31}$$

The corresponding Hamiltonian identically vanishes indicating that the system contains constraints. In the usual method of canonical quantization, one would replace the Poisson bracket by commutators and the phase space variables by operators acting on a Hilbert space. Because this is a system with second class constraints, the correct procedure is to instead construct Dirac brackets which are defined in terms of these constraints of the system, and then proceed with the usual techniques of canonical quantization [24],[25]. The constraints are

$$\phi_i = p_{x_i} + \epsilon_{ij}\frac{B}{2}x_j \approx 0. \tag{2.32}$$

These constraints may be strongly set to zero upon calculating the Dirac brackets in which case one finds a nonvanishing bracket among spacetime coordinates,

$$\{x^i, x^j\} = \frac{\epsilon^{ij}}{B}. \tag{2.33}$$

In the quantum theory, (2.33) becomes $[x^i, x^j] = \frac{i\hbar\epsilon^{ij}}{B}$. This limit corresponds to the projection of the state into the lowest Landau level, and it is readily seen that this introduces a noncommutativity among spacetime coordinates.

Something similar happens in string theory. In this case, one considers the action for open bosonic strings moving in a flat Euclidean, ten-dimensional space in the presence of a constant Neveu-Schwarz two-form field, B, and ending on Dp-branes [31]

$$S = \frac{1}{4\pi\alpha'}\int_\Sigma (d^2\sigma g_{ij}\partial_a x^i \partial_a x^j - 2\pi\alpha' B_{ij}\epsilon^{ab}\partial_a x^i \partial_b x^j). \tag{2.34}$$

Here, g_{ij} is the metric tensor of the background spacetime, $\partial_a = \frac{\partial}{\partial\sigma^a}$, and Σ is the string worldsheet parameterized by σ^a, $a = 1, 2$. x^i are the embedding functions into flat space and the string tension, T, Regge slope, α', and string length, l_s, are related by: $T = \frac{1}{2\pi\alpha'} = \frac{1}{2\pi l_s^2}$. B_{ij} can be thought of as a magnetic field on the brane. In a low-energy limit with B_{ij} fixed, the kinetic terms for the embedding functions of the string vanishes. The second term can be written as boundary term:

$$S_{\partial\Sigma} = -\frac{i}{2}\int_{\partial\Sigma} B_{ij} x^i \partial_t x^j, \tag{2.35}$$

where t is the coordinate parameterizing the boundary. This boundary string action is then analogous to the Landau action of electrons in a strong magnetic field. From there, one obtains a noncommutativity of the spacetime coordinates as in (2.33). Strictly speaking, the $x^i(t)$ only describe the boundary degrees of freedom of the open strings and not the motion of particles. Nonetheless, this analogy demonstrates how noncommutativity can arise in the context of string theory.

2.1.2 How matrices arise

The aim of superstring theory is to consistently formulate all of the fundamental interactions of nature. The fundamental degrees of freedom are internal oscillation quanta of the string, which corresponds to particles. The string spectrum contains gravitons and massless gauge fields and since it demands supersymmetry, the theory also accounts for fermions. At low energy, the string perturbation theory reproduces Einstein equations for closed strings and Yang-mills equations for open strings. At the Planck scale, the string interactions can not be described with perturbation theory alone; one needs to include the non-perturbative effects. There are multiple consistent string theories that are related to one another through dualities transformations. It is believed that these superstring models are limiting cases of a unified theory. The construction, which is hypothesized to contain all the current string theories, is called M-theory and is an eleven-dimensional theory. It's low energy limit is consistent with 11-dimensional supergravity and gives rise to an eleven-dimensional supermultiplet containing the metric $g_{\mu,\nu}$, gravitino,Ψ_μ^α and an antisymmetric tensor field, $A_{\mu,\nu,\lambda}$. It is currently unknown what the fundamental degrees of freedom of this theory will be away from the low energy limit.

One attempt at writing a dynamical formulation of M-theory is the matrix model of Banks, Fisher, Shenkar and Susskind (BFSS)[29]. A similar formulation was developed to provide a non-perturbative definition of type IIB superstring theory by Ishibashi, Kawai, Kitazawa, and Tsuchiya (IKKT)[30]. The degrees of freedom for both models are N x N Hermitian matrix elements. These matrix models then naturally employ the use of noncommutative geometry.

To see how matrices can arise in the open superstring theory, we can consider soliton-like solutions in the theory called p-branes. A p brane sweeps out a p+1 dimensional world volume and has finite tension and carries Ramond-Ramond (RR) charges. Although the perturbative string states are not usually charged with respect to the RR sector fields, solitons may carry electric and magnetic RR charges. Polchinski [32] showed that these states could be described as Dirichlet branes (or Dp-branes, "D" denoting Dirichlet and "p" indicating the number of dimensions) which are then an inherently nonperturbative feature of string theories. The ends of open strings can move along the Dp-brane and satisfy Dirichlet boundary conditions in the direction normal to the surface. In the low energy regime, the dynamics of superstring theory is dominated by massless string states. Low energy interactions of these massless degrees of freedom can be described with a Lagrangian and the result is that the dynamics of the Dp-brane is described by an effective field theory on it's world volume. Even-dimensional Dp-branes occur in type IIA theory and odd-dimensional ones appear in the type IIB theory.

The massless degrees of freedom of the open string ends are gauge fields and their superpartners. The effective theory for the low energy limit of an individual Dp-brane becomes a ten-dimensional U(1) gauge theory "dimensionally reduced" such that the fields are no longer dependent on the coordinates orthogonal to the Dp-brane world volume. For a Dp-brane parameterized by $\xi^0 ... \xi^p$, (p+1) degrees of freedom describe the components of a vector potential tangential to the Dp-brane. The other (9-p) components correspond to scalars [5] and are associated with the coordinates of the Dp-brane fluctuations. If we want to describe a state with an RR charge of N, this is equivalent to a stack of N Dp-branes where each brane carries a U(1) gauge field. Since strings can begin and end on different branes and are oriented, the number of degrees of freedom for a stack of N branes is proportional to N^2. When the branes are very close together all of these degrees of freedom are massless and the gauge symmetry of the effective theory becomes U(N) in the low energy limit [27]. The Lagrangian of this effective theory is

[5] They transform as scalars with respect to coordinate transformations on the world volume.

a reduction of the ten-dimensional supersymmetric U(N) gauge theory to one of (p+1) dimensions. The scalar fields describing the transverse coordinates of the Dp-brane in this case can be represented by N $\times$ N Hermitian matrices. The figure below depicts the situation for the case of two parallel Dp-branes.

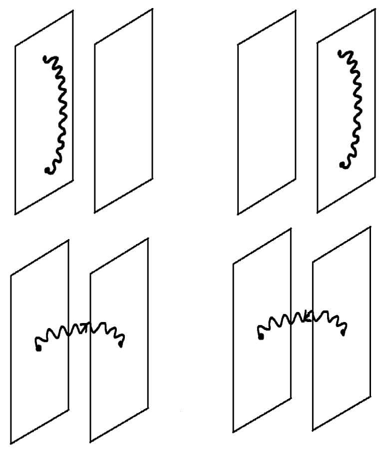

Figure 2.1: In the case of two bound parallel Dp-branes, the strings can end on the same brane or start on one and end on the other. There are 4 *massless* vector states for *coincident* branes, i.e., they form a U(2) group representation [28].

Having given a motivation of how noncommutative geometry and matrices arise in the context of string theory, we will look into more details of the matrix models of BFSS and IKKT, which will have a similar structure to the matrix models further studied in this thesis.

2.1.3 IKKT

The IKKT matrix model [30] is an effective action obtained from a 10-dimensional supersymmetric gauge theory by projecting the spacetime domain to a point. After the projection, the matrix elements are independent of space and time coordinates. Upon taking the matrix size to infinity, this zero dimensional matrix model is claimed to provide a non-perturbative definition of type IIB superstring theory. The Lagrangian for the IKKT model is a matrix analog of the superstring action in the Schild gauge. This formulation is equivalent to the Nambu-Goto formulation upon introducing an auxiliary field, $\sqrt{g(\sigma_0,\sigma_1)}$, which is a positive definite function. The classical Schild action is given by

$$S_{GS} = \int d^2\sigma[-\frac{\alpha}{4\sqrt{\mathsf{g}}}\{X_\mu, X_\nu\}^2 + \frac{i}{2}\bar{\psi}\Gamma^\mu\{X_\mu,\psi\}) + \beta\sqrt{\mathsf{g}}]. \tag{2.36}$$

The indices, μ,ν run from 0 to 9 and are raised and lowered with a flat background metric $\mathrm{diag}\,\eta^{\mu,\nu} = (+,-...-)$. The bosonic degrees of freedom are contained in $X_\mu(\sigma_1,\sigma_2)$ which are in a vector representation of SO(1,9). The fermionic field $\Psi_\alpha(\sigma_1,\sigma_2)$ carries a Spin(1,9) Majorana-Weyl spinor index, $\alpha = 1,...,16$. Poisson brackets are given by $\{X,Y\} = \epsilon^{ab}\partial_a X \partial_b Y$ and Γ^μ satisfy anticommutation relations $\{\Gamma^\mu,\Gamma^\nu\} = 2\eta^{\mu\nu}$. It can be checked that the action also has spacetime $\mathcal{N} = 2$ supersymmetry.

The superstring (2.36) action becomes a matrix model by substituting

$$X_\mu(\sigma_0,\sigma_1) \to A_\mu^{IJ} \tag{2.37}$$

$$\boldsymbol{\Psi}_\alpha(\sigma_0,\sigma_1) \to \Psi_\alpha^{IJ} \tag{2.38}$$

Here A_μ^{IJ} and Ψ_α^{IJ} are N $\times$ N Hermitian matrices, bosonic and fermionic respectively. The matrix elements can be constructed using

$$M(x,y)^{IJ} = \Sigma_{n,m} j(x,y)_{m,n}(J_{m,n})^{IJ} \tag{2.39}$$

where $j(x,y)_{m,n}$ are functions on $\mathbb{R}^2$ and provide a suitable basis for the function space we are constructing a matrix representation for. $J_{m,n}^{IJ}$ is a basis for gl(N). In particular, we have the following maps,

$$A_\mu^{IJ} = \int d^2\sigma N X_\mu M^{IJ} \tag{2.40}$$

$$X_\mu = \mathrm{Tr} A_\mu M. \tag{2.41}$$

In the $N \to \infty$ limit, we have that

$$\mathrm{Tr} \to \int d^2\sigma \mathrm{N} \tag{2.42}$$

$$[.,.] \to \frac{i}{N} \{.,.\} . \tag{2.43}$$

This limit may not exist for each algebra of functions. An explicit construction has been worked out for several cases such as the sphere and torus [28].

Using the above substitutions, the IKKT model is written as

$$S = \mathrm{tr}(\frac{\alpha}{4}[A_\mu, A_\nu]^2 + \frac{1}{2}\bar{\Psi}\Gamma^\mu[A_\mu, \Psi] + \beta) \tag{2.44}$$

where integration has been replaced with a matrix trace and Poisson brackets have been replaced with commutators. In a double scaling limit, $N \to \infty$ and $g^2 \to 0$ with Ng^2 fixed, (2.36) can be viewed as a "classical" limit of (2.44) where we may use the replacements (2.42 -2.43).

Multi-object solutions correspond to block diagonal forms for the A_μ and Ψ. In this block diagonal form, the action is really a sum over traces of each of the blocks. Each one of the traces is the Schild action in the classical limit which corresponds to a theory of multi-string states. Aside from its invariance under $\mathcal{N} = 2$ supersymmetry, (2.44) is invariant under U(N) gauge transformations.[6] They induce the infinitesimal

[6]These transformations become the symplectic transformations on the string world sheet in the limit $N \to \infty$.

variations: $\delta A_\mu = i[A_\mu, \omega]$, $\delta \Psi_\alpha = i[\Psi_\alpha, \omega]$, where ω are N $\times$ N Hermitean matrices with infinitesimal elements.

The classical equations of motion of the bosonic sector of (2.44) are obtained by extremizing the action with respect to variations in A_μ and are given by

$$[A^\mu, [A_\mu, A_\nu]] = 0. \tag{2.45}$$

Trivial solutions are expressed in the form of a diagonal matrix

$$A_\mu = \begin{pmatrix} x_\mu^1 & & \\ & \ddots & \\ & & x_\mu^N \end{pmatrix}. \tag{2.46}$$

These are the classical vacuum solutions of the theory and x_μ are interpreted as space-time coordinates of D-instantons[7]. These solutions were found to be stable against quantum corrections by using the standard background field method to show that the one loop quantum corrections to the classical solutions vanish [30]. In this calculation, the fermionic contributions to quantum corrections are necessary to cancel the bosonic contributions and provide even spacing of the eigenvalues of (2.46) and $\mathbf{R}^D$ symmetry. There are also Bogomol'nyi-Prasad-Sommerfield (BPS) solutions (those that preserve half the supersymmetry) and are also found to be stable against quantum corrections. Interaction among various objects in the theory is described with off diagonal blocks of the matrices that arise due to quantum corrections in the loop calculations. The interaction between the BPS solutions can also be calculated in the model and agrees with the expected behavior at long distance with the interactions in supergravity.

The action (2.44) is interpreted as an effective theory for the large N reduced ten-dimensional super Yang-Mills model (SYM) [34]. While this large N reduced SYM

[7] To consider other D-objects, one must take the N $\to \infty$ limit.

theory cannot be defined perturbatively, there is believed to be a nontrivial fixed point of the renormalization group of the theory such that the continuum limit about this point provides a nonperturbative definition of it. The conjecture is that this large N reduced ten-dimensional super Yang-Mills theory is a constructive definition of (2.44) and is equivalent to type IIB superstring theory.

2.1.4 BFSS Model

Compactification of M-theory on a circle of radius R leads to the theory of type IIA superstrings, which is governed by a 10-dimensional action. The parameter of M-theory is the Planck length, l_p and string theory is characterized by a dimensionless coupling g_s and a string tension $T = \frac{1}{2\pi\alpha'}$ where $\alpha' = \frac{l_s^2}{2}$ with l_s the string length scale. The parameters of M-theory and string theory are related by

$$R = g_s^{2/3} l_p \quad l_s = g_s^{-1/3} l_p \tag{2.47}$$

In this sense, M-theory is the strong coupling limit of IIA string theory and as $g_s \to \infty$ the radius tends to infinity and the theory becomes 11-dimensional. An infinite set of Kaluza-Klein modes appear in the ten-dimensional theory from each state in the gravity supermultiplet upon compactification. The massive modes in this tower of states are interpreted as nonperturbative objects in the superstring theory.[8] Precisely, the bound state of N D0-branes is associated to a graviton with momentum $p_- = N/R$ where N=(0,1,2...) and light cone coordinates have been introduced:

$$t = x^+ = (x^0 + x^{10}/\sqrt{2}) \tag{2.48}$$

$$x^- = (x^0 - x^{10}/\sqrt{2}). \tag{2.49}$$

[8]Their masses go like N/g_s.

Compactification is given by identifying $x^- = x^- + 2\pi R$ where R is the compactification radius.

The BFSS matrix model [29] claim is that the fundamental degrees of freedom of M-theory in the light cone gauge are D0-branes. The degrees of freedom are contained in nine N $\times$ N Hermitian bosonic matrices, $X^i(t)$ $i = 1, ..., 9$ and a spinor in nine spatial dimensions with 16 supercharges, $\theta^\alpha(t)\,(\alpha = 1, .., 16)$. Their dynamics is determined from the Lagrangian:

$$L = \mathrm{tr}\left\{\frac{1}{2R}(D_t X^i)^2 + \frac{R}{4}[X^i, X^j]^2 + \theta D_t \theta + iR\theta\gamma_i[X^i, \theta]\right\} \tag{2.50}$$

with nine-dimeniosnal Dirac matrices, γ_i satisfying the anticommutation relations $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$. This matrix Lagrangian is a fully *quantum model* as opposed to the effective action for D0-branes. This means that the Lagrangian does not only describe the lowest energy states, but it also describes all the quantum interactions once one considers the large N limit. The classical equations of motion to (2.50) are found by setting all off diagonal elements to zero such that

$$[X^i, [X^j, X^i]] = 0 \tag{2.51}$$

The matrices are related to world sheet coordinates just as $X^i(\sigma, \tau)$ are in string theory, with the exception that here the matrices include all the dynamical information and not just information about free string states. Multigraviton states correspond to block diagonal matrices and the off diagonal elements represent their interactions. If the separation between the gravitons is small, the off diagonal degrees of freedom can be integrated out of the action, but as the distance decreases these off diagonal pieces interact with one another which corresponds to gravitons scattering. The one loop effects reproduce those of 11-dimensional supergravity, and in principle all scattering cross sections in the theory can be obtained. The notion of spacetime only emerges when the gravitons are greatly separated and can not scatter.

2.2 Gauge Theory

A natural way to formulate a gauge theory on arbitrary noncommutative spaces is by making the coordinate operators transform under the adjoint action of the U(N) group. One can then introduce a gauge connection in an analogous way to how the covariant derivative is defined in commutative gauge theories [36]. Assume the noncommutative space is defined by an associative algebraic structure $\mathcal{A}_x$ with generators, $\hat{x}^i$, and the canonical structure, (1.1), where $\hat{\theta}_{ij} = \theta_{ij}$ and is independent of $\hat{x}_i$. (We shall assume this for the entire subsection.) A gauge connection, A^i is a one-form with values in $\mathcal{A}_x$ such that $\mathrm{A}^i = \mathrm{A}^{i\dagger}$. An infinitesimal gauge transformation of $\psi \in \mathcal{A}_x$ is given by $\psi \to \psi + \delta\psi$ where $\delta\psi(\hat{x}) = i\alpha(\hat{x})\psi(\hat{x})$ and $\alpha(\hat{x})$ is an infinitesimal function. Then

$$\delta(\hat{x}^i\psi) = i\hat{x}^i\alpha(\hat{x})\psi \tag{2.52}$$

which is not necessarily equal to $i\alpha(\hat{x})\hat{x}^i\psi$. Covariant coordinates can be introduced such that

$$\delta(\hat{X}^i\psi) = i\alpha(\hat{x})\hat{X}^i\psi \quad \text{i.e.,} \quad \delta(\hat{X}^i) = i[\alpha, \hat{X}^i] \tag{2.53}$$

which is what we expect from the transformation law for an ordinary gauge theory. For this we define

$$\hat{X}^i = \hat{x}^i + A^i(\hat{x}) \quad A^i(\hat{x}) \in \mathcal{A}_x \tag{2.54}$$

where the transformation properties of A^i can be derived as

$$\delta A^i = i[\alpha, A^i] - i[\hat{x}^i, \alpha]. \tag{2.55}$$

We have already seen that the commutator is a derivation, $[\hat{x}^i, f] = i\theta^{ij}\partial_j f$ where θ^{ij} is the inverse of θ_{ij} which means that

$$\delta A^i = i[\alpha, A^i] - \theta^{ij}\partial_j\alpha. \tag{2.56}$$

This is similar to the transformation of the *non-abelian* gauge potential in the *commutative* gauge theory. In that case, α and A^i are elements of some Lie algebra. We can define a tensor

$$F^{ij} = [\hat{X}^i, \hat{X}^j] - i\theta^{ij} \tag{2.57}$$

that transforms covariantly, $\delta F^{ij} = i[\alpha, F^{ij}]$.

Let $\alpha = u^i\theta_{ij}\hat{x}^j$. Then (2.56) is

$$\delta_\alpha A^i = u^j\partial_j A^i + u^j\theta_{ji} \tag{2.58}$$

which from the second term can be seen as just a translation of A^i. We see that the gauge symmetry group contains spacetime translations. That internal gauge transformations and spacetime translations cannot be seperated in noncommutative geometry is reminiscent of gravity theories and not commutative gauge theories.

There is a map between noncommutative and commutative Yang-Mills theories established by Seiberg and Witten [33]. That these two gauge theories are related is due to the fact that regularization used in quantum field theories is not unique. Some prescriptions of regularization of e.g., (2.34) (with background gauge fields added) lead to a noncommutative gauge theory while others lead to the usual commutative gauge theory. Since the regularization used should not alter the physics, there exists a map relating noncommutative gauge transformation on noncommutative gauge fields to commutative gauge transformations on commutative gauge fields called the Seiberg-Witten map. In order to see the relation, we will need to compare commutative and noncommutative gauge transformations. We will call the commutative gauge potential a_i and the non-commutative gauge potential A_i. ϵ is infinitesimal gauge parameter in the commutative case. The gauge transformation rules in each case are given by:

$$\delta_\epsilon a_i = \partial_i \epsilon + i[\epsilon, a_i]. \tag{2.59}$$

$$\delta A^i = i[\alpha, A^i]_\star - i[x^i, \alpha]_\star. \tag{2.60}$$

where $\star$ is the Moyal $\star$-product. We can expand (2.60) in orders of θ^{ij}. In this case, the second term of (2.60) can be written as a derivative

$$[x^i, \alpha]_\star = i\theta^{ij}\partial_j \alpha. \tag{2.61}$$

We would like to write the noncommutative gauge field and parameter in terms of the commutative ones. To do that, Seiberg and Witten proposed that the following consistency relation should hold for A^i and α which are dependent on a, ϵ, and θ only.

$$A_i(a) + \delta_\alpha A_i(a) = A_i(a + \delta_\epsilon a). \tag{2.62}$$

One can then construct a map order by order in θ such that the lowest order gives a commutative theory. For the Moyal $\star$-product and abelian a_i we find that

$$A_i = a_i - \frac{1}{2}\theta^{kl} a_k (\partial_l a_i + \partial_l a_i - \partial_i a_l) + \mathcal{O}(\theta^2) \tag{2.63}$$

$$\alpha = \epsilon - \frac{1}{2}\theta^{kl} a_k \partial_l \epsilon + \mathcal{O}(\theta^2) \tag{2.64}$$

The construction of the Seiberg-Witten map is not be unique and must be constructed for each associate algebra, $\mathcal{A}_x$.

3 CHERN-SIMONS INSPIRED MATRIX MODEL

Classically, black holes are objects which only absorb and do not emit radiation. In 1973, Bekenstein proposed that the laws of black hole mechanics have a similar structure to laws of thermodynamics [41]. Soon afterward, Hawking showed that when one considers quantum mechanical effects, black holes actually emit black body radiation to spatial infinity. It consists of all particle species with a temperature $T = \frac{\kappa}{2\pi}$ [2]. To see how this arises, one can consider a quantum field theory near the black hole horizon which constitutes a curved background. In a flat spacetime, general solutions to the equations of motion can be expanded in a complete, orthonormal set of modes. One can define positive and negative frequency modes with respect to the global timelike Killing vector and then associate them with creation and annihilation operators whose eigenstates all form a basis for the Fock space. In the flat spacetime, the number of particles and the vacuum state are invariant under Lorentz transformations and different observers agree on which state is the vacuum. In curved spacetime, where time translation invariance is not necessarily preserved under coordinate transformations, there is in general no timelike killing vector defined over all space, and modes can not be globally defined as having a positive or negative frequency. Since the vacuum is defined as the state annihilated by the negative frequency modes, different observers will not agree upon which state is the vacuum or its particle content. As a result, an observer not in free fall near the horizon will detect radiation from the black hole. This Hawking radiation has a constant temperature proportional to the surface gravity, κ, which is defined as the magnitude of acceleration with respect to Killing time of a stationary particle just outside the black hole horizon. Using this relation along with the first law of black hole mechanics, it is natural to associate an entropy to the black hole [2, 41]. The intriguing relation is that the maximum entropy is related to the surface area of the black hole and not the volume, a statement of the holographic principle. This is depicted in Figure 3.1. In 3+1 dimensions the entropy relation is given by the Bekenstein-Hawking relation, $S = \frac{A}{4\hbar G}$, where A is the area of the horizon.

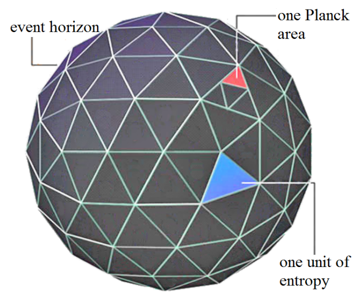

Figure 3.1: Each unit of entropy is associated with 4 units of Planck area [37].

In statistical mechanics, entropy is related to the number of microstates that make up the system under study. This should mean that there are microstates associated with the black hole, and it is believed that these microstates are associated with spacetime in some manner on the quantum scale. In this way, black holes are one of our most promising probes for studying quantum gravity. Different programs, such as string theory [45] and loop quantum gravity [46],[47] have reproduced the correct Bekenstein-Hawking entropy relation in some cases.[1]

General relativity in 2+1 dimensions - two spatial dimensions and one of time- has been studied as a toy model for investigating the quantum properties of spacetime. Even though there are no dynamical gravitational degrees of freedom, 2+1 gravity with a negative cosmological constant admits a black hole solution, known as the Banados-Teitelboim-Zanelli (BTZ) solution [38] . This solution is fully characterized by an Arnowitt-Deser-Misner (ADM) type mass, angular momentum and charge. Like the

[1] Extremal (and near extremal) black holes are considered in string theory.

Kerr black hole solution, a non-extremal rotating BTZ black hole has an inner and outer radius. It has been shown that this black hole can arise from collapsing matter [39]. For the BTZ black hole, whose horizon is one dimensional, the entropy goes as

$$S = \frac{\pi r_+}{2G} \tag{3.1}$$

with G the gravitational constant in 2+1 dimensions and r_+ the outer horizon radius.

In 1988, Witten showed that one could rewrite general relativity in 2+1 dimensions as a Chern-Simons theory, whose quantization is well studied [35]. The BTZ black hole solution can also be obtained in the Chern-Simons formulation [38],[40]. Carlip, and subsequently Banados, were able use the Chern-Simons theory [48],[50] to obtain the correct entropy associated to the BTZ black hole from first principles. These models use conformal field theory techniques[2] along with complicated boundary conditions to obtain the correct results. The degeneracy of boundary states are associated with black hole horizon microstates, but the counting relies on using a nonunitary theory which is not well understood. There are simpler derivations, but none detail which fields contain the relevant degrees of freedom contributing to the black hole entropy or where those degrees of freedom live. See ([52],[49],[53]) .

In what follows, we propose a matrix model inspired by a noncommutative Chern-Simons theory in order to account for the degrees of freedom associated with the entropy of the BTZ black hole. Since a commutative Chern Simons theory does contain gravity, it seems plausible that the matrix model would also contain gravity in the commutative limit. While Chern-Simons theory without a boundary is a trivial topological field theory, the introduction of boundaries or defects does allow for nontrivial degrees of freedom [42]. We introduce a 'boundary' in the matrix model and show that it can

[2]The boundary is two dimensional and the Chern-Simons theory naturally induces a Wess-Zumino-Witten action on the boundary.

account for the degenerate set of states associated with the expected entropy for the BTZ solution.

3.0.1 Noncommutative Chern-Simons

The noncommutative Chern-Simons theory [51] is given by 2 infinite dimensional square matrices X_i; $i = 1, 2$ which are covariant coordinates with units of distance and transform under the adjoint action of the U(N) gauge group. These coordinates are related to the noncommutative gauge potentials, A_i, of Chern-Simons by $X_i = A_i + x_i$ where derivations are given by $[x_i, \,] = \partial_i$. In what follows we consider only noncommutativity among x_i and time is kept as a continuous variable. The commutative Chern Simons Lagrangian in 2+1 dimensions is given by $L_{cs} = \frac{k}{4\pi} tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$, where A is defined in the Lie algebra and tr is the trace over their matrix representations. Here we consider the abelian, noncommutative matrix version of the Chern Simons theory, which when expressed in terms of the X_i (and also expressed in canonical form seperating the temporal and spatial components) can be written using an invariant trace

$$L_{cs}(X_i, \dot{X}_i) = \frac{k}{2\theta_0} \mathrm{Tr}\Big(\epsilon_{ij} D_t X_i X_j - 2i\theta_0 A_0\Big) \;, \tag{3.2}$$

where the covariant derivative is defined by

$$D_t X_i = \dot{X}_i + [A_0, X_i] \;, \tag{3.3}$$

and the dot denotes differentiation in the time t, which is assumed to be continuous. k and θ_0 are real constants. The former, which we assume to be positive, is known as the level, and here takes integer values.[43],[44]. Level quantization was a result of the fact that the Lagrangian is not invariant under gauge transformations, but rather changes by a time derivative. θ_0 is the noncommutativity parameter, and has units of length-squared. k and θ_0 will play different roles in the subsequent sections. A_0 is an infinite dimensional square matrix whose elements correspond to Lagrange multipliers. Reality for the Lagrangian requires A_0 to be antihermitean, while X_i can be hermitean or antihermitean. Our convention will be to take X_i antihermitean.

The action corresponding to (3.2) is $S = \int dt\, L^{(N)}_{CS}(X_i, \dot{X}_i)$. The equations of motion obtained by varying X_i and A_0 are

$$[X_i, X_j] = i\theta_0\epsilon_{ij}\mathbb{1} \tag{3.4}$$

$$D_t X_i = 0, \tag{3.5}$$

where $\mathbb{1}$ denotes the identity matrix. The equation of motion (3.4) defines the Heisenberg algebra, which implies that the space spanned by coordinates X_i is the Moyal-Weyl plane, with noncommutativity parameter θ_0.

The action is invariant under noncommutative gauge transformations which can be constructed. In the Hamiltonian formalism, gauge transformations are generated from the first class constraint[3], $F_{ij} = [X_i, X_j] - i\theta_0\epsilon_{ij}\mathbb{1} \approx 0$ which are associated with (3.4). See [24], [54] for Dirac's theory of quantization of gauge theories, i.e., when the symplectic form of a mechanical system is degenerate. The generator of gauge transformations is then given by

$$G(\Lambda) = \frac{-k}{2\theta_0}\epsilon_{ij} Tr\,\Lambda F_{ij}. \tag{3.6}$$

Infinitesimal variations are of the form

$$\begin{aligned} \delta_\Lambda X_i &= [X_i, \Lambda] \\ \delta_\Lambda A_0 &= D_t\Lambda\ , \end{aligned} \tag{3.7}$$

[3]First class constraints are those quantities that have vanishing Poisson bracket with other dynamical quantites and constraints in phase space.

where Λ is an infinite dimensional square matrix, with time-dependent matrix elements. The reality conditions for X_i and A_0 are preserved provided Λ is antihermitean. Since A_0 is nondynamical and each first class constraint removes two physical degrees of freedom from the matrix elements X_1 and X_2, no degrees of freedom remain in this noncommutative theory. That is, there is one first class constraint for each pair of matrix elements. The fact that this model is trivial is not surprising since the commutative Chern Simons theory also contains no degrees of freedom when there is no nontrivial topology or boundary. Since no degrees of freedom remain, one can not hope to account for the BTZ entropy in this scenario. In the commutative case, one needed to introduce a boundary or a nontrivial topology in order to support physical degrees of freedom. The analogy to a boundary in the matrix model may be to consider finite dimensional matrices which is done in the next section.

3.1 Matrix Chern-Simons

In order to study the matrix model in more detail, we consider the case of finite dimensional matrices. Here the covariant coordinates X_i; $i = 1, 2$ are N $\times$ N matrices. In the finite dimensional case, the Lagrangian must be modified so that equation of motion (3.4) is consistent. That is, taking the trace of both sides of (3.4) results in an inconsistent equation. This inconsistency is corrected if A_0 is traceless and takes values in the defining representation of the $su(N)$ Lie algebra. The Lagrangian in this case simplifies to

$$L^{(N)}_{cs}(X_i, \dot{X}_i) = \frac{k}{2\theta_0}\epsilon_{ij}\mathrm{Tr}D_t X_i X_j, \tag{3.8}$$

where Tr represents a standard matrix trace. Now instead of (3.4), variations of A_0 lead to

$$[X_i, X_j] = 0 \ , \tag{3.9}$$

while variations in X_i again give (3.5). The equation of motion (3.9) implies that the space spanned by spatial coordinates X_i is *commutative*, as opposed to what one gets from (3.4). (Here θ_0 no longer plays the role of a noncommutativity parameter.) It has been argued that commuting configurations do not support propagating degrees of freedom in four-dimensional gravity.[12] There should not be propagating degrees of freedom in a 2+1 gravity theory, however, and so it is reasonable to consider commuting configurations here.

Because the Lagrangian (3.8) does not contain the previous $\mathrm{Tr}A_0$ term, it is invariant under $SU(N)$ gauge transformations, with infinitesimal variations given by (3.7). Here Λ are *traceless* antihermitean matrices. Because the action no longer changes by a total time derivative, the Chern-Simons level, k, does not get quantized in this model. Since X_i has units of length, all we require is that k/θ_0 has units of inverse length-squared.

The counting of degrees of freedom in the finite Chern-Simons matrix model is best studied in the Hamiltonian formalism. There are 2 sets of primary constraints arising from the momenta given by

$$\Phi_k = (\pi_k) - \frac{k}{2\theta_0}\epsilon_{kj}(X_j) \approx 0 \qquad \text{where} \;\; (\pi_k)_{\mu\nu} = \frac{\partial L}{\partial(\dot{X}_k)_{\mu\nu}} = \frac{k}{2\theta_0}\epsilon_{kj}(X_j)_{\nu\mu} \tag{3.10}$$

$$\Phi_0 = (\pi_0) \approx 0 \quad \text{where} \;\; (\pi_0)_{\mu\nu} = \frac{\partial L}{\partial(\dot{A}_0)_{\mu\nu}} = 0. \tag{3.11}$$

The canonical Hamiltonian of the theory is given by

$$H_C = -\frac{k}{\theta_0}\epsilon_{ij}Tr(A_0 X_i X_j) \tag{3.12}$$

and the total Hamiltonian

$$H_T = H_C + \mu_0\Phi_0 + \mu_i\Phi_i, \tag{3.13}$$

where the μ_0, μ_i are functions of the coordinates and momenta. One must check if these primary constraints lead to other secondary constraints by checking the consistency of their time derivatives. This is done by computing their Poisson bracket with the total Hamiltonian. We find secondary constraints in such a way arising from (3.11).

$$\dot{\Phi}_0 = \frac{k}{\theta_0}\epsilon_{ij} X_i X_j \approx 0. \tag{3.14}$$

(3.10) does not produce any secondary constraints. Equation (3.14) become first class constraints which generates gauge transformations of the theory which we show below. One can work in the reduced phase space by setting the second class constraints strongly equal to zero and simultaneously using the new Poisson bracket of the theory which is given by

$$\{(X_i)_{\alpha\beta}, (X_j)_{\gamma\delta}\} = \frac{\theta_0}{k}\epsilon_{ij}\delta_{\alpha\delta}\delta_{\beta\gamma} \ , \tag{3.15}$$

where $\alpha, \beta, \gamma, \delta, ... = 1, ..., N$ are the matrix indices. This Dirac bracket is computed using the second class constraints and for two functions on phase space has the form

$$\{f, g\}_{DB} = \{f, g\}_{PB} - \sum_{a,b}\{f, \tilde{\phi}_a\}_{PB} M^{-1}_{ab}\{\tilde{\phi}_b, g\}_{PB}, \tag{3.16}$$

where M^{-1}_{ab} is the inverse matrix of the second class constraints: $\{\tilde{\phi}_a, \tilde{\phi}_b\}_{PB} = M_{ab}$ The first class constraints (3.14) generate $SU(N)$ gauge transformations

$$G(\Lambda) = -\frac{k}{2\theta_0}\epsilon_{ij}\mathrm{Tr}\Lambda[X_i, X_j], \tag{3.17}$$

where Λ is traceless. This is since $\{X_i, G(\Lambda)\} = [X_i, \Lambda]$. Using (3.15), they form a closed algebra

$$\{G(\Lambda), G(\Lambda')\} = G([\Lambda', \Lambda]) \tag{3.18}$$

and therefore are first class.

There are now N^2-1 first class constraints which should eliminate $2(N^2-1)$ degrees of freedom from the matrix theory. This means that we will have at least two degrees of freedom remaining amongst X_1 and X_2. Actually, there are more. It turns out that the assumption that the first class constraints each remove two degrees of freedom is incorrect. To count the number of physical degrees of freedom, one starts with the unconstrained $2N^2-$dimensional phase space spanned by the two matrices $X_i,\ i=1,2$. The traceless parts of these matrices, call them $X_i^{tl},\ i=1,2$, can be taken to be elements of the $su(N)$ Lie algebra in the defining representation. Using the $SU(N)$ gauge symmetry, one of them, say X_1^{tl}, can be rotated to the $(N-1)$-dimensional Cartan sub-algebra. (The result is unique up to Weyl reflections.) This corresponds to a gauge fixing. (Actually, it is only a partial gauge fixing, as the rotated X_1^{tl} are invariant under rotations by the Cartan generators.) From the gauge constraints, the remaining matrix X_2^{tl} must commute with the gauge fixed X_1^{tl}. If the latter spans all of the $su(N)$ Cartan-subalgebra (we call this the generic case), then X_2^{tl} must also be in the Cartan-subalgebra. So $2(N-1)$ phase space variables remain amongst $X_i^{tl},\ i=1,2$, after eliminating the gauge degrees of freedom. Upon including the $SU(N)$ invariant traces of X_1 and X_2, one then ends up with $2N$ independent degrees of freedom. They can be expressed in terms of the $SU(N)$ invariants $\mathrm{Tr}X_1^nX_2^m$, n and m being integers. The above argument shows that only $2N$ of them are independent in the generic case. For the example of $N=2$, we can take them to be

$$\mathrm{Tr}X_1\ ,\quad \mathrm{Tr}X_2\ ,\quad \mathrm{Tr}X_1^2,\quad \text{and}\quad \mathrm{Tr}X_2^2. \tag{3.19}$$

More generally, (3.19) correspond to a minimal set of independent degrees of freedom for the matrix model.

Let us examine the simplest case of $N = 2$. ($N > 2$ will be studied in detail in the following section.) The 2×2 antihermitean matrices X_1 and X_2 can be expressed as

$$X_1 = \sqrt{\frac{\theta_0}{2k}}\, p_\mu \tau_\mu \qquad X_2 = \sqrt{\frac{\theta_0}{2k}}\, q_\mu \tau_\mu \;, \qquad \mu, \nu, ... = 0, ..., 3 \tag{3.20}$$

where $\tau_0 = i\mathbb{1}$ and $\tau_{1,2,3} = i\sigma_{1,2,3}$. $\mathbb{1}$ and $\sigma_{1,2,3}$, respectively, denote the unit matrix and Pauli matrices. Then (3.15) correspond to canonical brackets for q_μ and p_μ,

$$\{q_\mu, p_\nu\} = \delta_{\mu\nu}. \tag{3.21}$$

The traces of X_i, which are proportional to q_0 and p_0, are $SU(2)$ invariants. The traceless parts of X_i, corresponding to $\vec{q} = (q_1, q_2, q_3)$ and $\vec{p} = (p_1, p_2, p_3)$, transform as vectors, so additional $SU(2)$ invariants are $\vec{q}^2$, $\vec{p}^2$ and $\vec{q} \cdot \vec{p}$, the dot denoting the scalar product. These invariants are not all independent since the constraint (3.9) means that the cross product of $\vec{q}$ and $\vec{p}$ vanishes. Excluding the special (non generic) cases where one of the vectors vanishes and the other is arbitrary, we get that $\vec{q}$ and $\vec{p}$ are parallel. Then there are a total of four independent gauge invariant quantities, q_0, p_0, $\vec{q}^2$ and $\vec{p}^2$, or equivalently, (3.19).

3.1.1 Rotational Symmetry

It has been argued that the horizon of the black hole should have a rigid rotational symmetry[4] [49]. Including the analogue of this symmetry in the commutative Chern-Simons theory was necessary for the entropy counting argument. In that case, the theory was allowed to have the topology of a cylinder with the corresponding rigid rotations being the diffeomorphisms that are generated from an infinitesimal vector field tangential to the boundary which carries only time dependence. Since we would like to associate

[4]Rigid rotations are those rotations that vary by a constant amount at each point. In Carlip's work they are generated by the lowest mode Virasoro operator, L_0.

our work to the black hole solution, we assume our action should also be invariant under the matrix analog of rigid rotations.

The matrix model can be modified to include an additional U(1) gauge symmetry which will act nontrivially on the spatial coordinates of the matrices and is thus analogous to a spatial rotation symmetry. As mentioned in the introduction, translations in the noncommutative theory are equivalent to gauge transformations. We need to write down the matrix analog of this symmetry as well as a consistent Lagrangian which includes this new symmetry. The transformations on the matrices are assumed to be analogous to rotations of components of a vector. Let $A_i, \quad i = 1, 2$ be a vector field in $\mathbb{R}^2$. In the commutative case, this rigid rotation is given by the Lie derivative and has the form $\delta A_i = \epsilon(t)(\mathrm{L}A_i + \epsilon_{ij}A_j)$ where $\mathrm{L} = \epsilon_{ij}x_i\partial_{x_j}$ is the usual angular momentum and x_i are coordinates in $\mathbb{R}^2$. The action of angular momentum in the matrix model is given by the derivation:

$$L_\Delta M = [\Delta, M], \tag{3.22}$$

where $\triangle$ is time independent and M is an arbitrary N $\times$ N matrix. The dynamical matrices then transform as

$$\delta_\epsilon X_i = \epsilon(t)(L_\Delta X_i + \epsilon_{ij}X_j). \tag{3.23}$$

The A_0 matrix is assumed to have a transformation of the form: $\delta_\epsilon A_0 = \epsilon(t)L_\Delta A_0 + \dot{\epsilon}(t)\Upsilon$. This means that the variation of the Lagrangian will be given by

$$\delta_\epsilon L_{CS} = \dot{\epsilon}(t)\frac{k}{2\theta_0}Tr\,(\epsilon_{ij}L_\Delta X_i X_j + X_i X_i + \epsilon_{ij}[X_i, X_j]\Upsilon). \tag{3.24}$$

This variation is a total derivative if Υ is given by $\Upsilon = iaX_iX_i - \Delta$ along with the constraint Tr($X_iX_i + \frac{i\Delta}{a}) = 0$ which can be imposed by adding a Lagrange multiplier term to $L^{(N)}_{CS}$

$$L_{CS} = \frac{k}{2\theta_0}\epsilon_{ij}TrD_tX_iX_j + \mu Tr\,(X_iX_i + \frac{i\Delta}{a}) \tag{3.25}$$

with

$$\delta_\epsilon A_0 = \epsilon(t)L_\Delta A_0 + \dot{\epsilon}(t)(iaX_iX_i - \Delta). \tag{3.26}$$

Now, both (3.23) and (3.26) leave the action invariant, with the variation of the Lagrange multiplier given by:

$$\delta_\epsilon\mu = -\dot{\epsilon}(t)\frac{k}{2\theta_0}. \tag{3.27}$$

The equations of motion for (3.25) are given by varying the action with respect to A_0, X_i and μ

$$[X_i, X_j] = 0 \tag{3.28}$$

$$D_tX_i + \frac{2\theta_0}{k}\mu\epsilon_{ij}X_j = 0 \tag{3.29}$$

$$Tr\,(X_iX_i + \frac{i\Delta}{a}) = 0. \tag{3.30}$$

(3.30) has nontrivial consequences. Upon restricting $i\mathrm{Tr}\Delta/a > 0$, it states that all matrix elements of X_i lie on the surface of a $2N^2 - 1$ dimensional sphere. (Recall that X_i are antihermitean.) However, from (3.15), one does not have the standard Poisson structure on a sphere. The constraint (3.30) implies that all matrix elements have a finite range, corresponding to the diameter of the sphere. This means that boundary conditions must be imposed in all directions in the phase space, making quantization problematic. [The situation is even worse for the case $\mathrm{Tr}\Delta = 0$, since then the constraint (3.30) says that all matrix elements of the antihermitean matrices X_i vanish!] This obstacle to quantization can be easily rectified by a simple modification of the reality conditions on the matrices X_i, which we choose below.

3.1.2 Observables

So far, we have constructed a Lagrangian which is invariant under SU(N) gauge transformations as well as spacetime rigid rotations, which we will refer to as Diff_0 gauge symmetry. In order to proceed to the quantum theory, one must differentiate between the physical and gauge degrees of freedom. All of the elements of the matrices, X_i, are not physical degrees of freedom. The generator of SU(N) gauge transformations is given in (3.17) and the generator of rigid rotations is

$$V_\Delta = \frac{k}{2\theta_0} Tr\,(\epsilon_{ij} L_\Delta X_i X_j + X_i X_i + \frac{i\Delta}{a}) \approx 0 \tag{3.31}$$

since it gives $\{X_i, V_\Delta\} = \delta_\epsilon X_i = \epsilon(t)(L_\Delta X_i + \epsilon_{ij} X_j)$. This generator forms a closed algebra with the SU(N) generator, $\{V_\Delta, G(\Lambda)\} = G([\Delta, \Lambda\})$, and together they correspond to N^2 first class constraints. That this algebra closes implies that the internal SU(N) transformations are coupled to the external rigid rotations, Diff_0. The combination defines the action of the semidirect product group, SU(N) $\otimes \mathrm{Diff}_0$. These constraints do not eliminate all the degrees of freedom, however. To see this, we can examine first the case for N=2 and then generalize to the case for N > 2.

The reality conditions for the matrices can be chosen in a particular way. In order for the constraint (3.31) to define an *unbounded* surface (which is still 2N-1 dimensional), one can choose the X_i such that a) the trace is real and b) the traceless part is antiHermitian. Λ and Δ are both antiHermitian, with Λ being time-dependent and traceless and Δ being constant.

In order to count the degrees of freedom left in the system, it is again instructive to start with the N=2 case. The SU(2) invariants $\vec{q}^2$, $\vec{p}^2$ and $\vec{q}\cdot\vec{p}$ still contain the Diff_0 gauge degree of freedom. The Diff_0 constraint coming from (3.30) in canonical coordinates is written as

$$q_0^2 + p_0^2 \approx \overrightarrow{q}^2 + \overrightarrow{p}^2 + d_0 \tag{3.32}$$

where $d_0 = \frac{k}{\theta_0 a}(-iTr\Delta)$. This Diff_0 constraint can be removed by fixing the gauge $\overrightarrow{q}^2 \approx 0$. Then from (3.32), $\overrightarrow{p}^2$ is constrained in terms of the remaining independent coordinates: q_0 and p_0 (or TrX_1, TrX_2). This particular gauge fixing makes it such that the new Dirac bracket is equal to the previous one: $\{q_0, p_0\} = -1$. A quadratic invariant is given by $q_0^2 + p_0^2$ and is the form of a harmonic oscillator Hamiltonian.

Generalizing to arbitrary N, the quadratic invariant can be written as

$$I^{(2)} = \frac{1}{N}((TrX_1)^2 + (TrX_2)^2) \tag{3.33}$$

where the invariants, $Tr\,X_1$ and $Tr\,X_2$ obey the following algebra

$$\{TrX_1, TrX_2\} = \frac{\theta_0 N}{k}. \tag{3.34}$$

A gauge fixing condition which eliminates the Diff_0 degree of freedom (the analog of $\overrightarrow{q}^2 \approx 0$ for general N) in this case is given by

$$\psi = TrX_2^2 - \frac{1}{N}(TrX_2)^2 \approx 0. \tag{3.35}$$

(3.33) is a quadratic invariant that has units of distance squared and will later be associated to the squared radius of the black hole. The algebra (3.34) is still preserved after the the gauge fixing. In addition, it's quantization gives the Heisenberg algebra:

$$[T\hat{r}X_1, T\hat{r}X_2] = i\frac{\theta_0 N}{k}. \tag{3.36}$$

These invariants can then be written in terms of lowering and raising operators:

$$T\hat{r}X_1 = \sqrt{\frac{\theta_0 N}{2k}}(a^\dagger + a), \qquad T\hat{r}X_2 = \sqrt{\frac{\theta_0 N}{2k}}(a^\dagger - a) \tag{3.37}$$

where $[a^\dagger, a] = 1$. The operator form of (3.33) can be written in terms of the number operator, $a^\dagger a$ with eigenvalues:

$$I_n^{(2)} = \frac{2\theta_0}{k}(n + \frac{1}{2})\,,\ n = 0, 1, 2.. \tag{3.38}$$

There is another way to explicitly isolate the degrees of freedom in the matrices X_i which will be a more convenient method for the analysis we would like to do in the case of N $>$ 2. To begin with here, we have 8 degrees of freedom in the matrices. To remove some of the SU(N) gauge freedom, we can first rotate the vector $\overrightarrow{p}$ to the third direction by requiring $p_1 = p_2 = 0$. From the constraint, (3.28), it means that $q_1 = q_2 = 0$. This leaves us with 4 of the original 8 degrees of freedom. Next, we impose the constraint (3.32) and fix the gauge to eliminate the Diff_0. degree of freedom. For this gauge fixing, we can choose $q_0 \approx 0$. The remaining physical degrees of freedom are then q_3, p_3 and they satisfy the Dirac bracket $\{q_3, p_3\} = 1$. The harmonic oscillator Hamiltonian associated with these invariants is $q_3^2 + p_3^2$, and the eigenvalues of this Hamiltonian are $2n + 1,\ n = 0, 1, 2...$

Now to generalize to the N x N case, it is convenient to express X_1 and X_2 in a Cartan-Weyl basis of U(N).

$$X_1 = \sqrt{\frac{\theta_0}{k}}\left(\frac{p_0 1}{\sqrt{N}} + i\sqrt{2}p_a H_a + ip_{-\overrightarrow{\alpha}}E_{\overrightarrow{\alpha}}\right) X_2 = \sqrt{\frac{\theta_0}{k}}\left(\frac{q_0 1}{\sqrt{N}} + i\sqrt{2}q_a H_a + iq_{-\overrightarrow{\alpha}}E_{\overrightarrow{\alpha}}\right). \tag{3.39}$$

The set $\{H_a,\, a = 1, ..., N-1\}$ spans the Cartan sub algebra which means that

$$[H_a, H_b] = 0. \tag{3.40}$$

The $E_{\overrightarrow{\alpha}}$ are root vectors that satisfy the following commutation relations:

$$[H_{a,} E_{\overrightarrow{\alpha}}] = \alpha_a E_{\overrightarrow{\alpha}} \tag{3.41}$$

$$[E_{\overrightarrow{\alpha}}, E_{\overrightarrow{\beta}}] = \begin{cases} \alpha_a H_a & \alpha + \beta = 0 \\ N_{\overrightarrow{\alpha}, \overrightarrow{\beta}} E_{\overrightarrow{\alpha}+\overrightarrow{\beta}} & \alpha + \beta \text{ is a root} \\ 0 & \alpha + \beta \text{ is not a root.} \end{cases}$$

A standard representation can be chosen such that

$$TrH_aH_b = \frac{1}{2}\delta_{a,b} \quad TrE_{\overrightarrow{\alpha}} E_{\overrightarrow{\beta}} = \delta_{\overrightarrow{\alpha}+\overrightarrow{\beta},0} \quad TrH_aE_{\overrightarrow{\alpha}} = 0. \tag{3.42}$$

Now, the Dirac brackets for the variables q and p are found from (3.15)

$$\{q_0, p_0\} = -1 \tag{3.43}$$

$$\{q_a, p_b\} = \delta_{a,b} \tag{3.44}$$

$$\{q_{\overrightarrow{\alpha}}, p_{\overrightarrow{\beta}}\} = \delta_{\overrightarrow{\alpha}+\overrightarrow{\beta},0}. \tag{3.45}$$

The first class constraints associated with the SU(N) gauge transformations can also be written in terms of the canonical variables, q and p:

$$\Phi_a = \sum_{\overrightarrow{\alpha}} \alpha_a q_{\overrightarrow{\alpha}} p_{-\overrightarrow{\alpha}} \approx 0, \tag{3.46}$$

$$\Phi_{\vec{\alpha}} = \sqrt{2}\sum_a \alpha_a (q_{-\vec{\alpha}} p_a - p_{-\vec{\alpha}} q_a) + \sum_{\vec{\beta} \neq \vec{\alpha}} N_{\vec{\alpha}-\vec{\beta},\vec{\beta}} q_{-\vec{\beta}} p_{\vec{\beta}-\vec{\alpha}} \approx 0. \qquad (3.47)$$

Just as was done in the N=2 case, one can fix some of the SU(N) gauge freedom by rotating the traceless part of X_1 to the SU(N) Cartan sub algebra. The freedom to rotate around the Cartan generators is not fixed and this choice corresponds to only a partial gauge fixing. This can be done by imposing the gauge fixing constraint $p_{\vec{\alpha}} \approx 0$. It follows from (3.47) that all $q_{\vec{\alpha}}$ also vanish. The gauge fixing conditions turn the first class constraints (3.46, 3.47) into second class constraints, and the Dirac bracket between the remaining degrees of freedom (q_0, p_0, q_a, p_a) is computed to be the same as in (3.43, 3.44). Explicitly, the Dirac bracket is given by

$$\{A, B\}_{DB} = \{A, B\}_{PB} + \sum_{\vec{\alpha}} \frac{1}{\sqrt{2}\alpha_a p_a}(\{A, \phi_{\vec{\alpha}}\}\{p_{\vec{\alpha}}, B\} - \{B, \phi_{\vec{\alpha}}\}\{p_{\vec{\alpha}}, A\}). \qquad (3.48)$$

Now the phase space has been reduced to 2N dimensions, but there is still the Diff_0 constraint to deal with. This constraint is again written as $q_0^2 + p_0^2 \approx \vec{q}^{\,2} + \vec{p}^{\,2} + d_0$ where $\vec{q}$ and $\vec{p}$ are now N-1 dimensional vectors. After applying this Diff_0 constraint and fixing the gauge so that $q_0 \approx 0$, there are 2(N-1) independent degrees of freedom. The Dirac bracket (3.43, 3.44) is once again preserved by this gauge fixing.

3.1.3 Degeneracy

The aim is to relate the degeneracy of the eigenvalues of the Hamiltonian $\vec{q}^{\,2} + \vec{p}^{\,2}$ to the density of states of the BTZ black hole. For the case N=2 there is no degeneracy; there is one eigenvalue for each state of the oscillator.

In the case N $>$ 2, $q_0^2 + p_0^2 \approx p_0^2 \approx \vec{q}^{\,2} + \vec{p}^{\,2} + d_0$ is a sum of N-1 harmonic oscillator Hamiltonians, and there will be a degeneracy in the eigenvalues. The operator analogue of (3.33) is written in terms of the number operators of each of the N-1 harmonic oscillators, $n_a = 0, 1, 2....$ Then the eigenvalues are $2\sum_{a=1}^{N-1} n_a + N - 1 + d_0$ and (3.38) becomes:

$$\frac{2\theta_0}{k}\left(\sum_{a=1}^{N-1} n_a + \frac{N-1+d_0}{2}\right). \tag{3.49}$$

Since the eigenvalues of (3.38) and (3.49) should be the same, we can find a relation which sets the value of the parameter d_0.

$$n = \sum_{a=1}^{N-1} n_a, \quad d_0 = 2 - N. \tag{3.50}$$

The degeneracy of the nth excited state is the same as that for the nth excited state of an isotropic N-1 dimensional harmonic oscillator, which is defined here in terms of bosonic raising and lowering operators. Then, the degeneracy of the nth excited state is what one would have from considering n identical bosons occupying N-1 positions. This is equivalent to partitioning a number, n, into parts in which no part can be greater than N-1. This is known to be defined as follows [[55]]

$$g_N^{(N)} = \sum_{k=1}^{N-1} p(n,k). \tag{3.51}$$

If we define the asymptotic limit as the size of the matrices $N \to \infty$ as well as the nth excited state $n \to \infty$, then the degeneracy is given by the Hardy-Ramanujan formula [56]:

$$g_N^{(N)} \to \frac{1}{4n\sqrt{3}} \exp{(\pi\sqrt{\frac{2n}{3}})}. \tag{3.52}$$

The entropy is related to the log of the number of miscrostates in a system. After taking the log and using the definition for n from (34), we find that the entropy of the nth excited state in the asymptotic limit is given by

$$S_n \sim \pi\sqrt{\frac{kI_n^{(2)}}{3\theta_0}}. \tag{3.53}$$

We now want to identify the quadratic invariant, (3.33), with the squared radius of the black hole, r_+^2 and the two constants of the theories: $\frac{k}{\theta_0}$ and $\frac{3}{4G^2}$. From this we obtain the usual entropy (3.1). This identification also sets the scale for the eigenvalues of (3.33), and hence r_+^2. It says that they are separated by $\frac{8}{3}G^2$, and that the smallest value for the horizon radius is $\frac{2}{\sqrt{3}}G$.

4 MATRIX MODEL COSMOLOGY

Solutions to the equations of motion of general relativity are known to contain spacetime singularities, points at which the metric becomes infinite. Often these singularities are due solely to the choice of the coordinate system and therefore do not have a physical meaning. In order to determine if the solution is regular at some point, one may construct diffeomorphism invariant quantities and analyze their behavior at this point. A spacetime can also be characterized as containing singularities (which are not due to the choice of coordinates) if it is geodesically incomplete. This means that the motion of a freely falling particle cannot be determined after a finite time upon reaching the singularity. One such singularity is the causal singularity occuring in the classical version of the Big Bang model. At the hypothetical time $t = 0$, timelike geodesics can not be extended to the past. Moreover, the universe at this time is believed to have zero spatial dimension, infinite density, infinite temperature and infinite spacetime curvature. It is believed that upon including quantum gravitational effects, these singularities can be resolved.

The standard models of cosmology are referred to as Friedman-Lemaitre-Walker (FLRW) models. These models are constructed to represent a spatially homogeneous and isotropic universe which can either be contracting or expanding. In polar coordinates, the metric is given by

$$ds^2 = -c^2dt^2 + a(t)^2\frac{dr^2}{1-kr^2} + r^2d\Omega^2, \quad \text{where } d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2. \tag{4.1}$$

k represents the curvature of the uniform 3-dimensional space that can in principle be either positive, negative or zero: k=0 corresponds to spatially flat solutions, $k > 0$ gives spatially closed solutions and k <0 gives hyperbolic 3-geometries. The spatial component of (4.1) is allowed to be time dependent in which all such dependence is contained in the dimensionless scale factor, $a(t)$. These models contain an overall six dimensional isometry group resulting from the conditions that the universe be completely spatially homogeneous and isotropic. The isotropic condition requires the spatial directions to contain an O(3) rotational symmetry. The spatially homogeneous condition requires that each spacelike 3-surface at constant time be invariant under translations in three dimensions. Einstein's field equations are used in order to find the explicit time dependence of the scale factor. In particular, given a mass density, pressure, and cosmological constant, (ρ, p, Λ, respectively) they lead to the Friedmann equations: $\frac{\dot{a}^2+kc^2}{a^2} = \frac{8\pi G\rho+\Lambda c^2}{3}$ and $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3}$, where dots represent derivatives with respect to time. Current observations indicate that the universe is best described by a uniformly expanding four-dimensional manifold which is spatially flat at each time slice.

Matrix models have been shown to contain gravity with spacetime geometry emerging dynamically, and solutions to these matrix models have been shown to include black holes and cosmological spacetimes [58],[59],[12],[60],[61],[62],[63],[64],[65]. As was shown in the previous chapter, matrix models are capable of describing black hole entropy which is a phenomena arising only once quantum mechanics is taken into consideration along with general relativity. Additionally, since these matrix models are related to nonperturbative effects in string theory, these models have promise to include these effects in the cosmological solutions and potentially contain a mechanism for controlling the classical singularities. A Newtonian type cosmology was shown to result from the BFSS matrix model [80]. This model, however, lacks manifest Lorentz invariance. We instead prefer to begin with an IKKT type model in which time and space are initially treated on the same footing and a covariant theory can result in the continuum limit. Being a definition of superstring theory, the IKKT model is originally constructed as a ten dimensional model. It has been shown that the rotational invariance of the nine spatial dimensions are spontaneously broken to SO(3), which acts with the defining representation on three spatial dimensions [64]. This gives us justification to study the model in less than ten dimensions.

Our aim is to first study rotationally invariant spacetime cosmologies emerging as solutions in the commutative limit of matrix models in d dimensions. For this, we must define the notion of rotational invariance in for the matrix solutions. The resulting solutions to the equations of motion will then be d-1 dimensional. As studying lower dimensional toy models proved to be fruitful in the previous chapter, we will again begin by considering general solutions with a two-dimensional commutative limit. Afterward we restrict our attention to those rotationally invariant models possessing four-dimensional solutions. In particular, we will examine the Lorentzian analog of the complex projective plane which arises as a four dimensional cosmological solution of an eight-dimensional action. In each case, we will be only concerned with the bosonic sector of IKKT type matrix models.

In general, exact matrix solutions are nontrivial to obtain, irregardless of the dimension. Obvious solutions are those defining a finite dimensional Lie algebra such as the fuzzy sphere for the three dimensional action. We will be able to study generalized classes of invariant solutions by expressing the matrix equations of motion in terms of recursion relations. In general we do not find exact solutions to the matrix model, but are able to interpret solutions in the commutative limit. We are able to find solutions in the commutative limit which resemble open, closed and static cosmologies. Such solutions are always endowed with a Poisson structure. Most impressively, we are able to find solutions which resolve cosmological singularities.

4.1 Three dimensional case

We first examine the bosonic sector of a Lorentzian IKKT-type matrix model in three spacetime dimensions. The dynamics for the three infinite-dimensional Hermitean matrices Y^μ, $\mu = 0, 1, 2$ is determined from the action

$$S(Y) = \frac{1}{g^2}\mathrm{Tr}\Big(-\frac{1}{4}[Y_\mu, Y_\nu][Y^\mu, Y^\nu] - \frac{2}{3}i\alpha\epsilon_{\mu\nu\lambda}Y^\mu Y^\nu Y^\lambda\Big) \,, \tag{4.2}$$

where a totally antisymmetric cubic term is added to the standard Yang-Mills term and α and g are constants. Our conventions are $\epsilon_{012} = 1$, and we raise and lower indices with the flat metric $\eta_{\mu\nu}$ =diag$(-1, 1, 1)$. The resulting equations of motion are

$$[[Y_\mu, Y_\nu], Y^\nu] - i\alpha\epsilon_{\mu\nu\lambda}[Y^\nu, Y^\lambda] = 0. \tag{4.3}$$

They are invariant under:

i) Lorentz transformations $Y^\mu \to L^\mu{}_\nu Y^\nu$, where L is a 3×3 Lorentz matrix,

ii) translations in the three-dimensional Minkowski space $Y^\mu \to Y^\mu + v^\mu \mathbb{1}$, where $\mathbb{1}$ is the unit matrix, and

iii) unitary 'gauge' transformations, $Y^\mu \to UY^\mu U^\dagger$, where U is an infinite dimensional unitary matrix.

We seek matrix solutions which are rotationally invariant in the 1-2 plane. In order to define a notion of rotational invariance for the matrix model solutions, we seek solutions which reduce to rotationally invariant solutions in the commutative limit. The commutative limit of a matrix model is defined in analogous fashion to the classical limit of a quantum system as described in the introductory section. This limit is taken by replacing the matrices Y^μ with commuting spacetime coordinates y^μ. y^0 and y^i, $i = 1, 2$ can be regarded as time and space coordinates, respectively. In addition, one replaces the commutator of functions of Y^μ by some Poisson bracket $\{\ ,\ \}$ of the corresponding functions of y^μ. For this one introduces a noncommutativity parameter Θ, and defines the commutative limit by $\Theta \to 0$. To lowest order in Θ, $[\mathcal{F}(Y), \mathcal{G}(Y)] \to i\Theta\{\mathcal{F}(y), \mathcal{G}(y)\}$ for arbitrary functions $\mathcal{F}$ and $\mathcal{G}$. In $2+1$ dimensions this is easily defined. The dynamical degrees of freedom in this case are contained in three infinite-dimensional Hermitean matrices Y^μ, $\mu = 0, 1, 2$, with 0 being the time index and 1 and 2 being spatial indices.

We can take rotationally invariant matrix configurations for (Y^0, Y^1, Y^2) to be those satisfying

$$[Y_+ Y_-, Y^0] = 0 \ , \tag{4.4}$$

where

$$Y_\pm = Y^1 \pm i Y^2. \tag{4.5}$$

In the commutative limit (4.4) goes to

$$\{(y^1)^2 + (y^2)^2, y^0\} = 0. \tag{4.6}$$

In order for this Poisson bracket to be satisfied, the spatial radius must be a function of only the time y^0 coordinate

$$(y^1)^2 + (y^2)^2 = a^2(y^0) \ , \tag{4.7}$$

which defines a rotationally invariant manifold embedded in three spacetime dimensions.

The commutative limit of the matrix equations was examined previously in [73] and a family of rotationally invariant solutions were obtained. The Poisson brackets on the three dimensional space spanned by y^μ are singular, and a function of the coordinates can be found which is central in the Poisson bracket algebra. Setting that function equal to a constant yields a two dimensional surface $\mathcal{M}_2$, upon which a nonsingular Poisson bracket can be defined. Similar arguments can be made to recover an even dimensional manifold starting with a d =odd dimensional matrix model. Say that τ and σ parametrize the two-dimensional surface, where τ is a time-like parameter and σ is space-like. We will assume that any time slice of $\mathcal{M}_2$ is a circle, $0 \leq \sigma < 2\pi$. In terms of the three embedding coordinates the surface is defined by the functions $y^\mu = y^\mu(\tau, e^{i\sigma})$. Since $\mathcal{M}_2$ is a two-dimensional surface the Jacobi identity is automatically satisfied, and for any two functions $\mathcal{F}(\tau, e^{i\sigma})$ and $\mathcal{G}(\tau, e^{i\sigma})$ on $\mathcal{M}_2$ we can write

$$\{\mathcal{F}, \mathcal{G}\}(\tau, e^{i\sigma}) = h\left(\partial_\sigma \mathcal{F} \partial_\tau \mathcal{G} - \partial_\tau \mathcal{F} \partial_\sigma \mathcal{G}\right), \tag{4.8}$$

where in general h is some function of τ and $e^{i\sigma}$.

Since the matrix model action (4.2) and the equations of motion (4.3) can be expressed in terms of commutators, their commutative limit can be expressed in terms of Poisson brackets. In order that all terms survive in the commutative limit, we need that α vanishes in the limit, more specifically, that it is proportional to Θ. We write as $\alpha \to +\upsilon\Theta$, with υ finite. Then the commutative limit of the action is

$$S_c(y) = \frac{1}{g_c^2} \int_{\mathcal{M}_2} d\mu(\tau, \sigma) \Big(\frac{1}{4}\{y_\mu, y_\nu\}\{y^\mu, y^\nu\} + \frac{\upsilon}{3} \epsilon_{\mu\nu\lambda}\, y^\mu \{y^\nu, y^\lambda\} \Big), \tag{4.9}$$

where g_c is the commutative limit of the coupling g and $d\mu(\tau, \sigma)$ is the integration measure on $\mathcal{M}_2$. The latter is required to be consistent with the cyclic trace identity,

$$\int_{\mathcal{M}_2} d\mu(\tau, \sigma) \{\mathcal{F}, \mathcal{G}\} \mathcal{H} = \int_{\mathcal{M}_2} d\mu(\tau, \sigma) \mathcal{F} \{\mathcal{G}, \mathcal{H}\}, \tag{4.10}$$

for arbitrary functions $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ on $\mathcal{M}_2$. We can then take

$$d\mu(\tau, \sigma) = d\tau d\sigma / h. \tag{4.11}$$

The commutative limit of the equations of motion (4.3) is given by

$$\{\{y_\mu, y_\nu\}, y^\nu\} - \upsilon\epsilon_{\mu\nu\rho}\{y^\nu, y^\rho\} = 0. \tag{4.12}$$

The dynamics retains its invariance under i) Lorentz transformations, ii) translations and iii) gauge transformations. Infinitesimal gauge variations have the form $\delta y^\mu = \Theta\{\Lambda, y^\mu\}$, where Θ again denotes the noncommutativity parameter and Λ is an infinitesimal function on $\mathcal{M}_2$.

We denote solutions to the equations of motion by $y^\mu = x^\mu(\tau, e^{i\sigma})$, and focus on solutions with an $SO(2)$ isometry group, associated with rotations in the 1-2 plane. For this we write the ansatz

$$\begin{bmatrix} x^0 \\ x^1 \\ x^2 \end{bmatrix} = \begin{bmatrix} \tau \\ a(\tau)\cos\sigma \\ a(\tau)\sin\sigma \end{bmatrix} . \tag{4.13}$$

Here we have introduced a factor $a(\tau)$ which is the radius at any τ-slice. The ansatz (4.13) is consistent with (4.7). The invariant interval on the surface is

$$ds^2 = -(1 - a'(\tau)^2)\, d\tau^2 + a(\tau)^2\, d\sigma^2 \ , \tag{4.14}$$

the prime denoting differentiation in τ. This gives the Ricci scalar

$$\mathtt{R} = \frac{2a''(\tau)}{a(\tau)\Big(1 - a'(\tau)^2\Big)^2}. \tag{4.15}$$

Rotational invariance in the 1-2 plane requires that we restrict h in (4.8) to being a function of only τ. In order to have a solution to (4.12), the functions a and h need to satisfy

$$\Big((aa'h)' + h - 2\upsilon\Big)\, h = 0 \qquad\qquad \Big(2ha' + ah' - 2\upsilon a'\Big)\, ah = 0. \tag{4.16}$$

From these equations it follows that $h^2\mathtt{g}$ is a constant of integration.$h(\tau)$ can be eliminated by setting $h = \frac{1}{\sqrt{-g}}$ and obtain a second order equation for the scale factor

$$\frac{a''}{a} = \left(\frac{a'}{a}\right)^2 - \frac{1}{a^2} + \frac{2\upsilon}{a}(1 - a'^2)^{\frac{3}{2}}. \tag{4.17}$$

This yields the integral of the motion

$$\mathcal{E} = a/\sqrt{1 - a'^2} - \upsilon a^2 \tag{4.18}$$

which was shown in [73] to be associated with the energy of a bosonic string. We obtain the following Friedmann-type equation for $a(\tau)$:

$$\left(\frac{a'}{a}\right)^2 - \frac{1}{a^2} = -\frac{1}{(\mathcal{E} + \upsilon a^2)^2}\ . \tag{4.19}$$

Solutions to (4.19) can be expressed in terms of inverse elliptic integrals, which are plotted in figure 1. For all the solutions plotted there (except the limiting case of $\upsilon = \frac{1}{2}$) we assume that a has a turning point at $\tau = 0$, i.e., $a(0) = 1$ and $a'(0) = 0$. The solutions describe closed, stationary and open spacetimes, the choice depending on values for υ [73]. Closed two-dimensional spacetimes, having initial and final singularities, occur for $\upsilon < \frac{1}{2}$. The limiting case of $\upsilon = \frac{1}{2}$ gives the static or cylindrical spacetime solution [72]. Open universe solutions correspond to $\upsilon > \frac{1}{2}$.

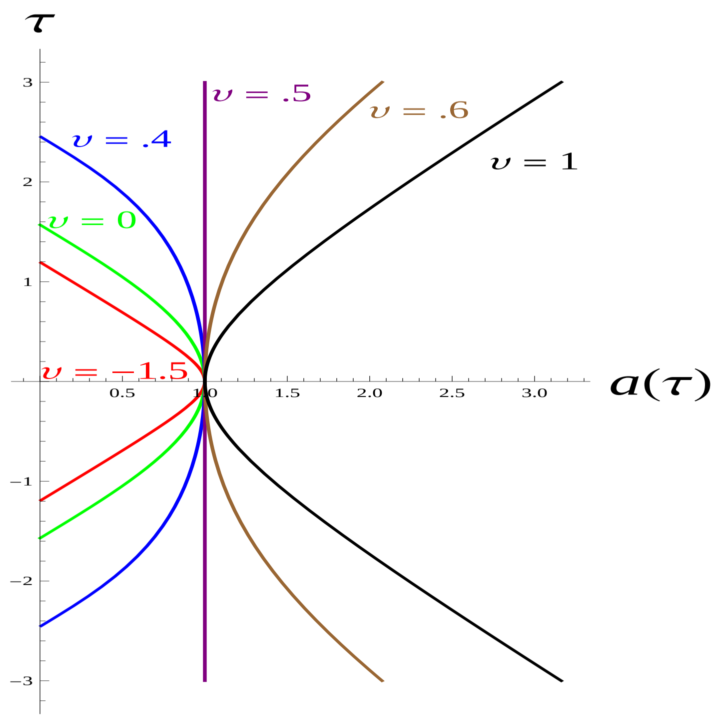

Figure 4.1: Numerical solution to (4.17) for $\upsilon = -1.5, 0, .4, .5, .6$ and 1. $\upsilon = .5$ and 1 correspond to the cylinder and de Sitter solutions, respectively. The boundary values are $a(0) = 1$ and $a'(0) = 0$.

Exact expressions for the solutions exist for different values of υ. They are:

a) For the case of $\upsilon = 0$, one has the simple expression

$$a(\tau) = \cos\tau \ , \qquad -\frac{\pi}{2} \le \tau \le \frac{\pi}{2} \ , \tag{4.20}$$

where once again we assumed $a(0) = 1$ and $a'(0) = 0$, which leads to singularities at $\tau = \pm\frac{\pi}{2}$. It defines the surface $(x^1)^2 + (x^2)^2 = \cos^2(x^0)$, with metric given by

$$ds^2 = \cos^2\tau\,(-d\tau^2 + d\sigma^2) \ , \tag{4.21}$$

and Ricci curvature $\mathtt{R} = -2\sec^4(\tau)$, the latter being singular at $\tau = \pm\frac{\pi}{2}$. The Poisson brackets of the embedding coordinates are

$$\{x^0, x^1\} = \frac{x^2}{\cos^2(x^0)} \qquad \{x^0, x^2\} = -\frac{x^1}{\cos^2(x^0)} \qquad \{x^1, x^2\} = \tan(x^0)\ , \tag{4.22}$$

where we used (4.8) and $h = \frac{1}{\sqrt{-g}}$.

b) For $\upsilon = \frac{1}{2}$ the solution is simply

$$a = 1. \tag{4.23}$$

The manifold is just a cylinder of unit radius with a flat metric tensor

$$ds^2 = -d\tau^2 + d\sigma^2. \tag{4.24}$$

Using (4.8) and $h = \frac{1}{\sqrt{-g}}$ one now gets the Poisson brackets

$$\{x^0, x^1\} = x^2 \qquad \{x^0, x^2\} = -x^1 \qquad \{x^1, x^2\} = 0\ , \tag{4.25}$$

which define the three-dimensional Euclidean algebra.

c) When $\upsilon = 1$ one gets

$$a(\tau)^2 = 1 + \tau^2\ , \tag{4.26}$$

corresponding to a de Sitter spacetime,

$$(x^1)^2 + (x^2)^2 - (x^0)^2 = 1. \tag{4.27}$$

The invariant measure is given by

$$ds^2 = -\frac{d\tau^2}{a(\tau)^2} + a(\tau)^2 d\sigma^2\ , \tag{4.28}$$

corresponding to yielding a constant positive Ricci curvature $\mathtt{R} = 2$. The Poisson brackets on the surface define the $su(1,1)$ algebra

$$\{x^0, x^1\} = x^2 \qquad \{x^0, x^2\} = -x^1 \qquad \{x^1, x^2\} = -x^0 \ . \tag{4.29}$$

Since the Poisson brackets for solutions *b)* and *c)* define Lie algebras, their noncommutative analogues are easy to obtain. One simply replaces the Poisson brackets by commutation relations. With the exception of these two cases, obtaining the matrix analogues of classical solutions is nontrivial. We give a procedure for finding 'rotationally invariant' matrix solutions in the following subsections.

4.1.1 Rotational Invariance

Here we search for matrix analogues of the rotationally invariant solutions of the previous section to the commutative equations of motion (4.12). Our aim is to obtain the spectra of the matrices which solve the equations, which then give lattice versions of the commutative solutions depicted in the plots in figure 4.1. After first defining the meaning of rotational invariance for the matrices, we obtain recursion relations for the spectra. Exact solutions to the recursion relations are discussed and additional remarks concerning finite dimensional solutions are made.

As a preliminary step it is convenient to write down an alternative expression for the commutative solutions of section 4.1. For this we utilize a different parametrization of the two dimensional manifolds. We replace τ by some other time coordinate t, which along with σ, satisfies the fundamental Poisson bracket

$$\{\sigma, t\} = 1 \ , \tag{4.30}$$

which has a simple noncommutative extension. The previous commutative solutions can now be written as $y^\mu = x^\mu(t, e^{i\sigma})$. We then regard x^0 and the scale factor, which we now denote by $\tilde{a}$, as functions of t, thereby replacing (4.13) with

$$\begin{bmatrix} x^0 \\ x^1 \\ x^2 \end{bmatrix} = \begin{bmatrix} x^0(t) \\ a(t)\cos\sigma \\ a(t)\sin\sigma \end{bmatrix} \tag{4.31}$$

Then the equations of motion (4.12) give

$$2\partial_t\tilde{a}\,(\partial_t x^0 - \upsilon) + \tilde{a}\,\partial_t^2 x^0 = 0 \qquad (\partial_t x^0 - \upsilon)^2 - \upsilon^2 + \partial_t(\tilde{a}\partial_t\tilde{a}) = 0 \tag{4.32}$$

The first equation implies that

$$k = \tilde{a}^2\,(\partial_t x^0 - \upsilon) \tag{4.33}$$

is independent of t. The second equation then says that the dynamics $\tilde{a}^2$ is determined by a simple force equation:

$$\frac{1}{2}\partial_t^2(\tilde{a}^2) = -\frac{k^2}{\tilde{a}^4} + \upsilon^2 \tag{4.34}$$

The solutions are characterized by k and the conserved 'energy' $\frac{1}{4}\Big(\partial_t(\tilde{a}^2)\Big)^2 - \frac{k^2}{\tilde{a}^2} - \upsilon^2\tilde{a}^2$. They are, of course, equivalent to those found previously. To see this one only needs to apply the reparametrization $t \to \tau = x^0(t)$.

Using (4.5) to define $Y_\pm$, the equations (4.3) can be written according to

$$\begin{aligned} [Y_+,[Y_-,Y^0]] + \frac{1}{2}[Y^0,[Y_+,Y_-]] \;-\; \alpha[Y_+,Y_-] \;&=\; 0 \\ [Y^0,[Y^0,Y_-]] + \frac{1}{2}[Y_-,[Y_-,Y_+]] + 2\alpha[Y^0,Y_-] \;&=\; 0 \end{aligned} \tag{4.35}$$

We wish to write down a rotationally invariant ansatz for the matrices Y^0 and $Y_\pm$ which reduces to (4.31) in the commutative limit. Different definitions are possible. We require that our choice satisfies (4.4). Our ansatz shall be expressed in terms of functions of two infinite dimensional matrices $\hat{t}$ and $e^{i\hat{\sigma}}$, and are the matrix analogues of t and $e^{i\sigma}$, respectively. The former is hermitean and the latter is unitary. The matrix analogue of the Poisson bracket (4.30) is the commutation relation

$$[e^{i\hat{\sigma}}, \hat{t}] = -\Delta e^{i\hat{\sigma}} \ , \tag{4.36}$$

where Δ is a central element with units of time which is assumed to be linear in the noncommutative parameter. $e^{i\hat{\sigma}}$ generates time translations $\hat{t} \to \hat{t} + \Delta$. Together $e^{i\hat{\sigma}}$ and $\hat{t}$ generate the algebra of the noncommutative cylinder [91],[92],[72].

For solutions to (4.35), which we denote by $Y^\mu = X^\mu$, we take

$$X_+ = X^1 + iX^2 = A(\hat{t})e^{i\hat{\sigma}} \qquad\qquad X^0 = X^0(\hat{t}). \tag{4.37}$$

This is consistent with our definition (4.4) of rotation invariance. Here we restrict X^0 and A to being real polynomial functions of $\hat{t}$. Then $A(\hat{t})$ and $X^0(\hat{t})$ are infinite dimensional hermitean matrices. In the commutative limit, the ansatz (4.37) agrees with the expression (4.31). After substituting the ansatz into (4.35) one gets

$$\Big(X^0(\hat{t}) - X^0(\hat{t}-\Delta) - \alpha\Big)A(\hat{t})^2 - \Big(X^0(\hat{t}+\Delta) - X^0(\hat{t}) - \alpha\Big)A(\hat{t}+\Delta)^2 = 0$$

$$\frac{1}{2}\Big(A(\hat{t}-\Delta)^2 + A(\hat{t}+\Delta)^2 - 2A(\hat{t})^2\Big) + \Big(X^0(\hat{t}) - X^0(\hat{t}-\Delta) - \alpha\Big)^2 - \alpha^2 = 0 \tag{4.38}$$

The first equation states that $\Big(X^0(\hat{t}) - X^0(\hat{t}-\Delta) - \alpha\Big)A(\hat{t})^2$ is invariant under discrete translations $\hat{t} \to \hat{t} + n\Delta$, $n =$integer, and is the matrix analogue of (4.33).

We next write down recursion relations for the eigenvalues of the matrices $X^0(\hat{t})$ and $A(\hat{t})$. The spectrum for the operator $\hat{t}$ is discrete, with equally spaced eigenvalues

$$t_n = t_0 - n\Delta \ , \qquad n \in Z \ , \tag{4.39}$$

where t_0 is real. This follows since from the commutation relations (4.36), $e^{2\pi i\hat{t}/\Delta}$ is a central element. It is a constant phase $e^{2\pi i t_0/\Delta}\mathbb{1}$ in any irreducible representation of the algebra, from which (4.39) results.

The eigenvalues of $X^0(\hat{t})$ and $A(\hat{t})$ are real and we denote them by

$$x_n^0 = X^0(t_n) \qquad\qquad a_n = A(t_n). \tag{4.40}$$

From (4.37), X_+ and X_- act as lowering and raising operators, respectively, on the corresponding eigenvectors. Since the eigenvalues of $X_1^2 + X_2^2 = \frac{1}{2}(X_+X_- + X_-X_+)$ are positive definite, we get that $a_n^2 + a_{n-1}^2 \geq 0$, for all n. However, since we want $A(\hat{t})$ to be hermitean, we get the stronger condition that

$$a_n^2 \geq 0\ , \tag{4.41}$$

for all n. From the equations of motion (4.38) we get the following recursion relations for the eigenvalues:

$$\begin{gathered}\Big(x_n^0 - x_{n+1}^0 - \alpha\Big)a_n^2 - \Big(x_{n-1}^0 - x_n^0 - \alpha\Big)a_{n-1}^2 = 0\\[2ex] \frac{1}{2}\Big(a_{n+1}^2 + a_{n-1}^2 - 2a_n^2\Big) + \Big(x_n^0 - x_{n+1}^0 - \alpha\Big)^2 - \alpha^2 = 0.\end{gathered} \tag{4.42}$$

From the first equation, $k = \Big(x_n^0 - x_{n+1}^0 - \alpha\Big)a_n^2$ is independent of n, and then from the second equation we get a recursion relation for just a_n

$$\frac{1}{2}\Big(a_{n+1}^2 + a_{n-1}^2 - 2a_n^2\Big) + \frac{k^2}{a_n^4} - \alpha^2 = 0\ , \tag{4.43}$$

which is valid provided a_n doesn't vanish. (4.43) is the lattice version of (4.34). Given the values for any neighboring pair of eigenvalues for A, we can determine the the entire series $\{a_n\}$. Then starting with one time eigenvalue, we can determine all of $\{x^0_n\}$ using $x^0_n - x^0_{n+1} = \alpha + k/a^2_n$.

Solutions are plotted in figure 4.2 for $\alpha = .5, .51$ and $.6$ with $k = .5$ and boundary values $a_0 = a_1 = 1$, $x^0_0 = 0$. $\alpha = .5$ corresponds to the noncommutative cylinder. $\alpha = .51$ and $.6$ are examples of discrete versions of open universe solutions. Another example of a discrete open universe is the noncommutative de Sitter solution which corresponds to $k = 0$.

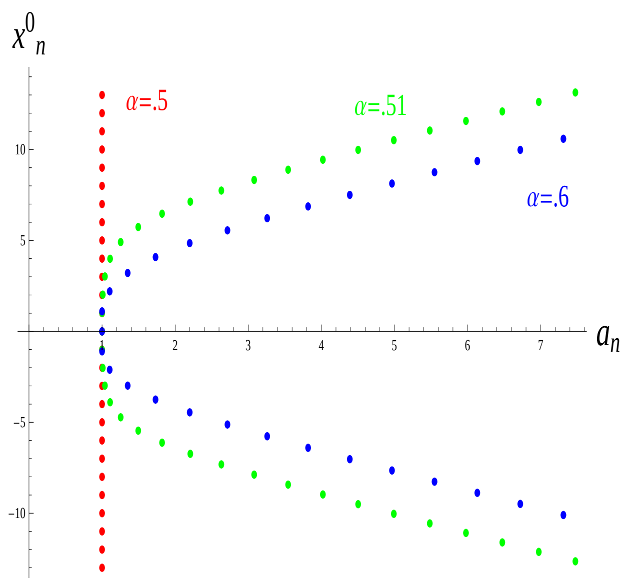

Figure 4.2: Solutions to the recursion relation (4.43) for $\alpha = .5, .51$ and $.6$ with $k = \big(x^0_n - x^0_{n+1} - \alpha\big)a^2_n$ fixed to be $.5$. $\alpha = .5$ corresponds to the noncommutative cylinder solution, while $\alpha = .51$ and $.6$ are examples of noncommutative analogues of open universe solutions. The initial conditions are $a_0 = a_1 = 1$ and $x^0_0 = 0$.

While figure 4.2 shows matrix analogues of the cylindrical and open spacetime solutions, here we are unable to obtain matrix analogues of closed spacetimes. Related to this issue is the absence of solutions having $\alpha < .5$ (or more generally, $|\alpha|< |k|$, along with initial conditions $a_0 = a_1 = 1$). For these cases, a^2_n decreases to zero as one goes away from the initial values. ($a^2_n = 0$ is analogous to zero radius in the continuous case;

i.e., a cosmological singularity.) As a_n^2 decreases to zero, the k^2/a_n^4 term dominates in the recursion relation (4.43), i.e., the leading term in the expression for a_{n+1}^2 goes like $-k^2/a_n^4 < 0$. Then for some n, a_n^2 becomes negative, which is inconsistent with hermiticity. Thus, either such solutions do not exist or there must be raising or lowering operators that kill all states with $a_n^2 < 0$.

We note that the analysis leading to recursion relations (4.42) and (4.43) is only valid for infinite dimensional solutions to the matrix equations, and moreover when the index n spans all positive and negative integers. (4.37) is not valid if this is not the case. Alternatively, if n does not span all positive and negative integers it still may be possible to write

$$X_+ = AU \ , \tag{4.44}$$

where A and U are diagonal and unitary matrices, respectively, and X^0 is a diagonal matrix. This is a generalization of the ansatz (4.37). Both (4.37) and (4.44) imply that X_+X_- commutes with X^0, and so they are consistent with the definition (4.4) of rotational invariance. Then X_+X_- and X^0 have common eigenvalues. In the case where (4.37) holds they are, respectively, a_n^2 and x_n^0. Even if (4.37) does not hold, it may still be possible that the recursion relations (4.42) for the eigenvalues x_n^0 and a_n^2 of X^0 and A^2, respectively, are valid after restricting the values of the label n in some fashion.

There are well known examples of rotationally invariant matrix model solutions which are exact solutions of the recursion relations (4.43). For example, a trivial solution of the recursion relation (4.43) is

$$x_n^0 = -2\alpha n + x_0^0 \qquad\qquad a_n = a_0 \tag{4.45}$$

where $k = \alpha a_0^2$ and $n \in Z$. x_0^0 and a_0 are real and here are identified with eigenvalues of X^0 and A for the noncommutative cylinder. The solution represents the discrete version of the constant solution for $a = 1$ in (4.23). The noncommutative cylinder solution $Y^\mu = X^\mu$ is defined by the commutation relations

$$[X^0, X_+] = 2\alpha X_+ \qquad\qquad [X_+, X_-] = 0 \;, \tag{4.46}$$

from which one recovers Poisson brackets (4.25) in the limit $\alpha \to 0$. x^0_n and a_n in (4.45) are the eigenvalues, respectively, of X^0 and the square root of X_+X_-, which is central in the algebra. The latter is constant in any irreducible representation of the algebra and is the radius-squared of the noncommutative cylinder. X_+ and X_- are raising and lowering operators, respectively, for the eigenvectors of X^0.

4.1.2 The question of finite dimensional solutions

One can ask whether or not there exist nontrivial finite dimensional matrix solutions of the equations of motion (4.3). This question is relevant for knowing whether or not there are matrix solution analogues of the closed spacetime cosmologies. The latter are expected to emerge upon taking the $N \to \infty$ limit of the $N \times N$ matrix solutions, along with initial and final singularities on the resulting spacetime manifold. So for example, one can ask if there is a matrix analogue of the closed spacetime solution (4.20) of the commutative equations of motion (4.12).

As stated above, if a_n^2 tends to zero, it becomes necessary to terminate the series generated by the recursion relations (4.43) in order to prevent a_n^2 from becoming negative. There must then exist a bottom or top state, which would correspond, respectively, to an initial or final singularity in the continuum limit. Any matrix analogue of a closed spacetime solution must have *both* a bottom and top state, and thus the matrix solution should be finite dimensional. In this regard, we have not been able to find any nontrivial finite dimensional matrix solutions to (4.3), and thus here we do not have matrix model analogues of the closed spacetime solutions of subsection 2.1; i.e., all the solutions of (4.17) with $\upsilon < \frac{1}{2}$.

Although we do not have a proof that there are no nontrivial $N \times N$ solutions, for arbitrary finite N, to the matrix equations (4.3), it is easy to show that no nontrivial solutions exist for the simplest case of $N = 2$. In that case we can set X^μ equal to a linear combination of Pauli matrices, one of which, say X^0, we can take up to a factor to be σ_3. (Terms in X^μ which are proportional to the identity matrix trivially solve the equations of motion.)

$$X^0 = \sigma_3 \qquad X^1 = u_i \sigma_i \qquad X^2 = v_i \sigma_i \ ,$$

where u_i and v_i are real. Here we are not making any additional restrictions such as rotational invariance. Upon substituting into the equations of motion (4.3) one gets

$$u_3 u_i + v_3 v_i - \Big(\vec{u}^2 + \vec{v}^2\Big)\delta_{i3} - \alpha\epsilon_{ijk} u_j v_k = 0$$

$$\vec{u}\cdot\vec{v}\, u_i - v_3\delta_{i3} - \Big(\vec{u}^2 - 1\Big) v_i - \alpha\epsilon_{ij3} u_j = 0$$

$$\vec{u}\cdot\vec{v}\, v_i - u_3\delta_{i3} - \Big(\vec{v}^2 - 1\Big) u_i + \alpha\epsilon_{ij3} v_j = 0$$

The only real solutions are $u_i = u_3\delta_{i3},\ v_i = v_3\delta_{i3}$, but these are trivial solutions since then all X^μ are proportional to σ_3. Thus there are no nontrivial 2×2 matrix solutions of the equations of motion (4.3)

We find that by adding a quadratic term we can find finite dimensional representations of solutions. The total matrix model action becomes

$$S_{\rm total}(Y) \quad = \quad S(Y) + \frac{\beta}{2g^2}{\rm Tr}Y_\mu Y^\mu$$

$$= \frac{1}{g^2}\mathrm{Tr}\Big(-\frac{1}{4}[Y_\mu, Y_\nu][Y^\mu, Y^\nu] - \frac{2}{3}i\tilde{\alpha}\epsilon_{\mu\nu\lambda}Y^\mu Y^\nu Y^\lambda + \frac{\beta}{2}Y_\mu Y^\mu\Big), \qquad (4.47)$$

where β is a real constant and we now denote the coefficient of the cubic term by $\tilde{\alpha}$ in order to distinguish it from the noncommutative parameter α appearing in the rotationally invariant classical solutions of the previous section. The matrix equations of motion now read

$$[[Y_\mu, Y_\nu], Y^\nu] - i\tilde{\alpha}\epsilon_{\mu\nu\lambda}[Y^\nu, Y^\lambda] = -\beta Y_\mu. \qquad (4.48)$$

The i) $2+1$ Lorentz symmetry of the background space, as well as iii) the unitary gauge symmetry, is preserved by the last term, but ii) translation symmetry is broken when $\beta \neq 0$.

This system contains new solutions, as well as some of the previous solutions (even when $\beta \neq 0$). Before discussing the matrix solutions, we first examine solutions in the commutative limit of the matrix model.

4.1.3 Commutative limit with quadratic term

We first write down the modification of the commutative equations of subsection 2.1. Now a much larger family of solutions exist. We shall give a (mostly) qualitative discussion of these solutions.

The commutative limit of the matrix model action can once again be expressed using the Poisson bracket (4.8) on some two-dimensional manifold $\mathcal{M}_2$. In order for the cubic term in the action to survive in the limit we again need for its coefficient to be linear in the noncommutativity parameter Θ, i.e., $\tilde{\alpha} \to +\upsilon\Theta$. The quadratic term in the action will survive in the limit provided that β goes like Θ^2, i.e., $\beta \to \omega\Theta^2$, with ω finite. Then (4.9) is replaced by

$$S_c(y) = \frac{1}{g_c^2}\int_{\mathcal{M}_2} d\mu(\tau,\sigma)\Big(\frac{1}{4}\{y_\mu, y_\nu\}\{y^\mu, y^\nu\} + \frac{\upsilon}{3}\epsilon_{\mu\nu\lambda}\, y^\mu\{y^\nu, y^\lambda\} + \frac{\omega}{2}y_\mu y^\mu\Big) \qquad (4.49)$$

where $d\mu(\tau,\sigma) = d\tau d\sigma/h$ is once again the invariant integration measure on $\mathcal{M}_2$. Not surprisingly, translational invariance is broken when $\omega \neq 0$. The resulting equations of motion are now

$$\{\{y_\mu, y_\nu\}, y^\nu\} - \upsilon\epsilon_{\mu\nu\rho}\{y^\nu, y^\rho\} = \omega y_\mu. \tag{4.50}$$

Upon substituting the rotationally invariant expression (4.13) into (4.50), we get the following equations for a and h, both of which are assumed to be functions of only τ

$$(aa'h)'h + h^2 - 2\upsilon h = \omega \qquad 2h(h-\upsilon)a'a + a^2h'h = \omega\tau, \tag{4.51}$$

the prime again denoting a derivative in τ. It follows that $h^2\mathsf{g} - \omega(a^2-\tau^2)$ is a constant of integration, where once again g is the determinant of the induced metric. We then get an explicit formula for h, and hence the integration measure

$$h = \sqrt{\frac{c_1 - \omega(a^2-\tau^2)}{(1-a'^2)a^2}}, \tag{4.52}$$

c_1 being the constant of integration. Here we see that the measure is not simply expressed in terms of the metric, except for the case $\omega = 0$ where we recover the result $h = \frac{1}{\sqrt{-g}}$ of the previous section. For all ω and υ, h can be eliminated from the differential equation for a, which can be written

$$a'' - \frac{a'^2}{a} + \frac{1}{a} - \frac{2\upsilon}{ah}(1-a'^2) - \frac{\omega}{h^2}\Big(\frac{\tau}{a}\Big)' = 0 \tag{4.53}$$

generalizing (4.17). The breaking of time translation symmetry when $\omega \neq 0$ implies the absence of a conserved energy, and consistent with that, we have not found a generalization of the quantity (4.18) which is conserved when $\omega = 0$.

There is now a large family of solutions, including those already discussed when $\omega = 0$. Among them are some exact solutions, all of which have h equal to a constant value.

1. There are two distinct dS^2 solutions to (4.51) of the form (4.26) when $\upsilon^2 + 2\omega > 0$ and $\omega \neq 0$. [Here we assume υ and ω are finite.] They yield the following constant values for h,

$$h_\pm = \frac{1}{2}(\upsilon \pm \sqrt{\upsilon^2 + 2\omega}) \; . \tag{4.54}$$

The solution is degenerate when $\upsilon^2 + 2\omega = 0$, and no de Sitter solution exists for $\upsilon^2 + 2\omega < 0$.

2. A dS^2 solution exists to the equations of motion in the limit υ, $\omega \to \infty$, with $\frac{\omega}{\upsilon}$ finite and nonzero. In this limit, the kinetic energy (or Yang-Mills) term is absent from the action (4.49). In this case

$$h = -\frac{\omega}{2\upsilon}. \tag{4.55}$$

If both the kinetic energy term and the quadratic term are absent from the action and only the totally antisymmetric term remains, i.e., $\upsilon \to \infty$ and $\frac{\omega}{\upsilon} \to 0$, then there are only trivial solutions to the equations of motion, $\{x^\mu, x^\nu\} = 0$. In the matrix model, all matrices X^μ commute in this case. This result does not generalize to higher dimensions where one can have nontrivial solutions of the equations of motion when only a totally antisymmetric term appears in the action, as we show in the next section.

3. Another solution, which exists only when $\omega \neq 0$, is a sphere, $(x^1)^2 + (x^2)^2 + (x^0)^2 = 1$, embedded in three-dimensional Minkowski spacetime [75].

The solution represents a closed spacetime cosmology with the initial and final singularity occurring at the latitudes where the spatial radius is not zero. The solution can be expressed in terms of the $su(2)$ Poisson bracket algebra.

In general, solutions to (4.51) are labeled by four independent parameters: υ, ω, c_1 (the integration constant in (4.52)) and the value of a' at some $\tau = \tau_0$. (The value of a at a given τ merely determines the overall scale.) The solutions can be obtained numerically and some novel results appear when $\omega \neq 0$. In figure 4 we plot τ versus $a(\tau)$ for $\upsilon = 0$ and different values of ω. A closed spacetime results for $\omega \leq 1$, a dumbbell shaped curve (with two maxima for $a(\tau)$) appears for $1 < \omega < 2$ and an open spacetime for $\omega \geq 2$.

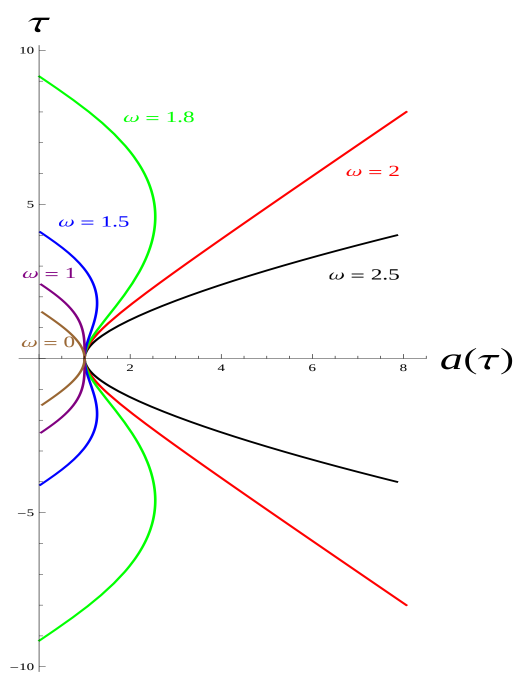

Figure 4.3: Numerical solutions to (4.51) for $\upsilon = 0$ and $\omega = 1,\ 1.5,\ 1.8,\ 2$ and 2.5. The boundary conditions are $a(0) = 1$ and $a'(0) = 0$.

Recursion relations

When $\beta \neq 0$, the right hand sides of the matrix equations of motion (4.35) no longer vanish and substitution of the ansatz (4.37) into the equations of motion (4.48) yields

$$\Big(X^0(\hat{t}) - X^0(\hat{t}-\Delta) - \tilde{\alpha}\Big)A(\hat{t})^2 - \Big(X^0(\hat{t}+\Delta) - X^0(\hat{t}) - \tilde{\alpha}\Big)A(\hat{t}+\Delta)^2 + \beta X^0(\hat{t}) = 0$$

$$\frac{1}{2}\Big(A(\hat{t}-\Delta)^2 + A(\hat{t}+\Delta)^2 - 2A(\hat{t})^2\Big) + \Big(X^0(\hat{t}) - X^0(\hat{t}-\Delta) - \tilde{\alpha}\Big)^2 - \tilde{\alpha}^2 - \beta = 0. \quad (4.56)$$

$\Big(X^0(\hat{t}) - X^0(\hat{t}-\Delta) - \tilde{\alpha}\Big)A(\hat{t})^2$ is not invariant under discrete translations $\hat{t} \to \hat{t} + n\Delta$, n =integer, except for $\beta = 0$. The recursion relations (4.42) for the eigenvalues (4.40) are now generalized to

$$\Big(x^0_n - x^0_{n+1} - \tilde{\alpha}\Big)a^2_n - \Big(x^0_{n-1} - x^0_n - \tilde{\alpha}\Big)a^2_{n-1} + \beta x^0_n = 0$$

$$\frac{1}{2}\Big(a^2_{n+1} + a^2_{n-1} - 2a^2_n\Big) + \Big(x^0_n - x^0_{n+1} - \tilde{\alpha}\Big)^2 - \tilde{\alpha}^2 - \beta = 0. \quad (4.57)$$

Once again, starting with any two neighboring eigenvalues for A^2 and one time eigenvalue x^0_n one can use the recursion relations to generate a matrix solution. As before solutions are only valid providing all $a^2_n \geq 0$. This means that either n spans all positive and negative integers or the series is terminated at some n (and then (4.37) no longer holds).

4.2 Four dimensional solutions

We consider five hermitean matrices Y^μ, with indices $\mu, \nu, .. = 0, 1, 2, 3, 4$, which are raised and lowered with the five dimensional Minkowski spacetime metric η =diag$(-1, 1, 1, 1, 1)$. We propose a five dimensional matrix action

$$S_{\rm total}(Y) = \frac{1}{g^2}{\rm Tr}\Big(-\frac{1}{4}[Y_\mu, Y_\nu][Y^\mu, Y^\nu] + \frac{4}{5}\alpha_5\,\epsilon_{\mu\nu\lambda\rho\sigma}Y^\mu Y^\nu Y^\lambda Y^\rho Y^\sigma + \frac{\beta}{2}Y_\mu Y^\mu\Big)\quad , \qquad (4.58)$$

where $\epsilon_{01234} = 1$ and g, α_5 and β are real constants. The dynamics for Y^μ is invariant under $4 + 1$ Lorentz transformations and unitary gauge transformations, but not translations. The resulting equations of motion are

$$[[Y_\mu, Y_\nu], Y^\nu] + 4\tilde{\alpha}_5\,\epsilon_{\mu\nu\lambda\rho\sigma}Y^\nu Y^\lambda Y^\rho Y^\sigma = -\beta Y_\mu. \qquad (4.59)$$

A fuzzy four sphere solution has been found for these equations when $\beta = 0$ and the background metric is Euclidean [67], [69],[70]. We propose that an analogous construction should be possible in Minkowski spacetime to obtain a noncommutative four dimensional de Sitter space. While other nontrivial solutions to this model are not known, we can show that large families of solutions exist after taking the commutative limit of this matrix model.

In order to take the commutative limit, it is convenient to write the fifth order term in the trace (4.58) using the commutator, $\frac{1}{5}\alpha_5\,\epsilon_{\mu\nu\lambda\rho\sigma}Y^\mu[Y^\nu, Y^\lambda][Y^\rho, Y^\sigma]$. Then we take a commutative limit in the usual way: replace matrices Y^μ by coordinate functions y^μ, now defined on a four dimensional manifold $\mathcal{M}_4$, and matrix commutators by $i\Theta$ times the corresponding Poisson bracket on $\mathcal{M}_4$ where Θ denotes the noncommutativity parameter. If all three terms in the action are to survive in the limit we need that α_5 goes to a finite value, $\alpha_5 \to \upsilon_5$ and, as before, $\beta \to \omega\Theta^2$, with υ_5 and ω finite. The limiting action in that case is

$$S_c(y) = \frac{1}{g_c^2}\int_{\mathcal{M}_4} d\mu_4\Big(\frac{1}{4}\{y_\mu, y_\nu\}\{y^\mu, y^\nu\} - \frac{\upsilon_5}{5}\epsilon_{\mu\nu\lambda\rho\sigma}\, y^\mu\{y^\nu, y^\lambda\}\{y^\rho, y^\sigma\} + \frac{\omega}{2}y_\mu y^\mu\Big)\ , \quad (4.60)$$

where $d\mu_4$ denotes the invariant integration measure, on $\mathcal{M}_4$. Variations of y^μ give the equations of motion

$$\{\{y_\mu, y_\nu\}, y^\nu\} + \upsilon_5\epsilon_{\mu\nu\lambda\rho\sigma}\{y^\nu, y^\lambda\}\{y^\rho, y^\sigma\} = \omega y_\mu. \quad (4.61)$$

Infinitesimal gauge variations have the form $\delta y^\mu = \Theta\{\Lambda, y^\mu\}$, where Λ is an infinitesimal function on $\mathcal{M}_4$.

We denote coordinates on a local patch of $\mathcal{M}_4$ by τ, σ and ξ^i, $i, j, ... = 1, 2, 3$. τ is time-like, while σ and ξ^i are space-like, the latter spanning a unit two-sphere $(\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 = 1$. A rotationally invariant ansatz for solutions $y^\mu = x^\mu(\tau, \sigma, \xi^i)$ to the equations of motion (4.60) is:

$$\begin{bmatrix} x^0 \\ x^i \\ x^4 \end{bmatrix} = \begin{bmatrix} \tau \\ a(\tau)\xi^i \sin\sigma \\ a(\tau)\cos\sigma \end{bmatrix}.$$

The spatial coordinates x^1, x^2, x^3, x^4 span a three-sphere of radius $a(\tau)$ at time slice,

$$\vec{x}^2 + (x^4)^2 = a^2(\tau)\ , \quad (4.62)$$

where $\vec{x}^2 = x^i x^i$, and the isometry group is $SO(4)$. For this one assumes $0 \le \sigma \le \pi$, with $\sigma = 0, \pi$ corresponding to the poles. The invariant interval on the surface is

$$ds^2 = -(1 - a'(\tau)^2)\, d\tau^2 + a(\tau)^2\,(d\sigma^2 + \sin^2\sigma ds^2_{S^2})\ , \quad (4.63)$$

where $ds^2_{S^2}$ is the invariant interval on the two-sphere and $d\sigma^2 + \sin^2\sigma ds^2_{S^2}$ is the invariant interval on the three-sphere. There is a gravitational source present, indicated by a nonvanishing Ricci scalar and Einstein tensor. They are, respectively,

$$\mathtt{R} = \frac{6(1 - a'(\tau)^2 + a(\tau)a''(\tau)}{a(\tau)^2\Big(1 - a'(\tau)^2\Big)^2} \tag{4.64}$$

and

$$\mathtt{G}_{\tau\tau} = \frac{3}{a(\tau)^2} \qquad\qquad \mathtt{G}_{\sigma\sigma} = -\frac{\Big(1 - a'(\tau)^2 + 2a(\tau)a''(\tau)\Big)}{\Big(1 - a'(\tau)^2\Big)^2}$$

$$\mathtt{G}_{\theta\theta} = \sin^2\sigma\ \mathtt{G}_{\sigma\sigma} \qquad\qquad \mathtt{G}_{\phi\phi} = \sin^2\sigma \sin^2\theta\ \mathtt{G}_{\sigma\sigma}\ , \tag{4.65}$$

θ and ϕ being the usual spherical coordinates on the two-sphere.

Since the manifold we are considering is four dimensional, it is possible to define a nonsingular symplectic structure. Although there is no nonsingular Poisson structure on the three sphere, we can write Poisson brackets which are consistent with three dimensional rotation symmetry, i.e., corresponding to rotations among the three spatial coordinates x^i. The fundamental Poisson brackets are

$$\{\sigma, \tau\} = h(\tau, \sigma) \qquad\qquad \{\xi^i, \xi^j\} = \kappa\epsilon_{ijk}\xi^k\ , \tag{4.66}$$

where κ is constant. The Jacobi identity is trivially satisfied. Here we allow for h to be a function of σ as well as τ. This ansatz along with (4.2) will allow for nontrivial solutions to the equations of motion.

Below we examine three different families of solutions to the equations of motion (4.61). In each case only two out of the three terms in the action (4.60) contribute :

1.) We first consider the limiting case where both ω, $\upsilon_5 \to \infty$, with $\frac{\omega}{\upsilon_5}$ and κ finite and nonvanishing. In this limit the first term in the action (4.60) (i.e., the Yang-Mills term) does not contribute. The equations of motion are then

$$\upsilon_5 \epsilon_{\mu\nu\lambda\rho\sigma}\{y^\nu, y^\lambda\}\{y^\rho, y^\sigma\} = \omega y_\mu. \tag{4.67}$$

They are solved by $a(\tau)^2 = \tau^2 + 1$ which defines the four dimensional de Sitter space dS^4

$$-(x^0)^2 + \vec{x}^2 + (x^4)^2 = 1. \tag{4.68}$$

The Poisson structure on this space is determined by the two parameters $\frac{\omega}{\upsilon_5}$ and κ. The solution for $h(\tau, \sigma)$ is

$$h(\tau, \sigma) = -\frac{\omega}{8\kappa\upsilon_5}\frac{\csc^2\sigma}{a(\tau)^2}\ , \tag{4.69}$$

From (4.69), the Poisson brackets of the embedding coordinates x^μ are

$$\{x^0, x^i\} = \frac{\omega}{8\kappa\upsilon_5}\frac{x^i x^4}{(\vec{x}^2)^{\frac{3}{2}}} \qquad \{x^4, x^i\} = \frac{\omega}{8\kappa\upsilon_5}\frac{x^i x^0}{(\vec{x}^2)^{\frac{3}{2}}}$$

$$\{x^0, x^4\} = -\frac{\omega}{8\kappa\upsilon_5}\frac{1}{\sqrt{\vec{x}^2}} \qquad \{x^i, x^j\} = \kappa\sqrt{\vec{x}^2}\,\epsilon_{ijk}x^k. \tag{4.70}$$

It can be checked that the Poisson bracket relations are consistent with the de Sitter space condition (4.68). The Poisson brackets are invariant under the action of the three-dimensional rotation group, although they do not preserve all the isometries of de Sitter space. More specifically, $SO(4,1)$ is broken to $SO(3) \times \mathcal{L}_2$, where $\mathcal{L}_2$ is the two-dimensional Lorentz group.

The Poisson brackets of the coordinate functions x^μ on dS^4 are not associated with a finite dimensional Lie algebra, and so its matrix analogue of the commutative solution is nontrivial. This is in contrast to the ds^2 solution in [74]. By changing the background metric to η =diag$(-1,1,1,1,-1)$ we can obtain a four dimensional anti-de Sitter space and by switching the background metric to η =diag$(1,1,1,1,1)$ we recover a four sphere [74].

2.) Here we set $\kappa = \upsilon_5 = 0$ and take $h = h(\tau)$. Now the second term in the action (4.60) does not contribute to the dynamics. The Poisson brackets are noninvertable in this case, and the equations of motion trivially reduce to the two dimensional system ([74]) with $\upsilon = 0$; i.e. $h(\tau)$ and $a(\tau)$ should satisfy

$$(aa'h)'h + h^2 = \omega \qquad\qquad 2h^2 a'a + a^2 h'h = \omega\tau. \tag{4.71}$$

There is a one parameter (not including integration constants) family of solutions which can be obtained numerically. σ, $0 \le \sigma < 2\pi$, is periodic and parametrizes the longitudes on the three-sphere, where $\sigma = 0$ and π denote the poles and correspond to coordinate singularities.

3.) Finally we consider $\omega = 0$, $\kappa \neq 0$. Now the third term in the action (4.60) does not contribute. If we set

$$h(\tau,\sigma) = f(\tau)\sin^2\sigma \ , \tag{4.72}$$

then the solution to (4.61) with spacetime index $\mu = 0$ is

$$f(\tau) = 2\kappa\upsilon_5 a(\tau)^2 + \frac{c_1}{a(\tau)^2} \ , \tag{4.73}$$

where c_1 is an integration constant. The remaining equations of motion, $\mu = i, 4$, are solved when $a(\tau)$ satisfies

$$a'(\tau)^2 = \left(\frac{\kappa a(\tau)}{f(\tau)}\right)^2 + 1 \tag{4.74}$$

This implies that $|a'(\tau)|$ cannot be less than one. So, for instance, $a(\tau)$ cannot have turning points and there can be no closed space time solutions. Moreover, from (4.63) the induced metric has a Euclidean signature, even though the background metric is Lorentzian. Solutions of this form have no two-dimensional analogues. They simplify in some limiting cases:

i.) In the case $\upsilon_5 \to 0$, one gets

$$a'(\tau)^2 = \frac{\kappa^2}{c_1^2}\, a(\tau)^6 + 1 \qquad\qquad f(\tau) = \frac{c_1}{a(\tau)^2} \tag{4.75}$$

$a(\tau)$ is then expressible in terms of inverse elliptic integrals.

ii.) The limit $\kappa \to 0$, $c_1 \neq 0$, gives a linearly expanding (or contracting) universe

$$a(\tau) = \pm\tau \qquad\qquad f(\tau) = \frac{c_1}{\tau^2}\ , \tag{4.76}$$

iii.) Another simplifying limit is $c_1 \to 0$, leading to

$$a'(\tau)^2 = \frac{1}{4\upsilon_5 a(\tau)^2} + 1 \qquad\qquad f(\tau) = 2\kappa\upsilon_5 a(\tau)^2\ , \tag{4.77}$$

which can be easily integrated

$$a(\tau) = \frac{1}{2}\sqrt{(2\tau + c_2)^2 - \frac{1}{\upsilon_5^2}}\ , \tag{4.78}$$

where c_2 is an integration constant. This solution describes an open spacetime. Here we must restrict the time domain to $|2\tau + c_2| \geq 1/|\upsilon_5|$. The solution for $a(\tau)$ goes asymptotically to τ and it is singular in the limit $\upsilon_5 \to 0$, as well as $\kappa \to 0$.

In general, solutions of (4.73) and (4.74) are parametrized by κ, υ_5 and the integration constant c_1. (The initial value for a just determines the overall scale.) Some examples of numerical solutions for $a(\tau)$ are plotted in figure 6 for different values for υ_5 and c_1 and fixed $\kappa = 1$. In one example, $c_1 = 0$ and $\upsilon_5 = 5$, there is an initial rapid inflation followed by a linear expansion. In contrast, the example, $c_1 = 1$ and $\upsilon_5 = 0$,a linear expansion is followed by a rapid inflation.

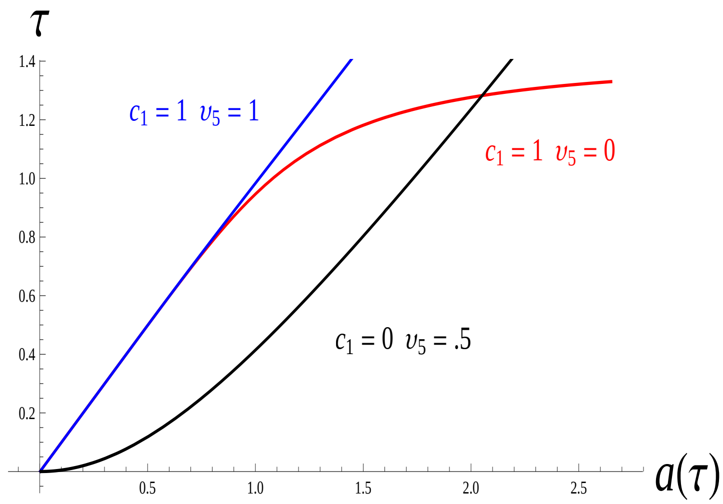

Figure 4.4: Numerical solution for (4.73) and (4.74) for $\kappa = 1$ and three different sets of values for υ_5 and c_1. τ is plotted versus $a(\tau)$, with initial condition $a(0) \approx 0$ in all cases. For the choice of $c_1 = 1$ and $\upsilon_5 = 1$, one gets an approximate linear expansion. The choice of $c_1 = 0$ and $\upsilon_5 = 5$ has an initial rapid inflation followed by a linear expansion, while the choice $c_1 = 1$ and $\upsilon_5 = 0$ has an initial linear expansion followed by a rapid inflation.

4.2.1 Perturbations about dS^4

Here we consider the four-dimensional de Sitter solutions which resulted upon taking the limit ω, $\upsilon_5 \to \infty$, with $\frac{\omega}{\upsilon_5}$ finite. The solutions have a Poisson structure which is determined by two finite nonvanishing parameters $\frac{\omega}{\upsilon_5}$ and κ. In this subsection we expand about the commutative dS^4 solutions, expressing perturbations in terms of commutative gauge and scalar fields. This requires finding the appropriate Seiberg-Witten map, which is given explicitly in the Appendix.

We again write the embedding coordinates y^μ according to $y^\mu = x^\mu + \Theta A^\mu$ where Θ is the perturbation parameter and the perturbations A^μ are now functions on dS^4. Locally then A^μ are functions of τ, σ and coordinates on S^2, which we take to be the usual spherical coordinates θ and ϕ, $0 < \theta < \pi$ and $0 \leq \phi < 2\pi$. The action (4.58) is gauge invariant, at least up to first order in Θ. So A_μ can be regarded as noncommutative gauge potentials up to first order in Θ. A Seiberg-Witten map can be constructed on dS^4, so that A_μ can be re-expressed in terms of commutative gauge potentials $\mathcal{A}_{\mathsf{a}}$, $\mathsf{a} = \tau, \sigma, \theta, \phi$, and a scalar field Φ. In order to produce the leading order correction to the action we need to obtain the Seiberg-Witten map up to first order. The result is given in (B.2) and (B.3) of the Appendix.

We next substitute $y^\mu = x^\mu + \Theta A^\mu$, along with (B.2) and (B.3), into the action $S_c(y)$. For the integration measure we take

$$d\mu_4 = \frac{d\mu_{S^2}}{h(\tau,\sigma)} = \frac{\sin\theta \, d\tau d\sigma d\theta d\phi}{h(\tau,\sigma)} = -\frac{8\kappa\upsilon_5}{\omega}\sqrt{-\mathsf{g}}\; d\tau d\sigma d\theta d\phi \;, \tag{4.79}$$

where $d\mu_{S^2}$ is the invariant measure on the sphere, $h(\tau,\sigma)$ is given in (4.69) and g is the determinant of the metric on dS^4. Thus

$$S_c(y) \propto \int \sqrt{-\mathsf{g}}\;\, d\tau d\sigma d\theta d\phi \Big(-\frac{\upsilon_5}{5}\epsilon_{\mu\nu\lambda\rho\sigma}\, y^\mu \{y^\nu, y^\lambda\}\{y^\rho, y^\sigma\} + \frac{\omega}{2} y_\mu y^\mu \Big) \tag{4.80}$$

After removing all of the total derivatives we arrive at the following simple expression for the leading order (i.e., quadratic) terms in the action $S_c(y)$ (up to a proportionality constant)

$$\Theta^2 \int \sqrt{-\mathsf{g}}\; d\tau d\sigma d\theta d\phi \left(\frac{\omega}{\upsilon_5}\, \frac{\mathcal{F}_{\tau\sigma}\Phi}{a(\tau)^2 \sin^2\sigma} + 8\kappa^2 \frac{\mathcal{F}_{\theta\phi}\Phi}{\sin\theta} - 12\kappa\Phi^2 \right) \,, \tag{4.81}$$

where $\mathcal{F}_{\mathtt{ab}} = \partial_{\mathtt{a}}\mathcal{A}_{\mathtt{b}} - \partial_{\mathtt{b}}\mathcal{A}_{\mathtt{a}}$ are the commutative field strengths. No kinetic energy terms appear for the scalar and gauge fields, which is not surprising since there were no kinetic energy term for y^μ in (4.80). (We therefore don't have to be concerned with the issue of the kinetic energy terms having opposing signs in this case.) The scalar field couples to the radial component of the electric and magnetic fields. The field equations imply that Φ is frozen to a constant value, while the field strengths are constrained by

$$\frac{\omega}{\upsilon_5}\frac{\mathcal{F}_{\tau\sigma}}{a(\tau)^2\sin^2\sigma} + 8\kappa^2\frac{\mathcal{F}_{\theta\phi}}{\sin\theta} = 24\kappa\Phi \tag{4.82}$$

So for example, in the absence of an electric field, the perturbations give rise to a magnetic monopole source with charge equal to $\frac{1}{4\pi}\int \mathcal{F}_{\theta\phi}\, d\theta\wedge d\phi = \frac{3\Phi}{4\pi\kappa}\int \sin\theta\, d\theta\wedge d\phi = \frac{3\Phi}{\kappa}$. The monopoles spontaneously breaks the de Sitter group symmetry down to the three-dimensional rotation group, due to the same symmetry breaking that is present in the Poisson brackets on the surface (4.70).

4.2.2 Rotationally Invariant Matrix Solutions

We previously wrote down an ansatz for rotationally invariant matrices for the three-dimensional matrix model and their resulting equations of motion. Here we propose to do the same in the five-dimensional case. Rotational symmetry in this case is applied to the three matrices X^i, $i = 1, 2, 3$ (and not also X_4). This is consistent with the $SO(3)$ symmetry of the solutions in subsection 5.1 of the commutative equations of motion. We show that the five matrix equations (4.59) reduce to three upon taking into account rotational symmetry, and in a special case reduce to the matrix equations of section 3.

A natural choice for the matrix analogues of the ansätse (4.2) and (4.66) is

$$\begin{bmatrix} Y^0 \\ Y^i \\ Y^4 \end{bmatrix} = \begin{bmatrix} t\otimes \mathbb{1} \\ u\otimes j_i \\ v\otimes \mathbb{1} \end{bmatrix}$$

where t, u, v are hermitean matrices and j_i generate the fuzzy sphere

$$[j_i, j_j] = i\alpha\kappa\, \epsilon_{ijk} j_k \tag{4.83}$$

Without loss of generality, we can choose its radius to be one, $j_i j_i = \mathbb{1}$. Upon substituting into the five matrix equations (4.59) one gets that t, u, v must satisfy

$$\begin{aligned}
-[[t,u],u] - [[t,v],v] + 4i\tilde{\alpha}_5\alpha\kappa\,[u^2,[u,v]]_+ &= -\beta t \\
-[[u,t],t] + [[u,v],v] - 4i\tilde{\alpha}_5\alpha\kappa\,[u^2,[t,v]]_+ + 4i\tilde{\alpha}_5\alpha\kappa\,[[u,v],[t,u]] + 2\alpha^2\kappa^2 u^3 &= -\beta u \\
-[[v,t],t] + [[v,u],u] + 4i\tilde{\alpha}_5\alpha\kappa\,[u^2,[t,u]]_+ &= -\beta v\ ,
\end{aligned}$$

where $[\ ,\ ]_+$ denotes the anticommutator.

While we have not found general solutions to these matrix equation, the system simplifies when $\kappa = 0$, or equivalently $\alpha = 0$. In that case the equations reduce to those of the three-dimensional system (4.48) with $(t, u, v) = (Y^0, Y^1, Y^2)$ and $\tilde{\alpha} = 0$. Therefore the recursion relations (4.57) can be applied in this case to numerically generate the spectra of matrices satisfying (4.37). The commutative limit of such solutions corresponds to solutions 2 of subsection 4.2.

4.3 $\mathbb{CP}^2$ Spacetime

We have so far been able to obtain matrix solutions in lower dimensions that resolved cosmological singularities as well as contained regions of Lorentzian signature in the commutative limit [75]. We have also constructed four dimensional cosmological solutions, but without corresponding matrix solutions. A natural generalization to a four dimensional fuzzy cosmology with a matrix solution is the tensor product $S_F^{2,L} \times S_F^2$, where $S_F^{2,L}$ is the fuzzy sphere with a Lorentzian region and S_F^2 is the fuzzy sphere in a Euclidean background. The computation follows that of [75]. After a modification of the IKKT type action, we may find another type of four dimensional solution to the matrix model which is the noncommutative analog to the complex projective space, $\mathbb{CP}^2$. This coset space is useful since it allows us to write down an exact matrix model solution. Embedding this coset space in a Minkowski background provides a four dimensional Lorentzian spacetime in the commutative limit with resolution of the cosmological singularity in the matrix model.

The *complex projective space* $\mathbb{CP}^n$ is the set of all complex 1-dimensional subspaces through $\mathbf{0}$ in $\mathbb{C}^{n+1}$. It is a compact, smooth 2n-dimensional manifold. We can define an equivalence relation on $\mathbb{C}^{n+1}\backslash\{\mathbf{0}\}$:

$$\mathbf{x} \sim \mathbf{y} \leftrightarrow \mathbf{x} = \lambda \mathbf{y} \quad \text{for some } \lambda \neq \mathbf{0} \text{ in } \mathbb{C} \text{ with } |\lambda| = 1. \tag{4.84}$$

A point in $\mathbb{CP}^n$ is a line through $\mathbf{0}$ in $\mathbb{C}^{n+1}$ and the set of all equivalence classes defines $\mathbb{CP}^n$.

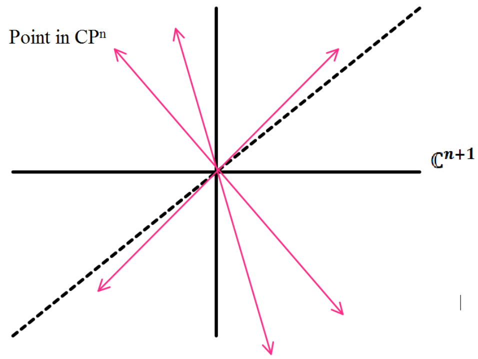

Figure 4.5: Each line in $\mathbb{C}^{n+1}$ through $\mathbf{0}$ constitutes a *point* in $\mathbb{CP}^n$. This can also be restricted to the sphere S^{2n+1}. The sphere S^{2n+1} is a fibration over $\mathbb{CP}^n$ with fibre S^1, i.e., each point in $\mathbb{CP}^n$ is represented by a circle in S^{2n+1}.

To define $\mathbb{CP}^2$ one starts with a three-dimensional complex vector space spanned by $z = (z_1, z_2, z_3)$, where $z_i \in \mathbb{C}$ are not all zero, and then makes the identification of z with γz, for all complex nonzero values of γ. $\mathbb{CP}^2$ can equivalently be defined as the space of $U(1)$ orbits on the unit 5$-$sphere S^5. The latter is spanned by z with $|z|^2 = z_i^* z_i = 1$, where i is summed from 1 to 3, and a point on the space of $U(1)$ orbits is $\{e^{i\beta} z,\ 0 \leq \beta < 2\pi\}$. Upon introducing the following Poisson brackets

$$\{z_i, z_j^*\} = -i\delta_{ij} \qquad \{z_i, z_j\} = \{z_i^*, z_j^*\} = 0\ , \tag{4.85}$$

one can generate the $U(1)$ orbits from the 5$-$sphere constraint

$$\mathcal{C} = z_i^* z_i - 1 \approx 0 \tag{4.86}$$

Infinitesimal variations δ_ϵ along an orbit are then

$$\delta_\epsilon z_i = \{z_i, \mathcal{C}\}\epsilon = -i\epsilon z_i \qquad \delta_\epsilon z_i^* = \{z_i^*, \mathcal{C}\}\epsilon = i\epsilon z_i^* \tag{4.87}$$

$\mathbb{CP}^2$ is also defined as $SU(3)/U(2)$, i.e., the space of adjoint orbits of $su(3)$ through λ^8, where λ^α, $\alpha = 1, 2, ..., 8$ are the Gell-Mann matrices, $\mathbb{CP}^2 = \{U\lambda^8 U^\dagger, U \in SU(3)\}$. Upon introducing

$$x^\alpha = \frac{\bar{z}\lambda^\alpha z}{|z|^2} \ , \tag{4.88}$$

one recovers the $su(3)$ Lie algebra from the Poisson bracket algebra on $\mathbb{CP}^2$. Using the commutator $[\lambda^\alpha, \lambda^\beta] = 2if^{\alpha\beta\gamma}\lambda^\gamma$ we get from (4.85) that

$$\{x^\alpha, x^\beta\} = \frac{2}{|z|^2} f^{\alpha\beta\gamma} x^\gamma \ , \tag{4.89}$$

after imposing the constraint (4.86). x^α are invariant under $z \leftrightarrow \gamma z$, $\gamma \in C$, and so they span a four-dimensional constrained surface, i.e., $\mathbb{CP}^2$, in $\mathbb{R}^8$. The constraints are contained in

$$x^\alpha x^\alpha = \frac{4}{3} \qquad\qquad d^{\alpha\beta\gamma} x^\beta x^\gamma = \frac{1}{3} x^\alpha \tag{4.90}$$

where $d^{\alpha\beta\gamma}$ are defined from the anticommutator of Gell-Mann matrices $[\lambda^\alpha, \lambda^\beta]_+ = \frac{4}{3}\delta_{\alpha,\beta}\mathbb{1}_3 + 2d^{\alpha\beta\gamma}\lambda^\gamma$, $\mathbb{1}_3$ being the 3×3 identity matrix. The constraints in (4.90) follow from the expression for x^α in (4.88).

The metric on $\mathbb{CP}^2$ is given by

$$ds_E^2 = \frac{4}{|z|^4}\Big(|z|^2 |dz|^2 - |\bar{z}dz|^2\Big) \ , \tag{4.91}$$

where $|dz|^2 = dz_i^* dz_i$ and $\bar{z}dz = z_i^* dz_i$. It is known as the Fubini-Study metric and is invariant under: $z \to \gamma z$ and $dz \to d\gamma\, z + \gamma dz$. The isometry of the metric is $SU(3)/Z_3$, with corresponding transformations: $z \to Uz$, $U \in SU(3)/Z_3$. The Fubini-Study metric is recovered from the embedding (4.88) of $\mathbb{CP}^2$ in the $\mathbb{R}^8$ target space, where one assumes a flat Euclidean metric tensor. That is, starting with the invariant length

$$ds_E^2 = dx^\alpha dx^\alpha \ , \tag{4.92}$$

and then substituting (4.88), one recovers (4.91).

The metric tensor in (4.91) can be re-expressed in terms of a pair of complex coordinates $\zeta_a = z_a/z_3$, $a = 1, 2$ (away from $z_3 = 0$), which are invariant under $z \to \gamma z$. Along with their complex conjugates, they span $\mathbb{CP}^2$ when $z_3 \neq 0$. In terms of these coordinates, the invariant length (4.91) becomes

$$ds_E^2 = 2\, g_{a\bar{b}}\, d\zeta_a d\zeta_b^* = \frac{4}{(|\zeta|^2+1)^2}\Big((|\zeta|^2+1)|d\zeta|^2 - |\bar{\zeta} d\zeta|^2\Big) \ , \tag{4.93}$$

where $|\zeta|^2 = \zeta_a^* \zeta_a$, $|d\zeta|^2 = d\zeta_a^* d\zeta_a$ and $\bar{\zeta} d\zeta = \zeta_a^* d\zeta_a$. It is well known to satisfy the sourceless Einstein equations with a positive cosmological constant, specifically $\Lambda = \frac{3}{2}$. From (4.85), the Poisson brackets are given by

$$\{\zeta_a, \zeta_b^*\} = -i(|\zeta|^2+1)(\zeta_a \zeta_b^* + \delta_{ab}) \qquad \{\zeta_a, \zeta_b\} = \{\zeta_a^*, \zeta_b^*\} = 0 \ , \tag{4.94}$$

Their inverse gives the symplectic two-form

$$\Omega = -\frac{i}{2}\, g_{a\bar{b}}\, d\zeta_a \wedge d\zeta_b^* \ , \tag{4.95}$$

which is also the Kähler two-form.

The invariant length and Kähler two-form can furthermore be expressed in terms of left-invariant Maurer-Cartan one forms ω_i on $SU(2)$, satisfying $d\omega_i + \epsilon_{ijk}\omega_j \wedge \omega_k = 0$.

For this take $\omega_i = \frac{i}{2}\mathrm{Tr}\,\sigma_i\, u^\dagger du$, where u is the $SU(2)$ matrix

$$u = \frac{1}{|\zeta|}\begin{bmatrix} \zeta_1^* & -i\zeta_2 \\ -i\zeta_2^* & \zeta_1 \end{bmatrix} \tag{4.96}$$

and σ_i are Pauli matrices. One can write

$$ds_E^2 = \frac{4(d|\zeta|)^2}{(|\zeta|^2{+}1)^2} + \frac{4|\zeta|^2}{(|\zeta|^2{+}1)}\,(\omega_1^2+\omega_2^2) + \frac{4|\zeta|^2}{(|\zeta|^2{+}1)^2}\,\omega_3^2 \tag{4.97}$$

and

$$\Omega = -2\,d\biggl(\frac{|\zeta|^2}{(|\zeta|^2{+}1)}\,\omega_3\biggr) \tag{4.98}$$

While the isometry of the metric tensor is $SU(3)/Z_3$, for any $|\zeta|\,(\neq 0)$-slice the metric tensor and symplectic two-form are invariant under $SU(2)\times U(1)/Z_2$. The latter symmetry is also present for the manifolds we obtain in section five. (On the other hand, the $SU(3)/Z_3$ isometry is broken for those manifolds.) The $SU(2)\times U(1)/Z_2$ transformations on u are of the form $u \to u' = v\,u\,e^{i\lambda\sigma_3}$, $v\in SU(2)$ and $\lambda \in R$, which leave ω_3 and $\omega_1^2+\omega_2^2$ invariant. We can parametrize the $SU(2)$ matrices in (4.96) by Euler angles (θ,ϕ,ψ) according to

$$\zeta_1 = e^{i\left(\frac{\psi+\phi}{2}\right)}\cos\frac{\theta}{2}\,|\zeta| \qquad\qquad \zeta_2 = e^{i\left(\frac{\psi-\phi}{2}\right)}\sin\frac{\theta}{2}\,|\zeta|\;, \tag{4.99}$$

where in order to span all of $SU(2)$, $0\le\theta<\pi$, $0\le\psi<2\pi$ and $0\le\phi\le 4\pi$. On the other hand, to parametrize the Maurer-Cartan one forms, we only need ϕ to run from 0 to 2π. In terms of the Euler angles, the metric is given by

$$ds_E^2 = \frac{4(d|\zeta|)^2}{(|\zeta|^2{+}1)^2} + \frac{|\zeta|^2}{(|\zeta|^2{+}1)}\,(d\theta^2+\sin^2\theta d\phi^2) + \frac{|\zeta|^2}{(|\zeta|^2{+}1)^2}\,(d\psi+\cos\theta d\phi)^2 \tag{4.100}$$

The Killing vectors $k_a,\ a = 1, ..., 4$, generating the $SU(2) \times U(1)/Z_2$ isometry group on any $|\zeta|\,(\neq 0)$-slice are expressed in terms of the Euler angles according to

$$k_1 \pm ik_2 = e^{\pm i\phi}\Big(\frac{\partial}{\partial\theta} \pm i\Big(\cot\theta\frac{\partial}{\partial\phi} - \csc\theta\frac{\partial}{\partial\psi}\Big)\Big) \qquad k_3 = \frac{\partial}{\partial\phi} \qquad k_4 = \frac{\partial}{\partial\psi} \tag{4.101}$$

4.4 Fuzzy $\mathbb{CP}^2$

We may think of fuzzy $\mathbb{CP}^2$ or $\mathbb{CP}^2_F$ as the quantization of $\mathbb{CP}^2$. For this one replaces the complex coordinates z_i and z_i^*, $i = 1, 2, 3$, by operators $a_i^\dagger$ and a_i,[17],[112] satisfying the commutation relations of raising and lowering operators

$$[a_i, a_j^\dagger] = \delta_{ij} \qquad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0 \tag{4.102}$$

In analogy with the 5−sphere constraint (4.86), one fixes the eigenvalue of the total number operator $a_i^\dagger a_i$ to some positive integer value n. This restricts $\mathcal{H}_n$ to be spanned by the $N = \frac{(n+2)!}{2n!}$ harmonic oscillator states $\{|n_1, n_2, n_3 >\}$, $n_i = 0, 1, 2, ...$, where $n = n_1 + n_2 + n_3$. The action of the raising and lowering operators is incompatible with this restriction, so $a_i^\dagger$ and a_i cannot generate the algebra of $\mathbb{CP}^2_F$. One can instead work with functions of the operators $a_i^\dagger a_j$, which do have a well defined action on the N−dimensional Hilbert space $\mathcal{H}_n$. Of course, $a_i^\dagger a_i$ acts trivially on $\mathcal{H}_n$. The remaining operators, $a_i^\dagger a_j - \frac{1}{3}\delta_{ij}a_k^\dagger a_k$, generate $SU(3)$ and are the noncommutative analogues of (4.88), which we can also write as

$$X^\alpha = \frac{1}{n}a_i^\dagger\lambda^\alpha_{ij}a_j\ , \qquad \alpha = 1, 2, ..., 8 \tag{4.103}$$

From them we recover the $su(3)$ Lie-algebra

$$[X^\alpha, X^\beta] = \frac{2i}{n}f^{\alpha\beta\gamma}X^\gamma \tag{4.104}$$

X^α acting on $\mathcal{H}_n$ generate an irreducible representation of $SU(3)$, which is uniquely specified by the values of the quadratic and cubic Casimirs, $X^\alpha X^\alpha$ and $d^{\alpha\beta\gamma}X^\alpha X^\beta X^\gamma$. They are contained in the following fuzzy analogues of the quadratic CP^2 constraints (4.90),

$$X^\alpha X^\alpha|_{\mathcal{H}_n} \quad = \quad \frac{4}{3}+\frac{4}{n} \tag{4.105}$$

$$d^{\alpha\beta\gamma}X^\alpha X^\beta|_{\mathcal{H}_n} \quad = \quad \left(\frac{1}{3}+\frac{1}{2n}\right) X^\gamma|_{\mathcal{H}_n}\ , \tag{4.106}$$

in addition to

$$f^{\alpha\beta\gamma}X^\alpha X^\beta|_{\mathcal{H}_n} \quad = \quad \frac{6i}{n}\ X^\gamma|_{\mathcal{H}_n} \tag{4.107}$$

The quadratic constraints (4.105-4.107) tend towards the commutative constraints (4.90) in the large n (or equivalently, large N) limit. (4.105) assigns a value to the quadratic Casimir, while for the cubic Casimir we then get

$$d^{\alpha\beta\gamma}X^\alpha X^\beta X^\gamma|_{\mathcal{H}_n}=\frac{4}{9}+\frac{2}{n}+\frac{2}{n^2} \tag{4.108}$$

The $\mathbb{CP}^2_F$ algebra is the algebra of $N\times N$ matrices which are polynomial functions of X^α, satisfying the constraints (4.105-4.107). The standard choice for the Laplace operator on $\mathbb{CP}^2_F$ is $\Delta_E=[X^\alpha,[X^\alpha,...]]$. Star products for $\mathbb{CP}^2_F$ are known.[111],[112] Under the star product, denoted by $\star$, the $\mathbb{CP}^2_F$ algebra is mapped to a noncommutative algebra of functions on $\mathbb{CP}^2$. For example, from (4.104), the images $\mathcal{X}_\alpha$ of the operators X_α under the map satisfy the star commutator:

$$\mathcal{X}^\alpha\star\mathcal{X}^\beta-\mathcal{X}^\beta\star\mathcal{X}^\alpha=\frac{2i}{n}f^{\alpha\beta\gamma}\mathcal{X}^\gamma \tag{4.109}$$

In the commutative limit $n \to \infty$, the star product of functions is required to reduce to the point-wise product (at zeroth order in $1/n$), and the star commutator of functions reduces to i times the Poisson bracket of functions (at first order in $1/n$). So for example, the left hand side of (4.109) goes to $\frac{i}{n}\{\mathcal{X}^\alpha, \mathcal{X}^\beta\}$ as $n \to \infty$, and in that limit, $\mathcal{X}^\alpha$ satisfy the same Poisson bracket relations as x^α in (4.89). Therefore χ^α can be identified with the $\mathbb{CP}^2$ embedding coordinates in the large n limit.

4.5 $\mathbb{CP}^2_F$ solutions to matrix models

4.5.1 Euclidean background

$\mathbb{CP}^2_F$ is easily seen to be a solution of a Yang-Mills matrix model with a Euclidean background metric. For this we introduce $M \times M$ matrices Y^α, $\alpha = 1, ..., 8$, whose dynamics is governed by the action [106]

$$S_E(Y) = \frac{1}{g^2}\mathrm{Tr}\Big(-\frac{1}{4}[Y^\alpha, Y^\beta]^2 + \frac{2}{3}i\tilde{\alpha} f^{\alpha\beta\gamma} Y^\alpha Y^\beta Y^\gamma\Big) , \tag{4.110}$$

where $\tilde{\alpha}$ is a real coefficient. The first term in the trace defines the Yang-Mills matrix action (which can be trivially extended to ten dimensions) appears in the IKKT matrix model.[30] It is invariant under rotations in the eight-dimensional Euclidean space. This $SO(8)$ symmetry is broken by the second term, which instead is invariant under the adjoint action of $SU(3)$, with infinitesimal variations $\delta Y^\alpha = 2if^{\alpha\beta\gamma}Y^\beta\epsilon^\gamma$, for infinitesimal parameters ϵ^α. Both terms are invariant under the common subgroup of rotations in the $\alpha = 1, 2, 3$ directions, as well as translations in the eight-dimensional Euclidean space.

The action (4.110) has extrema at

$$[[Y^\alpha, Y^\beta], Y^\beta] + i\tilde{\alpha} f^{\alpha\beta\gamma}[Y^\beta, Y^\gamma] = 0 \tag{4.111}$$

$\mathbb{CP}^2_F$ is a solution to (4.111). That is, we identify Y^α with $N \times N$ matrix representations of the X^α, defined in the previous section. For this we need to make the identification $\tilde{\alpha} = 2/n$, n being an integer such that $N = \frac{(n+2)!}{2n!} \leq M$.

4.5.2 Lorentzian background

The matrix action (4.110) was written in an eight-dimensional Euclidean ambient space. Here we change the ambient space to eight-dimensional Minkowski space, with metric tensor $\eta = \mathrm{diag}(1,1,1,1,1,1,1,-1)$. In order to find nontrivial solutions we also add a quadratic term to the action, which now reads

$$S_M(Y) = \frac{1}{g^2}\mathrm{Tr}\Big(-\frac{1}{4}[Y^\alpha, Y^\beta][Y_\alpha, Y_\beta] + \frac{2}{3} i\tilde{\alpha} f^{\alpha\beta\gamma} Y_\alpha Y_\beta Y_\gamma + \frac{\beta}{2} Y^\alpha Y_\alpha\Big) \ , \qquad (4.112)$$

with indices raised and lowered using η. The action is an extremum when

$$[[Y^\alpha, Y^\beta], Y_\beta] + i\tilde{\alpha} f^{\alpha\beta\gamma}[Y_\beta, Y_\gamma] + \beta Y^\alpha = 0 \qquad (4.113)$$

A simple solution $Y^\alpha = \bar{Y}^\alpha$ to (4.113) is CP^2_F, now written in a Lorentzian background:

$$\bar{Y}^\alpha = n\tilde{\alpha}\, X^\alpha \qquad (4.114)$$

Here $\tilde{\alpha}$ and β are constrained by

$$\beta = -6\tilde{\alpha}^2 \qquad (4.115)$$

For any fixed n, which defines a matrix representation, this solution is expressed in terms of only one free parameter, which sets an overall scale. This CP^2 solution is not invariant under all of $SU(3)$, since general transformations do not preserve the time-like direction of the background metric. On the other hand, the time-like direction is preserved under the adjoint action of the $SU(2) \times U(1)$ subgroup. In order for the Laplace operator associated with this solution to be consistent with the eight-dimensional Minkowski metric tensor η, we should take it to be $\Delta_M = [Y^\alpha, [Y_\alpha, ...]]$, rather than the Laplace operator on $\mathbb{CP}^2_F$.

A more general solution to (4.113) which is also invariant under $SU(2) \times U(1)$ is

$$
\begin{aligned}
\bar{Y}^i &= \frac{n\rho}{2} X^i\,, \qquad i = 1,2,3 \\
\bar{Y}^a &= v\,\frac{n\rho}{2} X^a\,, \qquad a = 4,5,6,7 \\
\bar{Y}^8 &= w\,\frac{n\rho}{2} X^8
\end{aligned}
\tag{4.116}
$$

where $v, w, \rho, \tilde{\alpha}$ and β are constrained by

$$
\begin{aligned}
v &= \frac{1}{2}\sqrt{\frac{\gamma + 5 + w - w^2 - w^3}{1+w}} \\
\frac{\tilde{\alpha}}{\rho} &= \frac{5 + w + 7w^2 - w^3 - \gamma}{4\,(1 + 4w - w^2)} \\
\frac{\beta}{\rho^2} &= -\frac{3\,(1 + 15w - 8w^3 - w^4 + w^5 + (1 + 2w - w^2)\,\gamma)}{4(1+w)\,(1 + 4w - w^2)}
\end{aligned}
\tag{4.117}
$$

and

$$\gamma = \sqrt{25 - 6w + 7w^2 + 4w^3 - 17w^4 + 2w^5 + w^6} \tag{4.118}$$

For any fixed n, this solution is determined by two parameters ρ and w, the former of which sets the overall scale. Again, here we assume the Laplace operator to be $\Delta_M = [Y^\alpha, [Y_\alpha, ...]]$. The solution is a one-parameter deformation of the previous CP^2_F solution, given by (4.114) and (4.115), and we can regard w as the deformation parameter. The previous solution is recovered for $w = 1$, since then (4.117) gives $v = 1$, $\tilde{\alpha} = \frac{\rho}{2}$ and $\beta = -\frac{3}{2}\rho^2$. v is real and finite for the domain $-1 < w \lesssim 1.32247$. v tends towards the lower bound $\approx .493295$ as w goes to the upper limit ≈ 1.32247, while it is singular in the limit $w \to -1$. v versus w is plotted for this domain in figure 1.

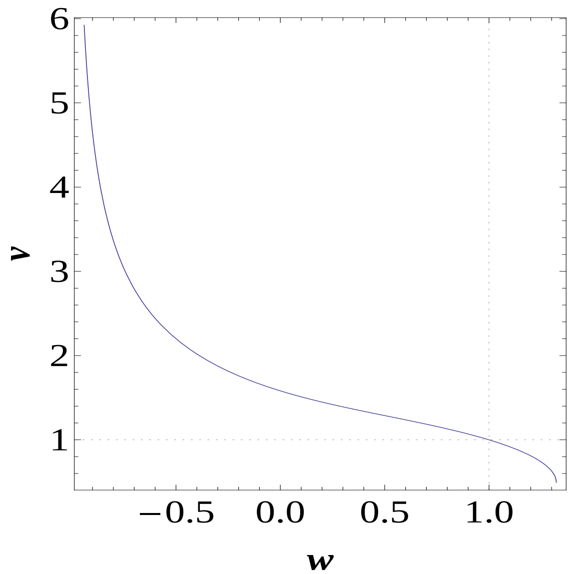

Figure 4.6: v versus w is plotted for the one-parameter family of deformed CP_F^2 solutions given in (4.116) and (4.117).

4.6 Commutative limit

The discussions in section four assume $N \times N$ representation for the $\mathbb{CP}_F^2$ solution (4.114), (4.115), and the deformed $\mathbb{CP}_F^2$ solution (4.116)-(4.118). Here we take the $N \to \infty$ limit of these solutions to reveal different spacetime manifolds. We begin with the undeformed $\mathbb{CP}_F^2$ solution (4.114), (4.115).

4.6.1 $\mathbb{CP}^2$ in a Lorentzian background

For convenience we first fix the scale of the solution (4.114), (4.115) by setting $\tilde{\alpha} = 1/n$ and $\beta = -1/n^2$. Thus $\bar{Y}^\alpha = X^\alpha$, for any n. For some star product we can introduce their corresponding symbols $\bar{\mathcal{Y}}^\alpha$, and so $\bar{\mathcal{Y}}^\alpha = \mathcal{X}^\alpha$, where $\mathcal{X}^\alpha$ satisfies the star commutator (4.109). Then in the $n \to \infty$ limit, $\bar{\mathcal{Y}}^\alpha$ obey the Poisson brackets (4.89), and constraints of the form (4.90). In the limit, $\bar{\mathcal{Y}}^\alpha$ can be expressed in terms of complex coordinates z as in (4.88), which once again span a four-dimensional manifold. However, now the manifold, strictly speaking, is not $\mathbb{CP}^2$. While we recover the $\mathbb{CP}^2$ constraints (4.90) and (4.89) in the commutative limit, the induced metric on the manifold cannot be the Fubini-Study metric (4.91). The latter followed from the Euclidean background metric tensor on $\mathbb{R}^8$, given in (4.92). Now the embedding matrices $\bar{Y}^\alpha$, and their symbols $\bar{\mathcal{Y}}^\alpha$, span eight-dimensional Minkowski space. The induced metric tensor on the surface is thus computed from the invariant length for the eight-dimensional Minkowski space,

$$ds_M^2 = d\bar{\mathcal{Y}}^\alpha d\bar{\mathcal{Y}}_\alpha = ds_E^2 - 2(d\mathcal{X}^8)^2, \tag{4.119}$$

where we assume $\bar{\mathcal{Y}}^\alpha = \mathcal{X}^\alpha$. Then by writing $\mathcal{X}^\alpha = \frac{\bar{z}\lambda^\alpha z}{|z|^2}$, one gets corrections to the Fubini-Study metric

$$ds_M^2 = ds_E^2 \; - \; \frac{2\left(d(\bar{z}\lambda^8 z)\right)^2}{|z|^4} \; - \; \frac{2\,(\bar{z}\lambda^8 z)^2\left(d|z|^2\right)^2 - d|z|^4\, d(\bar{z}\lambda^8 z)^2}{|z|^8} \tag{4.120}$$

In terms of the coordinates $\zeta_a = z_a/z_3$, $a = 1, 2$, which are invariant under $z \to \gamma z$, we get

$$\begin{aligned} ds_M^2 &= ds_E^2 \; - \; \frac{24|\zeta|^2}{(|\zeta|^2{+}1)^4}\,(d|\zeta|)^2 \\ &= 4\,\frac{(|\zeta|^2{-}1)^2 - 2|\zeta|^2}{(|\zeta|^2{+}1)^4}\,(d|\zeta|)^2 + \frac{4|\zeta|^2}{(|\zeta|^2{+}1)}\,(\omega_1^2 + \omega_2^2) \\ &+ \frac{4|\zeta|^2}{(|\zeta|^2{+}1)^2}\,\omega_3^2 \end{aligned} \tag{4.121}$$

,where the left-invariant one forms ω_i were defined previously in section two.

The metric tensor obtained here differs from that on $\mathbb{CP}^2$, and furthermore is not Kähler. On the other hand, the symplectic two-form remains unchanged, i.e. it is (4.98). $SU(3)/Z_3$ is no longer an isometry. Instead, the metric tensor (4.6.1) and symplectic two-form are invariant under $SU(2) \times U(1)/Z_2$, generated by the Killing vectors (4.101). A novel feature is that the metric tensor has variable signature. It has Euclidean signature for $0 < |\zeta|^2 < 2-\sqrt{3}$ and $|\zeta|^2 > 2+\sqrt{3}$, and Lorentzian signature for $2-\sqrt{3} < |\zeta|^2 < 2+\sqrt{3}$. The metric tensor, along with the Ricci scalar, is singular at the boundaries $|\zeta|^2 = 2 \pm \sqrt{3}$ between the regions, and so the boundaries define physical singularities. [There are also coordinate singularities located at $|\zeta| = 0$ and $|\zeta| \to \infty$, just as is the case with the CP^2 metric tensor given by (4.97).] Away from the singularities, the manifold is spatially homogeneous and axially symmetric at each point, and the invariant length (4.6.1) has a form which is similar to that of a Taub-NUT space (more specifically, the Taub region of Taub-NUT space since the coefficient of ω_3^2 is positive).

We now restrict to the Lorentzian region $2-\sqrt{3} < |\zeta|^2 < 2+\sqrt{3}$. $|\zeta|$ is a time parameter in this region, and one has the following properties:

a) There are time-like geodesics which originate at the initial singularity, which we choose to be at $|\zeta| = \sqrt{2-\sqrt{3}}$, and terminate at the final singularity at $|\zeta| = \sqrt{2+\sqrt{3}}$. The elapsed proper time along a geodesic with $\omega_1 = \omega_2 = \omega_3 = 0$ can be written as a function of $|\zeta|$

$$\tau(|\zeta|) = 2\int_{\sqrt{2-\sqrt{3}}}^{|\zeta|} \frac{\sqrt{-r^4+4r^2-1}}{(r^2+1)^2}\, dr \tag{4.122}$$

The the total proper time from the initial singularity to the final singularity is $\tau\left(\sqrt{2+\sqrt{3}}\right) \approx .672$.

b) From the volume of any time-slice, which can be constructed from the determinant of the metric, ${}^3g|_{|\zeta|}$, at the time-slice, one can assign a spatial distance scale a as a function of $|\zeta|$,

$$a(|\zeta|)^3 = \int \sqrt{{}^3g|_{|\zeta|}}\, d\theta d\phi d\psi = \frac{8\pi^2|\zeta|^3}{(|\zeta|^2+1)^2}\ , \tag{4.123}$$

where the integration is done over the time-slice, which can be parametrized by the Euler angles in (4.96). A novel feature of this spacetime is that the distance scale is nonvanishing at the time of the initial and final singularities, corresponding to $|\zeta| =$

$\sqrt{2-\sqrt{3}}$ and $|\zeta|= \sqrt{2+\sqrt{3}}$, respectively,

$$a(\sqrt{2-\sqrt{3}}) \approx 1.896 \qquad\qquad a(\sqrt{2+\sqrt{3}}) \approx 2.940$$

A plot of the normalized scale $a/a|_{\tau=0}$ as a function of the time τ from $\tau = 0$ (the time of the initial singularity) to the time of the final singularity appears in in figure 3 (solid curve). It is seen to grow and de-accelerate.

4.6.2 Deformed $\mathbb{CP}^2$ in a Lorentzian background

We can obtain a one-parameter family of spacetime manifolds, including the one obtained previously, by taking the commutative limit of the deformed $\mathbb{CP}^2_F$ solution (4.116)-(4.118). Here it is convenient to set $\rho = 2/n$. Then the symbols $\bar{\mathcal{Y}}^\alpha$ of the matrices $\bar{Y}^\alpha$ for the solution in (4.116) satisfy

$$\begin{aligned} \bar{\mathcal{Y}}^i &= \mathcal{X}^i\,, \qquad i = 1,2,3 \\ \bar{\mathcal{Y}}^a &= v\,\mathcal{X}^a\,, \qquad a = 4,5,6,7 \\ \bar{\mathcal{Y}}^8 &= w\,\mathcal{X}^8 \qquad , \end{aligned} \tag{4.124}$$

where $\mathcal{X}^\alpha$ again denote the symbols of the $\mathbb{CP}^2_F$ matrices. Recall v is real and finite for the domain $-1 < w \lesssim 1.32247$, while w is given in (4.117) and plotted in figure 1. In the $n \to \infty$ limit, we shall keep v and w fixed, which implies as before that $\tilde{\alpha}$ and β vanish in the limit, $\tilde{\alpha} \sim 1/n$ and $\beta \sim 1/n^2$. The invariant length in the eight-dimensional Minkowski space now reads

$$ds^2_M = d\bar{\mathcal{Y}}^\alpha d\bar{\mathcal{Y}}_\alpha = v^2 ds^2_E + (1-v^2)(d\mathcal{X}^i)^2 - (w^2+v^2)(d\mathcal{X}^8)^2\,, \tag{4.125}$$

where we substituted the commutative solution (4.124). Using the identities

$$(d\mathcal{X}^i)^2 = \frac{4\,|\zeta|^4}{(1+|\zeta|^2)^2}\,(\omega_1^2+\omega_2^2) + \frac{4\,|\zeta|^2}{(1+|\zeta|^2)^4}(d|\zeta|)^2 \qquad (d\mathcal{X}^8)^2 = \frac{12\,|\zeta|^2}{(1+|\zeta|^2)^4}(d|\zeta|)^2\;, \tag{4.126}$$

which follows from $\mathcal{X}^\alpha = \frac{\bar z \lambda^\alpha z}{|z|^2}$ and the previous definition of the left-invariant one forms ω_i, we now get

$$\begin{aligned} ds^2_M = 4\bigg(\frac{v^2(|\zeta|^2-1)^2 + (1-3w^2)|\zeta|^2}{(1+|\zeta|^2)^4}\bigg)(d|\zeta|)^2 + \frac{4|\zeta|^2(v^2+|\zeta|^2)}{(1+|\zeta|^2)^2}\,(\omega_1^2+\omega_2^2) \\ + \frac{4\,v^2|\zeta|^2}{(|\zeta|^2+1)^2}\,\omega_3^2 \end{aligned} \tag{4.127}$$

This expression reduces to (4.6.1) when $w = v = 1$. The symplectic two-form is again given by (4.98).

As in the previous case, the metric tensor and symplectic two-form are invariant under $SU(2)\times U(1)/Z_2$, generated by the Killing vectors (4.101). The induced metric tensor now has physical singularities at $|\zeta|=|\zeta_\pm|$, where

$$|\zeta_\pm|^2 = \frac{2v^2+3w^2-1 \pm \sqrt{(3w^2-1)(4v^2+3w^2-1)}}{2v^2}\;, \tag{4.128}$$

which using (4.117) are functions of only w. The singularities are plotted as a function of w in figure 2. There are two singularities for the domains $-1 > w > -\frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}} < w \lesssim 1.32247$, one singularity (at $|\zeta|=1$) for $w = \pm\frac{1}{\sqrt{3}}$, and none for $-\frac{1}{\sqrt{3}} < w < \frac{1}{\sqrt{3}}$. As before, they define the boundaries between regions of Euclidean signature and Lorentzian signature. (The regions of Lorentzian signature are shaded in the figure.) For the domain $-\frac{1}{\sqrt{3}} < w < \frac{1}{\sqrt{3}}$, the metric tensor in (4.127) has a Euclidean signature for all $|\zeta|^2$.

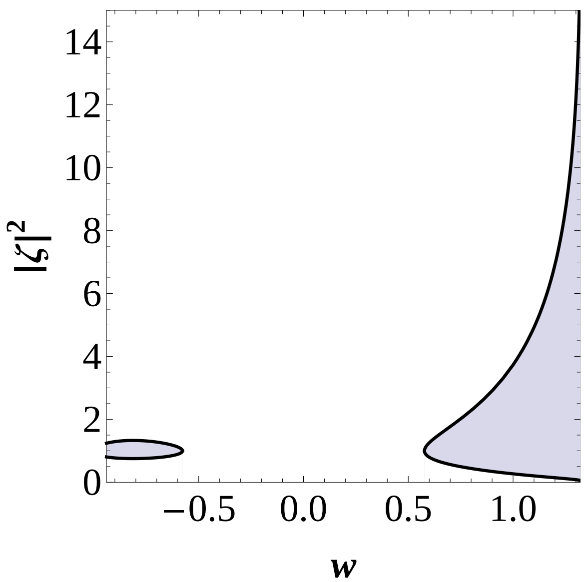

Figure 4.7: Singularities at $|\zeta|^2 = |\zeta_\pm|^2$ are plotted as a function of w using (4.128). They define boundarises between regions of Lorentzian signature (shaded) and Euclidean signature (unshaded).

Once again there are time-like geodesics which originate at the initial singularity, which we choose to be at $|\zeta| = |\zeta_-|$, and terminate at the final singularity at $|\zeta| = |\zeta_+|$. The generalization of the expression (4.122) for the elapsed proper time along a geodesic with $\omega_1 = \omega_2 = \omega_3 = 0$ can be written as

$$\tau(|\zeta|) = 2\int_{|\zeta_-|}^{|\zeta|} \frac{\sqrt{(3w^2-1)r^2 - v^2(r^2-1)^2}}{(r^2+1)^2}\, dr \tag{4.129}$$

The generalization of the expression (4.123) for the volume at a $|\zeta|$-slice, which we again denote by $a(|\zeta|)^3$, is

$$a(|\zeta|)^3 = \int \sqrt{{}^3g|_{|\zeta|}}\, d\theta d\phi d\psi = \frac{8\pi^2 v|\zeta|^3(|\zeta|^2+v^2)}{(|\zeta|^2+1)^3}\ , \tag{4.130}$$

We examine the region of Lorentzian signature for four different choices for w (and hence v), including the case $w = v = 1$ of the previous subsection, in figure 3. There we plot the normalized scale $a/a|_{\tau=0}$ as a function of the time τ, starting from $\tau = 0$ (the time of the initial singularity) to the time of the final singularity. In all cases the distance scale a is nonvanishing at the time of the initial and final singularities, and the scale grows and de-accelerates. The largest and longest expansion occurs when w takes its maximum value of ~ 1.3225, while the spacetime only exists for an instant for $w = \pm\frac{1}{\sqrt{3}}$.

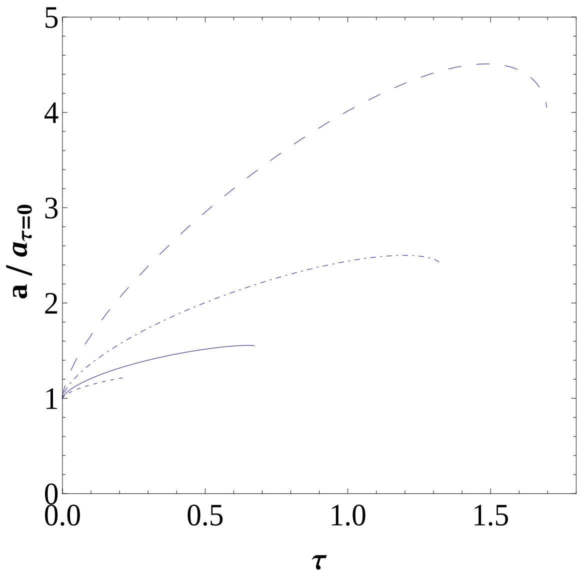

Figure 4.8: $a/a|_{\tau=0}$ as a function of the time τ from $\tau = 0$ (the time of the initial singularity) to the time of the final singularity for four different choices for w (and hence v): $w \approx 1.3225$ (large dashed curve), $w = 1.25$ (dot-dashed curve), $w = 1$ (solid curve) and $w = .75$ (small dashed curve).

5 CONCLUSION

Matrix models have been shown to describe geometry, gravity, and even nonperturbative aspects of string theory, and as such are candidates for a quantum theory of gravity. In particular, matrix models provide a simple way to incorporate dynamical noncommutative geometry into physical theories. Our motivation has been to use matrix models to study aspects of cosmological models and black holes that are believed to be related to effects of quantum gravity.

In chapter III, we introduced a BFSS type matrix model in order to study the BTZ black hole solution to 2+1 dimensional gravity. For this, we start with a noncommutative Chern Simons theory. The noncommutative Chern Simons theory can be expressed as a matrix model in which the dynamical variables are infinite dimensional matrices. In this case, the equations of motion correspond to the Moyal-Weyl plane and there are no physical degrees of freedom left in the model. If instead one begins with a finite version of this theory, which we call a matrix Chern Simons theory, there are physical degrees of freedom left in the model. This model contains an internal SU(N) gauge symmetry and can be extended to contain an external rigid rotation symmetry. This external symmetry proves to be important if one wishes to recover the BTZ entropy relation. This symmetry exactly corresponds to the first class constraint which gives us a degeneracy of eigenvalues of the quadratic invariant of the model. This quadratic invariant has units of distance squared and should thus be identified with the square of the radius of the black hole, which is also an invariant. It is the degeneracy of this invariant which we identify with the density of states of the black hole. In the asymptotic limit, the log of this degeneracy is equivalent to the entropy of the BTZ black hole.

The Chern Simons matrix model we have introduced successfully reproduced the BTZ entropy formula. Here, we have considered commuting configurations which are not normally considered in the literature. This model is also linear in the time derivatives as opposed to the Yang-Mills type of matrix models which are quadratic in the time derivatives. An important next step would be to obtain the commutative limit of this model in which one hopes to recover the BTZ geometry. Other matrix models have been shown to contain geometry in the commutative limit, including the 4 dimensional Schwarzschild solution [12], but it is not clear how such a limit should be taken in our case. An interesting addition to this model is to include a kinetic term to the Lagrangian. This is essentially a modification of the bosonic sector of the BFSS action. The work described above did not consider any fermionic degrees of freedom. Incorporating fermions may lead to the possibility of modeling a unitary mechanism for Hawking radiation in terms of a matrix model.

A similar framework may be possible in higher dimensional versions of the matrix model described above. For example, the Background Field (BF) model [57] is a topological field theory whose action has been shown to be equivalent to Einstein-Cartan action when one includes certain constraints. It may be possible to use the matrix analog of the BF model to count the entropy of the Schwarzschild black hole. This work indicates that matrix models could prove to be a useful means to study both commutative and noncommutative configurations and the geometry which results from them. These models allow us to consider directly the degrees of freedom in the theory which may be useful for considering gravity in the quantum regime.

We next turn our attention to using matrix models to study realistic cosmological models. Our motivation in chapter IV has been to find solutions to matrix models which resemble cosmological solutions of Einstein equations and which agree with observations of the Universe. We have studied two and four dimensional solutions to the bosonic sector of IKKT type matrix models, where we have written down a matrix analog of rotational invariance in these cases. We have found both exact matrix solutions as well as solutions in the commutative limit corresponding to open, closed and static universes. The solutions we study contain a distance scale which in some cases have desirable features. In the four dimensional case, for example, we obtain a solution that has an initial inflationary stage followed by a non-inflationary era.

In general, exact matrix solutions are nontrivial to obtain. A particular exact four dimensional solution that we have studied is the matrix version of the complex projective plane. From there we have found a Lorentzian analog of $\mathbb{CP}^2$ by embedding the solution in a higher dimensional Minkowskian background. The solution we obtain contains regions of Euclidean and Lorentzian signature separated by three dimensional singular regions. We interpret these singularities as cosmological singularities beyond which time itself no longer makes sense. This is demonstrated by both the infinite scalar curvature as well as time-like geodesics which are shown to be incomplete at the singularities. An interesting feature is that these singularities occur at non-zero spatial size. This solution has a distance scale which initially grows but eventually deaccelerates. This does not seem to resemble our actual universe which is observed to be continually expanding and accelerating. The commutative limit is described by a metric tensor and symplectic two-form on a spacetime manifold that are both invariant under $SU(2)$ x $U(1)/Z_2$. It is similar to the Taub region of a Taub-NUT spacetime.

While we have found solutions resembling cosmological solutions, we have not shown that Einstein's equations arise in a limit of the matrix model equations of motion. An important step in this work would be to understand how this limit may arise. None of the studies here have included fermionic degrees of freedom. It was necessary to include fermions in the IKKT and BFSS matrix models in order to obtain stable classical configurations. We have also not studied the stability against quantum effects for our solutions, but we do note that the quadratic terms explicitly break supersymmetry as well as translational invariance.

It may be possible to study the noncommutative aspects of the spacetime singularities in more detail. Noncommutative geometry is specified by a spectral triple in which the Dirac operator plays an important role in determining the noncommutative metric and a notion of distance between pure states. It is not known, however, how to define such a self-adjoint operator for a topological space which is non-compact, or a geometry described by a pseudo-Riemannian metric. The commutative limit of this solution contains regions of Euclidean signature as well, and it may be possible to study the Dirac operator corresponding to those regions. This may provide insight into understanding the noncommutative structure near the singularities.

It may be possible to find more realistic cosmological solutions from these or other matrix models. Noncompact noncommutative coset spaces are natural guesses for configurations which should have both an exact matrix solution and lead to four dimensional open universes in the commutative limit. In such a case, the commutative limit will have to be redefined since such a geometry will require infinite dimensional matrices to begin with. If these solutions also contain an initial non-zero spatial size, it may provide a mechanism for avoiding an initial inflationary era. Moreover, a similar mechanism for resolving singularities may be possible in resolving the singularities of black hole solutions.

6 APPENDIX

Here we obtain the Seiberg-Witten map up to first order in the noncommutativity parameter for the four-dimensional de Sitter solution of section IV.

We first obtain the zeroth order result. This is easy to determine by comparing the gauge transformation properties of the commutative gauge potentials $\mathcal{A}_{\mathtt{a}},\ \mathtt{a} = \tau, \sigma, \theta, \phi$ with those of the noncommutative potentials A_μ, using the Poisson brackets (4.66). The gauge variations of the former are simply $\delta\mathcal{A}_{\mathtt{a}} = \partial_{\mathtt{a}}\lambda$, λ being an infinitesimal commutative gauge parameter on dS^4, while the latter is given by $\delta A_\mu = \{\Lambda, x_\mu\} + \Theta\,\{\Lambda, a_\mu\}$, where Λ is an infinitesimal noncommutative gauge parameter. The result is

$$\delta A_\mu = -h\left(\partial_\tau\Lambda\partial_\sigma x_\mu - \partial_\sigma\Lambda\partial_\tau x_\mu\right) + \frac{\kappa}{\sin\theta}\left(\partial_\theta\Lambda\partial_\phi x_\mu - \partial_\phi\Lambda\partial_\theta x_\mu\right) + \Theta\delta A^{(1)}_\mu + \mathcal{O}(\Theta^2)\ , \tag{B.1}$$

where $h = h(\tau,\sigma)$ is given in (4.69). Then at zeroth order in Θ the commutative gauge potentials are tangent to dS^4, while an additional degree of freedom Φ is associated with perturbations normal to the surface. Thus the zeroth order result for A_μ and Λ is given in

$$\begin{aligned} A_\mu &= h\left(\mathcal{A}_\tau\partial_\sigma x_\mu - \mathcal{A}_\sigma\partial_\tau x_\mu\right) + \frac{\kappa}{\sin\theta}\left(\mathcal{A}_\theta\partial_\phi x_\mu - \mathcal{A}_\phi\partial_\theta x_\mu\right) + \Phi\, x_\mu \\ &\quad + \Theta A^{(1)}_\mu + \mathcal{O}(\Theta^2) \\ \Lambda &= \lambda + \Theta\Lambda^{(1)} + \mathcal{O}(\Theta^2) \end{aligned} \tag{B.2}$$

For the first order terms, $A_\mu^{(1)}$ and $\Lambda^{(1)}$, we demand consistency with the equation $\delta A_\mu = \{\Lambda, x_\mu\} + \Theta\{\Lambda, a_\mu\}$. A solution is

$$
\begin{aligned}
A_\mu^{(1)} &= \partial_\tau x_\mu\left(\frac{a^3 a'}{2}\Big(h^2\mathcal{A}_\tau^2 + \kappa^2\sin^2\sigma\mathcal{A}_\Omega^2\Big) + \frac{ha}{2}\partial_\tau\Big(\frac{h}{a}\mathcal{A}_\sigma^2\Big) - \frac{h\kappa}{\sin\theta}\Big(\mathcal{A}_\theta\mathcal{F}_{\sigma\phi} + \mathcal{A}_\phi\partial_\theta\mathcal{A}_\sigma\Big)\right) \\
&+ \partial_\sigma x_\mu\Bigg(h^2\mathcal{A}_\tau\mathcal{F}_{\sigma\tau} - \frac{h}{a}\mathcal{A}_\sigma\partial_\tau(ah\mathcal{A}_\tau) + \frac{1}{4}\partial_\sigma h^2\mathcal{A}_\tau^2 - \frac{\kappa^2\sin(2\sigma)}{4}\mathcal{A}_\Omega^2 \\
&+ \frac{h\kappa}{\sin\theta}\Big(\mathcal{A}_\theta\mathcal{F}_{\tau\phi} + \mathcal{A}_\phi\partial_\theta\mathcal{A}_\tau\Big)\Bigg) \\
&+ \frac{\kappa\,\partial_\theta x_\mu}{\sin\theta}\left(h\Big(\mathcal{A}_\tau\mathcal{F}_{\sigma\phi} + \cot\sigma\mathcal{A}_\tau\mathcal{A}_\phi - \frac{1}{a}\mathcal{A}_\sigma\partial_\tau(a\mathcal{A}_\phi)\Big) + \frac{\kappa}{2}\Big(\frac{\partial_\theta\mathcal{A}_\phi^2}{\sin\theta} - \cos\theta\mathcal{A}_\Omega^2\Big)\right) \\
&- \frac{\kappa\,\partial_\phi x_\mu}{\sin\theta}\left(h\Big(\mathcal{A}_\tau\mathcal{F}_{\sigma\theta} + \cot\sigma\mathcal{A}_\tau\mathcal{A}_\theta - \frac{1}{a}\mathcal{A}_\sigma\partial_\tau(a\mathcal{A}_\theta)\Big) + \frac{\kappa}{\sin\theta}\Big(\mathcal{A}_\theta\mathcal{F}_{\theta\phi} + \mathcal{A}_\phi\partial_\theta\mathcal{A}_\theta\Big)\right) \\
&- \frac{x_\mu}{2}\left(h^2\Big(a^2\mathcal{A}_\tau^2 - \frac{\mathcal{A}_\sigma^2}{a^2}\Big) + \kappa^2 a^2\sin^2\sigma\mathcal{A}_\Omega^2\right) \\
&- h\Big(\mathcal{A}_\tau\partial_\sigma(\Phi x_\mu) - \mathcal{A}_\sigma\partial_\tau(\Phi x_\mu)\Big) + \frac{\kappa}{\sin\theta}\Big(\mathcal{A}_\theta\partial_\phi(\Phi x_\mu) - \mathcal{A}_\phi\partial_\theta(\Phi x_\mu)\Big) \\
\\
\Lambda^{(1)} &= h\mathcal{A}_\sigma\partial_\tau\lambda - \frac{\kappa}{\sin\theta}\mathcal{A}_\phi\partial_\theta\lambda\,,
\end{aligned}
\tag{B.3}
$$

where we define $\mathcal{A}_\Omega^2 = \mathcal{A}_\theta^2 + \mathcal{A}_\phi^2/\sin^2\theta$. $\mathcal{F}_{\mathsf{ab}} = \partial_{\mathsf{a}}\mathcal{A}_{\mathsf{b}} - \partial_{\mathsf{b}}\mathcal{A}_{\mathsf{a}}$ are the commutative field strengths. In obtaining (B.3) we have used the explicit expression for the de Sitter solution, $a^2 = \tau^2 + 1$.

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