THREE ESSAYS ON RISK AND INSURANCE

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ABSTRACT

Previous writers have attempted to resolve the equity premium puzzle by employing a utility function that depends on current consumption minus (or relative to) past habit consumption. The first chapter points out that an individual’s current utility may also depend upon how well off in the recent past he or she had expected to be today. Hence we add the concept "expectation formation" to the utility modification term in a model with a habit-formation utility function. We apply the model to equity premium puzzle and find that it is able to fit the data with a relatively low coefficient of relative risk aversion. Furthermore we fit the model to 40-year rolling samples and find that the estimated coefficient of risk aversion does not vary much as the sample changes. Hence we conclude that the model is able to resolve the equity premium puzzle.

The second chapter presents a two-period model in which an individual can purchase insurance, save and borrow to protect herself against potential risk in the future. A model for insurance without the presence of capital market, and one for saving/borrowing without insurance are also discussed. We show that neither insurance nor precautionary saving/borrowing alone can generate a complete market analog, but both together can. We also show how optimal choices for insurance and saving/borrowing change when key factors in the environment change.

The third chapter incorporates the ideas of habit formation and reference-dependent preference with a two-period model for insurance, saving and borrowing. I compare the results with the ones with a standard expected utility, and find that this model helps to explain all the phenomena of over-insured, under-saving and over-borrowing at the same time.
DEDICATION

I dedicate my dissertation work to my loving parents, whose support and encouragement help me to make the decision to pursue this degree, and to go through this journey.

I also dedicate this dissertation to all my friends. They share all the wonderful times with me, and also bear those not so happy times with me. I can not be here without them all.

I dedicate this work to Dr. Harris Schlesinger, who was one of my co-advisors. His passing away still shocks me, but his attitude towards research and live will always inspire me.
LIST OF ABBREVIATIONS AND SYMBOLS

ARA  Absolute Risk Aversion.

CARA  Constant Absolute Risk Aversion.

CPI  Consumer Price Index.

CRRA  Constant Relative Risk Aversion.

DARA  Decreasing Absolute Risk Aversion.

FRED  Federal Reserve Economic Data.

IARA  Increasing Absolute Risk Aversion.

NIPA  National Income and Product Accounts.
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CHAPTER 1
LIFE WITH HABIT AND EXPECTATION: A NEW EXPLANATION OF EQUITY PREMIUM PUZZLE

1.1 Introduction

Since it was introduced by Lucas Jr (1978), the consumption based asset pricing model has attracted the attention of many economists. The model has been used to study various types of assets and evaluate their risks. One interesting result obtained from using this model is that of Mehra and Prescott (1985) who apply it to U.S. data for the period 1889-1978. During this period the average real annual return for the Standard & Poor 500 Index is about seven percent while the average real annual return for U.S. government issued short term Treasury Bills is about 0.8 percent. Applying the consumption asset pricing model to data with such a large gap between these two returns implies that consumers have a coefficient of risk aversion that is much higher than what most economists believed to be possible. This unexpectedly large difference in the average returns on the two assets is called the equity premium puzzle.

Since the large difference in returns is given, numerous efforts have been made to revolve the puzzle by modifying the assumed preference structure of the typical household. One important approach has been to use a utility function that exhibits habit formation rather than the typical CRRA utility function. However, Mehra and Prescott (2003) point out that the risk aversion parameter estimated by a model where utility is defined over the difference between current consumption and past consumption habit (Constantinides (1990), Campbell and Cochrane (1995), etc.) is not the actual coefficient of risk aversion, and the real risk aversion in this kind of model is still very high. Furthermore, a model where utility depends on the ratio of consumption relative to habit rather than difference between the two (Abel (1990)), which is not a trivial modification, also does not resolve
the puzzle.

In a model in which utility depends on the difference between current consumption and the habit level of consumption, current consumption must always be greater than the past habit level of consumption. This restriction reduces the amount of risk estimated by the model, because the model is not able to take into account the effect on utility of consumption falling below the level of habit consumption. Such a model therefore reduces the estimated equity premium and the required rate of return on the risky asset. As a result, the only way a model of this type can explain a high equity premium is through a relatively high degree of risk aversion. Furthermore, the requirement that consumption be above its habit level also implies that consumption in the future can not fluctuate as much as it would if consumption can fall below its habit levels. This means the model assumes households act as if there is less uncertainty about future consumption than exists in the data. This, in turn reduces the demand for saving as well as the demand for risk-free assets. As a result there must be a relatively high coefficient of risk aversion in order for the model to fit the data.

In this paper utility is a function of the ratio of consumption to a utility modification term that is somewhat different from the habit level of consumption. There are two advantages of choosing a ratio over a difference. One is, unlike the model where utility depends on the difference between consumption and habit, which by definition must be positive, in the ratio utility function, the level of consumption can fluctuate more freely according to its distribution. As a result, the effect on expected utility of consumption possibly falling below the level of habit are not excluded from the model (as they are in the model with difference utility). Hence, the model used here allows the level of risk to vary as much as is implied by the set of data being used, so that the calculated return on equity is based on its full risk. The second advantage is that in the model used here the estimated coefficient of relative risk aversion is the actual coefficient of risk aversion.

Along with habit, a household’s expectation of future income also affects its consumption choices. The permanent income hypothesis suggests that a household’s consumption is determined not only by its current income, but also by its expected future incomes.
Flavin (1981) points out that whenever a household observes that its current realization of income is greater than what it had anticipated, it revises upward its expectations of future income, and because this means it is revising upward its permanent income, it revises its consumption decisions accordingly. Furthermore, it is likely that a household's expectation about its future consumption has an impact on its current level of utility. For example, if someone expects to get a promotion but does not, then the resulting disappointment will cause her to have a lower level of utility even if current consumption has not declined. Hence, this paper proposes a new concept called "expectation formation", which is based on a household's recent expectation for its average level of current consumption. This term represents the expectation a household previously had for the level of consumption during the current period. Hence, the household has a utility modification term that depends not only on its past consumption history (habit), but also on its recent expectations for consumption during the current period. In an economy that is growing, the utility modification term of the average household grows over time, so in order to maintain a given level of utility, the household must have a level of consumption that is expected to be higher in the future. In order to realize the higher level of consumption, households must save more and this increase in the demand for saving allows a decrease in the risk-free rate, low enough to allow the model to fit the data with a coefficient of relative risk aversion of reasonable size.

The paper is organized as follows. Section 2 summarizes the U.S. historical data used by Mehra and Prescott (1985), as well as the new data set that is used in this paper. Section 3 introduces the model that contains both habit formation and expectation formation, and provides the intuition for these concepts. Section 4 compares and discusses the results from both data sets, then further divides the data set into two sets rolling samples, one of 40-year and the other of 25-year, and examines the performance of the model over these samples. Section 5 concludes the paper and provides a summary of main findings.
1.2 Data

Mehra and Prescott (1985) examine annual data for the period from 1889 to 1978. The annual real return on the Standard & Poor 500 stock index is used as a proxy for the real return on equity, while the real return on the 3-Month U.S. Treasury bill is used as the real risk-free rate. For the ninety-year sample period used by Mehra and Prescott (1985), the average risk-free rate is 0.8 percent with a standard deviation of 5.67, while the average equity premium is 6.18 percent with a standard deviation of 16.67.

Because the length of periods of expansion have been relatively long since the end of WWII (Cover and Pecorino (2005); Diebold and Rudebusch (1992)) in addition to the original data set, this paper also constructs and examines a new data set from 1945 to 2012, a sample period of sixty-eight years. For this updated sample, the average risk-free rate is only 0.4 percent, and the average equity premium is higher at 7.59 percent. Hence the gap between the average return on equity and the return on the risk-free asset is wider with the new data making it slightly more challenging to resolve the puzzle.

Finally, by combining the two data sets, this paper uses rolling samples to test how each of the estimated coefficients is affected by the way the key inputs change over time.

The S&P 500 price and dividend series and CPI are obtained from Robert Shiller Online Data. The risk-free rates (3-Month Treasure Bill) are obtained from FRED, and the U.S. consumption and population data are obtained from NIPA tables.

1.3 The Model

1.3.1 The Economy and Preferences

The economy follows the ones used by Mehra and Prescott (1985) and Abel (1990), where there is a single perishable consumption good and the representative household in the economy maximizes its life-time expected utility. We then define the utility function as:

\[ U_t = E_t \sum_{i=0}^{\infty} \beta^i \frac{(C_{t+i})^{1-\gamma}}{1-\gamma} \]

where \( C_t \) is the level of consumption during period \( t \), \( X_t \) is the utility modification term.
for period $t$, which affects the level of utility in a manner similar to what the habit term
does in the habit formation model. But as discussed below it depends on more than
just habit. As is the case in a CRRA utility function, $\gamma$ is the coefficient of relative risk
aversion, and $\beta$ is the subjective discount rate. According to (1.1), utility in any period
depends on the ratio of the level of consumption to the level of the utility modification
term.

Notice that according to (1.1), if we assume $X_t$ is the habit level of consumption,
the current level of consumption is not restricted to be above the current level of habit.
By relaxing this restriction, the model allows consumption to fluctuate as freely within
the model as is indicated by the data set, which in turn allows the dividend payment to
fluctuate freely. As a result, the amount of risk caused by holding equity is not under-
estimated.

The utility modification term of an agent is assumed to evolve according to the fol-
lowing process:

$$X_t = \alpha C_t^a + \delta [C_{t-1} - C_{t-1}^a]$$

where $C_t^a$ is the individual's expected average level of consumption at time $t$, which he
formed at time $t - 1$ with $C_t^a = (1 + \mu)C_{t-1}^a$ where $\mu$ is the average consumption growth
rate over the sample period.

According to equation (1.2) the utility modification term, $X_t$, consists of two parts.
The first, $\alpha C_t^a$, is the household’s expectation variable. This part of the utility modi-
fication term depends on the household’s own expected level of consumption, which is
assumed to increase each period at the same rate as average consumption in the economy.
For example, if a person has a job which pays a salary that he expects to increase, on
average, at a certain rate every year, it is reasonable for him to expect his consump-
tion, on average, to grow at the same rate, given that the salary is paid in terms of
the perishable consumption good. This part of the utility modification term comes from
the household expecting its consumption to increase by the average rate of consumption
growth. Because the household forms its expectation with the aggregate information it
has, it expects its level of consumption to increase at the long-run average growth rate
of consumption.
The second term in equation (1.2), $\delta[C_{t-1} - C_{t-1}^a]$, is an individual adjustment variable, which also serves as the habit variable\(^1\). Unlike previous literature on habit formation, habit persistence in this model does not directly enter the evolution equation as levels of past consumption, but as the difference between real consumption and expected real consumption for the last period. Thus the household adjusts its utility modification term according to how its consumption last period differed from what it had expected. If during period $t - 1$, consumption is lower than expected, then the utility modification term for period $t$ decreases. Likewise, if consumption during period $t - 1$ is higher than he had expected, then the utility modification during period $t$ increases. The main implication of this adjustment variable is that, given its level of consumption during period $t$, it causes the level of happiness that the household feels during period $t$ to increases as its consumption during period $t - 1$ declines relative to what it had expected for period $t - 1$.

During each period, $t$, a household makes its consumption decisions based on the information available to it during the period. There is one production unit producing the consumption good, and there is one equity traded competitively in the market, where the dividend payment of the equity is also the output of the production unit. The equity’s dividend payment is defined as:

$$D_{t+1}^e = g_{t+1}D_t^e$$

where $g_{t+1} \in (\lambda_1, \cdots, \lambda_n)$ is the growth rate of the dividend, and the dividend payment follows the same Markov chain as used by Mehra and Prescott (1985):

$$Prb[g_{t+1} = \lambda_j; g_{t+1} = \lambda_i] = \phi_{ij}$$

There is also a risk-free asset available in the market which pays one unit of the consumption good at the beginning of the next period independent of the states. The equilibrium consumption is the dividend payment at that period, so $\{C_t\} = \{D_t^e\}$ at every time period $t$, and consumption growth therefore follows the same Markov chain as above.

\(^1\)We use individual’s own consumption history in the equation, because as shown in Chen and Ludvigson (2009) and Grishchenko (2010), internal habit formation can describe the equity return data better than external one.
1.3.2 Asset Pricing and Equity Premium

Following the standard procedure for calculating the equity premium, from the first-order conditions, the conditional expectation of the product of the intertemporal marginal rate of substitution and the gross rate of return on any assets should be equal to 1:

\[ E_t(M_{t+1}R_{t+1}) = 1 \]  

(1.3)

where \( M_{t+1} \) is the intertemporal marginal rate of substitution. Define the returns on equity and risk free asset, respectively as,

\[ \frac{P^e_{t+1} + D^e_{t+1}}{P^e_t} = R^e_{t+1} \]

\[ \frac{1}{P^f_t} = R^f_{t+1} \]  

(1.4)

where \( P^e_t \) and \( P^f_t \) respectively are the prices for the equity and the risk-free asset.

\( M_{t+1} \) can be written as the following expression (Appendix A provides more details regarding this):

\[ M_{t+1} = \frac{\beta \left[ \frac{C^{1-\gamma}_{t+1}}{X_{t+1}} - \beta \delta \frac{C^{1-\gamma}_{t+2}}{X_{t+2}} \right]}{\beta \left[ \frac{C^{1-\gamma}_t}{X_t} - \beta \delta \frac{C^{1-\gamma}_{t+1}}{X_{t+1}} \right]} \]  

(1.5)

Because it is homogenous of degree one in consumption, the price of the equity as a function of consumption can be written as:

\[ P^e_t = w_i C_t \]

where \( w_i \) is a constant coefficient when state is \( i \).

Now define \( M_{ij} \) as the intertemporal marginal rate of substitution when the current state is \( i \), and the next state is \( j \), this implies:

\[ w_i = \sum_{j=1}^{n} \Phi_{ij}(w_j + 1)\lambda_j M_{ij} \]  

(1.6)

As is shown in Appendix B, assuming consumption during the last period is the same as last period’s expected average consumption, normalizing last period’s expected average consumption to 1 and using the assumption that expected average consumption grows at rate \( \mu \) yields the following expression for \( M_{ij} \):

\[ M_{ij} = \frac{\beta \left[ \frac{1}{n} \left( \frac{1}{\lambda_i \lambda_j} \left\{ \alpha (1 + \mu)^2 + \delta [\lambda_i - (1 + \mu)] \right\} \right)^{\gamma - 1} - \sum_{i=1}^{n} \frac{\beta}{\lambda_i \lambda_j} \left\{ \alpha (1 + \mu)^3 + \delta [\lambda_i \lambda_j - (1 + \mu)^2] \right\}^{\gamma - 2} \]  

(1.7)

\[ \frac{1}{\lambda_i} \left\{ \alpha (1 + \mu) \right\}^{\gamma - 1} - \frac{\beta}{\lambda_j} \left\{ \alpha (1 + \mu)^2 + \delta [\lambda_j (1 + \mu)] \right\}^{\gamma - 2} \]
The rate of return on equity when the current state is $i$ and the next state is $j$ is

$$r_{ij}^e = \frac{(w_j + 1)\lambda_j}{w_i} - 1$$

where the calculation of $w_i$ is provided in Appendix C.

The expected rate of return on the equity when the current state is $i$ is

$$r_i^e = \sum_{j=1}^{n} \Phi_{ij} r_{ij}^e$$

Combining (1.3), (1.4), (1.5) and (1.7) yields

$$P_i^f = \sum_{j=1}^{n} \Phi_{ij} M_{ij}$$

and the rate of return on the risk-free asset when current state is $i$ is

$$r_i^f = \frac{1}{P_i^f} - 1$$

Let $\pi \in R^n$ be the vector of stationary probabilities on $i$. The vector $\pi$ is the solution to the system of equations

$$\pi = \Phi^T \pi$$

with $\sum_{i=1}^{n} \pi_i = 1$ and $\Phi^T = \{\Phi_{ji}\}$.

So the expected rate of returns on the equity and on the risk-free asset are, respectively

$$R^e = \sum_{i=1}^{n} \pi_i r_i^e$$

and

$$R^f = \sum_{i=1}^{n} \pi_i r_i^f$$

The equity premium is then calculated as $R^e - R^f$.

1.3.3 The Results

Following Mehra and Prescott (1985) assume consumption growth follows a two-state Markov chain where:

$$\lambda_1 = 1 + \mu + \delta, \quad \lambda_2 = 1 + \mu - \delta$$

$$\Phi_{11} = \Phi_{22} = \Phi, \quad \Phi_{12} = \Phi_{21} = 1 - \Phi$$
where $\mu = 0.018$, $\delta = 0.036$, and $\Phi = 0.43$, the values chosen by Mehra and Prescott (1985).

To fit the model to the data, it is necessary to choose a value for the annual discount factor. Typically the chosen value has been close to one. But Weil (1989) points out the higher the chosen value of $\beta$, the lower the estimated value of the equilibrium risk-free rate, and that a value of $\beta$ above 1 (which is implausible) can therefore be the computer's solution to a risk-free rate puzzle. Most theoretical and empirical work, however, does not find a discount factor that is very close to 1. Frederick et al. (2002) provides an extensive literature review on discounting, and shows that most empirical estimates of discount rate give an annual discount factor around 0.9. More recent work by Meyer (2013) tests three data sets and also suggests a value of $\beta$ at 0.9. Based on these studies, this paper sets the value of the individual's discount factor constant at 0.9.

Using the above parameter values and Mehra and Prescott (1985) original data in which 0.8% is the average risk-free rate, and 6.18% is the average equity premium, the "expectation with habit model" yields a coefficient of relative risk aversion equal to approximately 2.96, which clearly meets the a-priori justification that the coefficient of risk aversion should not exceed $10^2$.

For the updated data which covers the period 1945-2012, the average growth rate of per capita consumption is 0.023, with a standard deviation equal to 0.022 and a first-order serial correlation coefficient (of the growth rate) equal to 0.21. These values imply the following model parameter values for this sample: $\mu = 0.023$, $\delta = 0.022$, $\Phi = 0.61$.

With this new set of parameters and maintaining the discount factor at 0.9, and with the risk-free rate and the equity premium matching their sample averages, the "expectation with habit model" yields a coefficient of risk aversion equal to only 2.98.

A further examination of the robustness of the "expectation with habit model" comes from applying it to 40-year-long rolling samples taken from the updated data set. If the model is truly robust, then subtracting one year from the beginning of the sample and adding one year to the end of the same will not cause much of a change in the estimated coefficient.

\[\text{"...they constitute an a-priori justification for restricting the value of relative risk aversion to be a maximum of ten..." Mehra and Prescott (1985).}\]
coefficients. This is important because some economists (Otrok et al. (2002)) have argued that estimates based on a model with habit formation are sensitive to slight changes in the values of the parameters. For each sample of 40 years the time discount factor is set at 0.9, and the values of the other parameters are set to match data in the sample (the growth rate of per capita consumption, its standard deviation and its first-order serial correlation coefficient). The result is presented in Figure 1.1.

Figure 1.1 shows the coefficients of habit and of expectation, and relative risk aversion estimated by the model over the 40-year rolling samples are very stable. So the model is able to produce both a low-enough and a consistent estimation of the level of risk aversion. At the same time, the habit coefficients and the expectation coefficients are also consistent over time.

1.4 Conclusion

The restriction that imposed in most previous researches, which requires the level of consumption to be above the level of habit, eliminate the possibility of certain "bad-enough" state, which will in turn under-estimate the amount of risk the equity bears.
Choosing a utility defined as ratio form over one defined as difference form not only relaxes this restriction and allows the model to calculate the return on equity based on its full risk, but also avoids the problem of generating the false estimation for the coefficient of relative risk aversion.

Meanwhile, introducing the expectation formation into the model enables us to reveal another important aspect which affects people’s preference. Along with habit, expectation also serves as a reference for people’s utility over consumption. People have expectations for having better lives in the future (given the overall consumption is growing over time). With an adjustment term served as the habit part in the utility modification evolution equation, the individual’s utility modification term depends on both his expectation for having a better future and his past consumption history. The individual’s level of utility modification will be growing over the long-run since the data suggests the per capita consumption is growing smoothly over the sample period, so in order to maintain the level of utility, the agent will need to have a higher level of consumption in the future. Under this circumstance, people’s demand for saving will increase, and the risk-free rate will be at a low level as the data suggested, so the model is able to explain the risk-free rate puzzle. The combination of both expectation formation and habit formation in the utility function enables us to better describe individuals’ preference. Furthermore, the results generated from the rolling samples show that the coefficients estimated by the model across time are consistent.
CHAPTER 2
PROTECTION AGAINST FUTURE RISK: A TWO-PERIOD MODEL FOR INSURANCE AND SAVING/BORROWING

2.1 Introduction

Previous insurance literature usually uses either a one-period model or a continuously-time model in which people make insurance decisions at the same time when potential losses occur. Under this framework, Mossin (1968) makes a famous statement: "a risk-averse individual will choose to fully insure at an actuarially fair premium." The intuition behind this theorem is straightforward: a risk-averse person will always prefer a path with certain outcome than one with same expected outcome but a positive variance (risk). This theorem has been investigated and proved for many different types of insurance (e.g. Schlesinger (1981) for deductible insurance and Schlesinger (2006) for upper-limit insurance).

However, insurances sold in the real world normally do not work that way. Almost all types of insurance (e.g. auto insurance, real estate insurance, health insurance, etc.) require people to purchase insurance in advance of the time when potential losses can occur (or the time when insurance companies would reimburse the individual if a loss indeed occurs).

Meanwhile, besides purchasing insurance, people also have other choices to smooth their consumption and protect themselves from risks in the future. Among these choices, saving and borrowing are commonly used in daily life. Hubbard et al. (1994) perform an empirical test to show that social insurance programs tend to decrease people’s precautionary saving. Kantor and Fishback (1996) use workers’ compensation as social insurance to show that it reduces both private insurance purchases and precautionary saving. The empirical results in Engen and Gruber (2001) suggest that there is a "crowd out" effect
of unemployment insurance on household saving. All three papers above use empirical data and do not allow individuals to make insurance and saving decisions freely. Dionne and Eeckhoudt (1984) present a two-period model which analyzes the optimal choice of insurance and saving. However, they do not allow borrowing, and use a bivariate inter-temporal utility (Dreze and Modigliani (1972)), thus do not get many definitive results. Briys (1986) and Somerville (2004) both study consumption and insurance in continuous time, and show that insurance decisions are "separable" if the premium is fair. All of these models assume that insurance premiums are paid in the period that potential losses occur (or the continuous-time analog of such an assumption).

Menegatti (2009) looks at two-period prevention models with prevention costs paid at date $t = 1$ and losses/benefits accruing at date $t = 2$. This inter-temporally separates benefits and costs. We look at paying a premium at date $t = 1$, and then incurring a potential loss (and potential indemnity) at date $t = 2$. Consumption takes place at both dates, yielding a savings motive.

Our paper is organized as follows. Section 2 begins by introducing the two-period framework, followed by discussing the cases for insurance only and precautionary saving/borrowing only (incomplete market analog). Then a model with both insurance and saving/borrowing (complete market analog) is presented. Section 3 provides the comparative statics, and shows how decisions for insurance and saving/borrowing change when insurance price changes. Section 4 shows the substitution effect between the two ways of protection. Section 5 concludes the paper.

### 2.2 The Models

#### 2.2.1 Model Setup

An individual receives endowments (incomes) of the amount $W_i$ in both current and future periods. She also faces a potential loss in the second period. We assume only two states are possible, and use a simple Bernoulli loss distribution. So either a loss of size $L$ occurs with probability $\pi$, or no loss occurs with probability $1 - \pi$.

The individual chooses the amount of insurance, $\alpha$, she wants to purchase at time 1,
which will reimburse her at time 2 if a loss state occurs. The price of insurance is denoted by \( P \), where \( P = (1 + \lambda)\pi L \), and \( \lambda \) is the loading factor of insurance premium. If \( \lambda = 0 \), insurance is actuarially fair. During the first period, the individual also makes decisions for saving and/or borrowing, which is denoted by \( S \), where \( S \) stands for saving if \( S > 0 \), and borrowing if \( S < 0 \). The subjective time discount rate and the gross risk-free return are \( \beta \) and \( R \), respectively, and the individual’s objective function is:

\[
V \equiv U(W_1 - \alpha P - S) + \beta[\pi U(W_2 - L + \alpha L + RS) + (1 - \pi)U(W_2 + RS)]
\]

Qualitative effects of \( \beta \) and \( R \) are well known. Interest rates yield a type of "pricing" that is "fair" or "unfair" (equal to one or not equal to one). Time discounting rates affect people’s decisions regarding either to consume more now (or more later). The simplest case is to have zero interest and zero impatience, so we assume for simplicity that \( \beta = R = 1 \) in most of our analysis.

### 2.2.2 Optimal Insurance without Saving/Borrowing

When \( S \) is set to 0, we have a model for insurance only. Consider the case when endowments (incomes) are the same across periods and equal to \( W \), the individual’s level of consumption in a current period is,

\[
C_1 = W - \alpha P
\]

and her levels of consumption in a future period with or without a total loss are, respectively,

\[
C_{2L} = W - L + \alpha L
\]

and

\[
C_2 = W
\]

The objective function then becomes:

\[
V \equiv U(W - \alpha P) + [\pi U(W - L + \alpha L) + (1 - \pi)U(W)]
\]

**Proposition 2.2.1.** Without ways to transfer wealth across periods, even when insurance is sold at fair price, individuals still purchase less than full insurance.

**Proof.** First-order condition with respect to \( \alpha \) is

\[
\frac{dV}{d\alpha} = -(1 + \lambda)\pi L U'(W - \alpha(1 + \lambda)\pi L) + \pi L U'(W - L + \alpha L)
\]

\[
= 0
\]

(2.1)
If the premium is actuarially fair (i.e. \( \lambda = 0 \)), from (2.1) we have
\[
U'(W - \alpha \pi L) = U'(W - L + \alpha L)
\]
which yields the optimal amount of insurance under fair premium: \( \alpha = \frac{1}{1+\pi} \). Notice that we do not require any particular utility form as long as the utility function is concave. \( \square \)

It is interesting to observe that the individual actually purchases less than full insurance even when insurance is sold at fair price, which implies Mossin’s Theorem is violated. The levels of consumption in different periods and states are illustrated below:

Figure 2.1: Levels of Consumption for Insurance Only

![Figure 2.1](image)

Note from Figure 2.1 that the no-loss wealth at date \( t = 2 \) is not permutable. Any choice of \( \alpha \) cannot affect this state. This is a type of personal market incompleteness. The 2nd best is to equate marginal utility of consumption at date \( t = 1 \) with expected marginal utility at date \( t = 2 \). Note that \( \alpha = 1 \) equates wealth in both states at date \( t = 2 \), but at too high of a cost.

The probability of loss, \( \pi \), affects people’s decisions even when the premium is fair. The individual buys less insurance as the probability of loss increases. At first glance, the finding appears unrealistic. However, as \( \pi \) goes up, the price of insurance also goes up, the agent needs to sacrifice more consumption at time 1 for the same amount of insurance purchased. Because the insurance price is fair, the amount of expected consumption gain in period 2 will increase by the same amount. Due to the concavity of the individual’s utility, the marginal cost of purchasing insurance will be higher than its marginal benefit. As a result, people will decrease their insurance purchases.

Now we allow different endowments for the two time periods in the model. Holding other factors the same, the individual’s objective function over the two periods then become
\[
V \equiv U(W_1 - \alpha P) + \pi U(W_2 - L + \alpha L) + (1 - \pi) U(W_2)
\]
Solving first-order condition yields the optimal amount of insurance under fair premium: \( \alpha = \frac{1}{1 + \pi} (1 + \frac{W_1 - W_2}{L}) \). Again the individual does not necessarily purchase full insurance when insurance is sold at a fair price. Still, the outcome in one of the two states in the second period is not affected by an individual’s choices, so we will again have a type of second best. However, more factors have influences on people’s decisions for insurance when the premium is fair. The difference between the two endowments and the potential loss size enter the equation for optimal insurance. Comparing to the optimal insurance level in the previous setting, we have a multiplier of \( 1 + \frac{W_1 - W_2}{L} \). The more initial wealth the individual has in period 1 compared to period 2, the more insurance she will purchase. Because the marginal cost of purchasing insurance becomes less than expected marginal benefit of doing so, she is willing to sacrifice more of her consumption in period 1 for the potential gain in wealth in period 2. On the other hand, a higher endowment in period 2 enables the individual to bear more potential loss, so the individual will purchase less insurance.

For the potential loss size \( L \), there are two cases. The first one is when \( W_1 > W_2 \). The higher the loss size is, the more expensive the insurance will be. To match marginal cost and expected marginal benefit, the extra amount of insurance (compare to previous setting where the endowments are the same) the individual will purchase become smaller. The other case is when \( W_1 < W_2 \). When the potential loss size is small, the individual will consider it as bearable, due to a higher level of initial wealth in period 2. So she will purchase even less insurance. When the potential loss size becomes larger, the individual will be willing to give up some of her consumption in period 1 to insure part of the potential loss in period 2, so she will purchase more insurance, but still less than what she would if the initial endowments were the same.

So what happens if the price becomes unfair?

**Proposition 2.2.2.** When insurance is the only way of protection against future risk, individuals always purchase less coverage when the insurance price becomes unfair.

**Proof.** First, let us consider the case with the same initial wealth in both periods. We
know that \( \alpha = \frac{1}{1+\pi} \) solves first-order condition when \( \lambda = 0 \), so when \( \lambda > 0 \),

\[
|\alpha - \frac{1}{1+\pi} (1 + \lambda)\pi L U'[W - \alpha(1 + \lambda)\pi L] > \pi L U'(W - L + \alpha L)
\]

because expected marginal benefit is the same as with a fair premium, but marginal cost is higher (Both marginal monetary cost and marginal utility are higher). In order to make RHS equal to LHS, we need \( U'[W - \alpha(1 + \lambda)\pi L] < U'(W - L + \alpha L) \), which can be achieved by reducing the amount of insurance purchased. So we know that the optimal insurance when the price is unfair will be some value of \( \alpha < \frac{1}{1+\pi} \).

It is straightforward to show that if insurance is sold with a positive loading, the optimal insurance will also decrease in the case when initial endowments are different across the two periods.

Even though unlike Mossin’s Theorem, the individual does not purchase full insurance under a fair price, she still decreases her insurance purchases when the price becomes unfair. Also, it is easy to show that we cannot rule out \( \alpha = 0 \), at which case we have a corner solution for the optimal insurance.

Note that in all cases where \( \alpha > 0 \), insurance decreases the gap between consumption in the first period and that in the loss state of second period, and the gap between consumption across two states of second period. However, it comes at a cost of increasing the gap between consumption in the first period and that in the no-loss state of second period.

2.2.3 Optimal Insurance and Optimal Saving/Borrowing

In order to smooth the levels of consumption across periods, we now introduce endogenous saving/borrowing into the model. In addition to consuming and purchasing insurance, the individual also decides how much she wants to save or borrow at period 1, and will withdraw or pay back that amount at period 2. Assuming endowments are the same across time, her current consumption is

\[
C_1 = W - \alpha P - S
\]

Her levels of future consumption with or without a total loss are, respectively,

\[
C_{2L} = W - L + \alpha L + S
\]
and \[ C_2 = W + S \]

Under the assumptions of zero time discounting rate and zero interest rate, the objective function is:

\[
V \equiv U(W_1 - \alpha P - S) + \pi U(W_2 - L + \alpha L + S) + (1 - \pi)U(W_2 + S)
\]

**Proposition 2.2.3.** When wealth can be transferred across periods, individuals purchase full insurance under fair price, and divide the cost of insurance between periods.

**Proof.** First-order conditions with respect to \( \alpha \) and \( S \) are, respectively,

\[
\frac{dV}{d\alpha} = -(1 + \lambda)\pi LU'(W - \alpha(1 + \lambda)\pi L - S) + \pi LU'(W - L + \alpha L + S) = 0
\]

and

\[
\frac{dV}{dS} = -U'(W - \alpha(1 + \lambda)\pi L) + \pi U'(W - L + \alpha L + S) + (1 - \pi)U'(W + S) = 0
\]

It is easy to see that second-order conditions hold, so we have a set of maximum solutions for optimal insurance and saving/borrowing. Solving first-order conditions for fair premium gives us \( \alpha = 1 \) and \( S = -\frac{1}{2}\pi L \). This time, just as Mossin (1968) states, an individual purchases full insurance when the premium is actuarially fair.

When initial endowments are different across periods, i.e. \( W_1 \neq W_2 \), it is straightforward to show that \( \alpha = 1 \) and \( S = \frac{W_1 - W_2 - \pi L}{2} \) solve both first-order conditions. The individual is able to equalize her consumption in different periods/states to \( \frac{W_1 + W_2 - \pi L}{2} \).

Since insurance is actuarially fair, expected lifetime wealth will not change by the choice of insurance. Full insurance \((\alpha = 1)\) equates consumption in loss state and no-loss state at date \( t = 2 \). Since \( \beta = R = 1 \), saving/borrowing is "fair" (perfect capital market), \( S = -\frac{1}{2}\pi L \) (borrow half the premium) equates consumption in \( t = 1 \) and \( t = 2 \).

By introducing endogenous saving/borrowing into the model, the individual now has more control over her wealth and levels of consumption across time and states, and the outcomes in every state are affected by the individual’s decisions. The outcomes are illustrated in Figure 2.2:
With a complete market analog, a first-best outcome is achieved and the individual can perfectly smooth her consumption across every date/state.

**Proposition 2.2.4.** Mossin’s Theorem is restored for a model with both insurance and saving/borrowing.

*Proof.* First, assume the individual still purchases full insurance under unfair price \((\alpha = 1)\), so
\[
\frac{dV}{dS}\bigg|_{\alpha=1} = -U'(W - (1 + \lambda)\pi L - S) + U'(W + S) = 0
\]
which yields \(S = -\frac{1}{2}(1 + \lambda)\pi L\) (borrow half the premium). A perfect capital market allows us to shift premium payment to \(t = 2\). With full insurance, income smoothing shifts half of the premium. However, if we plug this result back to first-order condition with respect to insurance
\[
\frac{dV}{d\alpha}\bigg|_{\alpha=1} = -(1 + \lambda)\pi LU'(W - \frac{1}{2}(1 + \lambda)\pi L) + \pi LU'(W - \frac{1}{2}(1 + \lambda)\pi L) < 0
\]
So full insurance is too much, which leads to the conclusion that purchasing full insurance is never optimal even with endogenous saving/borrowing. Combine this with the result we have previously, **Mossin’s Theorem** is restored.

The proof for different endowments across time is similar. In conclusion, **Mossin’s Theorem** can be proved for the model with both insurance and endogenous saving/borrowing. An individual purchases full insurance if and only if insurance is sold at fair price.

**Proposition 2.2.5.** An individual purchases full insurance if and only if both insurance and capital market are "fair" priced.
Proof. We already proved that when $\beta = R = 1$, individual purchases full insurance if and only if the insurance price is actuarially fair. Now we consider the case when $\beta$ and $R$ do not equal to one.

Once again, assuming the insurance price is actuarially fair, and the individual still purchases full insurance, so we have the following two equations from first-order conditions:

\[
\frac{dV}{d\alpha}\bigg|_{\alpha=1} = -\pi L U'(W - \pi L - S) + \pi L \beta U'(W + RS) = 0
\]

\[
\frac{dV}{dS}\bigg|_{\alpha=1} = -U'(W - \pi L - S) + R \beta U'(W + RS) = 0
\]

which obviously can not both be satisfied at the same time as long as $R \neq 1$. However, if the interest rate is zero, both conditions can be held regardless of the values of the time-discounting rate. So time impatience will only affect the levels of consumption across periods, thus the amount people save/borrow in the first period. 

This finding extends Mossin’s Theorem from the insurance market to the capital market, and shows that only when both markets are "fair" priced, people will purchase full insurance. Even though consumption across periods/states can still be equalized, unfair prices cause insurance (or sacrificing saving/increasing borrowing to fund insurance purchases) to generate a cost of lower aggregate consumption over time, so it is not optimal to purchase full coverage.

2.2.4 Optimal Saving/Borrowing without Insurance

To make things complete, we also consider a world with saving/borrowing, but no insurance. Let the individual face the same potential risk in the second period, and decide how much she would like to save or borrow in the first period. Again we begin with the same endowments across periods. The objective function is as follows:

\[
V \equiv U(W - S) + \pi U(W - L + S) + (1 - \pi)U(W + S)
\]

**Proposition 2.2.6.** Without insurance, individuals have to save more/borrow less to protect themselves from future risk. Even by doing so, they would be worse off than when insurance is available.
Proof. It is easy to show that if the probability of loss is 0 and the loss will never occur, the optimal saving for the individual will be 0. On the other hand, if the probability of loss is 1 and the loss state will always appear, the individual will save half of the loss size. And the optimal amount of saving will increase as the probability of loss increases, so we have \( 0 < S < \frac{1}{2}L \) when \( 0 < \pi < 1 \).

For the case when endowments are different across periods, we have \( \frac{W_1 - W_2}{2} < S < \frac{W_1 - W_2 + L}{2} \) when \( 0 < \pi < 1 \). In this case, the individual might borrow instead of save even when she faces a potential loss in the future, given that her endowment in the first period is lower than that in the second period.

Compare the results above with the ones we get from the model for insurance and endogenous saving/borrowing, we can see that the optimal amount of saving or borrowing is \(-\frac{1}{2}\pi L\) for the same endowments across periods and \(\frac{W_1 - W_2 - \pi L}{2}\) for different endowments, so with insurance as an additional choice, the individual will decrease her saving, or increase her borrowing in both cases. The combination of insurance and saving/borrowing offers the individuals the ability to further smooth consumption across periods and better protect themselves from potential shocks in the future. \(\square\)

2.3 Comparative Statics Analysis

2.3.1 The Insurance-Only Model

The Wealth Effect

What will happen to the optimal level of insurance if the level of wealth changes? Some previous literature classifies insurance as inferior good (e.g. Mossin (1968), etc.). However, in the real world wealthy people sometimes purchase more insurance than poor people. Schlesinger (2000) points out with greater wealth often comes greater risk, so people with more wealth no necessary purchase less insurance. Dionne and Eeckhoudt (1984) show that the derivative of optimal insurance with respect to wealth can be either positive, null or negative, and they claim that this finding reduces the importance of inferiority property observed in standard insurance literature. However, the sign is uncertain
and there is no clear condition for when the sign will change. So we want to see if our model can provide additional information about this issue.

Differentiate the first-order condition with respect to \( W \), we have:

\[
\frac{d\alpha}{dW} = \frac{PU''(W - \alpha P) - \pi LU''(W - L + \alpha L)}{P^2U''(W - \alpha P) + \pi L^2U''(W - L + \alpha L)}
\]

Because the second-order derivative of the utility is always negative, the denominator is always negative, and the sign of this equation depends only on the numerator. We know that \( PU'(W - \alpha P) = \pi LU'(W - L + \alpha L) \) from first-order conditions, and that \( PU'(W - \alpha P) \) and \( \pi LU'(W - L + \alpha L) \) will always be positive, so the sign of the derivative is the same as the sign of:

\[
-\left[ \frac{PU''(W - \alpha P)}{PU'(W - \alpha P)} - \frac{\pi LU''(W - L + \alpha L)}{\pi LU'(W - L + \alpha L)} \right]
\]

And this equation is simply \( ARA(W - \alpha P) - ARA(W - L + \alpha L) \).

If insurance is sold at a fair price, in which case the consumption level in the first period is equal to the consumption level in the second period, given a loss occurs. Then, the above derivative will always be zero. From this result, we have the following proposition:

**Proposition 2.3.1.** Changing initial wealth has no effects on the level of optimal insurance if the insurance price is actuarially fair.

If insurance is sold with a positive loading, people will decrease their insurance purchases. As a result, the consumption level in the first period will be higher than the consumption level in the second period, given a loss occurs. So the signs of the derivative will depend on the types of absolute risk aversion the individual exhibits.

**Proposition 2.3.2.** When initial levels of wealth are same across periods, changing them has no effects on/increases/decreases the level of optimal insurance, given the individual has a CARA/IARA/DARA utility.

**Proof.** If the individual has a CARA utility, this equation will always be zero. The marginal benefit and marginal cost always change at the same rate, and the individual will not become more or less risk averse as his wealth level changes, so the amount of
insurance she purchases will remain the same. Under CRRA utility, changing the wealth level would have no effects on the level of optimal insurance.

If the individual has an IARA utility, as the wealth level increases, the marginal cost of purchasing insurance increases at a higher rate than the marginal benefit does, and the individual becomes more risk averse. So optimal insurance increases as one’s wealth level increases.

If the individual has a DARA utility, as the wealth level increases, the marginal cost of purchasing insurance increases at a lower rate than marginal benefit does, and the individual becomes less risk averse. So optimal insurance decreases as wealth level increases. When the wealth level increases, people know they would have more to consume in both periods and become less risk averse, so they will decrease the amount of insurance purchased.

As for the case when initial wealth levels are different across time, the results are more straightforward.

Proposition 2.3.3. Increasing initial wealth in period 1 will increase individual’s insurance purchases while increasing initial wealth in period 2 will decrease her insurance purchases, regardless of the types of risk aversion the individual exhibits.

Proof. Differentiating the first-order condition with respect to \(W_1\), we have:

\[
\frac{d\alpha}{dW_1} = \frac{PU''(W_1 - \alpha P)}{P^2U''(W_1 - \alpha P)} = \frac{1}{P}
\]

which is always positive.

Differentiating the first-order condition with respect to \(W_2\) yields the following:

\[
\frac{d\alpha}{dW_2} = \frac{-\pi LU''(W_2 - L + \alpha L)}{\pi L^2U''(W_2 - L + \alpha L)} = -\frac{1}{L}
\]

which is always negative. 

\[\square\]
The intuition behind this is simple as increasing wealth level in period 1 decreases marginal cost of purchasing insurance, while increasing wealth level in period 2 decreases expected marginal benefit, so the individual will change her decision for insurance purchases accordingly.

Changing the Insurance Price and Its Effects

In previous literature, the effects on the demand for insurance when price changes are often ambiguous due to the uncertainty of the total effect of income effect and substitution effect. Little has been known for the total amount of money spent on protecting against potential risk, either. We want to see whether our model can provide additional insights regarding these topics.

**Proposition 2.3.4.** When insurance is the only way of protection against future risks, individuals always decrease the amount of insurance they purchase as the price of insurance increases.

**Proof.** The partial derivative for optimal insurance with respect to the insurance price premium is:

\[
\frac{d\alpha}{d\lambda} = \frac{\pi LU'(W - \alpha P) - (1 + \lambda)(\pi L)^2\alpha U''(W - \alpha P)}{(1 + \lambda)^2(\pi L)^2U''(W - \alpha P) + \pi L^2U''(W + \alpha L - L)}
\]

which is always negative.

Then what about the total amount spent on insurance? We differentiate it with respect to the insurance price premium and get

\[
\frac{d\alpha(1 + \lambda)\pi L}{d\lambda} = \pi L[\alpha - \frac{1}{\pi L} + \alpha(1 + \lambda)\text{ARA}(W - \alpha P) + \frac{1}{\pi}\text{ARA}(W + \alpha L - L)]
\]

Taking a close look at the above derivative can give us much information. As long as the size of loss is not extremely small, the derivative will be positive; Furthermore, as the loss size goes up, it will become more positive. From this we can have the following proposition:

**Proposition 2.3.5.** Individuals increase the amount of money they spend on insurance when the price of insurance goes up, given the loss size is not extremely small. As the loss size becomes larger, they will increase spending more rapidly.
And this is understandable, because we already know that people purchase less insurance as the price goes up, if they spend the same amount of money on it, the marginal cost of purchasing insurance in the first period stays the same. However, because the amount of insurance purchased is decreased, the marginal benefit in the second period increases. To equalize these two, more money will be spent on insurance.

2.3.2 Model with Both Insurance and Saving/Borrowing

Changing Wealth, Interest Rate, Loss Size, and the Probability of Loss

Let us consider the model with endogenous saving/borrowing, different initial wealth levels in the two periods, positive loading, and positive interest rate. Denote first-order conditions as \( f(\alpha, S, W_1, W_2, R, L, \pi) \) and \( g(\alpha, S, W_1, W_2, R, L, \pi) \), respectively. In order to simplify this notation, we also denote

\[
D = f_\alpha g_S - f_S g_\alpha
\]

which is always positive by second-order condition.

Then, we can obtain the effects of changes in initial wealth levels on the equilibrium value of optimal insurance. Combine the partial second-order derivatives with first-order conditions producing the following lemma:

Lemma 2.3.1.

\[
\frac{d\alpha}{dW_1} = \Phi_1 [ARA(W_2 + RS) - ARA(W_2 - L + \alpha L + RS)]
\]

\[
\frac{d\alpha}{dW_2} = \Phi_2 [ARA(W_2 + RS) - ARA(W_2 - L + \alpha L + RS)]
\]

where \( \Phi_1 \) and \( \Phi_2 \) are positive polynomials.

The proof of this lemma is provided in the Appendix.

Proposition 2.3.6. Increasing the wealth level of either period, or those of both periods will decrease/not change/increase the amount of optimal insurance when the agent’s preference exhibits DARA/CARA/IARA;
Under CARA or DARA, the optimal insurance decreases when interest rate increases, and increases when the potential loss size increases;

Optimal insurance decreases as the probability of loss increases when the individual exhibits CARA or IARA.

Proof. Because we have proved that people purchase less than full insurance with a positive loading, so \((W_2 + RS) > (W_2 + RS - L + \alpha L)\), which implies that the signs of the derivatives depend on the types of absolute risk aversion the agent has. The proofs for the changes of interest rate, potential loss size, and probability of loss are similar. □

This finding also shows that when endogenous saving/borrowing presences, DARA is both the necessary and the sufficient condition for insurance to be an inferior good.

Changing the Insurance Price

First, we examine how people change their insurance decisions when the insurance price changes. The partial derivative of insurance with respect to insurance premium is

\[
\frac{d\alpha}{d\lambda} = \Psi_1\left\{\text{ARA}(W - \alpha P - S)\alpha \pi L[\text{ARA}(W - L + \alpha L + RS)R - \text{ARA}(W + RS)]
\right. \\
- U'(W - \alpha P - S)[\text{ARA}(W - \alpha P - S) + (1 + \lambda)\pi R^2 \text{ARA}(W - L + \alpha L + RS)]
\\
+ R(1 - \pi R(1 + \lambda))\text{ARA}(W + RS)\}
\]

where \(\Psi_1\) is a positive polynomial.

There is a lot to learn from this result. If under CARA and assuming zero interest rate, the derivative is always negative. Even under DARA, the first term is positive but small, so more than likely, it will still be negative. As the loss size goes up, the first term becomes more positive and the second term becomes more negative, so the actual effect depends on the agent’s preference.

Then we study how people change their saving/borrowing decisions. The partial derivative of saving/borrowing with respect to insurance premium is

\[
\frac{dS}{d\lambda} = \Psi_2\left\{\text{ARA}(W - \alpha P - S) + \text{ARA}(W - L + \alpha L + RS)R
\right. \\
- \alpha L[1 - \pi R(1 + \lambda)]\text{ARA}(W - \alpha P - S)\text{ARA}(W - L + \alpha L + RS)\}
\]

where \(\Psi_2\) is a positive polynomial.
Again we can learn a lot from this result. If the loss size is small, people save more/borrow less to protect themselves. But if the loss size is big enough, as the price of insurance becomes unfair, people might save less/borrow more in order to fund their insurance purchases. However, as the price keeps going up, the third term becomes smaller, and eventually people will begin to borrow less/save more to protect themselves against potential future risk.

**Proposition 2.3.7.** Under both CARA and DARA, the total amount of resources people spend to protect themselves against potential future risk always increases as the price of insurance goes up.

**Proof.** The partial derivative of the total amount of money spent for protection against potential risk with respect to insurance premium is

\[
\frac{d[\alpha(1 + \lambda)\pi L + S]}{d\lambda} = \Psi_3\Psi_4[(1 + \lambda)(R - (1 - \pi))A\bar{R}(W - \alpha P - S)
+ R(1 - \pi)A\bar{R}(W - L + \alpha L + RS)]
+ \Psi_5[A\bar{R}(W - L + \alpha L + RS) - A\bar{R}(W + RS)]
\]

where \(\Psi_3, \Psi_4, \Psi_5\) are positive polynomials.

It is clear that under both CARA and DARA, the derivative is always positive. \(\Box\)

So with the ability to transfer wealth across periods, we now have a certain result for the effect of the insurance price changes. Furthermore, as the loss size goes up, the derivative becomes more positive, people increase their spending more rapidly.

### 2.4 Substitution Effect between Insurance and Saving/Borrowing

What is the relationship between purchasing insurance and saving/borrowing?

**Proposition 2.4.1.** When individuals save more and/or borrow less, they decrease both the amount of coverage and the amount of money spent on insurance.

**Proof.** First, we treat saving/borrowing as exogenous, then take the partial derivative of insurance with respect to saving/borrowing, which yields the following:

\[
\frac{d\alpha}{dS} = -\frac{f_S}{f_\alpha}
\]
which is obvious always negative.

As for the actual amount of money they spend on purchasing insurance, we have the following partial derivative:

\[
\frac{d\alpha(1 + \lambda)\pi L}{dS} = -(1 + \lambda)\pi L \frac{f_S}{f_0}
\]

which is also always negative.

So under this circumstance, insurance and saving/borrowing, as the two main ways to protect oneself from future risks, substitute each other.

2.5 Conclusion

The two-period model where people purchase insurance in advance of the time when a potential loss might occur better describes the time structure of real world insurance. In addition, endogenous saving/borrowing enriches people’s options, and enables them to smooth their levels of consumption across periods/states and to achieve a better total outcome.

There are several interesting findings generated from this structure. First of all, we show that Mossin’s Theorem is violated when there are not saving/borrowing options in the world. People purchase less than full insurance even under fair price, and the amount of insurance they purchased decreases as the probability of loss increases. However, people still decrease their insurance purchases when the price becomes unfair.

Once we introduce endogenous saving/borrowing into the model, people have the ability to smooth their levels of consumption across periods and states, and are able to achieve first-best outcome. Mossin’s Theorem is restored for this setting. Then we extend Mossin’s Theorem and proof that people purchase full insurance if and only if both insurance and capital market are "fair" priced. As long as we have actuarially fair insurance and zero interest rate, insurance decision is separable and time impatience only affects the decision for saving/borrowing.

Also, a world with saving/borrowing but not insurance can not provide a complete market analog either. People have to save more/borrow less than they would with insur-
ance purchases. Insurance and saving jointly allows enough degrees of freedom to equalize all three consumption amounts, but without "fair" insurance prices, we might not want to.

The comparative statics show how optimal insurance changes as environment factors (initial wealth in either or both periods, interest rate, loss size and probability of loss) change. Some new and intuitive results are found.

We also show that how insurance, saving/borrowing decisions change when the price of insurance changes, and that those two ways of protection substitute each other.

This paper only considers the type of coinsurance, and assumes there will either be a no loss state or a total loss state. Future research can be done for other types of insurance and more complicated loss settings.
CHAPTER 3
A TWO-PERIOD MODEL THAT EXPLAINS OVER-INSURED, UNDER-SAVING AND OVER-BORROWING AT THE SAME TIME

3.1 Introduction

When facing with risk and uncertainty in the future, there are two main ways to protect oneself: with insurance or with precautionary savings. So it is natural to think these two should move at the same direction in a society. However, it does not appear that way. Pashigian et al. (1966) find that individuals show propensity for lower deductibles than standard theory would suggest. Borch (1983) also shows that traveler, if insure his baggage, tends to take insurance for its full value. It seems like individuals in United States are extremely risk averse and are willing to pay the expensive premium to purchase more coverage. However, Americans are also known for their low level of savings and high level of debts.

Efforts have been made to explain these phenomena. Mossin (1968) provides three explanations for over-insured: Irrational behavior, uncertainty to the evaluation of the property in case of damage, and overestimation of the probability of loss. Ben-Arab et al. (1996) show that consumer who develops consumption habits over time purchases more insurance. Reasons for under-saving and over-borrowing are also provided by previous literature. Thaler (1994) suggest myopia, irrationality, and failures of households to enforce "mental accounting" could be the reasons for the low wealth accumulation. Hubbard et al. (1994) find that social insurance programs discourage saving by households. However, to the best of the authors’ knowledge, there is not an existing literature that explains these two contradictory phenomena together.

The majority of existing insurance literature uses expected utility to define agent’s preference. Others try to study the demand for insurance with habit formation (Ben-Arab
et al. (1996)) or apply prospect theory and reference-dependent preferences introduced by Kahneman and Tversky (1979). Sydnor (2010) shows that an expected utility model yields implausibly large measures of risk parameters for consumers’ choices toward costly low deductibles. Reference-dependent preference is mentioned as one of the potential explanations. Kőszegi and Rabin (2009) prove that reference-dependent preference can help to explain precautionary saving when facing a future uncertainty. However, Barseghyan et al. (2013) argue that probability weighting plays an essential role in explaining the aversion to risk manifested in deductible choices. Schmidt and Zank (2007) propose a linear cumulative prospect theory and use it to study insurance demand, but the only result they get is that the model implies a weakened variant of plunging. Marquis and Holmer (1996) show that model incorporates with prospect theory fits empirical insurance data better than standard expected utility model does.

In this paper, I will propose a two-period model with insurance, saving and borrowing, and incorporate it with the ideas of habit formation and reference-dependent preference. The remaining part of the paper is organized as follow: Section 2 introduces the model and explains why it is formed that way; Section 3 compares the demand for insurance, saving and borrowing with the one generated from the standard expected utility; Section 4 discusses the impacts of a positive interest rate; Section 5 concludes the paper.

3.2 Model

People are endowed with initial wealth $W$ in both periods. To protect themselves against the potential loss $L$ (with the probability of loss equal to $\pi$) in the future, they decide the amount of insurance, $\alpha$, they want to purchase, and how much they want to save or borrow in first period. I specify a person’s expected utility over two periods as:

$$V = U(W - \alpha P - S - X) + \pi U(W - L + \alpha L + S - X) + (1 - \pi) U(W + S - X)$$

where $P$ is the price of insurance, $S$ is saving if positive, borrowing if negative, and $X$ is the agent’s reference point, and is defined as follow:

$$X = \delta \left[ \frac{(W - \alpha P - S) + \pi(W - L + \alpha L + S) + (1 - \pi)(W + S)}{2} \right]$$

$$= \delta \left[ -\frac{\alpha P + \pi(-L + \alpha L)}{2} + W \right]$$
$0 < \delta < 1$ describe how reference-dependent the individual is. The reference point is formed endogenously as a fraction of the expected average level of consumption over two periods. Our specification of reference point combines the ideas of status quo, lagged status quo (or consumption habit), and expectation for future (Kőszegi and Rabin (2006, 2007) construct a reference-dependent utility directly from consumption utility in which the reference point is endogenously determined as rational expectations about outcomes). More specifically, during the first period, the reference point is determined by both the current level of consumption and the expected outcome in the future; During second period, the reference point is determined by both the past consumption and the recent expectation.

The intuition behind is straightforward. As proposed by Friedman (1957), people make their consumption decisions based on their permanent income, not on their temporary income. I dig one step further and ask ourselves, why do people act this way? The two main factors which affect people’s utility and how they make their consumption decisions could be their habits from the past and their expectations for the future. For instance, two students come from two families with different levels of income live in the same dorm room. Even though they share the exact same room, how they feel about the quality of the room might differ according to their different living experience when they grew up in their own families. On the other hand, let us assume these two students are twin brothers, so they share the exact same living experience until they graduate. Upon graduation they both find a job and need to make a loan and purchase a new home. Now, if one of them is going to become a doctor while the other will become a salesman in a mall. Do we expect them to purchase the houses that have similar value and features? The answer should be no. Even thought they have same habit from the past, because their expectations for their respective future differ so much, their purchasing decisions will differ as well. Meanwhile, the one who is going to purchase a fancy big house might begin to feel the dorm room unpleasant, while the one who is going to buy a tiny apartment might start to treasure the room more than before. Our model is able to capture all these feelings and preference of an individual.
3.3 Insurance and Saving/Borrowing Decisions

The first-order conditions with respected to \( \alpha \) and \( S \) are, respectively,

\[
\frac{dV}{d\alpha} = (-P + \delta \frac{P - \pi L}{2})U'(W - \alpha P - S - X) + \pi (L + \delta \frac{P - \pi L}{2})U'(W - L + \alpha L + S - X) \\
+ (1 - \pi)\frac{P - \pi L}{2}U'(W + S - X) \\
= 0
\]

and

\[
\frac{dV}{dS} = -U'(W - \alpha P - S - X) + \pi U'(W - L + \alpha L + S - X) + (1 - \pi)U'(W + S - X) \\
= 0
\]

When insurance price is fair, we can solve the first-order conditions and get the same results as the ones in the model without a reference point: \( \alpha = 1 \) and \( S = -\frac{1}{2} \pi L \). So the individual whose preference is affected by a reference point still purchases full insurance and borrow half of the premium under fair price, and Mossin’s Theorem still holds under this model setting.

More interestingly, what will happen when insurance is sold with a positive loading? From the first-order conditions we can get the following equation:

\[
[P - \delta(P - \pi L)]U'(W - \alpha^* P - S^* - X) = \pi LU'(W - L + \alpha^* L + S^* - X)
\]

And the same equation when there is not a reference point is:

\[
PU'(W - \alpha^{**} P - S^{**}) = \pi LU'(W - L + \alpha^{**} L + S^{**})
\]

where \( \alpha^*, S^* \), and \( \alpha^{**}, S^{**} \) are the optimal levels of insurance and saving or borrowing, with or without a reference point, respectively. Because \( 0 < [P - \delta(P - \pi L)] - \pi L < P - \pi L \), we can get the following result:

\[
0 < U'(W - L + \alpha^* L + S^* - X) - U'(W - \alpha^* P - S^* - X) \\
< U'(W - L + \alpha^{**} L + S^{**}) - U'(W - \alpha^{**} P - S^{**})
\]

As long as the agents are prudent (or, actually, we only need \( U''' \) to be non-negative), we have:

\[
0 < (W - \alpha^* P - S^* - X) - (W - L + \alpha^* L + S^* - X) \\
< (W - \alpha^{**} P - S^{**}) - (W - L + \alpha^{**} L + S^{**})
\] (3.1)
where the reference point will cancel out and we are left with the differences between the levels of consumption in first period and in second period given a loss occurs. Either an increase in insurance purchase or an increase in saving/a decrease in borrowing will help to narrow the gap between these two consumption levels. Notice that while changes in saving/borrowing will have no effects on the reference point, an increase in insurance purchase will decrease it. Recall the first-order condition with respect to saving:

$$\frac{dV}{dS} = -U'(W - \alpha P - S - X) + \pi U'(W - L + \alpha L + S - X) + (1 - \pi)U'(W + S - X) = 0$$

For this condition and the inequality (3.1) to hold at the same time, the individual need to increase his insurance purchases, and decrease his saving/increase his borrowing (compare to what he would do if his preference is not affected by a reference point).

So we can show that individual whose preference is affected by a reference point purchases more insurance, saves less or borrows more. These findings help to explain why people purchase more insurance than the standard expected utility theory suggested, as well as the low level of saving and high level of borrowing in United States.

### 3.4 Positive Interest Rate

Until this point, we only consider the case when interest rate across periods is assumed to be zero. Even though the Fed has been keeping the zero interest-rate policy for several years, we still would like to know what happens when the interest rate is positive. With a non-zero interest rate, we have the following model:

$$V = U(W - \alpha P - S - X) + \pi U(W - L + \alpha L + RS - X) + (1 - \pi)U(W + RS - X)$$

where \( R \) is the interest rate and we assume the rate for saving and borrowing is the same. Now \( X \), the agent’s reference point becomes:

$$X = \delta\left(\frac{W - \alpha P - S}{2} + \pi\left(W - L + \alpha L + \frac{RS}{2} + \frac{(1 - \pi)(W + RS)}{2}\right) + W\right)$$

Notice that with a positive interest rate, saving/borrowing now also affects the level of the reference point. The new fist-order conditions wih respected to \( \alpha \) and \( S \) become,
respectively,
\[
\frac{dV}{d\alpha} = (-P + \delta \frac{P - \pi L}{2})U'(W - \alpha P - S - X) + \pi(L + \delta \frac{P - \pi L}{2})U'(W - L + \alpha L + RS - X) \\
+ (1 - \pi)(\delta \frac{P - \pi L}{2})U'(W + RS - X)
\]
\[= 0\]

and
\[
\frac{dV}{dS} = -(1 + \delta \frac{R - 1}{2})U'(W - \alpha P - S - X) + \pi(R - \delta \frac{R - 1}{2})U'(W - L + \alpha L + RS - X) \\
+ (1 - \pi)(R - \delta \frac{R - 1}{2})U'(W + RS - X)
\]
\[= 0\]

Now assume the individual will purchase full insurance ($\alpha = 1$) with a positive interest rate. Then we have
\[
U'(W - \pi L - S - X) = U'(W + RS - X)
\]

Solve for $S$ we have $S = -\frac{1}{1+R} \pi L$, plug this result into the first order derivative with respect to $S$, we get
\[
\frac{dV}{dS} = [(1 - \delta)(R - 1)]U'(W - \frac{R}{1+R} \pi L - X) > 0
\]

because $R > 1$ and $\delta < 1$. So purchasing full insurance is not an optimal solution with a positive interest rate. The individual will purchase less than full insurance.

Again, more interestingly, when insurance is sold with an unfair price, from the first-order conditions we can get the following equation:
\[
[P - \delta(P - \pi L)]\frac{R + 1}{2R - \delta(R - 1)}U'(W - \alpha^{***} P - S^{***} - X) = \pi LU'(W - L + \alpha^{***} L + S^{***} - X)
\]

where $\alpha^{***}$ and $S^{***}$ are the optimal levels of insurance and saving or borrowing with a positive interest rate, respectively. Because $0 < \frac{R+1}{2R-\delta(R-1)} < 1$, we can get the following result:
\[
0 < U'(W - L + \alpha^* L + S^* - X) - U'(W - \alpha^* P - S^* - X) \\
< U'(W - L + \alpha^{***} L + S^{***} - X) - U'(W - \alpha^{***} P - S^{***} - X) \\
< U'(W - L + \alpha^{**} L + S^{**}) - U'(W - \alpha^{**} P - S^{**})
\]
As long as the agents are prudent (again, we only need $U'''$ to be non-negative), we have:

$$0 < (W - \alpha^*P - S^* - X) - (W - L + \alpha^*L + S^* - X)$$
$$< (W - \alpha^{***}P - S^{***} - X) - (W - L + \alpha^{***}L + S^{***} - X)$$
$$< (W - \alpha^{**}P - S^{**}) - (W - L + \alpha^{**}L + S^{**})$$ (3.2)

When the interest rate is positive, people will save more/borrow less and purchase less insurance. However, they still buy more insurance and save less/borrow more than they do without a reference point. So the result generated from this model is robust. It is consistent with the fundamental economic theory that a positive interest rate should encourage people to save and discourage people to borrow, yet still being able to show that people have the tendencies to over-insured and under-save/over-borrow at the same time.

### 3.5 Risk Aversion

Risk aversion plays an important role in people’s decision making process when dealing with risk. So how will different degrees of risk aversion affect the results we have under this model? Consider an agent who is more risk-averse, and his utility function can be describe as $F(\cdot)$, where $F(\cdot) = g[U(\cdot)]$, and $g(\cdot)$ satisfies the following properties: $g'(\cdot) > 0$, and $g''(\cdot) < 0$.

Mossin’s Theorem still holds, and when insurance is sold with a positive loading, from the first-order conditions we can get the following equation:

$$[P - \delta(P - \pi L)]U'(W - \alpha^rP - S^r - X)g'[U(W - \alpha^rP - S^r - X)]$$
$$= \pi LU'(W - L + \alpha^rL + S^r - X)g'[U(W - L + \alpha^rL + S^r - X)]$$

where $\alpha^r$ and $S^r$, are the optimal levels of insurance and saving or borrowing for the more risk-averse agent, respectively. With similar mathematical analysis as what we did in previous sections, we can get the following result:

$$0 < U'(W - L + \alpha^rL + S^r - X) - U'(W - \alpha^rP - S^r - X)$$
$$< U'(W - L + \alpha^rL + S^*) - U'(W - \alpha^*P - S^*)$$

So under the condition of prudent (or $U'''$ to be non-negative), we can show that a more risk-averse individual will purchase even more insurance and save even less/borrow
even more than a less risk-averse individual will. These finding are in line with what risk aversion theory implies and are consistent with most previous research.

3.6 Conclusion

There are many ways one can protect himself from the future risk. Two most popular ones are through insurance and through saving. It is naturally to assume that if a person, when facing a future risk, is willing to pay extra to buy more insurance than the theory-suggested optimal level, he or she should also be interested in increase/decrease his or her saving/borrowing. However, while the phenomenon of over-insured is prevalent, the level of saving/borrowing is surprisingly low/high in U.S. Previous literature offer various reasons for either of these two phenomena, but no a model or a theory that can explain both at the same time has been provided.

In this paper I proposed a theoretical model which incorporates the ideal of reference dependent preference, and use it to study people’s choices for insurance purchasing and saving/borrowing. I show that under this model, people will purchase more insurance than what a standard expected-utility model suggests, and at the same time decrease/increase their saving/borrowing.

Because a person’s utility is not only decided by his/her current level of consumption, but also by his/her average consumption across periods, the uncertainty in the future hurts more, thus the agent needs to purchase more insurance to protect himself/herself; Meanwhile, in order to keep his/her consumption levels more smooth across periods, the agent also needs to increase saving/decrease borrowing.

People are too complicated to be described by a simple rational expected utility model. The model I proposed is still an over-simplified one, but it can reveal more details in the decision-making process when people are dealing with future risk. More research can be done in the future, experiments and empirical data can be used to further test the robustness of this model.
REFERENCES


APPENDIX A
THE RETURNS OF ASSETS

We have the intertemporal marginal rate of substitution as:

\[ M_{t+1} = \beta \frac{U_{C_{t+1}}(C_{t+1}, X_{t+1})}{U_{C_t}(C_t, X_t)} \]  \hspace{1cm} (A.1)

substitute (1.1) into the equation (A.1), and we notice that the current level of consumption enters the equation for next period level of habit, so we have:

\[
M_{t+1} = \beta \frac{\partial \left[ \frac{C_{t+1}^{1-\gamma}}{X_{t+1}^{1-\gamma}} \right]}{\partial C_{t+1}} + \frac{\partial \beta \frac{C_{t+2}^{1-\gamma}}{X_{t+2}^{1-\gamma}}}{\partial X_{t+2}} \frac{\partial X_{t+2}}{\partial C_{t+1}} \\
M_{t+1} = \beta \frac{\partial \left[ \frac{C_{t+1}^{1-\gamma}}{X_{t+1}^{1-\gamma}} \right]}{\partial C_t} + \frac{\partial \beta \frac{C_{t+1}^{1-\gamma}}{X_{t+1}^{1-\gamma}}}{\partial X_{t+1}} \frac{\partial X_{t+1}}{\partial C_t}
\]

using the utility modification equation (1.2), we can get:

\[
M_{t+1} = \beta \left[ \frac{C_{t+1}^{1-\gamma}}{X_{t+1}^{1-\gamma}} - \beta \delta \frac{C_{t+2}^{1-\gamma}}{X_{t+2}^{1-\gamma}} \right] \\
M_{t+1} = \beta \left[ \frac{C_{t}^{1-\gamma}}{X_{t}^{1-\gamma}} - \beta \delta \frac{C_{t+1}^{1-\gamma}}{X_{t+1}^{1-\gamma}} \right] \hspace{1cm} (A.2)
\]

To maximize agent’s utility, for any assets with return \( R \), we have:

\[ E_t(M_{t+1}R_{t+1}) = 1 \]

Under definition, the returns for equity and risk-free asset are, respectively,

\[ R_{t+1}^e = \frac{P_{t+1}^e + D_{t+1}^e}{P_t^e} \]

and

\[ R_{t+1}^f = \frac{1}{P_t^f} \]

So we have the equations defining the returns of both assets as:

\[
\frac{P_{t+1}^e + D_{t+1}^e}{P_t^e} = \frac{C_{t+1}^{1-\gamma}}{X_{t+1}^{1-\gamma}} - \beta \delta \frac{C_{t+2}^{1-\gamma}}{X_{t+2}^{1-\gamma}} \\
\beta \left[ \frac{C_{t}^{1-\gamma}}{X_{t}^{1-\gamma}} - \beta \delta \frac{C_{t+1}^{1-\gamma}}{X_{t+1}^{1-\gamma}} \right]
\]

and

\[
\frac{1}{P_t^f} = \frac{C_{t+1}^{1-\gamma}}{X_{t+1}^{1-\gamma}} - \beta \delta \frac{C_{t+2}^{1-\gamma}}{X_{t+2}^{1-\gamma}} \\
\beta \left[ \frac{C_{t+1}^{1-\gamma}}{X_{t+1}^{1-\gamma}} - \beta \delta \frac{C_{t+2}^{1-\gamma}}{X_{t+2}^{1-\gamma}} \right]
\]
APPENDIX B

THE EXPRESSION FOR INTERTEMPORAL MARGINAL RATE OF SUBSTITUTION $M_{ij}$

Substitute habit equation (1.2) and Markov chain process, which governs consumption growth rate, into (A.2), we have:

\[
M_{ij} = \frac{\beta}{\sum_{i=1}^{n} \frac{\alpha_{C_{t-1}^{n}(1+\mu)^2+\delta[\lambda_{i}C_{t-1}^{n}C_{t-1}^{a}(1+\mu)]}}{(C_{t})^{\gamma}} - \frac{\beta\delta}{\sum_{i=1}^{n} \frac{\alpha_{C_{t-1}^{n}(1+\mu)^3+\delta[\lambda_{i}\lambda_{j}C_{t-1}^{n}C_{t-1}^{a}(1+\mu)]}}{(C_{t})^{\gamma}}} - \frac{\sum_{i=1}^{n} \Phi_{ji}(C_{t})^{\gamma-1}}{(C_{t})^{\gamma}}}
\]

We assume $C_{t-1} = C_{t-1}^{a}$ and normalize $C_{t-1}^{a} = 1$, and we time $(C_{t})^{\gamma-1}$ on both numerator and denominator, so we can simplify the expression for $M_{ij}$:

\[
M_{ij} = \frac{\beta}{\sum_{i=1}^{n} \frac{\alpha_{(1+\mu)^2+\delta[\lambda_{i}-(1+\mu)]}}{(C_{t})^{\gamma}}} - \frac{\beta\delta}{\sum_{i=1}^{n} \frac{\alpha_{(1+\mu)^3+\delta[\lambda_{i}\lambda_{j}-(1+\mu)^2]}}{(C_{t})^{\gamma}}}
\]

\[
= \frac{\beta}{\sum_{i=1}^{n} \frac{\alpha_{(1+\mu)}}{(C_{t})^{\gamma}}} - \frac{\beta\delta}{\sum_{i=1}^{n} \frac{\alpha_{(1+\mu)^2+\delta[\lambda_{i}(1+\mu)]}}{(C_{t})^{\gamma}}}
\]
APPENDIX C
THE CALCULATION FOR $W_I$

Applying the two-state assumption into (1.6), we have the following system of equations defining $w_i$:

$$w_1 = \sum_{j=1}^{2} \Phi_{1j}(w_j + 1)\lambda_j M_{1j}$$

$$w_2 = \sum_{j=1}^{2} \Phi_{2j}(w_j + 1)\lambda_j M_{2j}$$

Treat $M_{ij}$ as constants for now, this is a system of two linear equations with two unknowns, so we can solve for $w_i$ as functions of $M_{ij}$:

$$w_1 = \frac{\Phi_{11}\lambda_1 M_{11} + \Phi_{12}(w_2 + 1)\lambda_2 M_{12}}{1 - \Phi_{11}\lambda_1 M_{11}}$$

$$w_2 = \frac{\Phi_{21}\lambda_1 M_{21} \left[ \frac{\Phi_{11}\lambda_1 M_{11} + \Phi_{12}\lambda_2 M_{12}}{1 - \Phi_{11}\lambda_1 M_{11}} + 1 \right] + \Phi_{22}\lambda_2 M_{22}}{1 - \Phi_{21}\lambda_1 M_{21} \frac{\Phi_{12}\lambda_2 M_{12}}{1 - \Phi_{11}\lambda_1 M_{11}} - \Phi_{22}\lambda_2 M_{22}}$$

notice that I keep $w_2$ in the expression of $w_1$ for simplicity.
APPENDIX D
CHANGING INITIAL WEALTH AND ITS EFFECTS ON OPTIMAL INSURANCE

First we have the derivatives of optimal insurance over initial wealth levels in the two periods, respectively:

\[
\frac{d\alpha}{dW_1} = \frac{f_{sgw_1} - f_{w_1}g_s}{D}
\]

\[
\frac{d\alpha}{dW_2} = \frac{f_{sgw_2} - f_{w_2}g_s}{D}
\]

Substitute in the second-order derivatives and simplify, we get:

\[
\frac{d\alpha}{dW_1} = \frac{1}{D}\{\pi RLU''[W_1 - \alpha(1 + \lambda)\pi L - S][((1 + \lambda)\pi R - 1)U''(W_2 + RS - L + \alpha L) + ((1 + \lambda)R - (1 + \lambda)\pi R)U''(W_2 + RS)]\} \quad (D.1)
\]

\[
\frac{d\alpha}{dW_2} = \frac{1}{D}\{\pi LU''[W_1 - \alpha(1 + \lambda)\pi L - S][((1 + \lambda)\pi R - 1)U''(W_2 + RS - L + \alpha L) + ((1 + \lambda)R - (1 + \lambda)\pi R)U''(W_2 + RS)]\} \quad (D.2)
\]

From first-order condition with respect to insurance we have:

\[(1 + \lambda)\pi LU'[W_1 - \alpha(1 + \lambda)\pi L - S] = \pi LU'(W_2 + RS - L + \alpha L)\]

Substitute this equation into first-order condition with respect to saving/borrowing, we have:

\[1 - (1 + \lambda)\pi R]U'(W_2 + RS - L + \alpha L) = [(1 + \lambda)R - (1 + \lambda)\pi R]U'(W_2 + RS) \quad (D.3)\]

Divide (D.1) and (D.2) with either side of (D.3) and then time it back. Doing these yields:

\[
\frac{d\alpha}{dW_1} = \Phi_1[ARA(W_2 + RS) - ARA(W_2 - L + \alpha L + RS)]
\]

\[
\frac{d\alpha}{dW_2} = \Phi_2[ARA(W_2 + RS) - ARA(W_2 - L + \alpha L + RS)]
\]

where $\Phi_1$ and $\Phi_2$ are positive polynomials.