REAL-TIME HEAT FLUX ESTIMATION USING FILTER BASED SOLUTIONS FOR INVERSE HEAT CONDUCTION PROBLEMS

by

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ABSTRACT

Real-time heat flux measurement has great significance in numerous applications such as metal heat treating, quenching, fire safety tests, furnace operation, thermal therapy and more. Direct measurement of heat flux is not always possible and therefore, developing methods for accurate heat flux estimation using temperature data is desirable for several applications.

An Inverse Heat Conduction Problem (IHCP) has to be solved to estimate the unknown heat flux at the surface, knowing the temperature measurements in the interior points of the medium. This problem is mathematically ill-posed and a small error in temperature data result in large errors in estimated heat fluxes. The focus of this dissertation is developing solutions for IHCP’s using digital filter approach. A series of IHCP’s are closely investigated and filter based solutions are developed for each problem which allows real-time heat flux estimation. The characteristics of the filter form solutions are discussed in detail for each case. A series of articles are written to discuss these problems.

The articles in this dissertation provide real-time solutions for a series of IHCP’s with wide applications in industries. In particular, a filter based solution technique is developed for heat flux estimation in multi-layer mediums with temperature dependent material properties. While one application of this solution technique in using Directional Flame Thermometers is discussed in detail, there are many other applications for this solution method, e.g. in thermal therapy and tissue engineering (Articles 1-5). Furthermore, the concept of using artificial neural networks as
digital filters for real-time heat flux estimation is investigated in Article 6. Several test cases are demonstrated to study the applicability of ANN’s for solving linear and non-linear IHCP’s.

Finally, developing filter form solutions for two-dimensional IHCP’s with multiple unknown heat fluxes is presented in Article 7. This study not only provides a basis for heat flux estimation in heat treating processes in industries, but also proves the applicability of the filter solution for multi-dimensional mediums with more complexities and demonstrates the characteristics of this technique.
DEDICATION

This dissertation is dedicated to everyone who encouraged me, helped me and guided me through creating this manuscript. In particular, my parents, Kobra Kandy and Sadegh Najafi, who stood by me throughout the time taken to complete this work.
LIST OF ABBREVIATIONS AND SYMBOLS

\[ f \]  filter coefficients (coupled solution)

\[ \textbf{F}_1 \]  filter matrix (X22 case, layer 1)

\[ \textbf{F}_2 \]  filter matrix (X21 case, layer 2)

\[ g \]  filter coefficients (coupled solution)

\[ \textbf{G} \]  Green’s function

\[ \textbf{G} \]  filter matrix (X12 case)

\[ k \]  thermal conductivity, W/m-K

\[ L \]  thickness of the layer, m

\[ m_f \]  number of future time steps

\[ m_p \]  number of past time steps

\[ q \]  heat flux, W/m2

\[ S \]  sum of squares of the temperature error, K2

\[ t \]  time, s

\[ T \]  temperature, K

\[ x \]  spatial coordinate, m

\[ x' \]  dummy integration variable, Eqs. (1, 13)

\[ \textbf{X} \]  sensitivity matrix for unknown surface heat flux

\[ y \]  measured temperature at boundary \( x=L \)
$Y$ measured temperature at location $x=x_1$

$Z$ sensitivity matrix for measured temperature boundary condition at $x=L_2$

**GREEK/ROMAN**

$\alpha$ thermal diffusivity, $k/C$, $m^2/s$

$\alpha_T$ Tikhonov regularization parameter

$\beta$ eigenvalue

$\phi$ step response function for unit heat flux at $x=0$

$\eta$ step response function for unit temperature at $x=L$

$\tau$ integration variable, Eqs. (1, 13)

**SUBSCRIPTS**

0 surface location or reference value

$c$ reference value for non-dimensionalization

$i$ time index

$m$ eigenvalue index

$ss$ steady state

$i_{max}$ last time index

$X_{12}$ Cartesian heat conduction problem with type 1 and type 2 boundary conditions

$X_{21}$ Cartesian heat conduction problem with type 2 and type 1 boundary conditions

$X_{22}$ Cartesian heat conduction problem with type 2 and type 2 boundary conditions

**SUPERSCRIPT**

$\tilde{}$ dimensionless parameter
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I am pleased to have this opportunity to thank many colleagues, friends, and faculty members who have helped me with this research project. I am most indebted to Dr. Keith A. Woodbury, my advisor and the chairman of the dissertation committee for his continuous help and support. Special thanks to Prof. James V. Beck for his invaluable guidance on this project. I would also like to thank all of my committee members, Dr. Clark Midkiff, Dr. Robert Taylor, Dr. John Baker and Dr. Muhammad Sharif, for their support of both the dissertation and my academic progress.

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INTRODUCTION

An introduction to inverse heat conduction problems, heat flux measurement sensors as well as the outline of this dissertation are provided in this section.

Inverse Heat Conduction Problems

A “forward” heat conduction problem generally refers to the problem of calculating temperatures within a medium when boundary conditions are known. On the other hand, an “inverse” heat conduction problem is the problem of calculating the unknown active boundary condition (which causes the temperature response in the domain) when knowing temperature measurements in one or more points within the domain. Inverse heat conduction problems are mathematically ill-posed problems as they are extremely sensitive to the measurement errors [1]. Considering Fig. 1, the mathematical description of the “forward problem” can be given as below:

![Diagram](image)

Figure 1: a slab which is insulated at one end and heat flux is applied at the surface
Governing equation: \[
\frac{1}{\alpha} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}
\] (1)

Boundary conditions:
\[
-k \left. \frac{\partial T}{\partial x} \right|_{x=L} = 0, \quad -k \left. \frac{\partial T}{\partial x} \right|_{x=0} = q(t)
\] (2)

Initial condition:
\[
T(x, 0) = T_o
\] (3)

Unknown parameter:
\[
T(x, t) = ?
\] (4)

The mathematical statement of an “inverse problem” can be written as:

Governing equation:
\[
\frac{1}{\alpha} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}
\] (5)

Boundary conditions:
\[
-k \left. \frac{\partial T}{\partial x} \right|_{x=L} = 0, \quad T(x, t) = Y_j(t)
\] (6)

Initial condition:
\[
T(x, 0) = T_o
\] (7)

Unknown parameter:
\[
q(0, t) = -k \left. \frac{\partial T}{\partial x} \right|_{x=0} = ?
\] (8)

There are various solution methods for the IHCP’s which are based on solution of a linear problem expressed in the form of a convolution integral:
\[
T(x, t) = T_o + \int_0^t q(\lambda) \left( \frac{\partial \varphi(x, t-\lambda)}{\partial t} \right) d\lambda = T_o + \int_0^t \left[ \frac{\partial \varphi(x, t-\lambda)}{\partial \lambda} \right] q(\lambda) d\lambda
\] (9)

Some of these techniques are Function Specification Method (FSM), Tikhonov Regularization (TR), Conjugate Gradient Methods (CGM), Singular Value Decomposition (SVD), Mollification Method (MM) and Kalman Filtering (KF) [1]. Considering discrete time steps: \( t = \Delta t, 2\Delta t, ..., m\Delta t \), Eq. (9) can be represented in a matrix form as
\[
T = Xq
\] (10)
In Eq. 10, $X$ is the $m \times m$ sensitivity coefficient matrix. The $X$ matrix is a lower triangular matrix which can be determined using either exact or numerical solution techniques.

In most of IHCP solution techniques, solutions are found by minimizing the difference between the measured values $Y_i$ and the values from Eq. (10):

$$S = (Y - T)^T (Y - T)$$  \hspace{1cm} (11)

The main method used in this work is the Tikhonov regularization (TR) method. TR adds a regularization term ($\alpha_t$) to correct oscillations in the solution for $q$ to the sum of squares in Eq. (11) and convert the ill-posed problem to a near well-posed problem. The zeroth order TR can be given as:

$$q = \left[ X^T X + \alpha I \right]^{-1} X^T Y$$  \hspace{1cm} (12)

where $\alpha_t$ is the Tikhonov regularization parameter. TR is a whole time domain approach which means that this method applies to a full set of data over the entire time period.

The above equation can be also written as:

$$q = \left[ X^T X + \alpha I \right]^{-1} X^T Y = F Y; \quad F = \left[ X^T X + \alpha I \right]^{-1} X^T$$  \hspace{1cm} (13)

Where $F$ is the filter matrix. The filter matrix has several interesting characteristics [2]. The rows of the matrix, except for the first few and last few, have all the same values, but each row is shifted over by one time step. Likewise, all the columns are the same, but each successive one shifted down by one row. Also, the values on each row and column vanish to zero for small and large times. The structure of the filter matrix is shown below:
For dimensionless time step of 0.02, dimensionless location of $x=1$ and assuming, the filter matrix is calculated and few rows of the filter matrix are plotted in the figure below:

![Image of filter matrix](image)

\[
\mathbf{F} = \begin{bmatrix}
    f_0 & f_1 & f_2 & \ldots & f_{2-N} & f_{1-N} \\
    f_1 & f_0 & f_1 & \ldots & f_{2-N} & \vdots \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    f_N & f_{N-1} & \ldots & f_1 & f_0 & f_{-1} \\
    f_{N-1} & f_{N-2} & \ldots & f_2 & f_1 & f_0
\end{bmatrix} \tag{14}
\]

As seen, all of the rows look similar and there are only several previous/future time steps with non-zero values. The concept of the filter algorithm is that the solution for the heat flux at any time ($M$) is only affected by the recent temperature history and a few future time steps.

\[
\hat{q}_M = \sum_{j=-n}^{w} f_{M-j} Y_j \tag{15}
\]

Where $f_i$s are the filter coefficients which are the components of one row of the filter matrix. Alternatively, all the $f$-filter coefficients can be found at one time by setting all the $Y$ components
equal to zero except the \( m_f \) component of \( Y \) is set equal to one \( (Y_{mf}=1) \). The solution of the IHCP with this data gives the \( f \) coefficients.

Heat Flux Measurement

Heat flux measurement is needed in several industrial processes such as metal heat treating, quenching, fire safety tests, furnace operation and more. In high temperature environments, such as fires and furnaces heat flux measurement can be used to support control systems, fire safety tests and the development of engineering models. Currently, two major types of measuring sensors are used in standard fire tests for measuring the heat flux namely active and equilibrium sensors [3].

Active sensors measure the heat flux across a measured temperature difference, such as Gardon and Schmidt-Boelter gauges [4, 5]. These sensors have usually a relatively low response time and work based on the concept of measuring the temperature gradient within the sensor. For this purpose, water needs to continuously pass through the sensor to maintain a temperature gradient from the fire. As the result, the use of active sensors is limited to applications where adequate water is available and water tubes can be installed safely. Possible condensation of the unburned fuel or water on the surface of the sensor could be also problematic when using active sensors [6].

Active sensors are currently being used in variety of applications. However, they are relatively expensive, difficult to install and maintain (since water tubing should be protected), and less reliable than temperature measurement equipment [6]. Therefore, developing methods which allow heat flux estimation using temperature measurement data is desirable.
The second types of heat flux measurement sensors, equilibrium sensors, consist of two metal plates (Inconel) with an insulation layer (ceramic fiber) in between. Thermocouples are installed on the backside of each plate and covered with insulation material. The plate’s temperature increases quickly and reaching quasi-equilibrium with the fire environment. An Inverse Heat Conduction Problem (IHCP) has to be solved to estimate the unknown heat flux at the surface, knowing the temperature measurements in the interior points of the medium. Although equilibrium sensors have generally a slower response time compared with active sensors, they are relatively inexpensive and do not need water for operation. The easy installation makes them appropriate for different environments and variety of applications. Moreover, since the surface temperature is very close to that of the fire environment, there is no concern for condensation of water/unburned fuel and the associated uncertainties.

![Schematic of a DFT](image)

Figure 3: Schematic of a DFT

Different types of equilibrium sensors such as Plate Thermometers, Sandia Hemispherical Heat Flux Gage and Directional Flame Thermometers are tested and discussed in [7, 8, 9].

A schematic of a DFT is shown in Fig. 1. The original DFT design involved a thin metal disk mounted in a steel tube [10, 11]. To minimize heat loss from the unexposed surface of the disk,
multiple radiation shields and some ceramic fiber insulation were mounted behind the front disk. Sandia National Laboratories adapted DFTs for use in large pool fire and other tests. Their goal was to provide both transient and quasi-steady heat transfer measurements in various fire environments [12]. Analyzing DFT data over the entire test duration needs an inverse heat conduction code which uses two temperature measurement histories for estimating the net heat flux to the exposed surface ($q_{\text{front}}$, in Fig. 1). Presently, analysis of dynamic temperature data from the DFTs to compute heat flux information must be performed off-line at the conclusion of data-gathering using a inverse heat conduction code.

Numerous methods have been developed and applied for solving IHCPs. Several of these methods need the whole time domain data for the analysis and cannot be used for real-time applications. More attention has been attracted to approaches with real-time capability during recent years. Availability of a near real-time algorithm for accurate reduction of data will allow for continual monitoring of the furnace during operation. This will result in better furnace control and significant savings in energy and cost.

The focus of this dissertation is developing near real-time solutions for inverse heat conduction problems (IHCP’s), particularly in multi-layer mediums such as DFT case. The solution is developed through different steps and an article is written to explain each step.

The first article discusses a method of accounting for thermal action at the remote boundary using a second measured temperature history in a single layer medium. The measurement need not be at an actual boundary but can be at an interior point in the domain. The solution is then written in a digital filter form which allows near real-time heat flux estimation. In the second article, the solution is developed for a multi-layer problem. For this purpose one IHCP is solved for each layer and a coupled solution in the digital filter form is determined eventually. The
developed solution allows accounting for the known thermal contact resistance. The third article discusses an innovative method for interpolating filter coefficients to account for temperature dependent material properties and the nonlinear problem. The fourth article describes applying the methods developed for surface heat flux estimation when using DFT. Numerical examples as well as actual field data are used to demonstrate the developed solution technique. The fifth article explains developing a LabView based interface for DFT application which allows the end user with limited or no knowledge of IHCP’s to calculate surface heat fluxes. The sixth article provides an overview of using neural networks as a digital filter method for real-time heat flux estimation for both linear and non-linear problems. Article 7 studies developing filter based solution for IHCP’s in two-dimensional mediums with multiple unknown heat fluxes and provides a detailed discussion regarding the characteristics of the filter matrix through several numerical examples.

The solutions developed in this dissertation can be used for near real-time heat flux estimation in numerous applications. Accurate and real-time measurement of heat flux improves controllability over variety of thermal processes which includes but not limited to furnace control, fire safety tests, quenching, metal heat treating, thermal therapy and more.
Directional Flame Thermometer

Developing accurate engineering models for fire safety tests requires heat flux measurements in high-temperature environments, such as furnaces and fires. Active sensors (water-cooled sensors) are the direct reading Schmidt-Boelter or Gardon Gauge designs that are used commonly in different fire safety tests. However, using active sensors is not recommended in some applications (e.g., due to high temperatures, lack of cooling, soot deposition, cost consideration etc.).

Directional Flame Thermometer (DFT), is an equilibrium sensor (passive sensor) which was developed at Sandia National Laboratories, for measuring heat transfer in large pool fires. DFTs do not require a calibration factor. Some of the advantages of DFT’s are higher temperature capability, very rugged, relatively low cost, require no cooling, easy installation, and are not susceptible to fouling of the sensing surface. These characteristics ease the installation of DFT’s in a wide range of fire applications. DFT’s have been used for temperature and heat flux measurement in variety of applications such as furnaces, large pool fires, spill fires, rocket launch accidents, fire resistance tests, research of forest and wild land fires and others.

DFTs consist of two Inconel 600 plates with a layer of ceramic fiber insulation placed between. Two metal sheathed thermocouples attached to the unexposed faces of the plates. Inverse heat conduction analysis is used to calculate the heat fluxes from transient temperature measurements made with two installed thermocouples.

The desired unknown parameter is typically the incident radiative heat flux \( q_{inc,r} \) or the thermal exposure to the test item which is sum of the incident radiative heat flux and the convective flux. The incident heat flux is the measurement made by commercially available gages which is independent of the surface chosen. The net heat flux, which is measured by a DFT using an
inverse heat conduction code, is the net flux into the DFT (which is probably different from the
heat flux into the test item of interest). The incident radiative flux can be estimated from a DFT
by use of an energy balance.

The energy balance on any surface (DFT, test item, etc.) can be calculated as below:

\[ q_{net} = q_{inc,r} - q_{ref} - q_{emit} + q_{conv} \]  \hspace{1cm} (16)

Where

- \( q_{inc,r} \) = incident radiative heat flux
- \( q_{ref} \) = reflected radiative heat flux
- \( q_{emit} \) = emitted heat flux from surface
- \( q_{conv} \) = convective heat flux, assumed positive into the surface; \( q_{conv} \) is expressed as Newton's
Law of Cooling.

The first two terms on the right hand side of the above equation can be calculated as:

\[ q_{inc,r} - q_{ref} = \alpha_{DFT} q_{inc} \]  \hspace{1cm} (17)

where \( \alpha_{DFT} \) is the plate absorptivity. The emitted heat flux can be also found as:

\[ q_{emit} = e_{DFT} \sigma T_{DFT}^4 \]  \hspace{1cm} (18)

Where \( e_{DFT} \) is the plate emissivity and \( \sigma \) is the Stefan Boltzmann constant \((5.67 \times 10^{-8} \text{ W m}^{-2} \text{K}^{-4})\).

The convection term can be estimated as:

\[ q_{conv} = h (T_{DFT} - T_{env}) \]  \hspace{1cm} (19)

For late times, when the DFT temperature is close to the temperature in the environment, the
convective term will be negligible.
References


ARTICLE 1: FILTER SOLUTION OF INVERSE HEAT CONDUCTION PROBLEM USING MEASURED TEMPERATURE HISTORY AS REMOTE BOUNDARY CONDITION

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Abstract

The inverse heat conduction problem (IHCP) involves estimation of a surface heat flux from transient temperature measurements inside a heat conducting body. Commonly an insulated remote boundary or one with a known heat transfer coefficient is modeled. However, in many practical applications, the precise thermal condition at the remote boundary is not known. In this paper, a method of accounting for thermal action at the remote boundary using a second measured temperature history is presented. The measurement need not be at an actual boundary but can be at an interior point in the domain.

The IHCP solution herein is achieved through the filter coefficient method, which uses filter coefficients in a convolution summation. By using the filter technique, near real-time heat flux measurements can be continuously obtained in manufacturing settings to enhance...
productivity. Also the filter concept opens the way for the development of new scientific instruments that incorporate inverse problem methods.

Introduction

The inverse heat conduction problem (IHCP) is typically defined as the problem of using internal temperature measurements to find heat fluxes on the surface. Sometimes this specific task is termed a \textit{boundary IHCP} to emphasize recovery of the unknown surface heating action. The IHCP has been studied widely during last few decades, and notable texts have been authored by Beck et al. \cite{Beck1990}, Alifanov \cite{Alifanov2000}, Ozisik and Orlande \cite{Ozisik1999} and Murio \cite{Murio2004}, to name a few. Several different methods have been developed and demonstrated for solving IHCPs. All boundary IHCP solutions assume full knowledge of the governing equation and thermal material properties, and full knowledge of boundary conditions on all but the active surface under investigation.

Remote boundaries are typically assumed to be insulated or cooled with a known heat transfer coefficient. For example, Ijaz et al. \cite{Ijaz2009} consider estimation of heat flux histories on two faces of a two-dimensional domain. Their model assumes perfectly insulated surfaces on the other two faces, and temperature data are gathered on these insulated surfaces to drive the solution of the IHCP. Chen, et al. \cite{Chen2010} also consider a two-dimensional problem, with two faces insulated, but the third face at a homogeneous temperature condition. In their method, the temperature, and not the heat flux, is determined on the fourth face of the domain. Mulcahy et al. \cite{Mulcahy2012} consider a problem in an annular three-dimensional geometry and seek the heat flux history on the inner surface of the annulus as a function of angle $\theta$ and axial coordinate $z$. They assume that all the other surfaces are insulated. Chan \cite{Chan2013} performs an interesting study to determine the heat flux on a cylinder cooled by a jet. In this two-dimensional problem, measured
temperatures on the surface of the cylinder are used to compute the heat flux on the cylinder, but this is done through solution of the flow field around the cylinder. In this case, insulated and no-slip boundaries are imposed on the bounding walls of the domain. Khajehpour et al. [9] tackle two-dimensional problems with finite element analysis and obtain solutions by splitting the domain into sub-regions. However, they do assume the remote boundaries are either insulated or have known convection cooling.

Chen, et al. [10] consider a slightly different problem. Their analysis of a one-dimensional domain considers two subsurface sensors, but seeks to compute the heat flux at both exposed surfaces.

Since IHCPs are mathematically ill-posed, an appropriate regularization method needs to be applied in order to convert the original ill-posed problem to a nearby well-posed problem and achieve a solution for it. Tikhonov regularization (TR) [11,12] is a common technique to stabilize the IHCP. It is commonly applied over the whole time domain which means that the solution procedure needs to access all the observations at once [1].

More recently, efforts have been made to develop real-time or filter forms for processing temperature data to solve the IHCP. Khorrami, et al. [13] employ an interesting approach based on the premise that the heat flux component at a particular time is a linear combination of the temperature around the time of its occurrence. They use an artificial neural network to determine the required coefficients of the linear filter based on detailed numerical simulation of the target process. Deng and Hwang [14] also use artificial neural networks to generate the filter solutions for heat flux estimation based on temperature histories. Ijaz et al. [5] solve a transient IHCP by using Kalman filter. LeBreux et al. [15] use a combination of Kalman filtering and recursive least squares to achieve a real-time algorithm to determine heat flux history for industrial
applications. Lamm [16, 17] employs a sequential computation based on TR by using a recent subset of available data and a limited number of past predictions and observations. Cabeza, et al. [18] studied the filter effect of the function specification method and the truncated singular value decomposition method in a sequential form [19]. By examining the power spectral densities of the two methods, they concluded that the regularization methods act as band-pass filters. Most recently, Woodbury and Beck [20] study the structure of the TR problem and conclude that the method can be interpreted as a sequential filter formulation for continuous processing of data. They show that the computed heat fluxes using the whole domain solution and the filter coefficient solution are virtually the same for the constant-property solutions.

In this paper, a method is developed to incorporate the temperature measurement history from a second subsurface sensor as a remote boundary condition in an IHCP solution. The second measurement may, or may not, coincide with the physical boundary of the domain. Tikhonov Regularization is used to stabilize the solution and the resulting algorithm is written in filter form [20]. An example problem is considered and both the whole domain method and the sequential filter solution method are illustrated. The filter solution of the IHCP has a number of advantages including simplicity, continuous operation and application to moderate nonlinearity [21] which makes it an appropriate approach for real-time heat flux estimation. Measuring heat flux has a great significance in several scientific experiments as well as industrial applications such as monitoring manufacturing processes and fire safety tests. Using IHCP filter solution, instruments can be designed and built for near real-time heat flux estimation by measuring the temperature histories. It should be noted that, filter solution is really representing an IHCP solution algorithm, typically for a linear problem. However, when the material properties are
temperature dependent, the problem is no longer linear. Therefore, by “moderate nonlinearity”, the authors referring to the problems with temperature dependent material properties.

**Problem**

The IHCP solution typically starts by minimizing a sum of squares function, which is the sum of the squares of the difference between the measured and calculated temperatures at a location $x_1$. The calculated temperatures are functions of the unknown surface heat flux. Some form of regularization is needed, and the function specification method [1] or Tikhonov Regularization (TR) [11,12] are only two of many techniques. For the function specification method, the heat flux is found using a sum of squares function that uses both past and future information. The Tikhonov whole domain regularization method will have a sum of squares function over the total time domain. However, the nature of the IHCP is such that a given heat flux component is a linear function of only a limited number of past and future measured temperatures. This point is illustrated below.

For simplicity, at this point a sum of squares function over the total time domain will be used in this analysis. Furthermore this time domain is large enough so that there is long middle time region in which the very early conditions and the final time conditions have negligible effect upon the heat flux. This, too, is discussed further below.

The describing heat conduction equation is

$$\frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) = C \frac{\partial T}{\partial t}, \quad 0 < x < L, \quad t > 0$$

(1)

This equation permits the thermal conductivity, $k$, and the volumetric heat capacity, $C$, to be functions of temperature. However, in the derivation that follows, these properties are considered constants. It is found that the filter solution of the nonlinear IHCP for moderate changes in the
properties is sufficiently accurate [21]. In this case, the filter coefficients need to be found for different temperatures using corresponding material properties and then by applying linear interpolation, the values of filter coefficients can be found for each time step. Eq. 1 implies that the problem is one-dimensional Cartesian, side heat losses are not present and no internal heat sources are present. A consequence of all these observations is that it usually is not necessary to attempt to estimate the heat flux extremely accurately, such as to one or two percent of the maximum value. With no loss of generality, the initial temperature is considered to be zero,

\[ T(x,0) = 0 \]  

(2)

Figure 1 illustrates the one-dimensional domain and summarizes the nomenclature. The temperature history at a location \( x = L \) is measured and is used as a boundary condition. This measurement location may be at a physical boundary, or may be at a subsurface location within the domain.

Hence, the plate can be much thicker than the length \( L \) and heat may be flowing though the plane at \( L \).
The measured temperature boundary condition is expressed as

\[ T(L, t) = y(t) \quad \text{(3)} \]

The temperature is also measured at a point \( x_1 \), where \( 0 \leq x_1 < L \). This data serves as supplemental data required for solution of the IHCP. This measured data is written as

\[ \hat{T}(x_1, t) = Y(t) \quad \text{(4)} \]

where \( \hat{T} \) indicates an estimate of the temperature at \( x_1 \). It is an estimate since it is a measured value which has some errors. The location \( x_1 \) is between zero and \( L \) but, in practice, \( x_1 \) should be equal to or smaller than \( L/2 \), smaller being better.

The objective of the IHCP is to estimate the surface heat flux,

\[ -k \frac{\partial T}{\partial x}(0, t) = q(t) = ? \quad \text{(5)} \]

using information coming from the measured temperatures indicated by Eq. 4.

A sum of squares function over the complete time domain is

\[ S = \sum_{i=1}^{N} (Y_i - T(x_1, t_i))^2 \quad \text{(6)} \]

where \( T(x_1, t_i) \) is calculated and is a function of the unknown heat flux components. An expression is needed for this term. This temperature is a function of the heat flux and the given temperature at \( x = L \). These temperatures can be computed using a numerical method, such as Crank-Nicolson, or using an exact solution, which is used in this paper, but either one can be used. When using Tikhonov regularization another term is added to the sum of squares, such as the sum of squares of the heat flux components.
Temperature Computation Using an Exact Procedure

The temperature at \( x_1 \) is a function of the surface heat flux \( q(t) \) and the temperature at \( x = L \). One exact way to obtain the temperature uses Green’s functions [22]; an expression is

\[
T(x_1, t) = \frac{\alpha}{k} \int_{\tau=0}^{t} q(\tau) G_{X21}(x_1, 0, t-\tau) d\tau + \alpha \int_{\tau=0}^{t} y(\tau) \left( -\frac{\partial G_{X21}}{\partial x'}(x_1, L, t-\tau) \right) d\tau \quad (7)
\]

The Green’s function in this equation is for a boundary condition of the second kind (a gradient condition, also called Neumann) at \( x = 0 \) and the first kind (specified temperature, also called Dirichlet) at \( x = L \). The notation X21 denotes a Cartesian geometry with a boundary condition of the second kind at \( x = 0 \) and the first kind at \( x = L \). In using Eq. 7 with equally spaced times, it is necessary to specify the type of variation between adjacent times. At this point, the heat flux components, \( q_i \) and \( q_{i+1} \), and also between adjacent temperature components, \( y_i \) and \( y_{i+1} \), are specified as being constant between time points (step function); but they could also be linear, quadratic and cubic. The step variation can be described by

\[
q(t) = q_i, \quad t_i < t < t_{i+1} \quad (8 \text{ a})
\]

\[
t_i = i \Delta t \quad (8 \text{ b})
\]

A solution for the X21 case with a constant heat flux at \( x = 0 \) and zero temperature at \( x = L \) is [23]

\[
\frac{T(x, t)}{q_0 L/k} = \left( 1 - \frac{x}{L} \right) - 2 \sum_{m=1}^{\infty} \frac{e^{-\beta_m^2 a^2/t}}{\beta_m^2} \cos \left( \frac{\beta_m x}{L} \right) \quad (9)
\]

where \( \beta_m = (m - 1/2)\pi \), \( m = 1, 2, \ldots \) (The notation used to describe these solutions is sometimes referred to the BCHL notation, after the authors of Ref. 22. The BCHL notation for this case is X21B10T0. More information on the notation can be found in Refs 22 and 24.) For a constant
(step) heat flux input, the dimensionless temperatures at $x = x_1$ for times $t = \Delta t$, $2\Delta t$, ..., $i\Delta t$ are designated $\phi_1$, $\phi_2$, ..., $\phi_i$. These values are termed the response function and represent the dimensionless temperature rise at the measurement location due to a unit disturbance (step change) in the surface condition. More explicitly the $i$th value, $\phi_i$, is

$$
\phi_i = \frac{T(x_1, i\Delta t)}{q_0 L / k} = \left( 1 - \frac{x_1}{L} \right) - 2 \sum_{n=1}^{\infty} e^{-\frac{\beta_n^2 \Delta t}{L^2}} \frac{\cos \left( \beta_n \frac{x_1}{L} \right)}{\beta_n^2} \tag{10}
$$

Likewise, in consideration of a step change in temperature at the boundary $x=L$, the solution for a constant temperature, $T_0$, at $x = L$ is

$$
\frac{T(x, t)}{T_0} = 1 - 2 \sum_{n=1}^{\infty} e^{-\frac{\beta_n^2 \Delta t}{L^2}} \frac{\sin \left( \beta_n \left( 1 - \frac{x}{L} \right) \right)}{\beta_n} \tag{11}
$$

and the eigenvalues are the same as given below Eq. 9 (the BCHL notation for this case is X21B01T0). Analogous to $\phi_i$, the response function at $x_1$ for a constant temperature disturbance $T_0$ at $x = L$, denoted as $\eta$, is

$$
\eta_i = \frac{T(x_1, i\Delta t)}{T_0} = 1 - 2 \sum_{n=1}^{\infty} e^{-\frac{\beta_n^2 \Delta t}{L^2}} \frac{\sin \left( \beta_n \left( 1 - \frac{x_1}{L} \right) \right)}{\beta_n} \tag{12}
$$

Using Eqs. 10 and 12, the temperature at $x_1$ can be found for any time-variable heat flux at $x = 0$ and temperature variation at $x=L$ through scaling and superposition of the response functions. If the heat flux is piecewise constant, and the temperature variation at $x=L$ is piecewise constant, the result from the following procedure is exact. Since Eq. 7 indicates the $q(t)$ and $y(t)$ inputs can be added (because of linearity), the two components of Eq. 7 can be considered separately.
First, consider the temperature response at \( x_1 \) due to a piecewise constant heat flux at \( x=0 \). Using step (piecewise-constant) basis functions with an initial temperature equal to zero, the temperatures at \( i, i = 1, 2, \ldots, M \) are found by scaling the unit responses and superposition and can be written as

\[
T_{q,i} = T_q(x_1, \Delta t) = \frac{L}{k}(q_1 \phi_{i})
\]

\[
T_{q,2} = T_q(x_1, 2\Delta t) = \frac{L}{k}(q_1 \phi_{i} + q_2 \phi_{i})
\]

\[
\vdots
\]

\[
T_{q,M} = T_q(x_1, M\Delta t) = \frac{L}{k}\sum_{i=1}^{M} q_i \phi_{M-i}
\]

which can be written as

\[
T_{q,i} = q_1 X_i
\]

\[
T_{q,2} = q_1 X_2 + q_2 X_1
\]

\[
\vdots
\]

\[
T_{q,M} = q_1 X_M + q_2 X_{M-1} + \cdots + q_{M-1} X_2 + q_M X_1 = \sum_{i=1}^{M} q_i X_{M-i}
\]

where, in Eq. 13a,

\[
\Delta \phi_i = \phi_{i+1} - \phi_i
\]

Note that, in Eq. 14, \( \phi_0 = 0 \), so that \( \Delta \phi_0 = \phi_1 \). In Eq. (13a,b), the \( q \) subscript on \( T \) denotes that it is the contribution to the temperature at \( x_1 \) caused by the unknown heat flux.

Similarly the contribution to the temperature at \( x_1 \) due to the temperature history at \( x = L \) is

\[
T_{y,M} = \sum_{i=1}^{M} y_i \Delta \eta_{M-i}
\]

which can be written as
\[
T_{y,M} = \sum_{j=1}^{M} y_j Z_{M-j+1}
\]  

(15b)

Then the temperature at \(x_1\) caused by the heat flux at \(x = 0\) and the temperature at \(x = L\) is the sum of the two contributions:

\[
T_M = T_{q,M} + T_{y,M} = \frac{L}{k} \sum_{i=1}^{M} q_i \Delta \phi_{M-i} + \sum_{i=1}^{M} y_i \Delta \eta_{M-i} = \sum_{i=1}^{M} q_i X_{M-i+1} + \sum_{i=1}^{M} y_i Z_{M-i+1}
\]  

(16)

The step basis function representations (and also for other basis functions) for temperature given in Eq. 16 can be described by the matrix equation of

\[
T = X q + Z y
\]  

(17)

where

\[
T = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_N \end{bmatrix}, \quad q = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{bmatrix}, \quad X = \begin{bmatrix} X_1 & 0 & 0 & \cdots & 0 & 0 \\ X_2 & X_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ X_{N-1} & X_{N-2} & X_{N-3} & \cdots & X_2 & X_1 \end{bmatrix}
\]  

(18a,b,c)

\[
y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}, \quad Z = \begin{bmatrix} Z_1 & 0 & 0 & \cdots & 0 & 0 \\ Z_2 & Z_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Z_{N-1} & Z_{N-2} & Z_{N-3} & \cdots & Z_2 & Z_1 \end{bmatrix}
\]  

(19a,b)

The components of the \(X\) and \(Z\) matrices are related to the characteristic response functions by

\[
X_i = \frac{L}{k} \Delta \phi_{i-1}, \quad Z_i = \Delta \eta_{i-1}
\]  

(20a,b)

The units of \(X\) are temperature per unit heat flux; those for \(Z\) are temperature per unit temperature (dimensionless). The columns in the \(X\) matrix of Eq. 18 are the same as the sensitivity vectors in parameter estimation. The first column is for \(q_1\), the second for \(q_2\) and so on. Notice that these sensitivity coefficient vectors are the same as each successive column of
values is simply shifted down by one. Although there might be hundreds of heat flux components, their basic characteristics can be inferred from just examining the first column (the sensitivity vector). The effect of estimating multiple components can be inferred by shifting this curve (or vector) one unit.

When the columns of the $X$ matrix are identified as comprised of the sensitivity components, related important concepts of parameter estimation can be introduced. Two concepts are that the sensitivity vectors should 1) be large in magnitude and 2) be uncorrelated. (Actually, the scaled sensitivities are examined in which each sensitivity, $\frac{\partial T}{\partial q_i}$, is multiplied by $q_i$. This provides the same units of temperature so the sensitivities can be compared with each other and with the magnitude of the temperature rise itself.) These aspects of magnitude and uncorrelated sensitivity coefficients are examined below.

In light of the above comments, the scaled sensitivity coefficients for the heat flux components are examined. Note from Eq. (20a) that the $X_i$'s are the differences between successive values of the $\phi(i\Delta t)$ curve in Eq. 10 for a particular value of $\Delta t$ multiplied by the ratio $L/k$. Algebraic expressions for the scaled sensitivity coefficients can be found using Eqs. 10 and 20a. The first one is given by:

$$ q, \quad \frac{\partial T(x_i, \Delta t)}{\partial q_i} = q, X_i = q, \Delta \phi_0 = q, \phi_i = \frac{q, L}{k} \left( 1 - \frac{x_i}{L} \right) - 2 \sum_{n=1}^{\infty} e^{-\beta_n^2 \frac{L}{k}} \frac{\cos \left( \beta_n \frac{x_i}{L} \right)}{\beta_n^2} $$

All the other scaled sensitivity coefficients can be found as:

$$ q, \quad \frac{\partial T(x_i, i\Delta t)}{\partial q_i} = q, X_i = q, \frac{L}{k} (\phi_i - \phi_{i-1}) = q, \frac{L}{k} \left( 2 \sum_{n=1}^{\infty} \left( e^{-\beta_n^2 \frac{a(i-1)\Delta t}{L^2}} - e^{-\beta_n^2 \frac{a(i)\Delta t}{L^2}} \right) \frac{\cos \left( \beta_n \frac{x_i}{L} \right)}{\beta_n^3} \right) $$
where

\[ X_i = \frac{\partial T(x_i, \Delta t)}{\partial q_i} \]  \hspace{1cm} (21c)

Equations (21a,b) show that the sensitivity coefficients are functions of the sensor location, \( x_i \), and the time step in the data, \( \Delta t \). The behavior of these coefficients as functions of these two variables is investigated.

The scaled sensitivity coefficients are illustrated in Fig. 2 and Fig. 3. For convenience, dimensionless forms of the equations are treated by plotting

\[ \frac{q_i X_i}{q_i L / k} = \tilde{X}_i \]  \hspace{1cm} (22)

versus dimensionless time.

![Figure 2](image_url)

Figure 2. Components of the dimensionless sensitivity vector \( \tilde{X} \) for the heat flux at \( x = 0 \) for sensor at \( x_i/L = 0.5 \) for different time steps \( \alpha \Delta t/L^2 \).
Figure 2 shows the components of $\hat{X}$ for a sensor located at $x_1/L = 0.5$ for different values of dimensionless time step $\alpha \Delta t/L^2$. Figure 2 indicates that higher values of dimensionless time step result in higher sensitivity coefficients. From a parameter estimation point of view, this is highly desirable. However, the time step in the data also controls the resolution of the heat flux history, and larger time steps will only permit coarser piecewise constant representations of the heat flux. A balance between increased sensitivity and fidelity of the heat flux function must be struck.

Figure 3 shows the components of the sensitivity vector $\hat{X}$ for a fixed time step of $\alpha \Delta t/L^2 = 0.02$ for sensor locations at $x_1/L = 0.0, 0.25, 0.5, 0.75, \text{ and } 1.0$. The highest sensitivities result from sensor locations closest to the heated surface, and zero sensitivity results from sensors at $x_1 = L$.

![Graph](image.png)

Figure 3. Components of the sensitivity vector $\hat{X}$ for the heat flux at $x = 0$ for time step $\alpha \Delta t/L^2 = 0.02$ and various sensor locations $x_1/L = 0.0, 0.25, 0.50, 0.75, 1.0$
IHCP Solution Using Tikhonov Regularization

The whole domain Tikhonov regularization method [1,25] is used to solve the IHCP. Later the filter coefficients are found from the solution. The solution starts with a matrix form for the sum of squares with an added regularization term given by

\[
S = (Y - Xq - Zy)^T (Y - Xq - Zy) + \alpha_q H^T H q
\]  

(23)

\(S\) is minimized with respect to the parameter vector \(q\). The symbol \(Y\) is the temperature measurement vector at \(x_1\) and \(y\) is the measured boundary temperature at \(x=L\). The initial temperature is zero. The \(\alpha_q\) symbol is the Tikhonov regularization parameter which controls the degree of regularization (and bias!) introduced into the solution. Various orders of regularization can be selected and are implemented via the matrix \(H\). The first order method is selected here which uses first differences of the heat flux components. The matrix operator \(H\) is a forward difference approximation for the first derivative of \(q\) (see [1, page 139].) The estimated value heat flux vector, denoted \(\hat{q}\), is found by minimizing \(S\) with respect to \(q\) and is given by

\[
\hat{q} = [X^T X + \alpha_q H^T H]^{-1} X^T (Y - Zy) = F (Y - Zy) = FY + Gy
\]  

(24)

Note that this equation is dimensional. \(F\) has the same definition as the filter matrix of Ref. 20, however in the present application it is defined with the \(X\) matrix from the X21B10T0 case and Ref. 20 considers the X22B10T0 case. The units of \(F\) are heat flux per unit temperature (W/m²-K). In Eq. (24), the term \(Zy\) can be viewed as a modification or “correction” to the measured data \(Y\) to account for the contribution of the non-homogeneous boundary condition at \(x=L\) to the measured \(Y_i\) values. Recall that, although Eq. 24 is dimensional, the matrix \(Z\) is dimensionless. Finally, observe that this equation is a linear function of the measurement temperature vectors \(Y\) and \(y\). This is important for the filter method of analysis of the IHCP.
The concept of the IHCP filter is described in Ref. 1, but explained in more detail in Ref. 20. The basic idea is that the solution for the heat flux at any time is only affected locally by the recent temperature history and a limited number of future time steps. Temperature measurements in the distant past and far future do not have significant impact on the physics of the present.

Considering for the moment only the portion of the temperature disturbance at \( x_1 \) due to the heat flux at \( x = 0 \). The portion of the heat flux due to the measured temperature at \( x_1 \) for a particular time, \( t_M \), can be computed as

\[
\hat{q}_M \cdot Y = f_1 Y_{M + n_f} + f_2 Y_{M + n_f - 1} + \ldots + f_1 Y_{M} + \ldots + f_{M + n_f - 1} Y_{M - n_p - 1} + f_{M + n_f} Y_{M - n_p} = \sum_{i=1}^{m_f + m_p} f_i Y_{M + n_f - i} (25)
\]

where the \( f \)'s are elements of a filter vector, and \( m_f \) refers to the number of future time steps, and \( m_p \) refers to the number of past time steps. The elements of this vector are the non-zero entries of a column of the \( F \) matrix in Eq. 24. The \( F \) matrix for the X22B10T0 case (denoted \( F_{X22} \)) and its properties are described in detail in Ref. 20. The \( F \) matrix for the X21B10T0 case (denoted \( F_{X21} \)) has many similar properties, notably that, except for the first few and last few rows, the entries on each row are identical but shifted in time by one time step.

The entries of the middle column of the \( F_{X21} \) matrix are plotted in Fig. 4 for different sensor locations \( x_1 / L \). It is seen that the width or range of the filter, determined by the interval with non-zero values, increases as the sensor depth increases. This indicates that the number of time steps of data required for resolving the surface heat flux increases with distance of the sensor from the active surface. This is related to the fact that the magnitudes of the entries of the sensitivity matrix \( X \) decrease with distance from the surface. Sensor locations closer to the
surface will have higher sensitivity to the unknown heat flux and will result in a digital filter with the smallest width.

Figure 4. Dimensionless filter coefficients $f$ for the surface heat flux at $x=0$ for various sensor locations $x_i/L$. First order Tikhonov regularization $\alpha_T=1e^{-4}$, $m_f=m_p=25$.

As indicated by Eq. 24, an additional term is needed beyond Eq. 25 to account for the changing temperature at $x=L$. This term, $Gy = -FZy$, can also be written in a filter form as

$$\hat{q}_{M,j} = g_1 y_{M+m_j} + g_2 y_{M+m_j-1} + \cdots + g_m y_M + \cdots + g_{M+m_p-1} y_{M-m_p+1} + g_{M+m_p} y_{M-m_p} = \sum_{i=1}^{m_p+m_j} g_i y_{M+m_j-i} \tag{26}$$

where $\hat{q}_{M,j}$ is the estimate for the portion of the heat flux at $x=0$ due to the measured temperature at $x=L$. Many of the properties of the filter matrix $G$ are similar to those of the matrix $F$, in particular that the rows of the matrix, except for the first few and last few, have all the same values, but each row is shifted over by one time step. Likewise, all the columns are the same, but
each successive one shifted down by one row. Also, the values on each row and column vanish to zero for small and large times.

Figure 5 illustrates the middle column of the $G$ matrix for different sensor depths. The values plotted are the entries of the $g$ vector. The width or range of the $g$ filter is determined by the number of non-zero entries. It is seen in Fig. 5 that the width of the filter increases with smaller sensor depths. This peculiarity can be understood since the measured temperature is at the remote boundary ($x=L$), and sensor locations further from this active boundary are at smaller values of $x$. Figure 5 also indicates that the values of $g$ are predominantly negative.

![Figure 5. Dimensionless filter coefficients $g$ for the measured temperature at $x=L$ for various sensor locations $x_1/L$. First order Tikhonov regularization $\alpha_T=1.0\times10^{-4}$. $m_f=m_p=25$.](image)

The complete estimator for the surface heat flux at time $t_M$ is found as the sum of Eq. 25 and 26 as follows:
\[
q^*_M = \sum_{i=1}^{n_f + n_g} \left( f_i Y_{M+m_i-1} + g_i Y_{M+m_i-1} \right) =
\]
\[
f_1 Y_{M+m_1-1} + f_2 Y_{M+m_2-2} + \cdots + f_{m_f-1} Y_{M+m_f-1} + f_{m_f} Y_M + f_{m_f+1} Y_{M-1} + \cdots + f_{m_f+m_g} Y_{M-m_g},
\]
\[
+ g_1 Y_{M+m_1-1} + g_2 Y_{M+m_2-2} + \cdots + g_{m_g} Y_M + g_{m_g+1} Y_{M-1} + \cdots + g_{m_g+m_g} Y_{M-m_g}.
\]

(27)

Note that all the \(f\)-filter coefficients can be found at one time by setting all the \(Y\) and \(y\) components equal to zero except the \(m_f\) component, \(y_{m_f}\), in Eq. 27 is set equal to one. To get the \(g\)-filter coefficients (those for the \(y\) vector), the same procedure is followed with now all the components of \(Y\) and \(y\) equal to zero except \(y_{m_f} = 1\).

Figures 6 and 7 depict the \(f\) and \(g\) filter coefficients for the location \(x/L = 0.5\); the first order Tikhonov parameter value of \(\alpha_T\) is equal to 0.0001 in Fig. 6 and 0.01 in Fig. 7.

![Figure 6. Filter coefficients, \(f\) for \(Y(20) = 1\) and others zero and \(g\) for \(y(20) = 1\) and others zero.](image)

\(m_p = 20\) and \(m_f = 20\). \(\alpha T \Delta t / L^2 = 0.02\). \(\alpha_f = 0.0001\). \(\Sigma f = 2.002144\), \(\Sigma g = -1.999706\). Range of non-zero values about \(0.15 < \alpha t / L^2 < 0.6\).
Figure 7. Filter coefficients, \( f \) for \( Y(34) = 1 \) and others zero and \( g \) for \( y(34) = 1 \) and others zero. 

\[ m_p = 35 \text{ and } m_f = 35. \alpha_{\Delta t / L^2} = 0.02. \alpha_{\tau} = 0.01. \Sigma f = 2.002937, \Sigma g = -1.999715. \text{ Range of non-zero values about } 0.3 < \alpha_{\Delta t / L^2} < 1.1. \]

In both cases the filter coefficients for \( f \) is the largest, by about a factor of 10 in Fig. 6 and a factor of 4 in Fig. 7. Having \( f \) larger than \( g \) indicates that errors in the temperature boundary condition at \( x = L \) are less significant than those at \( x_1 \). Also note that the range of non-zero values in Fig. 6 is about 0.15 < \( \alpha_{\Delta t / L^2} \) < 0.6 or spans about 0.45 in dimensionless time. The corresponding range in Fig. 7 is from 0.3 < \( \alpha_{\Delta t / L^2} \) < 1.1 or spans about 0.8. Hence more regularization (i.e., larger \( \alpha_{\tau} \)), increases the range of the filter coefficients. Also, as \( \alpha_{\tau} \) is increased, the magnitude of the filter coefficients decreases. The magnitudes are important. For example, suppose that the heat flux is being estimated and there is a single error of 1 in the temperature at a given time. This would cause an error in the estimated heat flux for \( \alpha_{\tau} = 0.0001 \).
that briefly reaches 15 (the magnitude of $f$ in Fig. 6) and only about one-tenth of that for $\alpha = 0.01$, as shown by Fig. 7.

The maximum value of the absolute values of the filter coefficients increases as the Tikhonov parameter decreases. However, the sum of the $f$ filter coefficients is a constant for a given location $x_i$. This concept was explored in Ref. 20 for the X22 case and is described here for the X21 case. Information regarding the sums of the coefficients in Fig. 6 or Fig. 7 can be obtained by using the filter equation given by Eq. (27); this equation can be used for transient and steady state situations. Let it now be used for steady state and many measurements in time. These temperatures are constants which are here denoted by $Y_{ss}$ and $y_{ss}$.

Then Eq. (27) can be used to obtain

$$\hat{q}_M = \hat{q}_{ss} = \sum_{i=1}^{m} \left( f_i Y_{M+M-1} + g_i Y_{M+M-1} \right)$$

$$= \sum_{i=1}^{m} \left( f_i Y_{ss} + g_i y_{ss} \right) = Y_{ss} \sum_{i=1}^{m} f_i + y_{ss} \sum_{i=1}^{m} g_i \quad (28)$$

For steady-state heat conduction in a plate with fixed temperatures at $x$ and $L$, the heat flux is given by

$$q_{ss} = -k \frac{\Delta T}{\Delta x} = -k \frac{y_{ss} - Y_{ss}}{L - x} = k \frac{Y_{ss}}{L - x} - k \frac{y_{ss}}{L - x} \quad (29)$$

Equating the above two equations gives

$$Y_{ss} \sum_{i=1}^{m} f_i + y_{ss} \sum_{i=1}^{m} g_i = k \frac{Y_{ss}}{L - x} - k \frac{y_{ss}}{L - x}$$

$$Y_{ss} \left[ \sum_{i=1}^{m} f_i - \frac{k}{L - x} \right] = -y_{ss} \left[ \sum_{i=1}^{m} g_i + \frac{k}{L - x} \right] \quad (30)$$

Since $Y_{ss}$ and $y_{ss}$ are arbitrary in value, each of the terms inside the brackets is equal to zero. This then leads to
\[ \sum_{i=1}^{m} f_i = \frac{k}{L-x} \]
\[ \sum_{i=1}^{m} g_i = -\frac{k}{L-x} = - \sum_{i=1}^{m} f_i \]

For the special case of \( L = 1, x = L/2 \) and \( k =1 \), we obtain
\[ \sum_{i=1}^{m} f_i = 2, \quad \sum_{i=1}^{m} g_i = -2 \] (32)

Several observations can be made regarding the last two equations. First, they suggest a method of intrinsic verification \([22]\) of the computations. That is, regardless of the value of the Tikhonov regularization parameter, the same sum should be obtained. Furthermore, the value must be very close to those given. Second, the sum for the \( g_i \) values is the negative of the sum for the \( f_i \) values. A consequence of this relation is that the temperature difference is the only important aspect; hence, the initial temperature is not needed, which is true for both steady-state and transient conditions. Also, both the temperatures can be in \( ^\circ C \) or \( K \), for example.

Another type of insight from the filter coefficients can be obtained regarding the variances and standard deviations of the heat flux estimates. To obtain these estimates it is necessary to specify some statistical assumptions. Let the measurement errors be additive, have zero mean, have a constant variance \( \sigma^2 \) and be uncorrelated. Then the variance of Eq. (27) is
\[ \sigma_q^2 = V(q_M) = \sum_{j=1}^{m} (f_j^2 + g_j^2)\sigma^2 \] (33)

This result is independent of the subscript \( M \). Some results are shown in Table 1 for three different locations \( x_1/L = 0.25, 0.5 \) and 0.75 and for three Tikhonov 1st order regularization parameters of \( a_r = 0.0001, 0.001 \) and 0.01. The results in Table 1 are computed from Eq. (33) using an assumed measurement standard deviation of \( \sigma = 1.0 \). As shown in Table 1, as \( x_1/L \)
increases, the standard deviation of the estimated heat flux increases. In contrast, they decrease as the regularization parameter increases.

Table 1: Standard deviations in the estimated heat flux for additive measurement errors with zero mean, constant variance \( \sigma^2 \), and uncorrelated. Results computed using Eq. (33) with \( \sigma^2=1.0 \). Columns 3 and 4 show the contributions \( \sigma_{q_i} \) and \( \sigma_{q_j} \) of the measurements \( Y_i \) and \( y_i \) on the total standard deviation \( \sigma_q \) shown in column 5.

<table>
<thead>
<tr>
<th>( \frac{x_i}{L} )</th>
<th>( \alpha )</th>
<th>( \sigma_{q_i} )</th>
<th>( \sigma_{q_j} )</th>
<th>( \sigma_q )</th>
<th>( \sum_{j=1} g_j )</th>
<th>( \sum_{j=1} h_j )</th>
</tr>
</thead>
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<tr>
<td>0.25</td>
<td>0.0001</td>
<td>21.5939</td>
<td>4.7070</td>
<td>22.1010</td>
<td>1.3333</td>
<td>-1.3333</td>
</tr>
<tr>
<td>0.25</td>
<td>0.001</td>
<td>8.9798</td>
<td>2.7884</td>
<td>9.4028</td>
<td>1.3333</td>
<td>-1.3333</td>
</tr>
<tr>
<td>0.25</td>
<td>0.01</td>
<td>3.8136</td>
<td>1.5807</td>
<td>4.1282</td>
<td>1.3333</td>
<td>-1.3333</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0001</td>
<td>25.4757</td>
<td>2.5751</td>
<td>25.6055</td>
<td>2.0000</td>
<td>-2.0000</td>
</tr>
<tr>
<td>0.5</td>
<td>0.001</td>
<td>10.1786</td>
<td>1.6262</td>
<td>10.3077</td>
<td>2.0000</td>
<td>-2.0000</td>
</tr>
<tr>
<td>0.5</td>
<td>0.01</td>
<td>4.2242</td>
<td>1.0085</td>
<td>4.3429</td>
<td>2.0000</td>
<td>-2.0000</td>
</tr>
<tr>
<td>0.75</td>
<td>0.0001</td>
<td>28.6552</td>
<td>1.9841</td>
<td>28.7238</td>
<td>4.0000</td>
<td>-4.0000</td>
</tr>
<tr>
<td>0.75</td>
<td>0.001</td>
<td>11.7680</td>
<td>1.5304</td>
<td>11.8671</td>
<td>3.9999</td>
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<td>0.75</td>
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<td>5.0194</td>
<td>1.1761</td>
<td>5.1554</td>
<td>4.0007</td>
<td>-3.9997</td>
</tr>
</tbody>
</table>

Example

To illustrate the method to estimate the surface heat flux using a second subsurface sensor for the remote boundary condition, a finite domain of thickness \( 2L \) will be considered with a repeating triangular-in-time heat flux input at \( x = 0 \) and zero temperature at \( x = 2L \). The variation can be seen in Fig. 8 (which also includes results from the calculations, which will be explained below). Data for the remote temperature boundary is simulated from the solution at \( x=L \) (that is, half-way through the domain of \( 2L \)). “Measurements” are also extracted at \( x_1=L/2 \) to simulate the sensor for the IHCP.
The temperatures at $x = L/2$ and $L$ are shown in Fig. 9. The temperatures oscillate in an increasing damped manner and will eventually reach a quasi-steady state. The maximum temperature shown in Fig. 9 is about 43 in dimensionless temperature and the maximum surface temperature is about 50. These values are important to remember when introducing random errors; values small compared to these numbers are desired but should not be too small as to be unreasonable.
The IHCP problem is solved first with errorless data (to say, 10-digit accuracy). The temperature data illustrated by Fig. 9 at \( x = L/2 \) is for \( Y \) and the temperature at \( x = L \) is for \( y \). The whole domain Tikhonov method (Eq. (24)) and the sequential method based on filter coefficients coming from that method (Eq. (27)) are used and compared. The results for dimensionless time steps of 0.02 and \( \alpha_s = 0.0001 \) are shown in Fig. 8. Agreement between the Tikhonov whole-domain method and the filter analysis could hardly be better, it seems.

Note that, because of the width of the filter, values at the beginning and the end of the overall time range cannot be computed. The filter requires data from both past and future times, hence cannot be applied at the start, or at the finish. For this example, \( m_p = m_f = 25 \), so computations begin at the 26\(^{th}\) time step and end 25 steps from the final time. However, the filter
method is best suited for an online application where results are computed continuously from a stream of data, such as in a continuous industrial process, and for such an application the start and stop intervals are not important. Another case of interest is for relatively large simulated errors (additive, zero mean, constant variance, uncorrelated, and normally distributed). Normally distributed errors with a standard deviation of $\sigma=0.5$ were added to the exact temperatures in Fig. 9, and these “noisy” data are processed using the two methods. The results are seen in Fig. 10 which uses $\alpha_T = 0.01$ and $m_p=m_f=25$.

Notice that the filter method gives the exact same results as the full TR method, and this is confirmed by the calculation of the standard deviation of the error between the recovered heat flux and the exact heat flux, which is equal 2.40 in both cases. The exact heat flux is shown as a heavy line in Fig. 10 for comparison.

![Graph](image)

Figure 10. Comparison of whole domain Tikhonov regularization with associated filter coefficient results for the case of “large” random errors ($\sigma_{\text{error}}=0.5$, about 1% of maximum $T$).

$x_1=0.5; L=1; k=1; \alpha=1; T_1=1; q_0=1; \Delta t=0.02; \max=500; \alpha_{\text{Tr}}=0.01$;
The computer time to obtain results using the full Tikhonov regularization method is significantly more than that using the filter equation. Assuming that the filter coefficients are pre-computed and stored, the computer time required using full TR method is approximately 25 times more than the required time for the filter method. About 500 simultaneous linear algebraic equations must be solved in the Tikhonov procedure for this example; one equation for every unknown heat flux component. The computation associated with the filter equation is trivial compared to this calculation. Furthermore, whole-domain direct solution time using Eq. (24) increases more than linearly with the number of unknown heat flux components. Also, computer memory requirements increase as more and more cycles are considered in a batch mode of analysis. That is not true for the filter computation. Thousands of cycles can be analyzed using the filter concept in an almost real-time fashion. It is only necessary to have a limited “window” of measurements available for temperatures at time $t_M$ and a number of future and past temperatures relative to this time.

Conclusion
In this paper, a technique to account for the thermal action at a remote boundary in the IHCP solution using temperature data from a second sensor is outlined. Primary data needed for the solution is obtained from a sensor located at $x = x_1$, and supplemental data to account for the second boundary is measured at $x = L$, which may, or may not, correspond to a physical boundary. The IHCP solution is formulated into a filter representation. Using the filter technique, near real-time measurements can be continuously obtained, and this is suitable for industrial applications. The filter solution and a classical whole domain Tikhonov regularization solution are both applied to an example problem and the results match very well. The
calculation time for the filter method is significantly smaller than the whole domain method which makes it an appropriate approach when real-time estimation of heat flux is required.

References


ARTICLE 2: A FILTER BASED SOLUTION FOR INVERSE HEAT CONDUCTION PROBLEMS IN MULTI-LAYER MEDIUMS

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ABSTRACT

This paper presents a solution for the inverse heat conduction problem (IHCP) in a multi-layer medium based on solutions from individual layers separately. The approach allows for inclusion of known contact resistances between the layers. The temperature histories are assumed known at two points on the inner layer and the heat transfer rate at the far end of the outer layer is the desired unknown parameter. A step-by-step solution is proposed for solving this problem based on minimization of the sum-of-squared errors between the computed and known values and using Tikhonov regularization for stabilizing the solution. A Tikhonov digital filter solution is developed which allows near real-time heat transfer estimation in multi-layer application. The proposed method is tested via numerical experiments using exact solutions and ANSYS to generate synthetic data.

Keywords: Inverse heat conduction, heat flux measurement, multi-layer medium, filter solution.

Introduction
The inverse heat conduction problem (IHCP) is defined as the problem of estimating unknown surface conditions (temperature or heat flux) using internal temperature measurements. This problem arises in several industrial applications such as thermal manufacturing processes. The IHCP is an ill-posed problem due to the lack of continuous dependence of the solution on the data. In other words, an error in input data will result in a significant output error. Therefore, an appropriate regularization method needs to be applied to convert the ill-posed problem to a nearby well-posed problem which can be solved. Several techniques have been proposed and applied for solving IHCPs which can be found in references namely Beck [1], Alifanov [2], Ozisik and Orlande [3] and Murio [4]. Some of these methods include the least-square method with regularization, the sequential function specification, conjugate gradient method and numerical approaches [1].

Conduction through multi-layer mediums has been discussed in several references. Ozisik [5] discussed conduction in one dimensional composite media using different approaches including orthogonal expansions, Green’s functions and Laplace transform. The transient response of one-dimensional multilayered composite conducting slabs to sudden variations of the temperature of the surrounding fluid is studied by de Monte [6]. Lu et al. [7] developed an analytical method for solving multi-layer heat conduction problems using Laplace transform and separation of variables. They show that the result from their proposed closed form solution is in good agreement with numerical techniques. Haji-Sheikh and Beck [8] studied the temperature field in multi-dimensional, multi-layer bodies for the boundary conditions of the first, second and third
kind. A solution for transient heat conduction through a one-dimensional three-layer composite slab is presented by Sun and Wichman [9].

Unlike direct problems, the solution of IHCPs for a multi-layer medium is only discussed in a few studies. Al Najem and Ozisik [10] conducted an inverse heat conduction analysis for estimating the surface condition in composite layers based on a splitting-up procedure and nonlinear least-squares technique for the whole time domain. Ruan et al. [11] calculated the unknown boundary cooling condition and contact heat transfer coefficient for solidification of alloys based on the least square method and using Beck's future time method and a regularization technique to stabilize the solution. A study on the design of optimal transient heat conduction experiments on composite orthotropic materials is performed by Taktak et al. [12]. They considered several geometries for both 1-D and 2-D cases. Al-Najem [13] developed a method of analysis for determining surface conditions from the knowledge of the time variations of the temperature at the insulated boundary. He used two segmented polynomials in time for the unknown surface temperature. An inverse solution is then developed over the whole time domain using the splitting-up procedure.

Recently several research works have been performed to investigate real-time or filter forms for processing temperature data to solve IHCPs. A filter solution based on the idea of training neural networks is studied by Kowsari et al. [14]. Ijaz et al. [15], used a Kalman filter to solve a two-dimensional transient IHCP. Feng et al. [16] used Laplace transforms to relate the measured conditions at one end of a domain to the unknown conditions at the remote surface. Woodbury and Beck [17] studied the structure of the Tikhonov regularization problem and concluded that the method can be interpreted as a sequential filter formulation for continuous processing of data.
They show that the computed heat fluxes using the whole domain solution and the filter coefficient solution are virtually the same for the constant-property solutions.

In most of the IHCP studies, the heating condition on the remote boundary is assumed as an insulated surface or cooled with a known heat transfer coefficient, e.g. [15, 18, 19]. However, in practice such ideal conditions are not easy to attain. Most recently Woodbury et al. [20] developed a filter based solution to incorporate the temperature measurement history from a second subsurface sensor as a remote boundary condition in an IHCP solution. An example of such a problem in industry is Directional Flame Thermometer (DFT) which is an equilibrium heat flux sensor that is used to estimate the heat flux using temperature measurements [21].

In the present paper, a solution for the IHCP is proposed for a two-layer medium when the temperature measurement history is given in two interior locations of the inner layer. A step-by-step solution is proposed for solving this problem based on the minimization of the sum of the squared errors between the computed and known values and using Tikhonov regularization (TR) for stabilizing the solution. The resulting algorithm is written in filter form. The filter form solution can be used for near real time heat flux estimation. The proposed solution is then demonstrated through several numerical experiments. The filter solution of the IHCP has a number of advantages including simplicity, continuous operation and application to moderate nonlinearity [22] which makes it an appropriate approach for real time heat flux estimation in industrial applications. It is noteworthy that when the material properties are temperature dependent, the problem is no longer linear. For this type of problem, the filter coefficients for a range of temperatures can be calculated and then, using linear interpolation, the values of filter coefficients can be found for each time step based on the current level of temperature.
Problem definition

It should be noted that the method presented in this paper can be applied on a medium with more than two layers as long as there are two temperature measurements available on the inner layer. However, a two layer medium is considered to demonstrate the application of the proposed approach. Basically, an IHCP is solved for each layer, starting from the one with known temperature measurements, and the heat flux is estimated at the interface with the next layer. A schematic of a two-layer slab is shown in Fig. 1.

![Schematic of the two-layer problem](image)

Figure 1: Schematic of the two-layer problem

As seen, in Fig. 1, the temperature measurement histories are available in the innermost layer (layer 2) for \( x=x_1 \) and \( x=x_2 \) while no specific temperature/heat flux measurement is available on the other layer(s) (here, layer 1). The desired unknown parameter is the heat flux at the remote surface of the outer layer (layer 1, \( x=0 \)).
Solution Strategy for the Multi-Layer IHCP

Two IHCPs are solved separately for heat flux estimation at \( x=0 \) and eventually a coupled solution is derived for the two-layer problem. A schematic of the system is given in Fig. 1. The solution is started in the inner layer, where two temperature measurements are available at \( x_1 \) and \( x_2=L_1+L_2 \) and \( q_i \) is the unknown heat flux at the interface. After solving the first IHCP, the heat flux and temperature are both known at the interface \( (q_i \) and \( T_i) \). These values are used as boundary conditions to solve the second IHCP associated with the first layer, where \( q \) at \( x=0 \) (i.e., \( q_0(t) \)) is unknown. The analysis of each layer is explained in detail as below.

Inner Layer

The solution for the IHCP when the temperature measurement is given at two sub-surface locations is given by Woodbury et al. [20]. A similar approach is utilized here to analyze the second layer. Two temperature measurements histories are assumed available at \( x=x_1 \) and \( x=L_1+L_2 \) \((L_1 \leq x_1 \leq L_1+L_2)\) on layer 2. Note the the location \( L_1+L_2 \) may, or may not, coincide with a physical boundary of the domain:

\[
T(x_1, t) = Y(t)
\]

\[
T(L_1 + L_2, t) = y(t)
\]

The initial temperature is considered zero,

\[
T(x, 0) = 0
\]

The objective of the first IHCP is to estimate the heat flux \((q_i)\) at the interface between the two layers \((x=L_1)\),

\[
-k \frac{\partial T}{\partial x}(0, t) = q_i(t) = ?
\]
Solving the IHCP for the inner layer results in determining the heat flux at the interface with the next layer which then will be used as a boundary condition for solving the second IHCP.

The temperature at \( x_1 \) \( (L_1 \leq x_1 \leq L_1+L_2) \) is a function of the surface heat flux \( q_1(t) \) and the temperature at \( x = L_1+L_2 \). This dependency is illustrated through Green’s functions [23]:

\[
T(x_1,t) = \frac{\alpha_2}{k_2} \int_{\tau=0}^{t} q_1(\tau)G_{X21}(x_1,L_1,t-\tau)d\tau + \alpha_2 \int_{\tau=0}^{t} y(\tau) \left( -\frac{\partial G_{X21}}{\partial x} (x_1,L_1+L_2,t-\tau) \right) d\tau
\]  

(5)

The Green’s function in this equation is for a boundary condition of the second kind (heat flux) at \( x=L_1 \) and the first kind (temperature) at \( x=L_1+L_2 \). This solution is designated by Cole et al. [23] as X21B10T0, which is herein shortened to X21 for reference. The notation denotes a Cartesian geometry subjected to a type 2 condition at the first boundary and a type 1 condition at the second boundary, and that the first boundary has a step change in value while the second boundary is homogeneous, and that the initial condition is also homogeneous. It should be noted that the Green’s function solution is only included here to demonstrate the linearity of the problem and it is not used directly in the solution.

The connecting curves between the heat flux components, \( q_i \) and \( q_{i+1} \), and also between the adjacent components, \( y_i \) and \( y_{i+1} \) are considered as constant between points (step function):

\[
q(t) = q_j, \quad t_j < t < t_{j+1}
\]

(6)

\[
t_j = i\Delta t
\]

(7)

A solution for the X21 case with a constant heat flux, \( q_c \), at \( x = L_1 \) is

\[
\frac{T_{X21}(x,t)}{q_c L_2} = \left(1 - \frac{x-L_1}{L_2}\right) \sum_{m=1}^{\infty} \frac{\cos \left( \frac{\beta_m}{L_2} \frac{x-L_1}{L_2} \right)}{\beta_m^2} \exp \left( -\beta_m^2 \frac{\alpha_2 t}{L_2^2} \right)
\]

(8)

where \( \beta_m = (m-1/2)\pi \) is the \( m^{th} \) eigenvalue \( (m=1,2,3,\ldots) \).
Analogous to the above equations for a constant heat flux, the solution for a constant temperature, \( T_c \), at \( x = L_1 + L_2 \), with homogeneous boundary conditions elsewhere, is denoted \( X_{12B10}T_0 \) and shortened herein to \( X_{12} \):

\[
T_{X_{12}}(x,t) = \frac{T_c}{1 - 2\sum_{m=1}^{\infty} \frac{\sinh \left( \beta_m \left( \frac{x - L_1}{L_2} \right) \right)}{\beta_m} \exp \left( -\beta_m^2 \frac{\alpha t}{L_2^2} \right)}
\]

(9)

and the eigenvalues \( \beta_m = (m - 1/2)\pi \) are the same as for the \( X_{21} \) case.

The temperature at any location \( x_1 (L_1 \leq x_1 \leq L_1 + L_2) \) caused by the heat flux \( q_1 \) at \( x = L_1 \) and the temperature \( y \) at \( x = L_1 + L_2 \) is

\[
T_{m} = T_{x, m} + T_{y, m} = \sum_{i=1}^{M} q_{1,i} \Delta \phi_{M-i} + \sum_{i=1}^{M} y_{i} \Delta \eta_{M-i}
\]

(10)

where \( \phi \) and \( \eta \) are the response basis functions for the two cases. That is,

\[
\phi(x, t) = \frac{\partial T_{X_{21}}}{\partial q_c}, \quad \eta(x, t) = \frac{\partial T_{X_{12}}}{\partial T_c}
\]

(11)

Here \( \Delta \phi \)'s can be found as \( \Delta \phi = \phi(x_1, t_i) - \phi(x_i, t_{i-1}) \) [1], and \( \Delta \eta \), similarly.

Equation (10) can be described by the matrix equation of

\[
T = X_2 q_i + Z y
\]

(12)

where

\[
T = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix}, \quad q_i = \begin{bmatrix} q_{1,i} \\ q_{1,2} \\ \vdots \\ q_{1,n} \end{bmatrix}, \quad X_2 = \begin{bmatrix} X_{2,1} & 0 & 0 & \cdots & 0 & 0 \\ X_{2,2} & X_{2,1} & 0 & \cdots & 0 & 0 \\ X_{2,3} & X_{2,2} & X_{2,1} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ X_{2,n} & X_{2,n-1} & X_{2,n-2} & \cdots & X_{2,2} & X_{2,1} \end{bmatrix}
\]

(13)
The components of the $x$ and $z$ matrices are related to the response basis functions by

$$X_{2,i} = \Delta \phi_{i-1}, \quad Z_i = \Delta \eta_{i-1}.$$  \hspace{1cm} (15)

The columns in the $X$ matrix of Eq. (13) are the same as the sensitivity vectors in parameter estimation. The first column is for $q$ at $t_1$, the second for $q$ at $t_2$ and so on.

The IHCP solution for the inner layer (layer 2) is determined through minimization of the sum-of-squares between the computed (via Eq. (12) and measured ($Y$) values of temperature at $x=L_1+L_2$ and using Tikhonov regularization for stabilizing the solution:

$$S = (Y - X_2q_z - Zy)^T (Y - X_2q_z - Zy) + \alpha_y q_z^T H^T H q_z,$$  \hspace{1cm} (16)

The estimated value heat flux vector at the interface, $\hat{q}_{1,i}$, is found by minimizing $S$ with respect to $q_z$, and is given as:

$$\hat{q}_{1,i} = [X_2^T X_2 + \alpha_y H^T H]^{-1} X_2^T (Y - Zy)$$

$$\quad = F_2 (Y - Zy), \quad F_2 = [X_2^T X_2 + \alpha_y H^T H]^{-1} X_2^T$$  \hspace{1cm} (17)

where the $\alpha_y$ is the Tikhonov regularization parameter [1]. Here the subscript 2 for $X$ and $F$ refers to Layer 2. Note that this equation is dimensional. $F_2$ has the same definition as the filter matrix of Ref [20].

The value of $\hat{q}_{1,i}$ from Eq. (17) will be used as the first known boundary condition for the IHCP associated with the first layer and the second boundary condition is the temperature at the interface which can be calculated as below by using Eq. (12):
\[ \hat{T}_i = X_2 \hat{q}_i + Z y \]  

(18)

where, again, subscript 2 has been attached to \( X_2 \) to emphasize correspondence to layer 2.

By substituting \( \hat{q}_i \) from Eq. (17) in the above equation \( \hat{T}_i \) can be found as:

\[ \hat{T}_i = X_2 F_2 Y + (Z - X_2 F_2 Z) y \]  

(19)

The vector of heat fluxes at the remote surface \((x=0)\) is the unknown parameter in the second IHCP.

**Outer Layer**

The temperature at any location \( x \) \((0 \leq x \leq L_i)\) is a function of the surface heat fluxes \( q_0(t) \) and \( q_1(t) \). The results from the inner layer (layer 2) analysis determine \( q_1(t) \). One exact way to obtain the temperature uses Green’s functions \([23]\); for which an expression is

\[
T(L_i, t) = \frac{\alpha}{k_i} \int_{\tau=0}^{t} q_o(\tau) G_{x,22}(x, 0, t - \tau) d\tau + \frac{\alpha}{k_i} \int_{\tau=0}^{t} q_1(\tau) G_{x,22}(x, L_i, t - \tau) d\tau
\]

(20)

The Green’s function in this equation is for a boundary condition of the second kind (Neumann) at \( x = 0 \) and at \( x = L_i \) \( (X_{22}) \). Again, Eq. (20) merely establishes the linearity of the solution for the temperature at any point and is not used directly in the procedure.

A solution for the X22 case with a constant heat flux at \( x = 0 \) and zero heat flux at \( x = L_i \) (this is denoted the X22B10T0 case) is \([25]\):

\[
\frac{T_{x,22}(x, t)}{q_i L_i / k_i} = \frac{\alpha}{L_i^2} t + \frac{1}{3} - \frac{x}{L_i} + \frac{x^2}{2 L_i^2} - 2 \sum_{m=1}^{n_{w}} \frac{\cos(\gamma_m x / L_i)}{\gamma_m^2} \exp \left( -\gamma_m^2 \alpha / L_i^2 \right)
\]

(21)

where \( \gamma_m = m \pi \) is the \( m \)th eigenvalue \((m=1,2,3,\ldots)\).

Note the same solution applies for a zero heat flux at \( x = 0 \) and a constant heat flux at \( x = L_i \) through a simple change of variables, i.e., \( \xi = L_i - x \).
Analogous to Eq. (10), the temperature response at \( x_1 (0 \leq x_1 \leq L_1) \) can be found due to the heat flux histories \( q_0(t) \) and \( q_1(t) \). For assumed piecewise constant variation in these heat fluxes, the temperature can be computed from

\[
T_M = T_{q_0,M} + T_{q_1,M} = \sum_{i=1}^{M} q_{0,i} \Delta \varphi_{M-i} + \sum_{i=1}^{M} q_{1,i} \Delta \theta_{M-i}
\]  

(22)

where

\[
\varphi(x_1,t) = \frac{\partial T_{x,22}(x_1,t)}{\partial q_0}; \quad \theta(x_1,t) = -\frac{\partial T_{x,22}(L_1-x_1,t)}{\partial q_1}
\]

(23a,b)

The step basis function representation used here (and also others) for temperature given in Eq. (22) can be described by the matrix equation of

\[
T = X_0 q_0 + X_1 q_1
\]

(24)

where

\[
T = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix}, \quad q_0 = \begin{bmatrix} q_{0,1} \\ \vdots \\ q_{0,n} \end{bmatrix}, \quad q_1 = \begin{bmatrix} q_{1,1} \\ \vdots \\ q_{1,n} \end{bmatrix}
\]

(25 a,b,c)

\[
X_0 = \begin{bmatrix} X_{0,1} & 0 & \cdots & 0 & 0 \\ X_{0,2} & X_{0,1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ X_{0,n} & X_{0,n-1} & \cdots & X_{0,n-2} & X_{0,1} \end{bmatrix}
\]

(25d)

\[
X_1 = \begin{bmatrix} X_{L,1} & 0 & \cdots & 0 & 0 \\ X_{L,2} & X_{L,1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ X_{L,n} & X_{L,n-1} & \cdots & X_{L,n-2} & X_{L,1} \end{bmatrix}
\]

(25e)
The components of the $X_0$ and $x_i$ matrices are related to the response basis functions in Eq. (23) similar to the relation of $X_2$ and $Z$ in Eq. (15) to Eq. (11):

$$X_{0,i} = \Delta \varphi_{i-1}, \quad X_{L,i} = \Delta \theta_{i-1}.$$  \hspace{1cm} (25f)

The whole domain Tikhonov regularization method is used to solve the IHCP. Later the filter coefficients are found from the solution. The IHCP solution for Layer 1 starts with a matrix form for the sum of squares with an added regularization term given by

$$S = (\hat{T}_1 - X_o \hat{q}_0 - X_{L,1} \hat{q}_1)^T (\hat{T}_1 - X_o \hat{q}_0 - X_{L,1} \hat{q}_1) + \alpha_s \hat{q}_0^T H^T H \hat{q}_0$$ \hspace{1cm} (26)

This is minimized with respect to the parameter vector $q_0$. The symbol $\hat{T}_1$ is the estimated temperature vector at $L_1$ and $\hat{q}_1$ is the estimated heat flux (from the solution of the IHCP in Layer 2) at $x = L_1$. The initial temperature is zero. The $\alpha_s$ symbol is the Tikhonov regularization parameter.

The estimated value of the heat flux vector, denoted $\hat{q}_0$, is then given by

$$\hat{q}_0 = [X_o^T X_o + \alpha_s H^T H]^{-1} X_o^T (\hat{T}_1 - X_{L,1} \hat{q}_1)$$

$$= F_1 (\hat{T}_1 - X_{L,1} \hat{q}_1), \quad F_1 = [X_o^T X_o + \alpha_s H^T H]^{-1} X_o^T$$ \hspace{1cm} (27)

So far it is assumed that the contact resistance between the two layers is negligible. However, in many cases the contact resistance can be significant and neglecting it may result in considerable error. It is possible to easily accommodate any known interfacial contact resistance.

Assuming $T_{2,L_1}$ is the temperature at the interface on the inner layer (layer 2), $T_{1,L_1}$ is the temperature at the interface on the outer layer (layer 1) and $R_{int}$ is the known thermal contact resistance of the interface, the following equation can be written:

$$T_{1,L_1} = R_{int} q_1 + T_{2,L_1}$$ \hspace{1cm} (28)
Substituting above equation in Eq. (27):

\[
\hat{q}_0 = F_1 \left( R_{\text{inst}} \hat{q}_1 + \hat{T}_{2 \text{,e}} - X_{\text{e}} \hat{q}_1 \right)
= F_1 \left( \hat{T}_{2 \text{,e}} + (1R_{\text{inst}} - X_{\text{e}}) \hat{q}_1 \right)
\] (29)

Coupling of the solutions

To achieve a single expression for a two-layer IHCP, Eq. (17) is substituted in Eq. (27). An expression valid for perfect contact between the layers is:

\[
\hat{q}_0 = F_1 \hat{T}_1 - F_1 X_{\text{e}} (F_2 Y - F_2 Z y)
\] (30)

where \(\hat{T}_1\) is the temperature at the interface and can be found from Eq. (18). Substituting Eq. (18) in Eq. (30), an expression is:

\[
\hat{q}_0 = F_1 \hat{T}_1 - F_1 X_{\text{e}} (F_2 Y - F_2 Z y)
= F_1 (X_2 + X_{\text{e}}) F_2 Y + F_1 (Z - X_2 F_2 Z - X_{\text{e}} F_2 Z) y
\] (31)

Equation (31) can be used directly to calculate the heat flux at the remote surface on the first layer by using two sets of temperature measurements from the second layer.

To account for a known contact resistance, Eq. (31) can be written as:

\[
\hat{q}_0 = F_1 \left( R_{\text{inst}} \hat{q}_1 + \hat{T}_{2 \text{,e}} - X_{\text{e}} \hat{q}_1 \right)
= F_1 \left( R_{\text{inst}} + X_2 + X_{\text{e}} \right) F_2 Y + F_1 \left( -R_{\text{inst}} F_2 Z + Z - X_2 F_2 Z - X_{\text{e}} F_2 Z \right) y
\] (32)

As can be seen, Eq. (32) is very similar to Eq. (31) except for the additional terms for the contact resistance.

Since many available solutions for heat conduction problems in different geometries are in dimensionless form [25], it is useful to include the relation between dimensional quantities and dimensionless quantities by different thermal properties in the two regions. When each region is
non-dimensionalized using its own thickness and thermophysical properties, the dimensionless quantities, denoted with a tilde ‘~’, are:

\[ \tilde{F}_2 = \frac{L_2}{k_2} F_2, \quad \tilde{F}_1 = \frac{L_1}{k_1} F_1, \quad \tilde{X}_2 = \frac{k_2}{L_2} X_2, \quad \tilde{X}_0 = \frac{k_2}{L_1} X_0, \quad \tilde{X}_L = \frac{k_1}{L_1} X_L \]

\[ \tilde{Y} = \frac{1}{q_{ref} L_1 / k_1} Y, \quad \tilde{y} = \frac{1}{q_{ref} L_1 / k_1} y, \quad \tilde{R}_{int} = \frac{k_1}{L_1} R_{int} \]  

(33)

Note \( Z \) is dimensionless since it is \( \partial T_i / \partial T_{k_i} \), so

\[ \tilde{Z} = Z \]

After substitutions and simplifying the equation, Eq. (32) can be written as:

\[ \tilde{q}_0 = \frac{1}{q_0} \tilde{q}_0 = F_1 \left( \tilde{R}_{int} + \frac{L_2}{k_2} \tilde{X}_2 + \tilde{X}_L \right) \frac{k_2}{L_2} F_2 Y \]

\[ + F_1 \left( \tilde{R}_{int} \tilde{F}_2 Z + \frac{L_2}{k_2} \tilde{X}_2 \tilde{F}_2 Z - \tilde{X}_L \tilde{F}_2 Z \right) y \]  

(34)

where

\[ \tilde{L}_2 = \frac{L_2}{L_1} \text{ and } \tilde{k}_2 = \frac{k_2}{k_1} \]

Filter Form of the Solution

The concept of the filter algorithm is that the solution for the heat flux at any time is only affected by the recent temperature history and a few future time steps [20]. Equation (32) can be written in filter form as:

\[ \tilde{q}_m = \sum_{j=1}^{m_y + m_z} \left( f_j Y_{m_z+m_z-j} + g_j X_{m_y+m_y-j} \right) \]  

(35)

\[ f = \begin{bmatrix} f_1 & f_2 & \cdots & f_{m_y+m_z-1} & f_{m_y+m_z} \end{bmatrix} = \text{row} \left[ F_1 \left( R_{int} F_2 + X_2 F_2 + X_L F_2 \right) \right] \]

(36 a,b)

\[ g = \begin{bmatrix} g_1 & g_2 & \cdots & g_{m_y+m_z-1} & g_{m_y+m_z} \end{bmatrix} = \text{row} \left[ F_1 \left( -R_{int} F_2 Z + X_2 F_2 Z - X_L F_2 Z \right) \right] \]
where \( f \) and \( g \) have the same characteristics as filter coefficients, which are discussed in detail in Ref [20]. The rows of the filter matrix are similar but shifted in time. Each row corresponds to a specific time step. There are only limited non-zero values in each row and the rest of the components are zero. The meaning of “row()” in Eq. (36) designates a row in the middle of the indicated matrix. Note that these filter coefficients are not the same as filter factors suggested by Hansen [24].

Alternatively, all the \( f \)-filter coefficients can be found at one time by setting all the \( Y \) and \( y \) components equal to zero except the \( m_p + 1 \) component of \( Y \) is set equal to one (\( y_{m_p+1} = 1 \)). The solution of the IHCP with this data gives the \( f \) coefficients. Similarly, to calculate the \( g \)-filter coefficients (those for the \( y \) vector), the same procedure is followed with now all the components of \( Y \) and \( y \) equal to zero except \( y_{m_y+1} = 1 \).

Assuming \( x_f = L_1 \) and no contact thermal resistance, using the material properties given in Table 1, for the dimensionless time step of 0.01 and for first order Tikhonov regularization with parameter \( \alpha_r = 0.005 \), the 80\(^{th} \) rows of the matrices in Eq. (36) (the \( f \) and \( g \) filter coefficients) are plotted in Figs. 2 and 3.
Figure 2: $f$ filter coefficients ($t_d=0.01$, $\alpha_t=0.005$, X1C11B10T0 case)

Figure 3: $g$ filter coefficients ($t_d=0.01$, $\alpha_t=0.005$, X1C11B10T0 case)
Table 1: material properties for test case 1 and 2

<table>
<thead>
<tr>
<th>Properties</th>
<th>Layer 1</th>
<th>Layer 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volumetric Heat Capacity, kJ/m3K</td>
<td>3,642.3</td>
<td>3,829.4</td>
</tr>
<tr>
<td>Thickness, m</td>
<td>0.1</td>
<td>0.2</td>
</tr>
<tr>
<td>Conductivity (k), W/m-K</td>
<td>54</td>
<td>14.2</td>
</tr>
</tbody>
</table>

In these figures the current dimensionless time step is 0.8. As seen, the values of filter coefficients approach zero towards the both ends. As mentioned before, \( m_p \) and \( m_f \) are the number of required data points from previous and future time steps, respectively. It should be noted that the selection of \( m_p \) and \( m_f \) cannot be made arbitrarily as they depend on the boundary conditions, material properties, sensor location, time step and regularization parameter. To determine \( m_p \) and \( m_f \), one must examine the filter coefficients. \( m_p \) is the number of non-zero coefficients before the current time step and \( m_f \) is the number of non-zero coefficients after the current time step. It is noteworthy that here “zero” is not necessarily an exact value of zero, but it is very close to zero and significantly smaller than the other coefficients and therefore does not have any considerable effect in calculations. Here, it is assumed that any coefficient equal to or less than 0.0001 can be neglected, which results in good accuracy.

As seen in Figs 2 and 3, there are only a few non-zero values ahead of the current time step and this determines \( m_f \). A closer look at Fig. 2 and Fig. 3 reveal that \( m_f \) is 11 and 8 for \( f \) and \( g \) filter coefficients respectively. Therefore, the maximum of these two values, which is 11, should be used to achieve accurate results. The smaller \( m_f \) is, the closer to real-time the algorithm can operate. Similarly, \( m_p \) can be determined by close inspection of the filter coefficients. From Fig. 3, \( m_p \) is 30 and 40 for \( f \) and \( g \) respectively. So, the bigger value of 40 should be used for calculations. It should be noted that a bigger value of \( m_p \) does not affect how fast (closer to real-
time) the algorithm can operate as it only decides how many data points from previous time steps must be used to determine the heat flux at the current time.

For this problem (i.e., for fixed sensor locations) \( m_p \) and \( m_t \) will be different with different regularization parameter that is being used. Therefore, to accurately estimate the heat flux at the surface, the regularization parameter should be selected carefully. As discussed by Woodbury and Beck [20], appropriate selection of the order of magnitude of the regularizing parameter is important to achieve accurate results and minimize errors, but the precise selection of the regularization parameter is not important for effective estimation of heat flux. Also, when choosing a smaller regularization parameter, \( m_p \) and \( m_t \) will be smaller. The conclusion is, when using filter approach for real-time heat flux estimation, one need to have an idea of the problem that has to be solved (whether the heat flux is smoothly changing or has a lot of rapid variations) to pick the appropriate regularization parameter and corresponding \( m_p \) and \( m_t \) values.

**Results and discussion**

To demonstrate the solution method, a test case is generated using the exact solution for the two-layer one dimensional conduction problem [25]. The properties of the layers are given in Table 1. Note that the sensors locations in this test case are at \( x=L_1 \) and \( x=L_1+L_2 \) and the dimensionless time step is assumed as 0.01. The temperature data are generated using the solution for the direct problem for a step change in temperature at the surface (X1C11B10T0). The temperature profile for the two-layer slab is shown in Fig. 4. The value of \( \alpha \), is selected as 0.005 for this test case. This is determined by minimizing the RMS error between the estimated heat flux and known heat flux. Thermal contact resistance is assumed to be negligible for this test case.
Figure 4: Temperature history within the two layer slab due to step change in temperature (test case 1, X1C11B10T0)

It should be noted that the RMS error for each test case is calculated using following equation:

\[
E_{RMS} = \left( \sum_{i=1}^{n} \left( q_{exact,i} - q_{estimated,i} \right)^2 \right)^{1/2}
\]

Here, \( n \) is the number of time steps for which heat flux calculations have been made.

The heat flux at the surface is then calculated using the whole domain solution (Eq. (32)) and the filter solution (Eq. (35)) and compared with the results from the exact solution in Fig. 5.
As seen, the IHCP whole time domain results are well-matched with the filter solution results and they are both in a good agreement with the heat flux results from the exact solution. The average deviation (RMS) of the estimated heat flux from the exact solution is 3.011 W/m² and 3.0005 W/m² for the filter approach and the whole domain method respectively. It should be noted that the values of $m_p$ and $m_f$ are considered as 40 and 8 for this test case (as shown in Figs 2 and 3). To simulate measurement noise, a uniform random error (1% of the maximum temperature) is added to data and the heat flux calculated. The calculated heat flux when these errors are present is plotted in Fig. 6. It can be observed that the results from the filter solution and the whole domain solution are in a good agreement (RMS=0.4523 W/m²). The RMS between the estimated values from data with errors and the exact heat flux is 5.4499 W/m² and 5.4591 W/m² for the whole domain solution and the filter solution, respectively. The value of $\alpha_r$...
is 0.005 for this test case. The regularization parameter is determined by minimizing the RMS error between the estimated heat flux and known heat flux.

In the second test case, a step change in heat flux is applied to the surface of the first layer of a two-layer slab while the other boundary is insulated (X2C12B10T0). This test case is also generated using the exact solution for two-layer one dimensional heat conduction problem [25]. The material properties for this test case are given in Table 2.

<table>
<thead>
<tr>
<th>Properties</th>
<th>Layer 1</th>
<th>Layer 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volumetric Heat Capacity, kJ/m3K</td>
<td>3767.9</td>
<td>3542.0</td>
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<tr>
<td>Thickness, m</td>
<td>0.03</td>
<td>0.04</td>
</tr>
<tr>
<td>Conductivity (k), W/m-K</td>
<td>36</td>
<td>25</td>
</tr>
</tbody>
</table>

The temperature data at $x=L_1$ and $x = L_1+L_2$ is then obtained and used as inputs for the IHCP solution. The thermal contact resistance is neglected. The regularization parameter, $\alpha$, is
selected as 1e-5 for this test case as the result of minimizing the RMS error between the estimated heat flux and known heat flux. The dimensionless time step is set as 0.04 which is equivalent to 27.8 seconds. The values of $m_p$ and $m_f$ is selected as 40 as a result of close inspection of filter coefficients. The filter coefficients for this test case are shown in Fig. 7.

![Filter Coefficients](image)

**Figure 7**: non-zero filter coefficients for test case 2: (a) $f$ filter coefficients (b) $g$ filter coefficients

These coefficients are simply the 41$^{th}$ row of the $F$ and $G$ filter matrices for this problem. The heat flux at the surface is calculated by Eq. (31) and Eq. (35) and compared with the exact solution in Fig. 8.
Good agreement is observed between the results from the filter solution and the exact solution. The RMS error of the estimated heat flux with respect to the exact solution is calculated as 0.0043 W/m² and 0.0045 W/m² for the whole domain solution and the filter solution respectively. Similar to the previous test case, a uniform random 1% of the maximum temperature is added to the temperature data as an error and the heat flux is calculated by both whole time domain and filter solution. The regularization parameter is optimized as 0.1. The heat fluxes are demonstrated in Fig.9. The average RMS between the two methods is calculated as 0.0011 W/m². Also the RMS between the exact solution and the estimated heat flux by using whole time domain and the filter method is calculated as 0.0412 W/m² and 0.0411 W/m² respectively.

Figure 8: heat flux estimation for test case 2 (X2C12B10T0, t₁=0.04, α₁=1e-5)
In both of the previous test cases, the thermal contact resistance between the layers is negligible. In the third test case, the thermal contact resistance is known and accounted in the calculations. For this purpose, a two-layer geometry is developed in ANSYS and a known heat flux is applied to the surface of the outer layer (layer 1) while the temperature on the remote surface of the inner layer (layer 2) is maintained at 300 K. The material properties for this test case are given in Table 3.

<table>
<thead>
<tr>
<th>Properties</th>
<th>Layer 1</th>
<th>Layer 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volumetric Heat Capacity, kJ/m3K</td>
<td>4348.8</td>
<td>172.5</td>
</tr>
<tr>
<td>Thickness, m</td>
<td>0.003</td>
<td>0.004</td>
</tr>
<tr>
<td>Conductivity (k), W/m-K</td>
<td>18.9</td>
<td>0.078</td>
</tr>
</tbody>
</table>

It is assumed that the thermal contact resistance is 0.1 m^2K/W. It should be noted that this value for contact resistance is significantly higher than typical known values between two surfaces, however, it is purposely selected to examine the capability of the solution method in
accounting for contact resistance and clearly demonstrate the temperature difference between the two layers at the interface in presence of contact resistance.

The temperature measurements at the interface for both with/without thermal contact resistance cases are shown in Fig. 10.

![Figure 10: temperature at the interface for test case 3](image)

As seen, the temperature measurements at the interface in the presence of the contact resistance is considerably lower than the perfect contact condition. The temperature data is then used to estimate the heat flux at the surface of the outer layer. The time step for this test case is set at 2 seconds. Based on close inspection of filter coefficients, the values of $m_p$ and $m_f$ are considered as 15. These coefficients are demonstrated in Fig. 11. The regularization parameter, $\alpha_T$, is selected as 0.0001 which yielded the least RMS error with respect to the known heat flux values.
The estimated heat flux is calculated using Eq. (32) and Eq. (35) and plotted in Fig. 12. The estimated heat flux and the results from ANSYS simulation are well matched. The RMS error of the estimated heat fluxes from ANSYS simulation data is determined as 1097.6 W/m² for the whole time domain solution and 1112.7 W/m² for the filter solution.

In this case, a uniform random error of ±0.25% the maximum temperature (about +/–1.5 K) is added to the temperature data and the method is tested with the new data in presence of error. The same values for \( m_p \) and \( m_f \) are used and the heat flux is calculated as it is shown in Fig. 13.
Figure 12: estimated heat flux for test case 3 \((t=2\ \text{seconds}, \ \alpha_T=0.0001)\)

Figure 13: estimated heat flux for test case 3 - data with error \((t=2\ \text{seconds}, \ \alpha_T=0.2)\)
As can be seen, the filter method is able to estimate the heat flux with a reasonable error. The regularization parameter is selected as 0.2 for this test case to minimize the RMS error between ANSYS solution with respect to the whole time domain method and filter method (5484.3 W/m$^2$ 5505.3 W/m$^2$ respectively). Note that the calculated RMS values for this test case are significantly higher than the RMS values from previous test cases simply because the heat fluxes are much higher. The maximum and average heat fluxes for the third test case during the entire time are 115,000 W/m$^2$ and 49,522 W/m$^2$ respectively (compared with 441.95 W/m$^2$ and 73.63 W/m$^2$ for the first test case and 1 W/m$^2$ and 1 W/m$^2$ for the second test case). The calculated RMS errors for all of the test cases are summarized in Table 4.

<table>
<thead>
<tr>
<th>RMS error with respect to the exact solution (W/m$^2$)</th>
<th>Test case 1</th>
<th>Test case 2</th>
<th>Test case 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>w/o error</td>
<td>w/ error</td>
<td>w/o error</td>
</tr>
<tr>
<td>Filter Method</td>
<td>3.011</td>
<td>5.4499</td>
<td>0.0045</td>
</tr>
<tr>
<td>Whole Time Method</td>
<td>3.0005</td>
<td>5.4591</td>
<td>0.0043</td>
</tr>
</tbody>
</table>

Note that filter solution is a representation of the IHCP solution and in this case whole domain Tikhonov regularization method and therefore gives the same results.

### Conclusion

A strategy for solving one-dimensional IHCPs in multi-layer medium is developed. The method is discussed for a two-layer slab when the temperature measurement history is given in two interior locations of the inner layer. An IHCP is solved for each layer based on the minimization of the sum of the squared errors between the computed and known values and by using Tikhonov regularization (TR) for stabilizing the solution. The developed algorithm is then written in filter
form. The filter form solution allows near real time heat flux estimation which can be used in several industrial applications. The effect of thermal contact resistance is taken into account. The proposed solutions are demonstrated by numerical experiments. Three numerical test cases are considered and the heat flux at the surface is obtained using the proposed solutions. The results are compared with the exact solutions and good agreement between the results is observed in all three cases. The filter solution of the IHCP has a number of advantages including simplicity, continuous operation and application to moderate nonlinearity [21] which makes it a powerful approach for real time heat flux estimation in industrial applications.

References


ARTICLE 3: A FILTER FORM SOLUTION FOR NON-LINEAR INVERSE HEAT CONDUCTION PROBLEMS IN MULTI-LAYER MEDIUM

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Abstract

A solution for the non-linear inverse heat conduction problem (IHCP) in a two-layer medium is proposed and tested through numerical experiment. The temperature histories are considered to be known at two points on one layer and the heat transfer rate at the end of the layer exposed to a thermal environment is to be determined. A step-by-step solution is proposed for solving this problem based on the minimization of the sum of the squared errors between the computed and known values and by using of Tikhonov Regularization for stabilizing the solution. The solution is cast in digital filter form which allows a near real-time heat flux estimation in the multi-layer problem. The filter coefficients are determined for different temperatures. These data are used to train an artificial neural network (ANN) which calculates the filter coefficients based on the temperature at each time step. The ANN here serves to interpolate the filter coefficients to account for the temperature variation of the material properties. The proposed method is tested via numerical model developed in ANSYS and also the results are validated with the exact

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solution for constant properties. The filter algorithm can be used easily for near real-time heat flux estimation in industrial applications.

**Key words:** Inverse heat conduction, real-time heat flux measurement, temperature dependent material properties multi-layer medium, filter based solution

**Introduction**

The problem of estimating unknown surface conditions (temperature or heat flux) using internal temperature measurements is known as inverse heat conduction problem (IHCP). The IHCP is an ill-posed problem due to the lack of continuous dependence of the solution on the data. A small error in input data can significantly affect the outputs. Therefore, an appropriate regularization method needs to be applied to convert the ill-posed problem to a nearby well-posed problem which can be solved.

Methods include the least-square method with regularization, the sequential function specification, conjugate gradient method and numerical approaches [1].

Heat conduction in multi-layer mediums has been discussed in several papers. Ozisik [1] studied conduction in one dimensional composite media using orthogonal expansions, Green’s functions and Laplace transform. De Monte [2] studied the transient response of one-dimensional multi-layered composite conducting slabs to sudden variations of the temperature of the surrounding fluid. An analytical method for solving multi-layer heat conduction problems using Laplace transform and separation of variables is developed by Lu et al. [3]. They show that the result from their proposed closed form solution is in good agreement with numerical techniques. Haji-Sheikh and Beck [4] studied the temperature field in multi-dimensional, multi-layer bodies for the boundary conditions of the first, second and third kind. A solution for transient heat
conduction through a one-dimensional three-layer composite slab is proposed by Sun and Wichman [5].

A few studies discussed the solution of IHCPs in multi-layer medium. Al Najem and Ozisik [6] conducted an inverse heat conduction analysis using a splitting-up procedure and nonlinear least-squares technique for the whole time domain and estimated the surface condition in composite layers. Ruan et al. [7] used least square method and Beck's future time method for 1-D and 2-D geometries and calculated the unknown boundary cooling condition and contact heat transfer coefficient for solidification of alloys. The design of optimal transient heat conduction experiments on composite orthotropic materials is studied by Taktak et al. [8]. Al-Najem [9] developed a method of analysis for determining surface conditions from the knowledge of the time variations of the temperature at the insulated boundary. He used two segmented polynomial in time for the unknown surface temperature. An inverse solution is then developed over the whole time domain using the splitting-up procedure.

The necessity for real time heat flux measurement in variety of industrial applications attracted a lot of attentions to develop real-time solutions for IHCPs. A filter solution based on the idea of training neural networks is studied by Kowsari et al. [10]. Ijaz, et al. [11], used a Kalman filter to solve a two-dimensional transient IHCP. Feng et al. [12] used Laplace transforms to relate the measured conditions at one end of a domain to the unknown conditions at the remote surface. Woodbury and Beck [13] studied the structure of the Tikhonov regularization problem and concluded that the method can be interpreted as a sequential filter formulation for continuous processing of data. They show that the computed heat fluxes using the whole domain solution and the filter coefficient solution are virtually the same for the constant-property solutions.
While in most of the IHCP studies the remote boundaries is assumed as an insulated surface or cooled with a known heat transfer coefficient, e.g. [15, 14], this is not always the case in practice. Woodbury et al. [15] developed a filter based solution to incorporate the temperature measurement history from a second subsurface sensor as a remote boundary condition in an IHCP solution. In real world problems, the material thermal properties can greatly vary during the heating/cooling process due to significant temperature changes. This paper presents a filter based solution for two-layer mediums when the material thermal properties are temperature dependent. For this purpose, two IHCPs are solved (one for each layer) and a coupled solution is determined and tested to estimate the unknown heat flux at the surface of the front layer. The solution is then written in a digital filter form and filter coefficients are calculated for different temperatures and corresponding material thermal properties. An artificial neural network is then developed and trained to interpolate the filter coefficients at every time step. The proposed solution is then verified through several numerical experiments using exact solutions and ANSYS simulation. The filter solution of the IHCP has several advantages including simplicity, continuous operation and application to moderate nonlinearity [16] which makes it an appropriate approach for real time heat flux estimation in industrial applications.

Problem Description

A two-layer slab is considered to demonstrate the application of the proposed approach. A schematic of a two-layer slab is shown in Fig. 1. The temperature measurement histories are available for \( x=x_1 \) and \( x=x_2 \) on layer 2 while no specific temperature/heat flux measurement is available on layer 1. The heat flux at the remote surface of the first layer (\( x=0 \)) is unknown and to be determined by the proposed solution. An IHCP is solved for each layer, starting from the
one with known temperature measurements (layer #2 in here), and the heat flux is estimated at
the interface with the next layer.

Figure 1: Schematic of the two-layer problem

Solution Scheme
A schematic of the problem is given in Fig. 1. The solution is started in the second layer, where
two temperature measurements are available at $x_1$ and $x_2=L_1+L_2$ and $q_1$ is the unknown heat flux.
After solving the first IHCP, the heat flux and temperature are both known at the interface ($q_1$ and $T_1$). These values will be used as boundary conditions to solve the second IHCP associated
with the first layer, where $q$ at $x=0$ ($q_0$) is unknown. The analysis of each layer is explained in
detail as below.

Second Layer
The analysis starts with the second layer. The solution for the IHCP when the temperature
measurement is given at two sub-surface locations is given by Woodbury et al. [20]. A similar
approach is utilized in here to analyse the second layer. It is considered that the two temperature measurements histories are available at \(x = x_1\) and \(x = L_1 + L_2\) \((L_1 \leq x_1 \leq L_1 + L_2)\) on the layer 2:

\[
T(x_1, t) = Y(t) \tag{1}
\]

\[
T(L_1 + L_2, t) = y(t) \tag{2}
\]

The initial temperature is considered to be zero,

\[
T(x, 0) = 0 \tag{3}
\]

The objective of the first IHCP is to estimate the heat flux \((q_1)\) at the interface between the two layers:

\[
-k \frac{\partial T}{\partial x}(0, t) = q_1(t) = ? \tag{4}
\]

Solving the IHCP for the second layer results in determining the heat flux at the interface with the first layer which then will be used as a boundary condition for solving the second IHCP.

The temperature at \(x_1\) \((L_1 \leq x_1 \leq L_1 + L_2)\) is a function of the surface heat flux \(q_{10}\) and the temperature at \(x = L_1 + L_2\). This problem is known as X21B10T0, which is herein shortened to X21 for reference [17]. The notation denotes a Cartesian geometry subjected to a type 2 condition at the first boundary and a type 1 condition at the second boundary, and that the first boundary has a step change in value while the second boundary is homogeneous, and that the initial condition is also homogeneous.

The connecting curves between the heat flux components, \(q_i\) and \(q_{i+1}\), and also between the adjacent components, \(y_i\) and \(y_{i+1}\) are considered as constant between points (step function):

\[
q(t) = q_i, \quad t_i < t < t_{i+1} \tag{5}
\]

\[
t_i = i \Delta t \tag{6}
\]

A solution for the X21 case with a constant heat flux, \(q_c\), at \(x = L_1\) is
\[
T_{x_{21}}(x, t) = \left(1 - \frac{x}{L_2}\right) - \sum_{m=1}^{\infty} \frac{\cos \left(\beta_m \frac{x}{L_2}\right)}{\beta_m^2} \exp \left(-\frac{\beta_m^2 \alpha t}{L_2^2}\right)
\]  

(7)

where \( \beta_m = (m - 1/2)\pi, \ m = 1, 2, \ldots \).

Analogous to the above equations for a constant heat flux, the equation for a constant temperature, \( T_C \), at \( x = L_1 + L_2 \) is denoted \( X_{12B10T0} \) and shortened herein to \( X_{12} \):

\[
T_{x_{12}}(x, t) = 1 - 2 \sum_{m=1}^{\infty} \frac{\sin \left(\beta_m \frac{x}{L_2}\right)}{\beta_m^2} \exp \left(-\frac{\beta_m^2 \alpha t}{L_2^2}\right)
\]  

(8)

The temperature at any location \( x \) \((L_1 \leq x \leq L_1 + L_2)\) caused by the heat flux \( q_1 \) at \( x = L_1 \) and the temperature \( y \) at \( x = L_1 + L_2 \) is

\[
T_y = T_{y,M} + T_{y,M} = \sum_{i=1}^{M} q_i \Delta \phi_{y,i} + \sum_{i=1}^{M} y_i \Delta \eta_{y,i}
\]  

(9)

where \( \phi \) and \( \eta \) are the response basis functions for the two cases. That is,

\[
\phi(x, t) = \frac{\partial T_{x_{21}}}{\partial q_c}; \quad \eta(x, t) = \frac{\partial T_{x_{12}}}{\partial T_c}
\]  

(10)

The \( \Delta \phi \) 's can be found as [1]:

\[
\Delta \phi_0 = \phi, \quad \Delta \phi_1 = \phi_2 - \phi_1, \ldots, \quad \Delta \phi_i = \phi_{i+1} - \phi_i
\]  

(11)

Equation 9 can be described by the matrix equation of

\[
T = Xq + Zy
\]  

(12)

where
The components of the $x$ and $z$ matrices are related to the response basis functions by

$$X_i = \frac{L}{k} \Delta \phi_{i-1}, \quad Z_i = \Delta \eta_{i-1}$$  \hspace{1cm} (14)$$

where $\Delta \phi_i$ is defined in Eq. 11, and $\Delta \eta_i$ is defined analogously.

The columns in the $X$ matrix of Eq. 12 are the same as the sensitivity vectors in parameter estimation. The first column is for $q$ at $t_1$, the second for $q$ at $t_2$ and so on. Using whole domain Tikhonov regularization, the estimated value heat flux vector at the interface, $\hat{q}_1$, can be given as:

$$\hat{q}_1 = \left[ X_2^\top X_2 + \alpha \tau F_2^\top F_2 \right]^{-1} X_2^\top (Y - ZY) = F_1(Y - ZY)$$  \hspace{1cm} (15)$$

where the $\alpha$ is the Tikhonov regularization parameter [1]. Here the subscript 2 for $X$ and $F$ refers to Layer 2. Note that this equation is dimensional. $F_2$ has the same definition as the filter matrix of Ref [20]. The value of $\hat{q}_1$ from Eq. 15 will be used as a known boundary condition for the IHCP associated with the first layer. The heat flux at the remote surface ($x=0$) is the unknown parameters in the second IHCP.

First Layer

The temperature at any location $x_I$ ($0 \leq x_I \leq L_1$) is a function of the surface heat fluxes $q_0(t)$ and $q_1(t)$. The results from the middle layer analysis determine $q_1(t)$. A solution for the $X22$ case with a constant heat flux at $x = 0$ and zero heat flux at $x = L_I$ (this is denoted the $X22B10T0$ case) is [23]:
\[
T_{x,22}(x,t) = \frac{a t}{L_1^2} + \frac{1}{3} \frac{x}{L_1} + \frac{x^2}{2L_1^2} - 2 \sum_{n=1}^{\infty} \frac{\cos(\beta_n x / L_1)}{\beta_n^2} \exp\left(-\beta_n^2 \alpha t / L_1^2\right)
\]  

(16)

Note the same solution applies for a zero heat flux at \(x=0\) and a constant heat flux at \(x=L_1\) through a simple change of variables such as \(\xi = L_1 - x\). Analogous to Eq. (9), the temperature response at \(x_l (0 \leq x_l \leq L_l)\) can be found due to the heat flux histories \(q_{0}(t)\) and \(q_{1}(t)\). For assumed piecewise constant variation in these heat fluxes, the temperature can be computed from

\[
T_{w} = T_{w,0} + T_{w,1} = \sum_{i=1}^{M} q_{0i} \Delta \phi_{w,i} + \sum_{i=1}^{M} q_{1i} \Delta \theta_{w,i}
\]

(17)

where

\[
\phi(x_i, t) = \frac{\partial T_{x,22}(x_l, t)}{\partial q_i}; \quad \theta(x_i, t) = \frac{\partial T_{x,22}(L_l - x_i, t)}{\partial q_i}
\]

(18a,b)

The step basis function representation used here (and also others) for temperature given in Eq. (17) can be described by the matrix equation of

\[
T = X_0 q_0 + X_L q_L
\]

(19)

where

\[
T = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix}, \quad q_o = \begin{bmatrix} q_{01} \\ q_{02} \\ \vdots \\ q_{0n} \end{bmatrix}, \quad q_L = \begin{bmatrix} q_{L1} \\ q_{L2} \\ \vdots \\ q_{Ln} \end{bmatrix}, \quad X = \begin{bmatrix} X_1 & 0 & 0 & \cdots & 0 & 0 \\ X_2 & X_1 & 0 & \cdots & 0 & 0 \\ X_3 & X_2 & X_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ X_n & X_{n-1} & X_{n-2} & \cdots & X_2 & X_1 \end{bmatrix}
\]

(20a,b,c,d)

The components of the \(X_0\) and \(X_L\) matrices are related to the response basis functions in Eq. (18) similar to the relation of \(X\) and \(Z\) in Eq. (14) to Eq. (10).

The whole domain Tikhonov regularization method is used to solve the IHCP. Later the filter coefficients are found from the solution. The IHCP solution for Layer 1 starts with a matrix form for the sum of squares with an added regularization term given by
\[
S = (T_i - X_o \mathbf{q}_o - X_{t_1} \mathbf{q}_{1})^T (T_i - X_o \mathbf{q}_o - X_{t_1} \mathbf{q}_{1}) + \alpha_s \mathbf{q}_0^T H^T H \mathbf{q}_0
\]  

(21)

This is minimized with respect to the parameter vector \( \mathbf{q}_0 \). The symbol \( T_i \) is the temperature vector at \( L_I \) and \( \mathbf{q}_1 \) is the known heat flux (from the solution of the IHCP in Layer 2) at \( x = L_I \). The initial temperature is zero. The \( \alpha_s \) symbol is the Tikhonov regularization parameter.

The estimated value of the heat flux vector, denoted \( \hat{\mathbf{q}}_o \), is then given by

\[
\hat{\mathbf{q}}_o = (X_o^T X_o + \alpha_s H^T H)^{-1} X_o^T \left( T_i - X_{t_1} \mathbf{q}_1 \right) = \mathbf{F}_1 \left( T_i - X_{t_1} \mathbf{q}_1 \right)
\]  

(22)

**Coupling of the Solutions**

To achieve a single expression for a two-layer IHCP, Eq (15) is substituted in Eq (22). An expression is:

\[
\hat{\mathbf{q}}_o = \mathbf{F}_i T_i - \mathbf{F}_i X_{t_1} \left( \mathbf{F}_2 T_i - \mathbf{F}_2 Z y \right) = \mathbf{F}_i \left( \mathbf{I} - X_{t_1} \mathbf{F}_2 \right) T_i + \mathbf{F}_i X_{t_1} \mathbf{F}_2 Z y
\]  

(23)

where \( T_i \) is the temperature at the interface and can be found from Eq. 12:

\[
T_i = X_{t_1} \mathbf{q}_1 + Z y
\]

By substituting \( \mathbf{q}_1 \) in the above equation \( T_i \) can be found as:

\[
T_i = X_{t_1} \mathbf{F}_2 Y + (Z \cdot X_{t_1} \mathbf{F}_2 Z) y
\]  

(24)

Substituting Eq. 25 in Eq. 23, an expression is:

\[
\hat{\mathbf{q}}_o = \mathbf{F}_i \left( X_{t_1} \mathbf{F}_2 + X_{t_1} \mathbf{F}_2 \right) Y + \mathbf{F}_i \left( Z - X_{t_1} \mathbf{F}_2 Z - X_{t_1} \mathbf{F}_2 Z \right) y
\]  

(25)

Equation 25 can be used directly to calculate the heat flux at the remote surface on the first layer by using two sets of temperature measurements from the second layer.
Filter Form of the Solution

The concept of the filter algorithm is that the solution for the heat flux at any time is only affected by the recent temperature history and a few future time steps. Equation (25) can be written in filter form as:

\[ \hat{q}_k = fY + gy \]  \hspace{1cm} (26)

\[ f = \text{row}\left[ F_i \left( X_zF_z + X_i F_i \right) \right] \quad , \quad g = \text{row}\left[ F_i \left( Z - X_zF_zZ - X_i F_iZ \right) \right] \]  \hspace{1cm} (27)

where \( f \) and \( g \) have the same characteristics as filter coefficients, which are discussed in detail in Ref [20]. The meaning of “row()” in Eq. (27) designates a row in the middle of the indicated matrix. Note that these filter coefficients are not the same as filter factors suggested by Hansen [18].

All the \( f \)-filter coefficients can be found at one time by setting all the \( Y \) and \( y \) components equal to zero except the \( m_f \) component of \( Y \) is set equal to one \((Y_{mf}=1)\). The solution of the IHCP with this data gives the \( f \) coefficients. To get the \( g \)-filter coefficients (those for the \( y \) vector), the same procedure is followed with now all the components of \( Y \) and \( y \) equal to zero except \( y_{mf}=1 \).

Assuming \( x_i=L_i \), the dimensionless time step of 0.0052 (0.1 s) and for first order Tikhonov regularization with parameter \( \alpha = 0.0001 \), and material properties given in Table 1, the 150th rows of the matrices in Eq. (27) (the \( f \) and \( g \) filter coefficients) are plotted in figures 2 and 3.

**Table 1: material properties**

<table>
<thead>
<tr>
<th>Thickness, m</th>
<th>Layer 1 (Inconel)</th>
<th>Layer 2 (Ceramic Fiber)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>T= 300 K</strong></td>
<td>K= 14.9 W/m-K, ( \alpha = 3.9e-6 ) m²/s</td>
<td>K= 0.046 W/m-K, ( \alpha = 3.2e-7 ) m²/s</td>
</tr>
<tr>
<td><strong>T= 450 K</strong></td>
<td>K= 17.4 W/m-K, ( \alpha = 4.2e-6 ) m²/s</td>
<td>K= 0.065 W/m-K, ( \alpha = 3.9e-7 ) m²/s</td>
</tr>
<tr>
<td><strong>T= 800 K</strong></td>
<td>K= 22.6 W/m-K, ( \alpha = 4.9e-6 ) m²/s</td>
<td>K= 0.13 W/m-K, ( \alpha = 6.9e-7 ) m²/s</td>
</tr>
<tr>
<td><strong>T= 1000 K</strong></td>
<td>K= 25.4 W/m-K, ( \alpha = 5.3e-6 ) m²/s</td>
<td>K= 0.19 W/m-K, ( \alpha = 7.9e-7 ) m²/s</td>
</tr>
<tr>
<td><strong>T= 1200 K</strong></td>
<td>K= 28.2 W/m-K, ( \alpha = 5.57e-6 ) m²/s</td>
<td>K= 0.27 W/m-K, ( \alpha = 1.2e-6 ) m²/s</td>
</tr>
</tbody>
</table>
As can be seen, the filter coefficients can greatly vary as the temperature changes.
Filter Coefficients for the Case of Temperature Dependent Materials

If the material properties are temperature dependent, the filter coefficients are changing through the heating/cooling process as the temperature varies (as shown in Fig. 2 and 3). Therefore, it is necessary to find the filter coefficients at each time step with a particular temperature and use them accordingly. This can be done by finding the filter coefficients for a set of temperatures and linearly interpolate between those at each time step. However, this technique turned out to be time consuming. Therefore, a fast and accurate method is developed to find filter coefficients at each temperature by using Artificial Neural Networks (ANNs). ANN’s are computational models inspired from human brain system which has been successfully used in several engineering applications namely time series prediction, pattern recognition, function approximation, classification and more. They have also been used to solve direct and inverse heat conduction problems [19, 20, 21, 22, 23].

ANN consisted of a set of interconnected neurons that can evaluate outputs from inputs by feeding information through the network and adjusting the weights. In the present work (test case 2), a feed forward multi-layered network is used which consists of a layer of input neurons (including temperature data and time step), a layer of output neurons (filter coefficients) and two hidden layers. Data enter the network through the input nodes and going through a non-linear transformation. The output data are subsequently generated by the output nodes. The inputs of the network are the temperature and the time step. Since the filter coefficients beyond $m_p+m_f$ are all zero, the interpolations performed only for the time steps 1 through $m_p+m_f$. A schematic of the architecture of a three-layered neural network used in this work to calculate $f$ filter coefficients is shown in Figure 4. A similar structure is used to calculate $g$ filter coefficients.
Results and discussion

The developed method is verified via numerical experiments and ANSYS simulation. The verification of the method is described in this section. To verify the developed filter solution, a two layer slab is modelled in ANSYS. The material properties of the layers are given in Table 2.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Layer 1 (Steel)</th>
<th>Layer 2 (Aluminum)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density, kg/m$^3$</td>
<td>7833</td>
<td>2702</td>
</tr>
<tr>
<td>Specific Heat, J/kg.K</td>
<td>465</td>
<td>903</td>
</tr>
<tr>
<td>Conductivity ($k$), W/m.K</td>
<td>54</td>
<td>237</td>
</tr>
<tr>
<td>Thickness ($L$), m</td>
<td>0.01</td>
<td>0.02</td>
</tr>
</tbody>
</table>

To validate the ANSYS model, the results from ANSYS simulation is compared with exact solution [24] for constant material properties. The created model is meshed and the loads are applied on the geometry similar to the X1C11B10T0 case. A step change in temperature is applied at the front surface and the back surface temperature is kept at zero.

The sensor locations are at $x=L_1$ and $x=L_1+L_2$. The temperature data is also generated using the solution for the direct problem for a step change in temperature at the surface (X1C11B10T0) and used as inputs for the filter solution (Eq. 26) to determine the heat flux at the surface.

The calculated heat flux by the filter solution, ANSYS simulation and exact solution are compared in Fig. 5 and a good agreement can be observed between all three set of results.
It should be noted that the values of $m_p$ and $m_f$ are considered as 30 for this test case and the optimal regularization parameter is determined as 5E-12 through several trials of numbers with different order of magnitudes. Afterwards, a random error (0.5% of the temperature) is added to data and the heat flux is calculated. The calculated heat flux when error presents in the data is plotted in Fig. 6. It can be observed that the results are still in a good agreement with the exact solution. The regularization parameter is set as 1E-3. It should be noted that the first test case only examined the developed approach for constant material properties. The next test case has taken into account the temperature dependent material thermal properties.
In a second test case, a triangular heat flux profile is applied to the surface of the two-layer slab. It is assumed that geometry of the two-layer medium is similar to the previous test case. The material properties, however, are changing with temperature as it is shown in Table 1. The temperature histories are then obtained at the desired surfaces using ANSYS and used as inputs for the filter solution. The calculated temperature profiles are shown in Fig. 7.
Using filter coefficients for constant material properties, the heat flux profile is estimated and shown in Fig. 8. The Tikhonov parameter is considered as 0.0001 and the time step is 1 second.

![Heat flux estimation using filter method-assuming constant thermal properties](image)

As can be seen, the actual heat flux is significantly different from the estimated heat fluxes by the proposed method. The closest estimation is the one which uses average temperature of 864 K which still significantly off during the first 50 seconds, when the temperature is below the average temperature. This can clearly show the importance of accounting for variation of thermal properties due to temperature changes.

Next, using the developed neural networks, the filter coefficients are found for each time step based on the temperature. Each of the developed ANN’s for $f$ and $g$ includes 2 hidden layers, 31 inputs ($1 + m_p + m_f$) including one temperature and 30 time steps and 30 outputs ($m_p + m_f$) including the non-zero filter coefficients. The network trained using 10 set of samples ($T=300, 400, \ldots, T=1200$). The heat flux is then calculated using Eq. 26 and compared with the given heat flux at the surface in Fig. 9. As can be seen, the heat flux profile is very close to the
heat flux from the ANSYS model and can do significantly better than the calculated heat flux by the filter coefficients from average temperature.

![Graph of Heat Flux Estimation](image)

**Figure 9:** Heat flux estimation using filter solution and neural networks

**Conclusion**
A method for solving one dimensional non-linear IHCP in two-layer mediums with temperature dependent material thermal properties is developed and successfully tested. The method is discussed for a two-layer slab when the temperature measurement history is given in two interior locations of one layer. An IHCP is solved for each layer based on the minimization of the sum of the squared errors between the computed and known values and by using of Tikhonov Regularization (TR) for stabilizing the solution. The developed algorithm is then written in filter form. The filter form solution allows near real time heat flux estimation which can be used in several industrial applications. The proposed solutions are then validated by numerical experiments. Two numerical test cases are developed in ANSYS, the first one with constant
thermal properties and the second one with temperature dependent thermal properties. The heat flux at the surface is obtained using the proposed solution and compared with the ANSYS simulation and also exact solution. For the second test case, two neural networks are developed to estimate the filter coefficients $f$ and $g$ at each time step and its corresponding temperature. The filter coefficients are then used to calculate the heat flux at the surface by using the proposed solution. The results from the proposed solution is in a very good agreement with the ANSYS simulation. The results showed that when the temperature varies significantly, accounting for variation in thermal properties of the materials can greatly improve the accuracy of the estimated heat flux.

References


ARTICLE 4: REAL TIME MEASUREMENT OF HEAT FLUX BY DIRECTIONAL FLAME THERMOMETERS USING FILTER FORM IHCP METHOD

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Abstract
Real-time measurement of heat flux is an important challenge for several industrial applications, including furnace control. For efficient operation of high-temperature process furnaces, accurate and stable temperature measurements are needed. Directional Flame Thermometer (DFT) offers the ability to use both temperature and heat flux measurements for furnace control. Currently, analysis of dynamic temperature data from DFT to compute heat flux information must be performed off-line using the gathered temperature data and a full-non-linear inverse heat conduction problem (IHCP) analysis. Developing a near real-time algorithm for accurate reduction of the data will allow for continual monitoring of the furnace during operation. This will result in better furnace control and significant savings in energy and cost.

This paper provides a solution strategy based on the filter concept for the IHCP associated with DFT. The filter-based solution has the capability of heat flux estimation in near real-time. Two IHCPs are discussed and a coupled solution is proposed to estimate the unknown surface heat flux. The variation of thermal properties with temperature is taken into the account through interpolation of the filter coefficients computed at different temperatures. The solution procedure is validated by comparing the results with a numerical test case developed in ANSYS. Results are also computed using data from a physical experiment with DFT (see Reference [13]). The heat fluxes obtained are found in good agreement with those obtained from a full non-linear IHCP analysis.

**Keywords:** heat flux measurement, inverse heat conduction problem, non-linear filter solution, directional flame thermometer.

**Introduction**

Effective temperature control in furnaces needs accurate measurement of heat flux. Currently, two major types of measuring sensors are used in standard fire tests for measuring the heat flux, known as active (non-equilibrium) and passive (equilibrium) sensors [1].

Active sensors, such as Gardon and Schmidt-Boelter gauges, measure the heat flux across a measured temperature difference [2, 3]. These sensors work based on the concept of measuring the temperature gradient within the sensor. For this purpose, water needs to continuously pass through the sensor to maintain a temperature gradient from the fire. As the result, the use of active sensors is limited to applications where adequate water is available and water tubes can be installed safely. Possible condensation of the unburned fuel or water on the surface of the sensor could be also problematic when using active sensors [1].
The second type of heat flux measurement sensors, equilibrium sensors, generally consist of two metal plates with an insulation layer in between. Thermocouples are installed on the backside of each plate and covered with insulation material. The plate’s temperature increases quickly and reaches quasi-equilibrium with the fire environment. An Inverse Heat Conduction Problem (IHCP) is defined in order to determine the heat flux on the surface by using the measured temperature values. Equilibrium sensors are relatively inexpensive and do not need water for operation. The easy installation is an asset which allows use of Directional Flame Thermometer’s (DFT’s) for different environments and variety of applications. Moreover, since the surface temperature is close to the gas temperature, there is no concern for condensation of water/unburned fuel and the resulting uncertainties. Different types of equilibrium sensors such as Plate Thermometers, Sandia Hemispherical Heat Flux Gage and Directional Flame Thermometers (DFT) are tested and discussed in [4, 5, 6].

A schematic of a DFT is shown in Fig. 1. The original DFT design involved a thin metal disk mounted in a steel tube [7, 8]. To minimize heat loss from the unexposed surface of the disk, multiple radiation shields and some ceramic fiber insulation were mounted behind the front disk.

![Figure 1: Schematic of a directional flame thermometer](image-url)
Sandia National Laboratories improved DFTs for use in large pool fire and other tests. Their goal was to provide both transient and quasi-steady heat transfer measurements in various fire environments [9]. Samuel et al. [10] used DFTs to measure the heat flux in wildland–urban interface (WUI) fires. They emphasized that the limited access to water in such areas urges the use of DFTs. They used water cooled total heat flux sensors for a direct comparison of the heat flux obtained from the DFTs. Lam and Weckman [11] examined the steady state response of four heat flux gauges including DFTs under various radiative and convective conditions and compared the results. In another study, Sultan [12] investigated the performance of six different temperature sensors in fire resistance test. The result showed that all the sensors yield similar results after approximately 10 minutes.

Analyzing DFT data over the entire test duration needs an inverse heat conduction code which uses two temperature measurement histories for estimating the net heat flux to the exposed surface ($q_{\text{front}}$, in Fig. 1). Considering the wide range of temperature variation, the material properties of the DFT changes significantly which introduces non-linearity to the problem. Therefore, an accurate solution technique for IHCP associated with the DFT application, must be able to account for the variation temperature dependent material properties. Presently, analysis of dynamic temperature data from the DFTs to compute heat flux information must be performed off-line at the conclusion of data-gathering. Availability of a near real-time algorithm for accurate reduction of data will allow for continual monitoring of the furnace during operation. This will result in better furnace control and significant savings in energy and cost. Recently, Kokel et al. [13] developed a heat transfer model to provide the user with a simple forward solution methodology which allows close to real-time measurement of heat flux with DFT. The forward and inverse models were in good agreement with experimental data. They assumed
constant material properties for their model. Najafi et al. [14] developed a filter based solution for the near real time heat flux estimation by DFT by assuming constant material properties during the entire test duration.

IHCPs have been widely studied and several different methods used as the solution of these types of problems [15, 16, 17]. Recently several research works have been performed to investigate real-time or filter forms for processing temperature data to solve the IHCP. A filter solution based on the idea of training neural networks is studied by Kowsari et al. [18]. Ijazet al. [19], used a Kalman filter to solve a two-dimensional transient IHCP. Feng, et al. [20] used Laplace transforms to relate the measured conditions at one end of a domain to the unknown conditions at the remote surface. Most recently, Woodbury and Beck [21] studied the structure of the Tikhonov regularization problem and concluded that the method can be interpreted as a sequential filter formulation for continuous processing of data. They examined the filter based solution by two examples and compared the results with the whole domain method and observed that the results are the same for these constant-property solutions. It was also observed that while the correct selection of the order of magnitude of the regularization parameter is important, the results are not sensitive to the precise selection of it.

In the present paper, a filter form solution for the inverse heat conduction problem is developed for a two-layer medium considering the variation of material’s thermal properties with temperature. The procedure incorporates highly efficient and highly accurate exact analytical solutions for temperature distribution in the individual layers, valid for constant properties in the layers, which are extracted from the ExACT website [31]. The resulting algorithm is fast, accurate and highly computationally efficient for implementing in real-time.
For the two-layer problem, temperature is given at the boundaries of the middle layer of the DFT and the heat flux is unknown at the front surface. Two IHCPs are solved (one for the middle layer and one for the front plate) and a coupled solution is proposed and tested to estimate the unknown heat flux at the front plate. Importantly, the filter coefficients derived for constant properties can be interpolated based on the measured temperatures to account for non-linearity associated with temperature-dependent material properties. Several numerical experiments using exact solutions and ANSYS simulation are used to exercise the resulting non-linear IHCP filter form solution. Finally, the developed solution is applied to data from an actual experiment and the results compared with those from a full non-linear IHCP solver.

Problem Definition
Heat transfer problems can be categorized in two groups: forward problems and inverse problems. In a forward problem the initial conditions and the boundary conditions are known and the temperature for all points in the space-time domain is unknown. Mathematically, a forward problem is known as a well-posed problem because it has a unique and stable solution which depends continuously on the data. Unlike forward problems, inverse problems are ill-posed. The active boundary condition is unknown and, alternatively, the temperature history at one or more points inside the domain is available. In using DFT sensor for heat flux measurements, two sets of temperature histories are available at the boundaries of the middle layer. To estimate the unknown heat flux at the surface of the front plate, a solution strategy is developed as described in the next section.

DFT consists of three layers: two Inconel plates and one layer of insulation. It is assumed that the heat transfer is 1-dimensional (only x-direction). A schematic of DFT is shown in Fig. 1. The properties of the materials used in DFT are given in Table. 1.
Table 1: material properties for DFT

<table>
<thead>
<tr>
<th>Thickness, m</th>
<th>Layer 1 (Inconel)</th>
<th>Layer 2 (Ceramic Fiber)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.0031</td>
<td>0.019</td>
</tr>
<tr>
<td>T= 300 K</td>
<td>$k= 14.9$ W/m-K, $\alpha=3.9e^{-6}$ m$^2$/s</td>
<td>$k = 0.046$ W/m-K, $\alpha=3.2e^{-7}$ m$^2$/s</td>
</tr>
<tr>
<td>T= 450 K</td>
<td>$k = 17.4$ W/m-K, $\alpha=4.2e^{-6}$ m$^2$/s</td>
<td>$k = 0.065$ W/m-K, $\alpha=3.9e^{-7}$ m$^2$/s</td>
</tr>
<tr>
<td>T= 800 K</td>
<td>$k = 22.6$ W/m-K, $\alpha=4.9e^{-6}$ m$^2$/s</td>
<td>$k = 0.13$ W/m-K, $\alpha=6.9e^{-7}$ m$^2$/s</td>
</tr>
<tr>
<td>T= 1000 K</td>
<td>$k = 25.4$ W/m-K, $\alpha=5.3e^{-6}$ m$^2$/s</td>
<td>$k = 0.19$ W/m-K, $\alpha=7.9e^{-7}$ m$^2$/s</td>
</tr>
<tr>
<td>T= 1050 K</td>
<td>$k = 26.1$ W/m-K, $\alpha=5.31e^{-6}$ m$^2$/s</td>
<td>$k = 0.21$ W/m-K, $\alpha=1.04e^{-6}$ m$^2$/s</td>
</tr>
<tr>
<td>T= 1200 K</td>
<td>$k = 28.2$ W/m-K, $\alpha=5.57e^{-6}$ m$^2$/s</td>
<td>$k = 0.27$ W/m-K, $\alpha=1.2e^{-6}$ m$^2$/s</td>
</tr>
</tbody>
</table>

Solution Strategy of the IHCP

Two IHCP’s are solved separately for heat flux estimation on the front plate surface and eventually a coupled solution is derived for the two-layer problem associated with DFT. A schematic of the two layer problem is given in Fig. 2.

![Figure 2: Schematic of the two layer problem associated with DFT](image-url)
It should be noted that, in practice the front plate of the DFT is located towards the heat source (fire) and therefore the desired unknown value is the applied heat flux at the front plate. Consequently, the analysis is only performed on the middle layer and the front plate and the solution does not involve the back layer. The solution is started from the second layer (Ceramic Fiber), where two temperature measurements are available at two boundaries and \( q_i \) is the unknown heat flux. After solving the first IHCP, the heat flux and temperature are both known at the interface (\( q_1 \) and \( Y \)). These values will be used as boundary conditions to solve the second IHCP associated with the Inconel plate, where \( q_0 \) is unknown. The analysis of each layer is explained in detail as below.

Two IHCP’s are solved separately for heat flux estimation on the front plate surface and the middle layer and eventually a coupled solution is derived for the two-layer problem associated with DFT. It should be noted that, in practice the front plate of the DFT is located towards the heat source (fire) and therefore the desired unknown value is the applied heat flux at the front plate. Consequently, the analysis is only performed on the middle layer and the front plate and the solution does not involve the back layer. A schematic of the two layer problem is given in Fig. 2.

The solution is started from the second layer (Ceramic Fiber), where two temperature measurements are available at two boundaries and \( q_i \) is the unknown heat flux. After solving the first IHCP, the heat flux and temperature are both known at the interface (\( q_1 \) and \( Y \)). These values will be used as boundary conditions to solve the second IHCP associated with the Inconel plate, where \( q_0 \) is unknown. The analysis of each layer is explained in detail in below. A more general formulation for surface heat flux estimation in multi-layer medium when temperature sensors are located in the inner layer is presented by Najafi, et al. [22].
Herein, exact analytical solutions to the corresponding forward heat conduction problems are used as building blocks to construct the required IHCP solutions. These exact solutions are available at the ExACT website [31], which is a repository of efficient analytical solutions to diffusion problems. These algorithms are fast and accurate and highly suitable for development of a real-time algorithm for the IHCP.

Insulation layer

The temperature at \( x_1 \) (\( L_1 \leq x_1 \leq L_1 + L_2 \)) is a function of the surface heat flux \( q_1(t) \) and the temperature at \( x = L_1 + L_2 \).

Therefore, the boundary condition of the second kind is given at \( x = L_1 \) and the first kind at \( x = L_1 + L_2 \). This problem is designated by Cole, et al. [23] as X21B10T0, which is herein shortened to X21 for reference. The notation denotes a Cartesian geometry subjected to a type 2 condition at the first boundary and a type 1 condition at the second boundary, and that the first boundary has a step change in value while the second boundary is homogeneous, and that the initial condition is also homogeneous.

The heat flux components, \( q_i \), and temperatures, \( y_i \), are assumed constant between times (piecewise constant functions):

\[
q(t) = q_i, \quad t_i < t < t_{i+1}
\]

\[
t_i = i \Delta t
\]

A solution in Layer 2 for the X21 case with a constant heat flux, \( q_c \), at \( x = L_1 \) is

\[
\frac{T_{X21}(x,t)}{q_c L_2} = \left(1 - \frac{x}{L_2}\right) - \sum_{m=1}^{\infty} \frac{1}{\beta_m^2} \cos\left(\beta_m \frac{x}{L_2}\right) \exp\left(-\beta_m \frac{\alpha t}{L_2^2}\right)
\]

(3)
where \( \beta_m = (m - 1/2) \pi, \ m = 1, 2, ... \)

Similar to the above equations for a constant heat flux, the solution for a constant temperature, \( T_c \), at \( x = L_1 + L_2 \) is denoted \( X_{12B10T0} \) and shortened herein to \( X_{12} \) is:

\[
\frac{T_{X_{12}}(x,t)}{T_c} = 1 - 2 \sum_{m=1}^{\infty} \frac{1}{\beta_m} \sin \left( \beta_m \left( \frac{x}{L_2} \right) \right) \exp \left( -\beta_m^2 \frac{\alpha_{1t}}{L_2^2} \right) \tag{4}
\]

Note that the eigenvalues are the same as the \( X_{21} \) case. The temperature at any location \( x_1 \) \((L_1 \leq x_1 \leq L_1 + L_2)\) caused by the heat flux \( q_1 \) at \( x = L_1 \) and the temperature \( y \) at \( x = L_1 + L_2 \) is

\[
T_m = T_{q,m} + T_{y,m} = \sum_{i=1}^{M} q_{i,1} \Delta \phi_{M,i} + \sum_{i=1}^{M} y_i \Delta \eta_{M,i} \tag{5}
\]

where \( \phi \) and \( \eta \) are the response basis functions for the two cases. That is,

\[
\phi(x,t) = \frac{\partial T_{X_{21}}}{\partial q_c} \quad \eta(x,t) = \frac{\partial T_{X_{12}}}{\partial T_c} \tag{6a,b}
\]

Here \( \Delta \phi \)'s can be found as \( \Delta \phi_{i} = \phi(x_i,t_i) - \phi(x_i,t_{i-1}) \) \[15\], and \( \Delta \eta_{i} \), similarly.

Equation 5 can be written in the matrix form as

\[
T = Xq + Zy \tag{7}
\]

Here \( T \) is the temperature at any point \( x_1 \) \((L_1 \leq x_1 \leq L_1 + L_2)\). Where:

\[
T = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix}, \quad q = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}, \quad X = \begin{bmatrix} X_1 & 0 & 0 & \cdots & 0 & 0 \\ X_2 & X_3 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ X_n & X_{n-1} & X_{n-2} & \cdots & X_2 & X_1 \end{bmatrix} \tag{8 a,b,c}
\]

\[
y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad Z = \begin{bmatrix} Z_1 & 0 & 0 & \cdots & 0 & 0 \\ Z_2 & Z_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Z_n & Z_{n-1} & Z_{n-2} & \cdots & Z_2 & Z_1 \end{bmatrix} \tag{9 a,b}
\]
The components of the \( X \) and \( Z \) matrices are related to the response basis functions by
\[
X_i = \frac{L}{k} \Delta \phi_{i-1}, \quad Z_i = \Delta \eta_{i-1}
\]  

(10 a,b)

Using whole domain Tikhonov regularization, the estimated value heat flux vector at the interface, \( \hat{q}_i \), can be given as:
\[
\hat{q}_i = (X^T X + \alpha_h H^T H)^{-1} X^T (Y - Z y)
\]
\[
= F_2 (Y - Z y)
\]

(11)

Where \( Y \) and \( y \) are the vectors of temperature measurements at \( x=L_1 \) and \( x=L_1+L_2 \) respectively. The subscript 2 for \( X \) and \( F \) refers to Layer 2 and \( H \) is the first order regularization matrix. Note that this equation is dimensional. \( F_2 \) has the same definition as the filter matrix of Ref [21, 22, 24].

Front Inconel plate

The temperature at any location \( x_L (0 \leq x_L \leq L_1) \) is a function of the surface heat fluxes \( q_0(t) \) and \( q_1(t) \). The results from the middle layer analysis determined \( q_1(t) \) as given in Eq (11).

For this layer, the boundary condition of the second kind (Neumann) is given at \( x=0 \) and at \( x=L_1 \) (X22).

A solution for the X22 case with a constant heat flux at \( x=0 \) and zero heat flux at \( x=L_1 \) (this is denoted the X22B10T0 case) is [23]:
\[
\frac{T_{x_{22}}(x,t)}{q_L / k} = \frac{a t}{L_1^2} + \frac{1}{3} - \frac{x}{L_1} + \frac{x^2}{2 L_1^2} - 2 \sum_{m=1}^{\infty} \frac{1}{\gamma_m^2} \cos \left( \frac{\gamma_m x}{L_1} \right) \exp \left( -\gamma_m^2 a t \frac{L_1^2}{L_1^2} \right)
\]

(12)

where \( \gamma_m = m \pi \) is the \( m^{th} \) eigenvalue \((m=1,2,3,\ldots)\).

Note the same solution applies for a zero heat flux at \( x=0 \) and a constant heat flux at \( x=L_1 \) through a simple change of variables such as \( \xi = L_1 - x \).
Analogous to Eq. (5), the temperature response at \(x_1(0 \leq x_1 \leq L_1)\) can be found due to the heat flux histories \(q_0(t)\) and \(q_1(t)\). For assumed piecewise constant variation in these heat fluxes, the temperature can be computed from

\[
T_M = T_{q_0,M} + T_{q_1,M} = \sum_{i=1}^{M} q_{0,i} \Delta \varphi_{M,i} + \sum_{i=1}^{M} q_{1,i} \Delta \theta_{M,i}
\]  

(13)

where

\[
\varphi(x_1,t) = \frac{\partial T_{X,X_2} (x_1,t)}{\partial q_x}; \quad \theta(x_1,t) = -\frac{\partial T_{X,X_2} (L_1 - x_1,t)}{\partial q_x}
\]

(14a,b)

Eq. (13) can be described by the matrix equation of

\[
T = X_0 q_0 + X_L q_L
\]

(15)

where

\[
T = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix}, \quad q_0 = \begin{bmatrix} q_{01} \\ q_{02} \\ \vdots \\ q_{0n} \end{bmatrix}, \quad q_L = \begin{bmatrix} q_{L1} \\ q_{L2} \\ \vdots \\ q_{Ln} \end{bmatrix}
\]

(16a,b,c,d)

\[
X = \begin{bmatrix} X_1 & 0 & 0 & \cdots & 0 & 0 \\ X_2 & X_1 & 0 & \cdots & 0 & 0 \\ X_3 & X_2 & X_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ X_n & X_{n-1} & X_{n-2} & \cdots & X_2 & X_1 \end{bmatrix}
\]

The components of the \(X_0\) and \(X_L\) matrices are related to the response basis functions in Eq. (15) similar to the relation of \(X\) and \(Z\) in Eq. (10) to Eq. (7).

The whole domain Tikhonov regularization method is again used to solve the IHCP. The estimated value of the heat flux vector, denoted \(\hat{q}_0\), is then given by

\[
\hat{q}_0 = [X_0^T X_0 + \alpha \gamma H^T H]^{-1} X_0^T (Y - X_{Lq} q_1) = F_1 \left( Y - X_{Lq} q_1 \right)
\]

(17)
Coupling of the solutions

The heat flux at the interface \((x=\ell_i)\), \(\dot{q}_i\), is determined by solving the IHCP for the middle layer (Layer 2). The solution to this problem is given in Eq. (11). By substituting \(\dot{q}_1\) into Eq. (18), the following result is obtained:

\[
\dot{q}_a = F_f Y + F_f X \ell_i (F_f - F_f Z Y) = F_f \left( I + X \ell_i F_f \right) Y - F_f X \ell_i F_f Z Y
\]

Equation (18) is dimensional: \(F\) has units of \(W/m^2\cdot K\), \(X\) has units of \(K\cdot m^2/W\), and \(Z\) is dimensionless. Equation (18) demonstrates that the heat flux at the surface of the outer layer can be found from the measurements on the boundaries of the inner layer.

Equation (18) can be also written in a dimensionless form. The relation between dimensional quantities and dimensionless quantities by different thermal properties in the two regions are defined as below. The dimensionless quantities, denoted with a tilde ‘~’, are:

\[
\begin{align*}
\tilde{F}_2 & = \frac{L_2}{k_2} F_2, \quad \tilde{F}_1 = \frac{L_1}{k_1} F_1, \quad \tilde{X}_2 = \frac{k_2}{L_2} X_2, \quad \tilde{X}_0 = \frac{k_1}{L_1} X_0, \quad \tilde{X}_L = \frac{k_1}{L_1} X_L, \\
\tilde{Y} & = \frac{1}{q_{ref} L_1 / k_1} Y, \quad \tilde{y} = \frac{1}{q_{ref} L_i / k_i} y
\end{align*}
\]

Note \(Z\) is dimensionless since it is \(\partial T_{i_s} / \partial T_{\ell_i}\), so

\[
\tilde{Z} = Z
\]

The non-dimensional form of the equation can be written as:

\[
\tilde{\dot{q}}_a = \tilde{F}_f \left\{ I + \tilde{X} \ell_i \tilde{F}_f \right\} \tilde{Y} - \left\{ \tilde{F}_f \tilde{X} \ell_i \tilde{F}_f \tilde{Z} \right\} \tilde{y}
\]
Filter form of the solution

The concept of the filter algorithm is that the solution for the heat flux at any time is only a function of the recent temperature history and a few future time steps. Equation (18) can be written in filter form as below:

\[ \dot{q}_h = F \hat{Y} + G \hat{y} \]  

(21)

where \( F = F_I (I + X_{F_1} F_{1}) \) and \( G = -F_1 X_{F_1} F_{1} Z \).

Here, \( F \) and \( G \) are the filter matrices associated with the coupled solution and both of them have the same characteristics as filter matrix which is discussed in detail in Ref [21]. All of the rows of the filter matrix (except for the first few and last few rows) are similar but shifted in time. Also, the number of non-negligible terms of each row is limited \((m_p + m_f)\). Note that it is assumed that any filter coefficient less than \( \epsilon = 5 \times 10^{-5} \) is negligible. The number of non-negligible coefficients before and after the current time step is denoted as \( m_p \) and \( m_f \) respectively. Therefore, the heat flux at the current time step is only a function of temperature data associated to the time steps with non-negligible filter coefficients (several previous time step and a few future time steps) and can be found as:

\[ \dot{q}_M = \sum_{j=1}^{m_p + m_f} \left( f_j Y_{M + m_f - j} + g_j Y_{M + m_f - j} \right) \]  

(22)

where,

\[ f = \begin{bmatrix} f_1 & f_2 & \cdots & f_{m_p + m_f - 1} & f_{m_f} \\ g = \begin{bmatrix} g_1 & g_2 & \cdots & g_{m_p + m_f - 1} & g_{m_f} \end{bmatrix} \]

\[ = row \left( I + X_{F_1} F_{1} \right) \]

\[ = row \left( -F_1 X_{F_1} F_{1} Z \right) \]

(23 a,b)

Here \( f \) and \( g \) are vectors of filter coefficients associated with one row of \( F \) and \( G \) matrices respectively. Note that all the rows of a filter matrix are similar but shifted in time (except the
first few rows and last few rows). Therefore, it really doesn’t matter one would pick which row to use as the filter coefficients as long as it is not one of the very first or last rows. A good and easy practice is to pick the middle row. Note that these filter coefficients are not the same as filter factors suggested by Hansen [25].

Considering the material properties of an actual DFT given in Table 1, time step of 5 seconds and Tikhonov regularization parameter $\alpha_r = 0.001$, the “f” and “g” filter coefficients are determined for different temperatures. The calculated filter coefficients are plotted versus the time index in Figs. 3 and 4 for different temperatures.

![Figure 3: f filter coefficients for different temperatures](image-url)
As seen in Figs. 3 and 4, for each temperature, the values of filter coefficients approach to zero towards both ends. As mentioned, $m_p$ is the number of non-negligible coefficients before the current time step and $m_f$ is the number of non-negligible coefficients after the current time step. Any coefficient less than $\varepsilon = 5 \times 10^{-5}$ is assumed to be negligible.

The selection of $m_p$ and $m_f$ cannot be made arbitrarily as they depend on the boundary conditions, material properties, sensor location, time step and regularization parameter. To determine appropriate $m_p$ and $m_f$, one must examine the filter coefficients. As seen in Figs. 3 and 4, there are only a few non-negligible values ahead of the current time step and this determines $m_f$. The less is the number of required future time steps for calculating the heat flux at the current time step, the closer the algorithm can operate to the real time. A closer look at Fig. 3 and Fig. 4 show that $m_f$ is 4 and 3 for $f$ and $g$ filter coefficients respectively. Therefore, the maximum of these two values, which is 4, should be used to achieve accurate results. Similarly, $m_p$ can be
determined by close inspection of the filter coefficients. From Fig. 3, \( m_p \) is 3 and 40 for \( f \) and \( g \) respectively. So, the bigger value of 40 should be used for calculations. It should be noted that a bigger value of \( m_p \) does not affect how fast (closer to real-time) the algorithm can operate as it only decides how many data points from previous time steps must be used to determine the heat flux at the current time.

The regularization parameter, \( \alpha \), should be also selected carefully in order to achieve an accurate estimation of the heat flux at the surface. Several methods are available for selecting regularization parameter, including L-curve [26, 27], generalized cross validation [28] and discrepancy principle [29]. Woodbury and Beck [21] suggested that the optimal selection of the regularization parameter by minimizing the heat flux error yields results similar to the classical Morozov principle [29]. They showed that appropriate selection of the order of magnitude of the regularizing parameter is important to minimize errors, but the precise selection of the regularization parameter is not important for effective estimation of heat flux. When choosing a smaller regularization parameter, \( m_p \) and \( m_f \) will be smaller. The conclusion is, when using filter approach for real-time heat flux estimation, one need to have an idea of the problem that has to be solved (whether the heat flux is uniform or has a lot of sudden changes) to pick the appropriate regularization parameter and corresponding \( m_p \) and \( m_f \) values. This will be discussed further in the examples.

As seen, for the given geometry, material properties and sampling time step of 5 seconds in DFT application, the temperature data from four future time steps (\( m_f=4 \)) is needed to do the calculations (including the current time step). It means that the algorithm can report the heat flux with a 15 seconds delay. Also, it is determined that for accurate calculation of heat flux, the temperature data from 40 previous time steps (\( m_p=40 \)) is needed.
Accommodating temperature-dependent properties

As seen in Figs. 3 and 4, the filter coefficients vary considerably as the temperature changes. Beck [30] showed that when the material properties are temperature dependent, one can interpolate filter coefficients at each time step based on the measured temperature to achieve accurate results and handle the non-linearity of the problem. Since the temperature greatly varies in DFT application, it is necessary to find the filter coefficients at each time step with a particular temperature and use them accordingly. This can be done by finding the filter coefficients for a set of temperatures and interpolate between those at each time step. Here, the filter coefficients for 300 K, 400 K, …, 1300 K are calculated. At each time step, the associated filter coefficients for the temperature at the current time is calculated using linear interpolation. The obtained filter coefficients are used to estimate the heat flux. Later it is shown in the test cases that accounting for temperature dependent material properties can greatly improve the accuracy of the estimated heat flux values.

Heat Flux Estimation

Test Cases

In order to demonstrate the developed filter solution, a numerical test case is developed in ANSYS. To validate the ANSYS solution methodology, the results from a preliminary ANSYS simulation is compared with the exact solution [31] for constant material properties. For this purpose, a two-layer medium is created in ANSYS. This model is then meshed and the loads are applied on the geometry corresponding to the X1C11B10T0 case. The properties of the layers are given in Table 1, the sensor locations are considered the same as the DFT case and the material properties are considered similar to DFT materials at 1050 K. For this test case, it is assumed that the material properties are constant. The temperature data are also generated using
the solution for the direct problem for a step change in temperature at the surface (X1C11B10T0) and used as inputs for the filter solution (Eq. 22) to determine the heat flux at the surface. The calculated heat flux by the filter solution, ANSYS simulation and exact solution are compared in Fig. 5 and a good agreement can be observed between all three set of results. Again, the main purpose of this test case is to validate the ANSYS model with the exact solution and show that the filter approach well matches with these two sets of results.

![Figure 5: surface heat flux versus time: comparison of the results from exact solution, ANSYS simulation and Filter Solution (test case 1- X1C11B10T0)](image)

It should be noted that, for this test case, the values of $m_p$ and $m_f$ are considered as 80 and 15 respectively, the time step is 0.2324 seconds (dimensionless time step 0.0005). Following the suggestion of Ref [21] the regularization parameter is selected as $10^{-2}$ after several trials to find the best value by minimizing the RMS error between the estimated heat fluxes by the filter solution and heat fluxes from the exact solution. The RMS error can be calculated as below:
\[
E_{RMS} = \left( \sum_{i=1}^{n} \frac{(q_{exact,i} - q_{estimated,i})^2}{n} \right)^{\frac{1}{2}}
\]  

(24)

Here, \( n \) is the number of time steps for which heat flux calculations have performed. Note that, prior to calculation of the unknown heat flux, the regularization parameter should be optimized using a known heat flux and the corresponding temperatures. In other words, if we already know that the unknown heat flux that is to be estimated is almost constant all the times, then one may use a simple step function (using exact solution), calculate the temperatures, feed them into the inverse solution and adjust \( \alpha \) to minimize the RMS error between the known exact values and the estimated values using the filter method. Later, the adjusted \( \alpha \) will be used as the optimal regularization parameter to estimate the unknown heat flux. Similarly, if one knows that the heat flux will have several changes or it varies smoothly, a triangular heat flux function or a parabolic function can be used for calculating the optimal Tikhonov regularization parameter. For the test cases in this paper, the parabolic function can be used for the first case and triangular function can be used for the second case and the third case (DFT field data) to optimize \( \alpha \) prior to calculation of the unknown heat fluxes.

It can be observed that the results from the filter method is in a good agreement with the exact solution. The RMS error between the exact solution and the estimated heat flux using the filter method is 159.6 W/m\(^2\) which is significantly smaller than the peak value of 13,238 W/m\(^2\).

In a second test case, a triangular heat flux profile is applied to the front surface of the two-layer slab while the back surface of the second layer is maintained at 300 K. It is assumed that the material properties of the two-layer medium are temperature dependent as given in Table 1. The temperature histories are then obtained at the desired surfaces using ANSYS simulation and
used as inputs for the filter solution. The calculated temperature profiles are shown in Fig. 6. Note that these temperatures are calculated using ANSYS.

![Figure 6: Temperature measurements on front and back side of the second layer (test case 2)](image)

The heat flux is calculated from the computed temperatures using Eq. 22 and compared with the given heat flux at the surface in Fig. 7, which shows the results when constant material properties for various temperatures are used. The regularization parameter is set as 0.01, the time step is one second and the number of previous and future time steps are 150 and 8 respectively. It means that with this condition, the algorithm can calculate the heat flux with 7 seconds delay. As seen, there is a considerable difference between the estimated heat flux using constant material properties and the exact input to ANSYS simulation.

Next, the filter coefficients are interpolated at each time step and used accordingly to account for temperature dependent material properties. The estimated heat flux using this method is shown in Fig. 8 and compared with the ANSYS simulation inputs and linear filter results for average temperature of 691 K.
Figure 7: Calculated surface heat flux for test case 2 with constant material properties at different temperatures (sampling time: 1s, $m_p=40$, $m_c=4$, $\alpha_t=10^{-2}$)

Figure 8: Calculated surface heat flux for test case 2 by accounting for temperature dependent material properties (non-linear filter), ANSYS inputs and linear filter for constant material properties at average temperature of 691 K (sampling time: 1s, $m_p=150$, $m_c=8$, $\alpha_t=10^{-2}$)
The RMS error between the calculated heat flux values using filter method (linear and non-linear) and the ANSYS data is calculated and presented in Table 2. As expected, the RMS error for the nonlinear solution is significantly lower than linear solutions. This clearly shows the importance of accounting for the temperature dependent material properties. It should be noted that the value of Tikhonov regularization parameter is optimized as 0.01 for all cases which yielded the minimum RMS value.

Table 2: calculated RMS error between the estimated heat fluxes using the developed filter method and the exact values (for test case 2) / results from IHCP1D (for the DFT case)

<table>
<thead>
<tr>
<th>Test Case 2</th>
<th>Temperature (K)</th>
<th>RMS Error (W/m²)</th>
<th>Linear</th>
<th>Non-Linear</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test Case 2</td>
<td>300</td>
<td>9213.0</td>
<td>4727.4</td>
<td>2484.7</td>
</tr>
<tr>
<td>Test Case 2</td>
<td>500</td>
<td>48426</td>
<td>7,973</td>
<td></td>
</tr>
<tr>
<td>Test Case 2</td>
<td>800</td>
<td>691 (T_avg)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Test Case 2</td>
<td>1000</td>
<td>3834.8</td>
<td>7,973</td>
<td></td>
</tr>
</tbody>
</table>

DFT Data Analysis

The temperature data is obtained from an experimental work by Sultan et al [12]. The temperature profiles at the front and backside of the DFT are shown in Fig. 9. The temperature data are substituted in the coupled formulation to determine the heat flux at the front plate of the DFT.

The filter coefficients are calculated as described in the previous section and shown in Fig. 3 and Fig. 4. Using the filter method (Eq. 22), the heat flux into the front face of the plate is calculated and shown in Fig. 10 for constant material properties. By accounting for temperature dependent material properties, the heat flux is estimated using the filter coefficient method and compared with the results from IHCP1D in Fig. 11.
Figure 9: temperature measurement data for DFT field test

Figure 10: calculated surface heat flux for DFT field data- constant material properties at different temperatures (sampling time: 5s, $m_i=40$, $m_f=4$, $\alpha=10^{-7}$)
A good agreement between the results from the two methods is observed. This shows the practicality of using filter approach for real time measurement of the heat flux instead of using the whole time method. It should be noted that in this case, the Tikhonov regularization parameter is selected as $10^{-7}$, $m_f=4$ and $m_p=40$. The value of $m_f$ specifies the delay for heat flux estimation which is 15 seconds for this simulation, as discussed earlier. The RMS error between the results from filter method (linear and non-linear) and results from IHCP1D is listed in Table 2 (DFT case). Again, it is clear that the non-linear solution leads to a much lower RMS error which results in significantly higher accuracy.
Conclusion

In the present paper, a filter based solution is developed for heat flux estimation in Directional Flame Thermometer. For this purpose, two IHCPs are investigated: one for the middle insulation layer and one for the front Inconel plate, and a coupled solution is then proposed and written in the filter form to estimate the unknown heat flux. The filter method allows accounting for the temperature-dependent material properties.

The method is tested using two numerical examples one with constant material properties and one with temperature dependent material properties. Afterwards, using experimental data for an actual DFT test in a furnace, the heat flux is calculated at the surface. It was shown that for sampling time of 5 seconds and the given geometry and material properties for the DFT, the developed algorithm can perform heat flux estimation in a near real-time fashion with only 15 seconds delay. The results demonstrate good agreement with the IHCP1D analysis. The filter solution of the IHCP has several advantages including simplicity, continuous operation and application to nonlinear problems (temperature dependent material properties), which makes them an appropriate approach for real time heat flux estimation in industrial applications and in this particular case, DFTs.

References


ARTICLE 5: DEVELOPING AN INVERSE HEAT CONDUCTION ANALYSIS TOOL FOR
REAL TIME HEAT FLUX ESTIMATION IN DIRECTIONAL FLAME THERMOMETER
APPLICATION

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Abstract

Accurate measurement of heat flux and temperature can significantly affect the energy usage in several industrial applications, including furnace operation, metal processing, fire safety tests and more. Directional Flame Thermometers, or DFTs, offer the ability to use both temperature and heat flux measurements for furnace control. Currently, analysis of dynamic temperature data from the DFTs to compute heat flux information must be performed off-line at the conclusion of data-gathering by using software tools such as IHCP1D. Availability of a near real-time algorithm for accurate reduction of the data will allow for continual monitoring of the furnace during operation. This will result in better control over the process and significant savings in energy and cost. In this paper, a filter form of the inverse heat conduction algorithm is developed for utilization in DFTs. The algorithm is based on linearized solutions of the direct heat equation,

and non-linear effects introduced by temperature dependent thermal properties are accounted for by interpolating of the resulting filter coefficients. The developed method is tested through several numerical experiments and also ANSYS model. A graphical user interface is developed in LabVIEW to provide a friendly interface for the end user. The temperature data measured by thermocouples on the DFT are transmitted to the computer through data acquisition card and the developed tool in LabView display the heat flux in a near real time fashion.

*Keywords: inverse heat conduction, filter solution, directional flame thermometer, real-time heat flux estimation.*

**Introduction**

Real time heat flux measurement can significantly improve the controllability of processes such as metal heat treating, quenching, fire safety tests, furnace operation and more. Heat flux measurement sensors (active sensors) are rather expensive and less reliable than temperature measurement instruments. Also, heat flux measurement equipment cannot be easily installed and maintained in all applications, and therefore, using heat flux estimation techniques based on temperature measurements is highly desirable for several cases[1].

Equilibrium sensors such as directional flame thermometers (DFT) can be used for this purpose. They generally consist of two metal plates with an insulation layer in between. Thermocouples are installed on the backside of each plate and covered with insulation material. The plate’s temperature increases quickly and reaching quasi-equilibrium with the fire environment. Equilibrium sensors are relatively inexpensive and easy to install and maintain. DFTs can be used to measure both temperature and heat flux in fire environment.

A schematic of a DFT is shown in Fig. 1. The original DFT design involved a thin metal disk mounted in a steel tube [2, 3]. Sandia National Laboratories improved DFTs for use in large pool of...
fire and other tests. Their goal was to provide both transient and quasi-steady heat transfer measurements in various fire environments [4].

Samuel et al. [5] used DFTs to measure the heat flux in wild land–urban interface (WUI) fires. They emphasized that the limited access to water in such areas urges the use of DFTs. They used water cooled total heat flux sensors for a direct comparison of the heat flux obtained from the DFTs. Lam and Weckman [6] examined the steady state response of four heat flux gauges including DFTs under various radiative and convective conditions and compared the results. In another study, Sultan [7] investigated the performance of six different temperature sensors in fire resistance test. The result showed that all the sensors yield similar results after approximately 10 minutes. Kokel et al. [8] developed a heat transfer model to provide the user with a simple forward solution methodology for heat flux estimation by DFT with a moderate error, assuming the material properties are constant.
Estimating the unknown heat flux (boundary condition) at the surface when temperature measurements are known in the interior points of the medium is a class of problems known as inverse heat conduction problems (IHCP's). Analyzing DFT data over the entire test duration needs an inverse heat conduction code which uses two temperature measurement histories for estimating the net heat flux to the exposed surface ($q_{\text{front}}$, in Fig. 1). Currently, analysis of dynamic temperature data from the DFTs to compute heat flux information must be performed off-line for the entire test data. Developing a near real-time algorithm allows continuous monitoring of the furnace during operation. This will provides a better furnace control which can result in significant savings in energy and cost.

IHCPs have been widely studied and a number of approaches are developed as the solution of these types of problems [9, 10, 11]. Recently several studies focused on investigating real-time processing of temperature data to solve the IHCP. A filter solution based on the idea of training neural networks is studied by Kowsari et al. [12]. Ijaz et al. [13], used a Kalman filter to solve a two-dimensional transient IHCP. Feng, et al. [14] used Laplace transforms to relate the measured conditions at one end of a domain to the unknown conditions at the remote surface. Woodbury and Beck [15] studied the structure of the Tikhonov regularization problem and concluded that the method can be interpreted as a sequential filter formulation for continuous processing of data. They examined the filter based solution by two examples and compared the results with the whole domain method and observed that the results are the same for these constant-property solutions. It was also observed that while the correct selection of the order of magnitude of the regularization parameter is important, the results are not sensitive to the precise selection of it.
In this paper, a filter form of the inverse heat conduction algorithm is developed for utilization in DFTs. The temperature is given at the boundaries of the middle layer and the heat flux is unknown at the front surface of the DFT. Two IHCPs are solved (one for the middle layer and one for the front plate) and a coupled solution is proposed to estimate the unknown heat flux at the front plate using temperature measurement histories. The solution is then written in a digital filter form which allows near real time heat flux estimation at the front surface. To account for the temperature dependent material properties, the filter coefficients are calculated for different temperatures and step interpolation is used to calculate the filter coefficients at each time step and corresponding temperature. Several numerical experiments using exact solutions and ANSYS simulation are used to examine the developed solution. A graphical user interface is developed in LabVIEW to provide a friendly interface for the end user. The temperature data measured by thermocouples on the DFT are transmitted to the computer through data acquisition card and the developed tool in LabView display the heat flux in a near real time fashion. Finally, the developed solution is applied on the data from a furnace experiment and the results compared with those from a full non-linear IHCP solver.

Problem Definition
Heat transfer problems can be categorized in two classes of forward problems and inverse problems. Determining temperature for all points in the space-time domain when the initial conditions and the boundary conditions are known is recognized as a forward problem. When the active boundary condition is unknown and measurement of the temperature history at one or more interior points of the domain is available, the problem is called inverse problem. Unlike forward problems, inverse problems are mathematically ill-posed and an appropriate form of regularization is needed to be applied on the problem to yield a solution.
In using DFT for heat flux measurements, two sets of temperature measurements are available at the boundaries of the middle layer. To estimate the unknown heat flux at the surface of the front plate, a solution strategy is developed as described in the next section. DFT consists of three layers: two Inconel plates and one layer of insulation. It is assumed that the heat transfer is 1-dimensional (only x-direction). The properties of the materials in DFT are given in Table 1.

<table>
<thead>
<tr>
<th>Thickness, m</th>
<th>Layer 1 (Inconel)</th>
<th>Layer 2 (Ceramic Fiber)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T = 300 K</td>
<td>k = 14.9 W/m-K,</td>
<td>k = 0.046 W/m-K,</td>
</tr>
<tr>
<td></td>
<td>α = 3.9e-6 m²/s</td>
<td>α = 3.2e-7 m²/s</td>
</tr>
<tr>
<td>T = 450 K</td>
<td>k = 17.4 W/m-K,</td>
<td>k = 0.065 W/m-K,</td>
</tr>
<tr>
<td></td>
<td>α = 4.2e-6 m²/s</td>
<td>α = 3.9e-7 m²/s</td>
</tr>
<tr>
<td>T = 800 K</td>
<td>k = 22.6 W/m-K,</td>
<td>k = 0.13 W/m-K,</td>
</tr>
<tr>
<td></td>
<td>α = 4.9e-6 m²/s</td>
<td>α = 6.9e-7 m²/s</td>
</tr>
<tr>
<td>T = 1000 K</td>
<td>k = 25.4 W/m-K,</td>
<td>k = 0.19 W/m-K,</td>
</tr>
<tr>
<td></td>
<td>α = 5.3e-6 m²/s</td>
<td>α = 7.9e-7 m²/s</td>
</tr>
<tr>
<td>T = 1200 K</td>
<td>k = 28.2 W/m-K,</td>
<td>k = 0.27 W/m-K,</td>
</tr>
<tr>
<td></td>
<td>α = 5.57e-6 m²/s</td>
<td>α = 1.2e-6 m²/s</td>
</tr>
</tbody>
</table>

**Solution Method**

Two IHCP’s are solved separately for the middle layer and the front layer to estimate heat flux on the front plate surface. The two solutions are combined and a coupled solution is determined for the two-layer problem associated with DFT. It should be noted that, in practice the front plate of the DFT is located towards the heat source (fire) and therefore the desired unknown value is the applied heat flux at the front plate. As the result, the analysis is only performed on the middle layer and the front plate and the solution does not involve the back layer. A schematic of the two layer problem is given in Fig. 2.
The solution is started from the second layer (Ceramic Fiber), where two temperature measurements are available at two boundaries and $q_1$ is the unknown heat flux. After solving this first IHCP, the heat flux and temperature are both known at the interface ($q_1$ and $Y$). These values will be used as boundary conditions to solve the second IHCP associated with the Inconel plate, where $q_0$ is unknown. The analysis of each layer is explained in detail as below.

**Insulation Layer**

The temperature at $x_1(L_I \leq x_1 \leq L_I + L_2)$ is a function of the surface heat flux $q_1(t)$ and the temperature at $x = L_I + L_2$.

Therefore, the boundary condition of the second kind is given at $x = L_I$ and the first kind at $x = L_I + L_2$. This problem is designated by Cole, et al. [16] as X21B10T0, which is herein shortened to X21 for reference. The notation denotes a Cartesian geometry subjected to a type 2 condition at the first boundary and a type 1 condition at the second boundary, and that the first boundary
has a step change in value while the second boundary is homogeneous, and that the initial condition is also homogeneous.

It is considered that the connecting curves between the heat flux components, \( q_i \) and \( q_{i+1} \), and also between the adjacent components, \( y_i \) and \( y_{i+1} \) are specified as being constant between points (piecewise constant functions):

\[
q_i(t) = q_i, \quad t_i < t < t_{i+1}
\]

(1)

\[
t_i = i\Delta t
\]

(2)

A solution in Layer 2 for the X21 case with a constant heat flux, \( q_c \), at \( x = L_I \) is

\[
\frac{T_{X21}(x, t)}{q_c L_2 L_2^2} = \left(1 - \frac{x}{L_2}\right) - \sum_{m=1}^{\infty} \frac{\cos \left(\frac{x}{L_2} \beta_m \frac{x}{L_2} \right)}{\beta_m^2} \exp \left(-\frac{\beta_m a t}{L_2^2}\right)
\]

(3)

where \( \beta_m = (m - 1/2)\pi, \ m = 1, 2,... \)

Analogous to the above equations for a constant heat flux, the equation for a constant temperature, \( T_c \), at \( x = L_I + L_2 \) is denoted X12B10T0 and shortened herein to X12:

\[
\frac{T_{X12}(x, t)}{T_c L_2} = 1 - 2 \sum_{m=1}^{\infty} \frac{\sin \left(\frac{x}{L_2} \beta_m \frac{x}{L_2} \right)}{\beta_m^2} \exp \left(-\frac{\beta_m a t}{L_2^2}\right)
\]

(4)

The temperature at any location \( x_1(L_I \leq x_1 \leq L_I + L_2) \) caused by the heat flux \( q_I \) at \( x = L_I \) and the temperature \( y \) at \( x = L_I + L_2 \) is

\[
T_m = T_{q,m} + T_{y,m} = \sum_{i=1}^{M} q_{i,\Delta} \phi_{M - i} + \sum_{i=1}^{M} y_{i,\Delta} \eta_{M - i}
\]

(5)

where \( \phi \) and \( \eta \) are the response basis functions for the two cases. That is,

\[
\phi(x, t) = \frac{\partial T_{X21}}{\partial q_c}; \quad \eta(x, t) = \frac{\partial T_{X12}}{\partial T_c}
\]

(6 a,b)
The $\Delta \phi$'s can be found as [15]:

$$\Delta \phi_i = \phi_{i+1} - \phi_i$$  \hspace{1cm} (7)

Equation 5 can be described by the matrix equation of

$$T = \mathbf{X} q + \mathbf{Z} y$$  \hspace{1cm} (8)

where

$$\mathbf{T} = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$\mathbf{Z} = \begin{bmatrix} Z_1 & 0 & 0 & \cdots & 0 & 0 \\ Z_2 & Z_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Z_n & Z_{n-1} & Z_{n-2} & \cdots & Z_2 & Z_1 \end{bmatrix}$$  \hspace{1cm} (9 a,b,c,d,e)

The components of the $\mathbf{x}$ and $\mathbf{z}$ matrices are related to the response basis functions by

$$X_i = \frac{L}{k} \Delta \phi_{i-1}, \quad Z_i = \Delta \eta_{i-1}$$  \hspace{1cm} (10)

where $\Delta \phi_i$ is defined in Eq. (7), and $\Delta \eta_i$ is defined analogously.

The columns in the $\mathbf{x}$ matrix of Eq. (9) are the same as the sensitivity vectors in parameter estimation. The first column is for $q_1$, the second for $q_2$, and so on. Using whole domain Tikhonov regularization, the estimated value heat flux vector at the interface, $\hat{q}_1$, can be given as:

$$\hat{q}_1 = [X_2^T X_2 + \alpha \mathbf{H}^T \mathbf{H}]^{-1} X_2^T (Y - Z y) = \mathbf{F}_2 (Y - Z y)$$  \hspace{1cm} (11)

Here the subscript 2 for $\mathbf{X}$ and $\mathbf{F}$ refers to Layer 2 (insulation layer). Note that this equation is dimensional. $\mathbf{F}_2$ has the same definition as the filter matrix of Ref [21].
Front Inconel Plate

The temperature at any location \( x_1 \) \((0 \leq x_1 \leq L_1)\) is a function of the surface heat fluxes \( q_0(t) \) and \( q_1(t) \). The results from the middle layer analysis determined \( q_1(t) \) as given in Eq (11).

For this layer, the boundary condition of the second kind (Neumann) is given at \( x = 0 \) and at \( x = L_1 \) (X22).

A solution for the X22 case with a constant heat flux at \( x = 0 \) and zero heat flux at \( x = L_1 \) (this is denoted the X22B10T0 case):

\[
T_{x,22}(x,t) = \frac{\alpha t}{k} + \frac{x}{L_1^2} - 2\sum_{m=1}^\infty \frac{\cos(\beta_n x / L_1)}{\beta_n^2} \exp(-\beta_n^2 \alpha t / L_1^2)
\]

(12)

Note the same solution applies for a zero heat flux at \( x = 0 \) and a constant heat flux at \( x = L_1 \) through a simple change of variables such as \( \xi = L_1 - x \).

Analogous to Eq. (5), the temperature response at \( x_1 (0 \leq x_1 \leq L_1) \) can be found due to the heat flux histories \( q_0(t) \) and \( q_1(t) \). For assumed piecewise constant variation in these heat fluxes, the temperature can be computed from

\[
T_{x} = T_{x,M} + T_{x,M} = \sum_{i=1}^M q_{x,i} \Delta \varphi_{M_{x_{-i}}} + \sum_{i=1}^M q_{x,i} \Delta \theta_{M_{x_{-i}}}
\]

(13)

where

\[
\varphi(x_1,t) = \frac{\partial T_{x,22}(x_1,t)}{\partial q_x}; \quad \theta(x_1,t) = -\frac{\partial T_{x,22}(L_1 - x_1,t)}{\partial q_x}
\]

(14a,b)

The step basis function representation used here (and also others) for temperature given in Eq. (13) can be described by the matrix equation of

\[
T = X_0 q_0 + X_{L_1} q_{L_1}
\]

(15)

where
The components of the $X_0$ and $x_L$ matrices are related to the response basis functions in Eq. (14) similar to the relation of $X$ and $Z$ in Eq. (10) to Eq. (6).

The whole domain Tikhonov regularization method is used to solve the IHCP. Later the filter coefficients are found from the solution. The IHCP solution for Layer 1 starts with a matrix form for the sum of squares with an added regularization term given by

$$S = (Y - X_0 q_0 - X_L q_L)^T (Y - X q - X_L q_L) + \alpha \beta q_0^T H \beta q_0$$

This is minimized with respect to the parameter vector $q_0$. The symbol $Y$ is the temperature measurement vector at $L_1$ and $q_1$ is the known heat flux (from the solution of the IHCP in Layer 2) at $x = L_1$. The initial temperature is zero. The $\alpha \beta$ symbol is the Tikhonov regularization parameter. The estimated value of the heat flux vector, denoted $\hat{q}_0$, is given by

$$\hat{q}_0 = [X_0^T X_0 + \alpha \beta H^T H]^{-1} X_0^T (Y - X_L q_1) = F_1 (Y - X_L q_1)$$

**Coupling of the Solutions**

The heat flux at the interface ($x=L_1$), $\hat{q}_1$, is determined by solving the IHCP for the middle layer (Layer 2). The solution to this problem is given in Eq. (11). By substituting $\hat{q}_1$ into Eq. (18), the following result is obtained:

$$\hat{q}_0 = F_1 Y - F_1 X_{L_1} (F_2 Y - F_2 Z y)$$

$$= F_1 (I - X_{L_1} F_2) Y + F_1 X_{L_1} F_2 Z y$$
Equation (19) demonstrates that the heat flux at the surface of the outer layer can be found from the measurements on the boundaries of the inner layer.

Filter Form of the Solution

The concept of the filter algorithm is that the solution for the heat flux at any time is only a function of the recent temperature history and a few future time steps. Equation (19) can be written in filter form as below:

\[ \hat{q}_s = fY + gy \]  
\[ f = \text{row} \left( F_i \left( I - X_i F_z \right) \right) \]  
\[ g = \text{row} \left( F_i X_i F_z Z \right) \]  

Where \( f \) and \( g \) have the same characteristics as filter coefficients, which are discussed in detail in Ref [21]. The meaning of “\( \text{row()} \)” in Eq. (21) designates a row in the middle of the indicated matrix. Note that these filter coefficients are not the same as filter factors suggested by Hansen [17].

All the \( f \)-filter coefficients can be found at one time by setting all the \( Y \) and \( y \) components equal to zero except the \( m_f \) component of \( Y \) is set equal to one \( (Y_{mf}=1) \). The solution of the IHCP with this data gives the \( f \) coefficients. To get the \( g \)-filter coefficients (those for the \( y \) vector), the same procedure is followed with now all the components of \( Y \) and \( y \) equal to zero except \( y_{mf}=1 \).

Considering the material properties of an actual DFT given in Table 1, time step of 5 seconds and Tikhonov regularization parameter \( \alpha = 0.0001 \), the “\( f \)” and “\( g \)” filter coefficients are determined for different temperatures. The calculated filter coefficients are plotted versus the time index in Figures 3 and 4.
Figure 3: $f$ filter coefficients for different temperatures

Figure 4: $g$ filter coefficients for different temperatures
Developing Labview interface

In this section the developed software tool in LabView is described. LabView provides lots of features for data logging and also developing friendly graphical user interface. The solution method for the IHCP is coded in MATLAB and then entered in LabView using script nodes. Figure 5 shows the main interface (front panel) of the tool. The goal is to develop a software tool using LabView interface which can be easily used by the end user for heat flux estimation via DFT. The program can calculate the heat flux at the surface in both offline and online modes. As can be seen, there are three series of inputs, two of which needed to be filled out for each run. The "general settings" include the Tikhonov regularization parameter, the rate of data logging and end time are needed to be provided for both online and offline analysis. For offline analysis, the temperature data for the entire test duration are supposed to be loaded from an excel spreadsheet and the path of the excel file along with the associated columns and rows are needed to be entered in for the offline analysis.

![Image of the tool's interface](image)

Figure 5: front panel of the tool developed for heat flux estimation
For the online analysis, physical channels from the data logging equipment as well as the thermocouple type and maximum/minimum values of temperature are needed to be defined. Also, the value of $m_f$ (which is assumed same as $m_p$ in here) needed to be provided by the user. This value specifies the delay for the results. It should be noted that as of now the values of $m_p$, $m_f$ and $\alpha$ are considered as inputs for the software tool, however, an optimization algorithm will be added to the software to find the optimal values for these parameters. This will significantly eases the use of the software tool for the end user with limited or no knowledge of IHCP solution methods.

Results and Discussions
In order to verify the proposed filter solution, a test case is developed using ANSYS. To validate the ANSYS model, the results from ANSYS simulation is compared with exact solution[18] for constant material properties. For this purpose, a two-layer medium with a geometry similar to DFT is created in ANSYS. The created model is then meshed and loads are applied on the slab similar to the X1C11B10T0 case [16]. A step change in temperature is applied at the front surface while the back surface temperature is maintained at zero. The properties of the layers are given in Table 1. The material properties are considered similar to DFT materials at 1050 F. The temperature data is also generated using the solution for the direct problem for a step change in temperature at the surface (X1C11B10T0) and used as inputs for the filter solution (Eq. 20) to determine the heat flux at the surface.
The calculated heat flux by the filter solution, ANSYS simulation and exact solution are compared in Fig. 6 and a good agreement can be observed between all three sets of results. The value of heat flux converges to 11.03 W/m².

For this test case, the value of Tikhonov regularization parameter is considered as 0.001 and the dimensionless time step is 0.002. It should be noted that filter algorithm cannot calculate the heat flux for the last $m_f$ time steps, since not enough data from future time steps exists. Also, the calculated heat flux by the filter algorithm for the first few time steps may not be in a very good agreement with the data from Exact solution (or simulation) due to the fact that the $m_p$ previous time steps are not available and therefore a constant value for temperature is assumed to calculate the initial time steps before $m_p^{th}$ time step. The average deviation (RMS) of the estimated heat flux by the filter solution from the exact solution (excluding the first 9 seconds) is calculated as 0.559 W/m².

A random error (0.5% of the temperature) is added to the temperature data and the heat flux is calculated. The calculated heat flux when error presents in the data is plotted in Fig. 7.
It can be observed that the results from the filter solution are still in a good agreement with the exact solution and ANSYS simulation. The average deviation (RMS) of the estimated heat flux from the exact solution is calculated as 17.6 W/m$^2$.

In a second test case, a triangular heat flux profile is applied to the surface of the DFT model in ANSYS. The variation of thermal properties of the material versus temperature is taken into account for this test case. The temperature histories are then obtained at the desired surfaces and used as inputs for the filter solution. The temperature data is then fed into the developed tool and the heat flux is calculated. The value of $\alpha$ is considered as 0.001 and $m_p=m_f=25$ and the time step is 1 second. The temperature histories and estimated heat flux is shown in Fig. 8.
Figure 8: results from the developed tool for test case 2

The calculated heat flux is then exported to Excel and compared with the actual heat flux from ANSYS simulation. The comparison is shown in Fig. 9. As can be seen, a good agreement can be observed between the results.
DFT Field Data

A final test case is used to assess the application of the developed tool with actual DFT field test data [19]. The temperature measurement data, is available at every 5 seconds during the entire test duration which is 3500 s. The temperature data fed into the developed tool in an online fashion. The value of $\alpha_t$ is set as 0.001 and $m_f$ and $m_p$ are specified as 30 which means that the estimation of surface heat flux at each time step requires temperature data from 30 previous and 30 future time steps. The calculated heat flux by the developed tool and the temperature histories are shown in Fig. 10.

Figure 9: comparison of the estimated heat flux from the developed tool and ANSYS simulation (test case 2)
The calculated heat flux profile is then exported from the tool to an Excel spreadsheet.
A comparison between the results from the developed approach and the results from a full non-linear IHCP1D software [20] is shown in Fig. 11. As can be seen, there is a very good agreement between the results except for the first few time steps.

Conclusion

A new software tool is designed and developed for online heat flux estimation in DFT application. A solution method is developed and coded in MATLAB to solve the nonlinear inverse heat conduction problem associated with the DFT. The solution is then written in the digital filter form which allows near real-time heat flux estimation at the front surface. The variation of material properties with temperature is taken into account by interpolation filter coefficients at each time step and corresponding temperature. A graphical user interface is then developed using LabView and integrated with the MATLAB code. Data logging features are also considered for the developed tool. The software tool can read the temperature data from excel sheet or directly from the DFT's thermocouples and calculate the heat flux in a near real time fashion. The developed tool is then tested via several test cases including exact solution, ANSYS simulation and DFT field data and a good agreement between the results is observed for all of the test cases.

References


[20] IHCP1D (software), Beck Engineering Consultants, Okemos, MI.
ARTICLE 6: ONLINE HEAT FLUX ESTIMATION USING ARTIFICIAL NEURAL NETWORKS AS A DIGITAL FILTER APPROACH

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Abstract
Surface heat flux estimation using temperature measurement data from the interior points is known as inverse heat conduction problem (IHCP). Several methods have been developed as solutions of IHCP’s including analytical and numerical techniques. Digital filter representation for IHCP solution [1,2] is one of the methods which can be used for near real-time heat flux estimation. In this study, Artificial Neural Network (ANN) is utilized as a digital filter, for near real-time heat flux estimation using temperature measurements. Considering temperatures as the inputs and heat flux as the output, the weights can be interpreted as filter coefficients. The proposed approach is used for both constant and temperature dependent material properties. The method developed is tested through several test cases using exact solutions and numerical models. The results show that ANN can be used as a digital filter method for near real-time surface heat flux estimation. The advantageous and disadvantages of the method are also discussed.

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Keywords: Real time heat flux estimation, intelligent algorithms, artificial neural network, digital filter representation

Introduction

Real-time heat flux measurement has great significance in numerous industrial applications such as metal heat treating, quenching, fire safety tests, furnace operation, and more. Heat flux measurement instruments are expensive, difficult to install and maintain and less reliable than temperature measurement equipment. Therefore, heat flux estimation methods using temperature measurements are preferred approach for several applications. An inverse heat conduction problem (IHCP) must be solved to estimate the unknown heat flux at the surface, knowing the temperature measurements at interior points of the medium. Several methods have been developed and applied for solving IHCPs \[2\]. Using ANN as an approach to solve IHCPs has been also studied in several papers. Most of these studies used the whole time domain data for training and applying the network. Raudensky et al. \[3\] used ANN for coupled parameter and function specification in IHCPs. Jambunathan et al. \[4\] used neural networks to determine the convective heat transfer coefficients. Krejsa et al. \[5\] presented a summary of efforts to achieve solution of the IHCP using ANNs for both whole time domain mapping and sequential mapping. They concluded that ANN can be used to solve IHCPs and show that using an adaptive linear network cannot account for the noise in the input data while the back propagation network can handle the noise better. Sablani et al. \[6\] presented ANN models for calculating the convective heat transfer coefficient at the surface of a cube and semi-infinite plate using temperature
measurements inside the medium. Evaluation of heat flux distribution generated by a flame gun in a cylindrical coordinate system is studied by Hao et al. [7].

More attention has been attracted to approaches with real-time capability during recent years. Ijaz et al. [8] used a Kalman filter to solve a two-dimensional transient IHCP. They used the results from the adaptive estimator developed for transient heat flux estimation at the boundary in two-dimensional heat conduction domain with heated and insulated walls. Feng et al. [9] used Laplace transforms to relate the measured conditions at one end of a domain to the unknown conditions at the remote surface. ANN is also used for real-time heat flux estimation in few studies. A filter solution based on the idea of training neural networks is studied by Kowsari et al. [10]. Deng and Hwang [11] used neural network to compute the temperature distribution in forward heat conduction problems and solved inverse heat conduction problems using a back propagation neural network to identify the unknown boundary conditions. They reported that the proposed neural network analysis method can solve forward heat conduction problems and is capable of estimating the unknown parameters in inverse problems with acceptable error. Khorrami et al. [12] used a linear neural network for real-time estimation of multi-component heat flux.

Digital filter representation is one of the methods which can be used for real-time heat flux estimation using available temperature measurements. The idea of the filter algorithm is that the solution for the heat flux at any time is only affected by the recent temperature history and a few future time steps. Woodbury and Beck [1] studied the structure of the Tikhonov regularization problem and concluded that the method can be interpreted as a sequential filter formulation for continuous processing of data. They show that the computed heat fluxes using the whole domain solution and the filter coefficient solution are virtually the same for the constant-property
solutions. The idea of using digital filter representation method is further developed for IHCPs using measured temperature history as remote boundary condition [13] and in multi-layer domains [14,15].

The primary goal of this study is to use ANNs as a digital filter method for real-time heat flux estimation and compare some aspects of this approach with the digital filter representation of the Tikhonov regularization method. ANN consists of a set of interconnected neurons that can evaluate outputs from inputs by feeding information through the network and adjusting the weights [16,17,18]. In this work, a one-dimensional slab, initially at zero temperature, and with temperature-independent properties, with the perfectly insulated boundary at one end and subject to a change in heat flux at the other end is considered (this case is called X22B10T0 in ref [19]). The ANN is trained using a triangular heat flux input and corresponding temperatures. The trained network is tested afterwards through several test cases using exact solutions for different heat flux profiles. The calculated weights on the network are compared with the digital filter coefficients determined from the Tikhonov regularization technique.

The ANN approach is also tested for mediums with temperature dependent material properties and with the assumption of same boundary conditions mentioned above. One of the strength of the digital filter approach is the capability of handling non-linear problems by interpolating the filter coefficients [20]. It is shown that ANN can also be used to perform this task using both linear and non-linear transfer functions for the hidden layer.

Inverse Heat Conduction Problem (IHCP)

A forward heat conduction problem is known as determining the temperature through the domain when knowing the boundary conditions. In an inverse problem, on the other hand, the
temperature data are given for one or more points within the domain and the active boundary conditions are unknown.

In this paper, a 1D slab is assumed which is insulated at \( x=L \) and a heat flux is applied at \( x=0 \). A schematic of the problem is shown in Fig. 1. This is denoted as X22B-0T0 case in Cole et al. [19]. The mathematical statement of this problem can be given as:

\[
\frac{1}{\alpha} \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left( \frac{\partial T}{\partial x} \right), \tag{1}
\]

\[-k \left. \frac{\partial T}{\partial x} \right|_{x=L} = 0, \quad T(x,0) = T_0 \tag{2 a,b}\]

The unknown boundary condition:

\[q(0,t) = -k \left. \frac{\partial T}{\partial x} \right|_{x=0} \tag{3}\]

A solution for the X22B10T0 case with a constant heat flux at \( x = 0 \) and zero heat flux at \( x=L \) is [19]:

\[
\frac{T_{x22}(x,t)}{q \sqrt{L/k}} = \frac{\alpha t}{L^2} + \frac{1}{3} - \frac{x}{L} + \frac{x^2}{2L^2} - 2 \sum_{\gamma_m} \frac{\cos(\gamma_m x / L)}{\gamma_m^2} \exp \left( -\gamma_m^2 \alpha t / L^2 \right) \tag{4}
\]

where \( \gamma_m = m \pi \) is the \( m \)th eigenvalue (\( m=1,2,3,\ldots \)).
The method employed in this section is the filter coefficients representation for Tikhonov regularization method. This method is discussed extensively in Ref [1].

Knowing the temperature measurements at one point through the domain \( x=x_I \), the heat flux on the surface can be found using following equation:

\[
\hat{q} = F Y
\]

(5)

Where \( F \) is the filter matrix and \( Y \) is the vector of temperature measurements at \( x_I \). The filter matrix can be given as:

\[
F = (X^T X + \alpha I)^{-1} X^T
\]

(6)

where \( X \) is the sensitivity matrix and \( \alpha \) is the Tikhonov regularization parameter. The filter matrix has several interesting characteristics [1]. All of the rows of the filter matrix have the

Figure 1: schematic of the problem
same entries but are shifted in time, except for the first few terms and last few terms of each row.

This structure is shown in the equation below:

\[
\mathbf{F} = \begin{bmatrix}
  f_0 & f_{-1} & f_2 & \cdots & f_{2-N} & f_{1-N} \\
  f_1 & f_0 & f_{-1} & \cdots & f_{2-N} \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  f_{w-2} & f_{w-3} & \cdots & f_1 & f_0 & f_{-1} \\
  f_{N-1} & f_{N-2} & \cdots & f_2 & f_1 & f_0 
\end{bmatrix}
\]

(7)

The filter matrix is symmetric about the diagonal and has a Toeplitz structure [21]. Also, for the problem which is being discussed in this paper, \(X22B10T0\), the summation of the values in each row of the filter matrix is zero [1].

The dimensionless parameters used in this paper are defined as:

\[
\tilde{t} = \frac{\alpha t}{L^2} \\
\tilde{x} = \frac{x}{L} \\
\tilde{T} = \frac{T}{q_0 L/L}
\]

(8 a,b,c)

For dimensionless time step of 0.02, and \(\tilde{x} = 1\) and assuming \(\alpha_0 = 1.0 \times 10^{-6}\), the filter matrix is calculated and few rows of the filter matrix are plotted in Fig. 2.
As seen, at each time step, only few future ($m_f$) and previous time steps ($m_p$) have non-negligible values. In other words, the heat flux at each time step is only a function of temperature at few previous and future time steps and is independent from the rest of the time domain. Another interesting fact about the filter matrix is the sum of each row of the $F$ matrix is about zero which is related to the geometry and boundary conditions of this problem [1]. It should be noted that the regularization parameter, $\alpha_0$, is arbitrarily selected for this case as no heat flux estimation was performed. In later test cases, the $\alpha_0$ is selected to minimize the RMS error between the predicted and known heat flux. This will be discussed in the results section.

Neural Networks
ANN’s are computational models, inspired from human brain system, which has been successfully used in several engineering applications such as time series prediction, pattern recognition, function approximation, classification and more. They have also been used to solve forward and inverse heat conduction problems [22, 23, 24, 25]. ANN consists of a set of
interconnected neurons (nodes) that can evaluate outputs from inputs by feeding information through the network and adjusting the weights. There are several different types of ANN's namely, multi-layer perceptron (MLP), radial basis function network (RBF), adaptive linear network (ADALINE) and more [16,17].

The focus in this work is on MLP structure. A schematic of a simple MLP network with three inputs, one hidden layer containing four nodes and two outputs is shown in the Fig 3.

Figure 3: general structure of a neural network

In an MLP network, the nodes at each layer are fully connected to the nodes in the next one. Each node in one layer connects with a specific weight to every node in the next layer. The network's performance can be changed by adjusting these weights through the training process. In the training procedure, weights of the network are adjusted to find the best performance which maps the inputs to the outputs. The trained network with adjusted weights will be used to calculate the unknown output using the given inputs. There are two categories of training methods known as supervised methods and unsupervised methods. A commonly used supervised method is the back propagation method which is used in this paper. In using this method, the
The goal is to minimize the mean-squared errors between the network's output and the target values over all the example sets which are used for training the network.

The $i^{th}$ neuron in the $n^{th}$ layer is denoted as $v_i^n$ and assuming the network consisted of one input layer, $N$ hidden layers and one output layer, the output of the neural network can be calculated as below [11]:

$$v_i^n = \sum_j (W_{ij}^{n,n-1}v_j^{n-1} + b_i^n) \quad (9)$$

$$y_i^n = \phi_i^n(v_i^n) \quad (10)$$

Where $y_i^n$ is the output from the $i^{th}$ unit of the $n^{th}$ layer, $b_i^n$ is the bias and $\phi_i^n$ is the transfer function. It should be noted that there are several types of transfer functions that may be used for neural network applications. The selection of the suitable transfer function is important both in terms of complexity and performance of the network. The transfer function can be linear (e.g. “purelin’ in MATLAB) or non-linear (such as tan-sigmoid or “tansig” in MATLAB). The linear transfer function has less complexity and can perform well for several applications including solving IHCP’s, as will be demonstrated later.

The structure selected in this paper is a MLP with one hidden layer which contains one node. The smaller number of nodes results in a shorter training procedure and a simpler structure of the network and for those reasons it is preferred. The inputs are vectors of temperatures for $m_p+m_t$ time steps. The structure of the network is shown in Fig. 4.
The neural network toolbox in MATLAB is used for training the network. The transfer functions used for both of the output layer and hidden layer is selected as a linear function (“purelin”) for the case of constant material properties while for the temperature dependent material properties both linear and non-linear transfer functions are used for the hidden layer and results are compared.

Results and Discussions

As mentioned, two cases are investigated: (I) constant material properties (linear problem) (II) temperature dependent material properties (non-linear problem).

(I) Constant Material Properties

The geometry and boundary conditions are set as described before (X22B10T0). All the parameters are non-dimensional for this case. In order to train the network, a non-dimensional triangular heat flux is used. The exact solution is utilized to calculate the corresponding temperatures. Clearly, it is preferred to use smaller values of $m_f$, since using a smaller number of future time steps means the heat flux can be estimated closer to real-time. Also, it eases the process of training and simulation of the network as it has less inputs. Here, it is assumed that
Note that when using neural network, one needs to prescribe \( m_f \) and \( m_p \) to successfully train the network prior to using it for heat flux estimation. However, in ordinary filter solution technique the choice of \( m_p \) and \( m_f \) is not free and depend on the number of non-negligible filter coefficients before and after the current time step. The transient temperature data at \( \tilde{x} = 1 \) corresponds to the triangular heat flux function and the triangular heat flux function used to train the network are shown in Fig. 5 a,b.

There are total of 50 time steps (0.04 to 2) for this case. The input vector has 8 entities including temperature data from 4 previous time steps (\( m_p = 4 \)) and 4 future time steps (\( m_f = 4 \)). For example, the temperature data for time step 1 to 8 are used to calculate the heat flux at time step 5, the data for time steps 2 to 9 are used to determine the heat flux at time step 6 and so on. The 42 sets of inputs/output are used to train, test and validate the network. It should be noted that the 42 sets of data are simply the data during the time interval of 0.16 to 1.84, as inadequate data are available from time steps outside this interval, assuming \( m_p = m_f = 4 \). over the training data set yields an increase in accuracy over a data set that has not been introduced to the network yet. Finally, the testing set is used only for testing the final solution in order to confirm the actual predictive performance of the network. The trained network can now be used to predict the triangular heat flux using temperature data. Later in this section, the similarity between the adjusted weights and the filter coefficients will be discussed.
Figure 5: (a) dimensionless triangular heat flux used for training (b) transient temperature data corresponds to the triangular heat flux function
The training procedure used 70% of this data, while 15% of the available data are used for test and 15% is utilized for validation. The training data set is used to adjust the weights on the neural network. The validation set is used to minimize possible over-fitting by verifying that any increase in accuracy.

In the next step, the network is tested via two different heat flux profiles: quartic function [26] and step change. This is to find out whether the trained network by one function is capable of predicting the heat flux defined by a different function. The same time step, of course, is used (0.04) and eight sequential values are given to the network to estimate one heat flux component. It should be noted that for every test cases, the Root Mean Square (RMS) error between the exact solution and the predicted heat flux from the ANN is calculated using following equation:

\[
E_{RMS} = \left( \sum_{i=1}^{n} \frac{(q_{exact,i} - q_{estimated,i})^2}{N} \right)^{\frac{1}{2}}
\]  

(11)

For the quartic heat flux (test case 1), the temperature at \( \tilde{x} = 1 \) is plotted in Fig. 6 (a) which is calculated by the exact solution and then fed into the network. The heat flux is applied on the surface from dimensionless time 0.1 to 1.9. The predicted heat flux profile using the trained network and the heat flux from the exact solution are shown in Fig. 6 (b). It can be observed that the ANN was able to successfully predict the heat flux in this case. The RMS value between the ANN predicted results and the exact values for dimensionless heat flux is calculated as 9.67E-04.
Figure 6: (a) transient temperature data corresponds to the dimensionless quartic heat flux function (b) comparison of ANN estimated heat fluxes with exact solution (test case 1)
The step function is selected for the next test case (test case 2). A step change in heat flux applies on the domain at $x=0$ from dimensionless time 0.4 to 1.6. The corresponding temperature profile at $\bar{x} = 1$ is shown in Fig. 7 (a). The heat flux is then calculated using the trained ANN. The results from ANN are compared with the exact solution in Fig. 7 (b). As seen, the results are matched for most of the domain except near the sharp changes of heat flux. The RMS between the exact solution results and ANN predicted values is determined as 0.024. It is seen that ANN can be successfully used for surface heat flux estimation.

In next step, the focus is on the adjusted weights of the ANN and possible comparison with corresponding digital filter coefficients. The final adjusted weights and biases of the network which was trained and used for the previous test cases are given below:

Input layer weights: [0.5404, -2.3571, 4.1305, -1.8832, -6.7181, 12.0098, -5.2514, -0.4706]

Input layer bias: 0.0086

Output layer weight: -14.8130

Output layer bias: -0.8792

Note that these values are not unique and may vary every time that one trains the network. However, the network can always successfully be trained using the same values for $m_f$ and $m_p$. Interestingly, similar to the filter coefficient method, the summation of the weights of the trained ANN is very close to zero. This has been tested through over 50 trials and training processes. Every time the summation is checked and it was very close to zero.
Figure 7: (a) transient temperature data corresponds to the dimensionless step heat flux function (b) comparison of ANN estimated heat fluxes with exact solution (test case 2)
Another interesting fact is that the number of future time steps ($m_f$) needed for using Tikhonov filter coefficients method is more than what is needed for the ANN. For all of the above test cases the values of $m_p$ and $m_f$ were set on 4. However, the minimum number of future time steps that has to be used when using filter coefficient method is 10. This can be realized by close observation of the filter matrix in Fig. 8. As seen, there are 9 non-zero values after current time step ($m_f=10$, including the current time step) and 5 non-zero values before the current time step ($m_p=5$). Therefore, using filter coefficient for this specific geometry, boundary condition, time step and sensor location, the estimated heat flux values can be reported with a delay as long as 9 time steps.

![Figure 8: filter coefficients (time step=0.04, α_t=1E-8)](image)

For the step function and quartic test cases and assuming $m_f=10$, the RMS error between results from filter approach and exact solution is calculated as 0.0631  and 3.6793e-04 (with α_t=1E-6)
respectively. These values for the ANN simulation and $m_f=4$ were calculated as 0.024 and 9.67E-04 respectively.

It seems that ANN can perform faster by using smaller values of $m_f$. This feature of ANN approach can be used to minimize the temperature data required from future time steps.

Figure 9 shows a comparison between the results calculated by ANN, filter coefficient method and the exact solutions for the quartic heat flux. An error in the amount of 1% of the maximum temperature is added to the temperature data to further analyze these methods.

![Image of a graph showing comparison between ANN, digital filter and exact solution results for the quartic heat flux.](image)

**Figure 9:** comparison between ANN, digital filter and exact solution results for the quartic heat flux (data with 1% error)

For this test case, $m_f$ is assumed as 5 for the ANN, since the result with 4 future time steps had significant errors. Note that the selection of $m_f$ and $m_p$ plays an important role in achieving
accurate results. Although these parameters must be specified by the network designer, it is clear that they cannot be selected completely arbitrarily. If $m_f$ and $m_p$ are too small, or even too large, the network may not be trained successfully. The RMS values of the exact solution with respect to the ANN and digital filter solution results are calculated as 0.0446 and 0.0.0171 respectively. It should be noted that the value of $a_0$ is optimized to be $5e^{-4}$ for this test case. This value is determined by minimizing the RMS error between the known heat flux and estimated values.

Similar analysis is performed for the step function test case. The RMS error with respect to the exact solution is calculated as 0.1004 and 0.0987 for the ANN approach and filter method respectively. Figure 10 demonstrates this test case.

Figure 10: comparison between ANN, digital filter and exact solution results for the step heat flux (data with 1% error)
As seen, both of the methods were able to successfully estimate the heat flux when error
presents in the temperature data. Note that the heat flux for the last $m_f$ time steps cannot be
calculated by either of these methods.

\(\textbf{(II) Temperature dependent material properties}\)

When the material properties are temperature dependent, the problem is no longer linear. The
digital filter method offers the capability of accurately calculate the heat flux by interpolating
filter coefficients [20, 27]. In this section, the application of ANN as a digital filter is
investigated for the non-linear problem. Similar to the linear case, a triangular heat flux function
and the corresponding temperatures are used to train the network. This time both linear
(“purelin”) and non-linear transfer function (“tansig”) for the hidden layer are used and the
results are compared. Also, this test case is dimensional and the width of the slab is assumed to
be 0.01 m. It is assumed that the material used for this test case is Inconel and the material
properties are given in Table 1.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
\textbf{Temperature (K)} & \textbf{Conductivity (W/m-K)} & \textbf{thermal diffusivity (m$^2$/s)} \\
\hline
300 & 14.9 & 3.90E-06 \\
450 & 17.4 & 4.20E-06 \\
800 & 22.6 & 4.90E-06 \\
1000 & 25.4 & 5.30E-06 \\
1050 & 26.1 & 5.31E-06 \\
1200 & 28.2 & 5.57E-06 \\
\hline
\end{tabular}
\caption{material properties of Inconel at different temperatures}
\end{table}

A model is developed and simulated in ANSYS and temperature data are obtained to be used
for training and testing the network. A random heat flux is applied to the surface and the
temperature at $\tilde{x} = 1$ is calculated using ANSYS. These data are then used to train the network.
Figure 11: (a) temperature data used as input for training the network for the non-linear test case (b) heat flux data for training the network for the non-linear test case
The temperature and heat flux data used for training the network are shown in Fig. 11 (a,b). Note that it is assumed that $m_p=m_f=4$ and time step is 1 second.

An arbitrarily generated heat flux is then applied to the surface and temperature data are fed into the trained network. The temperature data are shown in Fig. 12 (a). The estimated heat flux using ANN is depicted in Fig. 12 (b). As seen, ANN successfully predicted the transient heat flux. The RMS error between the exact values and estimated values is calculated as 8.42 E+3 W/m$^2$ when using “tansig” and 8.55 E+3 W/m$^2$ when using “purelin” as the transfer function of the hidden layer. In order to compare the RMS values of the dimensionless linear problem and the dimensional non-linear problem, the heat flux values from the dimensional case can be non-dimensionalized when dividing by an average heat flux over the time domain which is 89,440 W/m$^2$. Using the non-dimensionalized heat fluxes, RMS error is calculated as 0.0942.

To further analyze this approach, the performance of the trained network is assessed for a problem when cooling occurs. So far, both training and testing the network have been performed for a heating problem. In the final scenario, it is assumed that the previous test case is reversed. Meaning that the slab is at 728 K and cools down to 300 K. The purpose of this test case is to investigate whether the network which is trained by data from a heating problem can be used for a cooling problem or not and how accurate it can perform.

Figure 13 (a,b) shows the temperature and heat flux for the cooling test case. As seen, the heat flux estimation by the ANN for this case is not as good as the heating scenario. The RMS error for this case is calculated as 13,250 W/m$^2$ which can be non-dimensionalized when diving by the average heat flux through the time domain. The non-dimensionalized RMS error is determined as 0.148.
Figure 12: (a) temperature data used for testing the non-linear case
(b) ANN predicted heat fluxes for the non-linear problem
Figure 13: (a) temperature data for the non-linear test case with cooling on the surface (b) estimated heat flux by ANN for the non-linear test case with cooling on the surface
Conclusion
The focus of this study was using multi-layer perceptron artificial neural networks (ANNs) for real-time heat flux calculation based on the idea of digital filter coefficients. The results showed that ANN can be trained by a known set of temperature and heat flux data and the trained network can be successfully used to calculate different heat flux profiles. It was concluded that for the geometry and boundary conditions that were assumed for this work, the ANN needs smaller number of data from future/previous time steps in order to calculate the heat flux at a certain time step, which makes it a faster approach than the Tikhonov digital filter coefficients. It was shown that for both constant material properties and temperature dependent material properties, it is possible to use the ANN approach with a linear transfer function. It is important that the training temperature/heat flux data set cover the range of the test cases, especially for the non-linear case to achieve accurate results. It was observed that the network can perform significantly better if the training and testing follow a similar pattern, meaning that a network which is trained by a set of data associated with a heating procedure will perform better if used to predict heat fluxes for a heating process and may not be used for accurate estimation of heat fluxes in a cooling process.

References


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Abstract

Heat flux measurement is essential in several industrial applications. While direct measurement of heat flux is not a simple task and sometimes is not even possible, measuring temperature is much easier and viable in variety of applications. Heat flux estimation using temperature measurement data requires solving inverse heat conduction problems (IHCP’s). In the present paper, a real-time solution for two-dimensional inverse heat conduction problem is presented. It is assumed that multiple unknown heat fluxes are applied at the bottom of a plate ($y = 0$) and the plate is insulated over other surfaces. Temperature sensors are located at the top of the plate. The number of temperature sensors has to be equal or greater than the number of unknown heat fluxes. A two-dimensional inverse heat conduction problem with multiple unknowns needs to be solved in order to estimate the heat fluxes using temperature measurement data. A solution is developed based on minimization of sum of the squares of the errors between the estimated temperatures and known values with respect to the unknown heat fluxes. Tikhonov regularization is applied to overcome the ill-posedness of the problem and achieve a

\footnote{Hamidreza Najafi, Keith A. Woodbury, James V. Beck, Real Time Solution for Inverse Heat Conduction Problems in a Two-Dimensional Plate with Multiple Heat Fluxes at the Surface, to be submitted to the International Journal of Heat and Mass Transfer.}
stable solution. The solution is then written in a digital filter form which allows near real-time heat flux estimation. Two numerical experiments are developed in ANSYS and used to demonstrate the performance of the proposed solution. The presented solution can be used to calculate heat fluxes in a near real-time fashion in a variety of applications including metal quenching. Real-time and accurate measurement of heat flux improves controllability of numerous industrial processes which lead to energy and cost savings.

**KEY WORDS:** two-dimensional IHCP, digital filter method, real-time solution, heat flux estimation.

1. Introduction
Surface heat flux estimation is crucial in multiple industrial applications including metal heat treating, quenching and furnace control. While measuring temperature is usually an easy task by using thermocouples, direct measurement of heat flux is not always straightforward and may not be even possible. Therefore, developing methods for estimating heat flux using temperature data has attracted a lot of attentions. Calculating surface heat flux using temperature data requires solving Inverse Heat Conduction Problems (IHCP’s). While forward problems are categorized as well-posed problems, IHCP’s are mathematically ill-posed, meaning that any small error in the input data can greatly affect the results. Appropriate regularization techniques have to be used in order to achieve accurate and stable solutions. Different methods have been developed for solving IHCP’s using analytical and numerical techniques. Some of these methods are the least-square method with regularization, the sequential function specification, conjugate gradient method and numerical approaches [1, 2]. Tikhonov regularization (TR) is an approach for stabilizing IHCP’s. TR has been mostly used as a whole time domain method which requires the data from all observations at once which means that all the heat flux components are simultaneously calculated for the whole time domain [3]. Recently,
Woodbury et al. [4, 5] developed Tikhonov digital filter for IHCP’s which allows near real-time heat flux estimation using temperature measurement data.

The filter concept is that heat flux at each time is only a function of temperature data from several previous time steps and a few future time steps and is independent from the rest of the time domain. This technique is further developed by Najafi et al. [6] for solution of IHCP’s in one-dimensional multi-layer mediums.

While several papers focused on solution of IHCP’s in one-dimensional mediums, fewer studies focused on IHCP solutions in two-dimensional problems. Imber [7] presented an approximate method using Laplace transform to solve IHCP’s in two-dimensional cylindrical geometries. Dowding and Beck [8] used a sequential gradient method for two-dimensional IHCPs. The hybrid application of the Laplace transform, finite difference method and the least-squares scheme with a sequential-in-time concept is studied by Chen et al. [9] for estimating the unknown surface temperature from temperature data measured at some locations in a plate. Osman et al. [10] developed a combined function specification and regularization method to solve a two-dimensional IHCP in an arbitrarily shaped body. Monde et al. [11] also used Laplace transform to develop a solution for two-dimensional IHCP in a finite rectangular body with one and two unknowns. Woodfield et al [12] further developed this technique for applications involving data with strong fluctuations and moving heat flux source. Wang et al. [13] presented a new technique for solving two-dimensional steady IHCP’s using decentralized fuzzy inference. A sequential method is proposed by Yang et al. [14] based on finite difference method for calculating the boundary condition in the two-dimensional inverse hyperbolic conduction problems. García et al. [15] used a sequential SVD (singular value decomposition) method and studied the two-
dimensional (in Cartesian or axisymmetric cylindrical coordinates) and non-linear IHCP in irregular-shaped bodies with temperature-dependent thermal properties. Yang et al. [16] proposed a method based on linear least-squares method for two-dimensional IHCP’s. A solution strategy is developed by Wang et al. [17] based on finite difference and finite volume method combined with weight coefficient to solve a two-dimensional IHCP with a non-homogeneous term and unknown Neumann boundary condition.

The focus of this paper is developing a real-time solution for IHCP’s in two-dimensional medium with multiple unknown heat fluxes using digital filter form of Tikhonov regularization method. It is assumed that multiple unknown heat fluxes are applied at the bottom of a plate (y = 0) while the plate is insulated over other surfaces. Temperature sensors are located at the top of the plate and the number of temperature sensors has to be equal or greater than the number of unknown heat fluxes. A solution is developed based on minimization of sum of the squares of the errors between the estimated temperatures and known values with respect to the unknown heat fluxes. Tikhonov regularization is used to achieve a stable solution. The solution is then written in a digital filter form which allows near real-time heat flux estimation. The characteristics of the filter matrix are discussed. Two numerical experiments are developed in ANSYS and used to demonstrate the performance of the proposed solution. The developed solution can be used to calculate heat fluxes in a near real-time fashion in variety of applications which lead to energy and cost savings.

2. Problem Statement

In this section, the two-dimensional forward problem is first briefly explained and then the corresponding IHCP discussed in detail.

2.1. Forward Problem
Figure 1 shows the schematic of a plate which is partially heated from the bottom surface. As seen, the plate is insulated over all surfaces except a part of the bottom surface where it is being heated \((0<x<L_1)\).

This problem can be denoted as \(X^{2B00}Y^{2B}(x5)0T0\). Where, \(X\) and \(Y\) denote the \(x\)- and \(y\)-directions, respectively; “22 represents boundary conditions of the first kind at both \(x=0\) and \(L\); B is a boundary condition modifier such that \(X^{2B00}\) denotes zero heat flux (insulated) at \(x=0\) and \(L\); and \(Y^{2B}(x5)0\) denotes a step change in the temperature in the \(x\)-direction at the \(y=0\) surface and a zero heat flux at \(y=W\). \(T0\) refer to the fact that the problem is transient and the initial temperature is 0. This numbering system is described in [18]. A schematic of this problem is depicted in Fig. 1.

![Figure 1: Schematic of the plate being heated at the bottom surface](image)

The mathematical statement of the problem can be described by:
\[ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}, \quad 0 < x < L, \quad 0 < y < W \]  

(1)

\[ \frac{\partial T}{\partial x}(0, y, t) = 0, \quad \frac{\partial T}{\partial x}(L, y, t) = 0, \quad \frac{\partial T}{\partial y}(x, W, t) = 0 \]  

(2)

\[ -k \frac{\partial T}{\partial y}(x, 0, t) = \begin{cases} q_0, & 0 < x < L_1 \\ 0, & L_1 < x < L \end{cases} \]  

\[ T(x, y, 0) = 0 \]

(3)

The solution of the problem given by Eqs. (1), (2) and (3) using Green’s functions is [18]:

\[ T(x, y, t) = \frac{q_0 \alpha}{k} \int_{\tau=0}^{t} \int_{x=0}^{L_1} G_{x,22}(x, x', t - \tau) dx' G_{y,22}(y, 0, t - \tau) d\tau \]

(4)

After integration, a solution to this problem can be written as:

\[ T(x, y, t) = T_i(t) + T_{i,x}(x, t) + T_{i,y}(y, t) + T_{i,xy}(x, y, t) \]

(5)

As seen, the above equation consists of four parts. These parts can be written as below:

First part (X2B1T0):

\[ T_i(t) = -\frac{q_0 W}{k} \frac{\alpha t}{L_1 W} \]

(6)

This part of solution is a lumped plate with a net constant heat flux which is denoted X2B1T0. It is given by Eq. (6),

\[ T_i(t) = \frac{q_0 W}{k} \left( \frac{L_1 \alpha t}{L W^2} \right) = \frac{q_0 W L_1}{k} t_w = \frac{q_0 W}{k} L_1 T_x \tilde{x}_{2B1T0}(t_w) \]  

(7a)

\[ T_{x,2B1T0}(t_w) = t_w \]  

(7b)

Second part (X2B00T0Gx5):
\[
T_{x}(x,t) = \frac{q_{0}W}{k} \frac{\alpha}{W^{2}} \left\{ \left(-L_{1} + 2 \sum_{n=1}^{\infty} \frac{\cos(\beta_{n}x) \sin(\beta_{n}L_{1})}{\beta_{n}} e^{-\frac{\beta_{n}^{2}L_{1}}{L}} \right) - \rho_{0} \right\} du = \frac{q_{0}W}{k} L_{1} T_{x22B00T0Gx5}(x,t_{L})
\]

\[
= \frac{q_{0}W}{k} L_{1} \left[ \frac{1}{12} \left\{ \frac{x_{0}^{3} - 3x_{0}^{2} + 2x_{0}}{x_{0}^{3} - 3\text{sign}(x_{0})x_{0}^{2} + 2x_{0}} \right\} \right]
\]

\[
= \frac{q_{0}W}{k} L_{1} \left[ \frac{1}{3} \epsilon_{1}^{3} - 3 \epsilon_{1} + 2 \epsilon_{1} \right]
\]

Third part (Y22B10T):

\[
T_{y}(y,t) = \frac{q_{0}W}{k} \alpha \frac{L_{1}}{W^{2}} \left\{ 1 + 2 \sum_{n=1}^{\infty} \frac{\cos(\eta_{n}y)}{\eta_{n}^{2}} e^{-\frac{\eta_{n}^{2}y}{w}} \right\} du
\]

\[
= \frac{q_{0}W}{k} L_{1} \left[ \frac{y^{2}}{2} - \frac{y}{3} + \frac{1}{2} - 2 \sum_{n=1}^{\infty} \frac{\cos(\eta_{n}y)}{\eta_{n}^{2}} e^{-\frac{\eta_{n}^{2}y}{w}} \right]
\]

Fourth part:

\[
T_{x}(x,y,t) = \frac{q_{0}W}{k} \alpha \frac{L_{1}}{W^{2}} \left\{ \begin{array}{l}
\frac{1}{3} \epsilon_{1}^{3} - 3 \epsilon_{1} + 2 \epsilon_{1} \left\{ 1 - \text{sign}(x-L_{1}) - 2L_{1} \right\} \left[ 1 - \text{sign}(x-L_{1}) \right] \left[ e^{-\eta_{1}^{2}(1-\epsilon_{1}^{2})} - e^{-\eta_{1}^{2}(1-\epsilon_{1}^{2})} \right] \right\}
\end{array}
\]

\[
= \frac{q_{0}W}{k} L_{1} \left[ \frac{1}{3} \epsilon_{1}^{3} - 3 \epsilon_{1} + 2 \epsilon_{1} \left\{ 1 - \text{sign}(x-L_{1}) \right\} \left[ e^{-\eta_{1}^{2}(1-\epsilon_{1}^{2})} - e^{-\eta_{1}^{2}(1-\epsilon_{1}^{2})} \right] \right]
\]

Note that \( \frac{x}{L} = \frac{L_{1}}{L}, \ \beta_{n} = m\pi, \ m = 1,2,\ldots \) and \( \frac{y}{W}, \ \eta_{n} = n\pi, \ n = 1,2,\ldots \)

Other dimensionless terms can be given as:
The solution to the above problem is the building block of the problem which is demonstrated in Fig. 2. Here the heat flux is applied at a section on the bottom surface of the plate:

Using the superposition principle, the solution to the problem shown in Fig. 2 can be expressed as follows. Let the solution for partial heating over the surface from $x=0$ to $x=L_1$, given in Eqs. (5-10), be denoted as $T_{0\rightarrow L_1}(x, y, t)$. Then the solution for constant heating over the surface between $x_j$ and $x_{j+1}$ can be found as:

$$T_{x_j \rightarrow x_{j+1}}(x, y, t) = T_{0\rightarrow x_j}(x, y, t) - T_{0\rightarrow x_{j+1}}(x, y, t). \quad (12a)$$

Likewise, the temperature response at any point $(x, y)$ due to a series of $p$ such pulses on the surface can be found by summing the contributions from each pulse:

$$T(x, y, t) = \sum_{i=1}^{p} T_{(i-1)\rightarrow i}(x, y, t). \quad (12b)$$

### 2.2. Inverse Problem

It is assumed that multiple heat fluxes are applied at the bottom surface of the plate and plate is insulated over other surfaces. The temperature sensors are located at the top surface of the plate. The number of
temperature sensors \((J)\) has to be equal or greater than the number of unknown heat fluxes \((p)\) in order to achieve a unique solution.

Figure 3: schematic of the test case developed in ANSYS (for Example I: \(L/W=10\) and for Example II: \(L/W=2\))

Figure 3 demonstrates a schematic of the problem. The unknown heat flux should be calculated using known temperature data from the sensors. This is described mathematically in below:

\[
-k \frac{\partial T}{\partial y} (x,0,t) = q = \text{unknown} = \begin{cases} 
q_1(t), & 0 < x < x_1 \\
q_2(t), & x_1 < x < x_2 \\
\vdots \\
q_p(t), & x_{p-1} < x < x_{p+1} 
\end{cases} \\
T(x_j, W, t) = \text{known} \quad j = 1, \cdots, J 
\]

The solution to the discrete problem can be given in matrix form as below:

\[
T = Xq 
\]

where \(X\) is the sensitivity matrix and \(q\) is the heat flux. These matrices can be written as:

\[
T = \begin{bmatrix} 
T(1) \\
T(2) \\
\vdots \\
T(N_x) 
\end{bmatrix}, \quad T(i) = \begin{bmatrix} 
T_i(1) \\
T_i(2) \\
\vdots \\
T_i(N_y) 
\end{bmatrix} 
\]

(15 a,b)
Here, \( M \) refers to the current time step and \( N_t \) is the total number of time steps. The matrix of the heat fluxes is defined as:

\[
\mathbf{q} = \begin{bmatrix} q(1) \\ q(2) \\ \vdots \\ q(N_t) \end{bmatrix}, \quad \mathbf{q}_i = \begin{bmatrix} q_1(i) \\ q_2(i) \\ \vdots \\ q_p(i) \end{bmatrix}, \quad q_j(i) = q_{s_{j-1} \to s_j}(t_i) \quad (17 \, a,b)
\]

And finally the \( \mathbf{X} \) matrix (matrix of sensitivity coefficients) can be given as [1]:

\[
\mathbf{X} = \begin{bmatrix} a(1) & 0 & 0 & \cdots & 0 \\ a(2) & a(1) & 0 & \cdots & 0 \\ a(3) & a(2) & a(1) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a(N_t) & a(N_t-1) & \cdots & a(1) \end{bmatrix}, \quad \mathbf{a}(i) = \begin{bmatrix} a_{11}(i) & a_{12}(i) & \cdots & a_{1p}(i) \\ a_{21}(i) & a_{22}(i) & \cdots & a_{2p}(i) \\ \vdots & \vdots & \ddots & \vdots \\ a_{j1}(i) & a_{j2}(i) & \cdots & a_{jp}(i) \end{bmatrix},
\]

\[
a_{jk}(i) = \frac{\partial T(x_j, t_i)}{\partial [q_k(1)]} \quad (18a,b,c)
\]

The sum of the squares of the errors between the estimated temperatures and known temperatures plus the regularization term for time and space can be found as:

\[
S = (\mathbf{Y} - \mathbf{T})^\top (\mathbf{Y} - \mathbf{T}) + \alpha_T [\mathbf{H} \mathbf{q}]^\top [\mathbf{H} \mathbf{q}] + \alpha_s [\mathbf{H_s} \mathbf{q}]^\top [\mathbf{H_s} \mathbf{q}] \quad (19)
\]

Where \( \alpha_T \) and \( \alpha_s \) are the regularization parameters used for time and space terms. \( \mathbf{H} \) and \( \mathbf{H_s} \) can be given as:

\[
\mathbf{H} = \begin{bmatrix} \mathbf{-I} & \mathbf{I} & 0 & \cdots & 0 \\ 0 & \mathbf{-I} & \mathbf{I} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mathbf{0} \end{bmatrix}, \quad \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \quad (20a,b)
\]

\[
\mathbf{H_s} = \begin{bmatrix} \mathbf{\overline{H}} & 0 & 0 & \cdots & 0 \\ 0 & \mathbf{\overline{H}} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mathbf{\overline{H}} \end{bmatrix}, \quad \mathbf{\overline{H}} = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{bmatrix} \quad (21a,b)
\]
Minimizing the sum of the squares of the errors given in Eq. 19 with respect to \( q \) results in:

\[
\hat{q} = \left[ X^T X + a^T H^T H + a^T H_{x}^T H_{x} \right]^{-1} X^T Y \tag{22}
\]

**Filter form of the solution**

The concept of filter solution suggest that heat flux at each time step is only a function of temperature measurements from several previous time steps and a few future time steps and is independent from the rest of the time domain.

The heat flux vector in Eq. (22) can be also written as below:

\[
\hat{q} = F Y \quad , \quad F = \left[ X^T X + a^T H^T H + a^T H_{x}^T H_{x} \right]^{-1} X^T \tag{23a,b}
\]

Here, \( F \) is the filter matrix that has several interesting characteristics. Note that in Eq. (23a), the dimensions of \( \hat{q} \) is \((N_t \times P) \times 1\). The dimension of \( F \) and \( Y \) matrices are \((N_t \times P) \times (N_t \times J)\) and \((N_t \times J) \times 1\) respectively. The structure of the filter matrix \( (F) \) can be shown as below:

\[
F = \begin{bmatrix}
    f_0 & f_{-1} & f_{-2} & \cdots & f_{2-N} & f_{-N} \\
    f_1 & f_0 & f_{-1} & \cdots & f_{2-N} & \vdots \\
    \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
    f_{N-2} & f_{N-3} & \cdots & f_1 & f_0 & f_{-1} \\
    f_{N-1} & f_{N-2} & \cdots & f_2 & f_1 & f_0
\end{bmatrix} \quad \tag{24}
\]

Note that here each component of the \( F \) matrix (i.e. \( f_n \) where \( n=(1-N_t), \ (2-N_t), \ldots, \ 0, \ 1, \ 2, \ldots(1-N_t), \ (N_t-1) \) ) is a \( P \times J \) block of entities and therefore, each row of the \( F \) matrix is a block of \( P \)-rows and \( N_t \times J \) columns. As seen in Eq. (24), the sub-matrices in rows of filter matrix are identical but shifted in time.
To describe the characteristics of the filter matrix, an example case is investigated in detail. It is assumed that four heat fluxes are applied at the bottom surface of a plate \((P=4)\) and four temperature sensors \((J=4)\) are used to measure temperature at the top of the plate. The dimensionless time step is defined as 0.05 and it starts from 0.05 and ends at \((N_t=40)\). For this case, \(F\) is a 160 × 160 matrix. Each block of four \((P)\) rows of this matrix are being repeated but shifted in time. To demonstrate this, the 41\textsuperscript{th}, 42\textsuperscript{th}, 43\textsuperscript{th} and 44\textsuperscript{th} rows (associated with 11\textsuperscript{th} time step) as well as the 105\textsuperscript{th}, 106\textsuperscript{th}, 107\textsuperscript{th} and 108\textsuperscript{th} rows (associated with 27\textsuperscript{th} time step) of the filter matrix are plotted and shown in Fig. 4. As seen, the rows of each of the blocks are identical to one another (e.g. values of filter coefficients on 41\textsuperscript{th} row and 105\textsuperscript{th} row are identical).

![Figure 4: rows of the filter matrix associated with the unknown heat fluxes](image)

To describe the characteristics of the filter matrix, an example case is investigated in detail. It is assumed that four heat fluxes are applied at the bottom surface of a plate \((P=4)\) and four temperature sensors \((J=4)\) are used to measure temperature at the top of the plate. The dimensionless time step is defined as 0.05 and it starts from 0.05 and ends at \((N_t=40)\). For this case, \(F\) is a 160 × 160 matrix. Each block of four \((P)\) rows of this matrix are being repeated but shifted in time. To demonstrate this, the 41\textsuperscript{th}, 42\textsuperscript{th}, 43\textsuperscript{th} and 44\textsuperscript{th} rows (associated with 11\textsuperscript{th} time step) as well as the 105\textsuperscript{th}, 106\textsuperscript{th}, 107\textsuperscript{th} and 108\textsuperscript{th} rows (associated with 27\textsuperscript{th} time step) of the filter matrix are plotted and shown in Fig. 4. As seen, the rows of each of the blocks are identical to one another (e.g. values of filter coefficients on 41\textsuperscript{th} row and 105\textsuperscript{th} row are identical).
Note that in $F$ matrix $1^{st}$, $5^{th}$, $9^{th}$, ..., $n+4^{th}$ rows are associated with $q_1$, $2^{nd}$, $6^{th}$, $10^{th}$, ..., $n+4^{th}$ are associated with $q_2$; and the coefficients associated with $q_3$ and $q_4$ can be written similarly. Also, in each row of the $F$ matrix, each column represents one of the temperature sensors. In other words $1^{st}$, $5^{th}$, $9^{th}$, ..., $n+4^{th}$ columns are associated with first sensor, $2^{nd}$, $6^{th}$, $10^{th}$, ..., $n+4^{th}$ columns are associated with the second sensor and so on so forth.

It is interesting to see that filter coefficients associated to $q_1$ are similar to $q_3$ and filter coefficients associated to $q_2$ are similar to $q_4$. That is because their locations are symmetric on the plate. Note that every four point along the $x$ axis in Fig. 4 actually represents four sensor locations for only one time step. To clarify on this point, a three-dimensional graph of filter coefficients is plotted in Fig. 5.

![Figure 5](image)

**Figure 5**: filter coefficients for Example I ($\alpha_t=1E-4$ and $\alpha_s=1E-5$, dimensionless time step: 0.05)

Here, the $41^{th}$, $42^{th}$, $43^{th}$ and $44^{th}$ rows of $F$ matrix are plotted. As seen, for $q_1$, the largest filter coefficients are associated with the temperature at $J_1$ location and as the sensor location gets far from
the location of the heat source \( q_1 \), the filter coefficients are getting smaller which is intuitive. This basically proves the idea that the heat flux at each point is only a function of temperature readings from nearby points. This is also true about filter coefficients for \( q_2 \), \( q_3 \) and \( q_4 \).

As seen, the values of filter coefficients approach to zero towards both ends. The number of non-negligible filter coefficients before and after the current time steps are denoted as \( m_p \) and \( m_f \) respectively.

The idea of the filter algorithm is that the solution for the heat flux at any time is only affected by the recent temperature history and a few future time steps [4]. Equation (23a) can be written in filter form as:

\[
\hat{q}_{p,M} = \hat{q}_{p,i}(t_M) = \sum_{i=1}^{(m_p \times m_i) \times J} \left( f_{p,i} Y_{m_{i-1} (m_i \times J) - i} \right)
\]

(25)

Here, \( f_{p,i} \) refers to the filter coefficients in the \( p^{th} \) column from one row of \( F \) associated to the \( p^{th} \) unknown heat flux. Also, \( M \) denotes the current time step. Since all the rows of the filter matrix are identical but shifted in time, instead of using the whole matrix \( F \) to calculate the heat fluxes (Eq. (23a)), the non-negligible terms from one row of this matrix can be used. Therefore, one can use the non-negligible terms from one row of the filter matrix (\( f_1, f_2, \ldots, f_{(m_p \times m_i) \times J} \)) associated to the \( p^{th} \) unknown heat flux and multiply those in the corresponding temperature values from temperature sensors and use the summation of these terms to calculate each of the unknown heat fluxes (\( q_1, q_2, \ldots, q_p \)) at a specific time step (\( M \)).

It should be noted that the selection of \( m_p \) and \( m_f \) should not be made arbitrarily as they depend on the boundary conditions, material properties, sensor location, time step and regularization parameter. The filter coefficients must be closely observed to determine \( m_p \) and \( m_f \).
As seen in Fig. 5, there are only a few non-zero values ahead of the current time step and this determines $m_f$. By a closer look at Fig. 5, it can be observed that $m_f$ and $m_p$ are both equal to 9. The smaller $m_f$ is, the closer to real-time the algorithm can operate. It should be noted that $m_p$ and $m_f$ are not necessarily equal for all different problems. Also, it is important to emphasize that a bigger value of $m_p$ does not affect how fast (closer to real-time) the algorithm can operate as it only decides how many data points from previous time steps must be used to determine the heat flux at the current time.

3. Results and Discussion
In this section two numerical examples are developed in ANSYS and the results are compared with the developed filter solution. For this purpose, a plate is modeled and meshed and four heat fluxes are applied to the plate at the bottom surface while it is insulated over other surfaces. It is also assumed that the temperature can be measured by four temperature sensors installed at the top surface of the plate. Figure 3 shows a schematic of the created model in ANSYS. The aspect ratio (W/L) that is used for the Example I and Example II is 0.1 and 0.5 respectively.

3.1. Example I
The heat fluxes are applied at the bottom surface and temperature is measured at the top surface using four sensors. The dimensionless time step is defined as 0.05 and it starts from 0.05 and ends at 2. The filter coefficients for this test case are already shown in Figs. 4 and 5. The heat fluxes and calculated temperatures are demonstrated in Figs. 6 and 7. These temperature data are used as inputs for the developed filter method (Eq. (25)). As already discussed, by a closer look at Fig. 5, $m_f$ and $m_p$ can be determined as 9.
The regularization parameter can also affect the values of $m_p$ and $m_f$. As discussed by Woodbury and Beck [4], selection of the order of magnitude of the regularizing parameter is important to achieve accurate results and minimize errors, but the precise selection of the regularization parameter is not important for effective estimation of heat flux. Also, when choosing a smaller regularization parameter, $m_p$ and $m_f$ will be smaller. The conclusion is, when using filter approach for real-time heat flux estimation, one need to have an idea of the problem that has to be solved (whether the heat flux is uniform or has a lot of sudden changes) to pick the appropriate regularization parameter and corresponding $m_p$ and $m_f$ values.
Using Eq. (25) and assuming \( m_p = m_f = 9 \) and regularization parameters are set as \( \alpha_t = 1E-4 \) and \( \alpha_s = 1E-5 \), the heat fluxes are calculated by using the temperature data. The estimated surface heat fluxes at different locations are shown in Fig. 8.

As seen in Fig. 8, the filter method is able to calculate the heat fluxes accurately. The average RMS error between estimated values and exact values of heat fluxes is determined as 0.0208, 0.0175, 0.0122, 0.0061 for \( q_1 \), \( q_2 \), \( q_3 \) and \( q_4 \) respectively. As mentioned, here the Tikhonov regularization parameter is selected as \( \alpha_t = 1E-4 \) and \( \alpha_s = 1E-5 \) for minimizing the RMS error. The RMS error is calculated using following equation:

\[
E_{\text{RMS}} = \left( \frac{1}{n} \sum_{i=1}^{n} (q_{\text{exact},i} - q_{\text{estimated},i})^2 \right)^{1/2}
\]  

(26)
Where $n$ is the number of time steps for which heat flux estimation is performed.

To further investigate the performance of the proposed method, a uniform random error in the amount of 0.5% of the maximum temperature is added to the temperature data. The estimated heat fluxes when error presents in the data is shown in Fig. 9. Figure 9 is plotted in a three-dimensional coordinate system to provide all the estimated heat fluxes at different locations on a single graph. Here the regularization parameter is selected as $\alpha_t = \alpha_s = 1 \times 10^{-4}$ and the RMS error between the estimated values and exact heat fluxes are 0.0369, 0.0210, 0.0120 and 0.0091 for $q_1$, $q_2$, $q_3$ and $q_4$ respectively. As seen, the developed method can estimate the unknown heat fluxes on the surface using temperature data in a near real-time fashion.
3.2. Example II

In the second example, variety of heat flux profiles are applied at the bottom surface of the plate. Here, \( q_1 \) is a triangular shape, \( q_2 \) is a parabolic function, \( q_3 \) is built from straight lines and \( q_4 \) has a random shape. The aspect ratio for this example is defined as \( W/L = 0.5 \) (five times more than the first example). Figure 10 shows the heat fluxes applied to the bottom surface and Fig. 11 demonstrates the corresponding temperatures at the sensors location (top of the plate) calculated by ANSYS. The time step and time range is assumed to be same as the first example (\( 0.05 < t < 2 \)).
Figure 10: heat fluxes applied to the bottom surface (Example II)

Figure 11: temperature measurements by the sensors at different locations
The filter coefficients associated to this example when regularization parameter is selected as $\alpha_t = 1E-4$ and $\alpha_s = 1E-5$ are shown in Fig. 12. As seen, similar to the first example, the filter coefficients have specific patterns: The largest coefficients for each heat flux is associated with the temperature data from the closest sensor to the area that the heat flux is applied. For this example the number of non-negligible coefficients is determined as $m_p = m_f = 10$ by close inspection of the filter coefficients.

![Figure 12: filter coefficients for Example II (\(\alpha_t = 1E-4\) and \(\alpha_s = 1E-5\) and dimensionless time step: 0.05)](image)

It is assumed that a random uniform error in the amount of 0.5% of the maximum temperature is added to the temperature data to investigate the performance of the developed technique under conditions when error presents in data. Using Eq. 24 the heat fluxes are calculated and plotted in Fig. 13. The RMS error between the known heat fluxes and estimated heat fluxes is calculated as 0.0364, 0.0274, 0.0175 and 0.0220 for $q_1$, $q_2$, $q_3$ and $q_4$ respectively. As seen, the proposed filter solution successfully calculated the unknown heat fluxes. Unlike the solution techniques that use data from the whole time domain, the presented method allows for near real-time heat flux estimation by using data only from a
number of previous time steps and a few future time steps. For Example II, the temperature data from 10 time steps ahead of the current time \( (m_f) \) and 10 time steps prior to the present time \( (m_p) \) is needed to calculate the heat flux at each time which allows near-real time heat flux estimation.

![Graph showing estimated heat flux values when error presents in the data \( (q_1, q_2, q_3 \text{ and } q_4) \)](image)

Figure 13: estimated heat flux values when error presents in the data \( (q_1, q_2, q_3 \text{ and } q_4) \)

The simplicity, capability of online heat flux estimation as well as application to moderate nonlinearity \([19, 20]\) are some of the characteristics of the presented filter method which makes it a powerful approach for real time heat flux estimation in industrial applications.

4. Conclusions

In the present paper, a filter based solution is developed for solving a two-dimensional IHCP with multiple unknown heat fluxes. It is assumed that a plate is insulated over all of the surfaces except the bottom surface. Multiple heat fluxes are applied at the bottom surface of the plate. Temperature sensors
are located on the top surface of the plate. The number of temperature sensors has to be equal to or greater than the number of unknown heat fluxes. The solution is developed based on minimization of sum of the squares of the errors between the estimated temperatures and known values with respect to the unknown heat fluxes. Tikhonov regularization is used to stabilize the solution. The characteristic of the filter matrix are discussed in detail. Two numerical examples are developed in ANSYS and used to demonstrate the performance of the proposed solution. It is shown that the surface heat fluxes can be estimated accurately in a near real-time fashion. The developed technique can be used to improve and optimize several industrial processes such as quenching and metal heat treating.

References


CONCLUSION

The focus of this dissertation is developing filter form solutions for inverse heat conduction problems (IHCP’s) that can be used for real-time heat flux estimation using temperature measurements. While temperature measurement is generally easy and doesn’t require expensive instrument, direct measurement of heat flux can be challenging or impossible in some cases. Filter form solutions in this dissertation allow near real-time heat flux estimation using temperature measurement data. Solutions are developed based on the digital filter representation of Tikhonov regularization method. The developed techniques are summarized as below:

Article 1 presents a filter form solution for a one-dimensional IHCP in a one layer medium when temperature measurements are available at two points within the medium. This technique can be used for surface heat flux estimation in similar conditions in industries, e.g. when there is no access to the surface.

Article 2 uses the results from Article one and further developed the technique for a multi-layer medium. This paper presents a real-time solution for IHCP’s in multi-layer mediums when temperature measurements are available at two points within the inner layer. The characteristics of the filter matrix in such a problem are discussed in detail. It is assumed that the material properties are constant in this paper.
Article 3 discusses a non-linear IHCP in multi-layer mediums where material properties are changing with temperature. Different schemes for interpolating filter coefficients are explained and it is shown that the developed technique can be successfully applied for non-linear problems.

Article 4 uses all the concepts developed in Article 1-3 and focuses on a specific application of these techniques in industry: heat flux measurement using Directional Flame Thermometer (DFT). This paper presents a computationally effective solution for calculating surface heat flux in a near real-time fashion using temperature data obtained by thermocouples in a DFT. One application of this technique is real-time measurement of heat flux in a furnace which improves controllability over the heating process that results in energy and cost savings.

Article 5 introduces a LabView based interface for the solution given in Article 4. This interface allows end user with limited or no knowledge of inverse problems to easily calculate surface heat flux by defining the time step.

Article 6 studies application of artificial neural networks as digital filters for real-time heat flux estimation in linear and non-linear problems. It is shown that ANN’s can be successfully trained to estimate surface heat flux for IHCP’s in a slab with constant material properties as well as temperature dependent material properties.

Article 7 developed a filter form solution for a two-dimensional IHCP with multiple unknown heat fluxes. It is assumed that multiple heat fluxes are being applied at the bottom surface of a plate while other surfaces are insulated. The characteristics of the filter matrix in such a problem are discussed in
detail and several test cases are used to demonstrate the application of this method. In practice, this technique can be used to measure heat fluxes in quenching process.