FREE INVERSE SEMIGROUPOIDS AND THEIR INVERSE SUBSEMIGROUPOIDS

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ABSTRACT

Semigroupoids are generalizations of semigroups and of small categories. In general, the quotient of a semigroupoid by a congruence is not a semigroupoid and homomorphisms of semigroupoids can also behave badly. We define certain types of congruences and homomorphisms that avoid this problem. We then investigate inverse semigroupoids which are semigroupoids in which each element has a unique inverse. A free inverse semigroupoid has a (symmetric) basis, and it turns out to be unique. Using the immersion of graphs from Stallings folding, we introduce the Stallings kernel. We use this to study the structure of free inverse semigroupoids and their inverse subsemigroupoids. We show that closed inverse subsemigroupoids of a free inverse semigroupoid are to some extent similar to subgroups of a free group. In particular, there are analogues of the Nielsen-Schreier theorem and Howson's theorem. In contrast to the situation in a free group, every finitely generated closed inverse subsemigroupoid of a free inverse semigroupoid F has finite index (whether or not F is finitely generated).

DEDICATION

I am lucky enough to have been given the supportive gift of amazing people in my life, without all of them, this journey would not have been completed. Thank you to...

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LIST OF ABBREVIATIONS AND SYMBOLS

$\lceil x \rceil$	The smallest subgraph containing x
$\equiv$	The unique minimal inverse semigroupoid congruence
$\mathrm{FIS}(\Gamma)$	Free Inverse Semigroupoid of Γ
$\mathrm{FIS}(\Gamma, v)$	Vertex inverse semigroup based at v
$\mathrm{FIS}(\Gamma, V_0)$	Full inverse subsemigroupoid of $\mathrm{FIS}(\Gamma)$ with vertex set $V_0 \subseteq V(\Gamma)$
$\Gamma^{(n)}$	The set of all paths of length n in Γ
$\langle \Delta \rangle$	The subsemigroupoid generated by Δ
$\leq$	Natural partial order
$\leftrightarrow_R$	Symmetric rewriting relation induced by R
$\mathrm{MT}(\Gamma)$	Munn Tree over Γ
$\mathrm{sd}(x)$	Subdivision map
$\searrow$	Elementary Reduction
$\simeq$	The unique minimal groupoid congruence
$\mathrm{St}(v, \Gamma)$	Star of a vertex $v \in V(\Gamma)$
$A(x, y)$	Subsemigroupoid with vertex set $\{x, y\}$
$E(S)$	The set of idempotents of S
$M(\Gamma)$	Finite connected subgraph of Γ
$s(x)$	Source of x
$t(x)$	Target of x
$V(\Gamma)$ or $\Gamma^{(0)}$	Vertex set of Γ
$X^{\uparrow}$	Closure of a subset X

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This journey was one of the most important and valuable experiences in my life.

$$\underset{n\in\mathbb{N}}{(\text{Thank you})^n}, \forall \text{ you} \in \{\text{Awesome People}\}.$$

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CHAPTER 1

INTRODUCTION

The motivation for this research comes from group theory and inverse semigroup theory. Inverse semigroups were introduced in the 1950's by Preston in the UK and Wagner in the former Soviet Union as algebraic analogues of pseudogroups of transformations. Since then there has been much work done on inverse semigroups with some important applications in formal language theory, theoretical computer science, C^*-algebras, tilings, solid-state physics, model theory, and linear logic ([4] and [9]).

Many aspects of geometric and combinatorial group theory can be adapted to inverse semigroups. My research develops the theory of inverse semigroupoids that are defined by weakening the axioms of a groupoid in the same way that the axioms of groups are weakened to define inverse semigroups. In particular, we aim to develop free inverse semigroupoids and highlight some of their connections with inverse semigroups and groups. Additionally, we want to recast the ideas of Stallings' work in a more combinatorial way and apply them more systematically to the inverse subsemigroupoid structure of free inverse subsemigroupoids. We find this approach especially beneficial in expounding the structure of the inverse subsemigroups of inverse semigroups. One objective of this work is to extend the results of Margolis and Meakin [7] to closed inverse subsemigroupoids of a free inverse semigroupoid in general.

We will introduce some notations throughout the paper. The second chapter revisits some definitions and basic properties from inverse semigroups and directed graphs. Then Chapter 3 presents the essentials of inverse semigroupoid theory, concentrating particularly on the congruences and homomorphisms of semigroupoids as our main observation. We first give the definition of a semigroupoid using a directed graph. While semigroupoids are generalizations of semigroups, inverse semigroupoids are generalizations of both inverse semigroups and groupoids. The precise definition of an inverse semigroupoid is given in Chapter 3. The quotient of a semigroupoid by a congruence is not always a semigroupoid, and homomorphisms of semigroupoids and inverse semigroupoids can also behave badly. For example, Lallement's lemma [3] does not generalize completely and homomorphic images of inverse semigroupoids may fail to be inverse semigroupoids. We define certain types of congruences, called *associative congruences*, and homomorphisms that avoid this problem.

By generalizing the definition of the natural partial order on an inverse semigroup, the natural partial order in an inverse semigroupoid is defined. We will be primarily interested in the inverse subsemigroupoids of inverse semigroupoid A that are closed in this natural partial order in a certain sense; see the precise definition in Section 3.3. In the last section of Chapter 3, we give a brief introduction to inverse subsemigroupoids of an inverse semigroupoid, while later we will focus on closed inverse subsemigroupoids.

The free objects in the category of inverse semigroupoids are interesting in themselves and they form the foundation for studying inverse semigroupoids combinatorially, which is discussed in Chapter 4. Starting with a free semigroupoid with involution and using a symmetric rewriting relation, we give presentations of semigroupoids with involution. A free inverse semigroupoid has a (symmetric) basis, which is a graph; and it turns out to be unique. We characterize the idempotents in a free inverse semigroupoid.

For immersions of graphs, we are able to show that the induced homomorphism of free inverse semigroupoids of graphs maps closed subsets to closed subsets. When we look at the special case of free inverse semigroupoids whose basis graphs have a single vertex, we find these are precisely the free inverse semigroups. The structure of the closed inverse subsemigroups of a free inverse semigroup is studied in [7].

Chapter 5 contains detailed canonical forms of free inverse semigroupoids. We show that Munn Tree and Scheiblich normal form both generalize to free inverse semigroupoids. Thus the word problem, whether distinct words may represent the same element, of finitely generated free inverse semigroupoids is solvable. Stallings introduced an extremely useful notion of folding of graphs [17] and used this concept to find the structure of subgroups of a free group. Given a graph Δ, we denote the free inverse semigroupoid on Δ by $\mathrm{FIS}(\Delta)$. In Chapter 6, we incorporate Stallings foldings with our construction of the immersion representing a closed inverse subsemigroupoid of $\mathrm{FIS}(\Delta)$ and the following observation.

Let $f\colon \Gamma \to \Delta$ be any map of graphs. Then there is a smallest equivalence relation R on the graph Γ such that the natural quotient map $\nu\colon \Gamma \to \Gamma/R$ factors through the composition of any finite sequence of foldings prescribed by f. Moreover, f factors through the quotient map $\nu\colon \Gamma \to \Gamma/R$ via a unique map of graphs $g\colon \Gamma/R \to \Delta$ and g is an immersion of graphs (Theorem 6.1.1). We call this equivalence relation R the *Stallings kernel* of f and give a construction of it in Section 6.1. Together with the concept of subdivisions, this leads us to important results in Chapter 7.

In Section 7.1, we show that a closed inverse subsemigroupoid A of $\mathrm{FIS}(\Delta)$ can be represented by an immersion of graphs $g\colon \Gamma \to \Delta$ and a subset V_0 of the vertex set of Γ (Theorem 7.1.2); moreover, this representation is essentially unique (Theorem 7.1.5). In particular, A is isomorphic to the full subsemigroupoid $\mathrm{FIS}(\Gamma, V_0)$ with vertex set V_0 of the free inverse semigroupoid $\mathrm{FIS}(\Gamma)$ (Corollary 7.1.3). Since the full subsemigroupoid $\mathrm{FIS}(\Gamma, V_0)$ with vertex set V_0 of $\mathrm{FIS}(\Gamma)$ is a rather well understood closed inverse subsemigroupoid of $\mathrm{FIS}(\Gamma)$, this representation of A turns out to be very useful.

Note that the fundamental group of a graph Δ based at the vertex v is the full subgroupoid $\pi_1(\Delta, v)$ with vertex set $\{v\}$ of the fundamental groupoid $\pi_1(\Delta)$, and $\pi_1(\Delta, v)$ is a free group. Similarly the full subsemigroupoid $\mathrm{FIS}(\Delta, v)$ with the singleton vertex set $\{v\}$ of the free inverse semigroupoid $\mathrm{FIS}(\Delta)$ is an inverse semigroup. However, $\mathrm{FIS}(\Delta, v)$ need not be a free inverse semigroup; it is easy to see that it is only free when the component of Δ containing v has only one vertex. Nevertheless, these fundamental inverse semigroups of graphs have many properties analogous to those of fundamental groups of graphs. For instance, we have the following analogue of the Nielsen-Schreier theorem: Every closed inverse subsemigroup of the fundamental inverse semigroup $\mathrm{FIS}(\Delta, v)$ of a graph Δ based at a vertex v is itself isomorphic to the fundamental inverse semigroup of some graph based at some vertex (Corollary 7.1.6).

The analogue of Howson's theorem (the intersection of two finitely generated closed inverse subsemigroups of a fundamental inverse semigroup of a graph is also finitely generated, provided that the intersection is nonempty) and the notion of the index of a subgroup of a group generalize in a natural way to closed inverse subsemigroupoids of an inverse semigroupoid; see Section 7.3. The much more general analogue of Howson's thorem is: If A_1 and A_2 are any finitely generated closed inverse subsemigroupoids of a free inverse semigroupoid $\mathrm{FIS}(\Delta)$ and $A_1 \cap A_2$ is nonempty, then the closed inverse subsemigroupoid $A_1 \cap A_2$ is also finitely generated (Corollary 7.2.2).

Finally, we show that some elementary properties about the index of a subgroup hold in general for the index of a closed inverse subsemigroupoid. For instance, the intersection of two closed inverse subsemigroupoids of finite index in an inverse semigroupoid has finite index (Corollary 7.3.8). Also a closed inverse subsemigroupoid of finite index in a finitely generated inverse semigroupoid is itself finitely generated (Corollary 7.3.7). The converse holds in the case of a free inverse semigroupoid: Every finitely generated closed inverse subsemigroupoid of a free inverse semigroupoid F has finite index, whether or not F is finitely generated (Corollary 7.3.6). This is certainly in contrast to the situation in a free group.

CHAPTER 2

BASIC DEFINITIONS AND PRELIMINARY RESULTS

In this chapter, we give a brief account of inverse semigroup theory with its basic properties. We will also establish the notation for later use.

2.1 Inverse Semigroups

We start by defining some elementary concepts related to inverse semigroups using definitions and notations in [4]. Inverse semigroups are defined to be semigroups S satisfying the following two conditions:

1. S is *regular*. This means that for every element $a \in S$ there is an element b, called an *inverse* of a, satisfying $a = aba$ and $b = bab$.

2. The idempotents of S commute.

After inverse semigroups were introduced, Liber [5] proved the following theorem.

Theorem 2.1.1. *A regular semigroup is inverse if and only if every element has a unique inverse.*

Thus, a semigroup S is said to be inverse if for each $x \in S$ there exists a unique element x^{-1} such that

$$x = xx^{-1}x \text{ and } x^{-1} = x^{-1}xx^{-1}.$$

The element of x^{-1} is called the *inverse* of x. An *inverse subsemigroup* of an inverse semigroup is a subsemigroup which is closed under taking inverses. An idempotent in a semigroup is an element e such that $e^2 = e$. The set of idempotents of S is denoted by $E(S)$. Note that inverse semigroups in general have many of idempotents: if $x \in S$ then xx^{-1} and $x^{-1}x$ are both idempotents of S. Thus,

Theorem 2.1.2. *All groups are inverse semigroups, and an inverse semigroup is a group if and only if it has a unique idempotent.*

An inverse semigroup with identity is called an *inverse monoid* and an inverse semigroup with zero is called an *inverse semigroup with zero.*

Each inverse semigroup S is equipped with a natural partial order relation $\leq$ on S defined by

$$\text{for } a, b \in S, a \leq b \text{ if and only if } a = eb \text{ for some } e \in E(S).$$

A *congruence* ρ on a semigroup S is an equivalence relation on S that respects to semigroup multiplication:

$$\text{for } a, b, c, d \in S, \text{ if } a \;\rho\; b \text{ and } c \;\rho\; d \text{ then } ac \;\rho\; bd.$$

Lallement's lemma shows that if ρ is a congruence on a regular semigroup S and $\rho(a)$ is an idempotent in the quotient S/ρ then $a \;\rho\; e$ for some idempotent $e \in S$.

We can formulate this property in terms of homomorphic images.

Proposition 2.1.3. *If $f : S \to T$ is a surjective homomorphism of semigroups and S is an inverse semigroup, then T is an inverse semigroup.*

Proof. It is clear that the image of a regular semigroup is regular, and so T is regular. Let $e_1, e_2 \in E(T)$. By Lallement's lemma, we can find $a_1, a_2 \in E(S)$ such that $f(a_i) = e_i$ for $i = 1, 2$. Since f is a homomorphism and S is an inverse semigroup, $e_1e_2 = e_2e_1$ and so T is an inverse semigroup. □

Thus, if ρ is a congruence on S where S is an inverse semigroup, then S/ρ is an inverse semigroup.

Of particular interest is the congruence σ on an inverse semigroup S defined by $a \;\sigma\; b$ if and only if there exists a $c \leq a, b$ for all $a, b, c \in S$. This congruence is called the minimal group congruence on S. S is called *E-unitary* if and only if σ is idempotent-pure; i.e. given $e \leq a$ where $e \in E(S)$ implies $a \in E(S)$. Let σ be the minimal group congruence on S. Then S/σ is called the maximal group homomorphic image of S.

2.2 Directed Graph

A *directed graph* consists of two sets $\Gamma^{(0)}$, $\Gamma^{(1)}$ and two functions $s, t\colon \Gamma^{(1)} \to \Gamma^{(0)}$. The elements of $\Gamma^{(0)}$ are called *vertices* and the elements of $\Gamma^{(1)}$ are called *edges* of Γ. If $x \in \Gamma^{(1)}$, then the vertices $s(x)$ and $t(x)$ are called the *source* and *target* of x, respectively. For convenience, we identify a directed graph Γ with its edge set, $\Gamma^{(1)}$, and $V(\Gamma) = \Gamma^{(0)}$ for its vertex set. A *sub-directed graph* Σ of a directed graph Γ is a pair of subsets $\Sigma \subseteq \Gamma$ and $V(\Sigma) \subseteq V(\Gamma)$ such that $s(\Sigma) \subseteq V(\Sigma)$ and $t(\Sigma) \subseteq V(\Sigma)$. In this case, Σ, equipped with the restrictions of the source and target maps of Γ, is itself a directed graph.

A *map of directed graphs* is a function $f\colon \Gamma \to \Delta$ between directed graphs consists of two functions $f^{(0)}\colon V(\Gamma) \to V(\Delta)$ and $f = f^{(1)}\colon \Gamma \to \Delta$ that respects sources and targets: $f^{(0)}(s(x)) = s(f(x))$ and $f^{(0)}(t(x)) = t(f(x))$ for all $x \in \Gamma$.

A *path* of length $n \geq 1$ in a directed graph Γ is an n-tuple $(x_1, \ldots, x_n)$ of consecutive edges of Γ; i.e., each $t(x_i) = s(x_{i+1})$. Each vertex $v \in V(\Gamma)$ is a *trivial path* (path of length n=0) in Γ. Paths of length one are identified with the edges of Γ. The source and target maps are extended to paths in Γ: If $p = (x_1, \ldots, x_n)$ is a nontrivial path in Γ, then $s(p) = s(x_1)$ and $t(p) = t(x_n)$. For a trivial path $v \in V(\Gamma)$, we define $s(v) = t(v) = v$.

The set of all paths and the set of all nontrivial paths are denoted respectively by

$$\Gamma^* = \bigcup_{n \geq 0} \Gamma^{(n)} \text{ and } \Gamma^+ = \bigcup_{n > 0} \Gamma^{(n)}$$

where $\Gamma^{(n)}$ is the set of all paths of length n. We equip both of these sets with the partial binary operation of concatenation of consecutive paths. Then Γ^* forms a small category whose identities are the trivial paths, and Γ^+ is a so called semigroupoid, which we will discuss in the next chapter.

CHAPTER 3

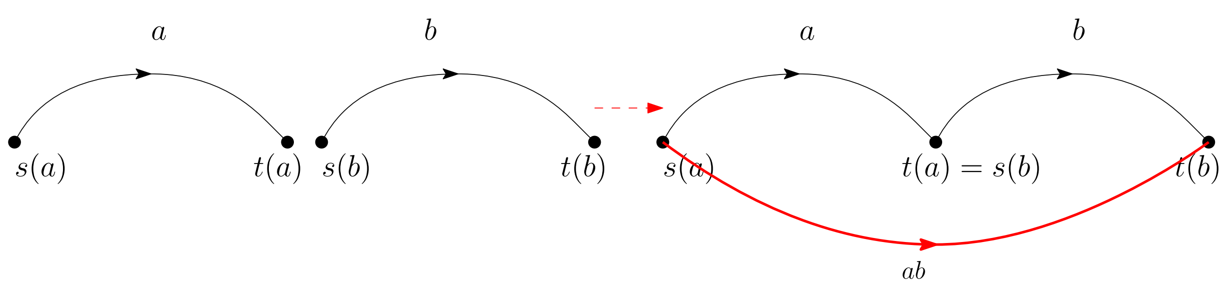
INVERSE SEMIGROUPOID THEORY

In this chapter, we trace the development of inverse semigroupoid theory from semigroupoids to inverse semigroupoids. We also examine congruences and homomorphisms of semigroupoids. In the last section of this chapter, we introduce inverse subsemigroupoids of an inverse semigroupoid, which will be our primary result in this paper.

3.1 Semigroupoids

Semigroupoids are generalizations of semigroups, so that a semigroup can be thought of as a semigroupoid whose underlying graph has a single vertex, and thus its edges are loops at the vertex. A *semigroupoid* is a directed graph A whose edge set is endowed with a partial binary operation $A^{(2)} \to A^{(1)}$ (denoted by juxtaposition $(a, b) \to ab$) satisfying the following conditions: For all $a, b, c \in A$,

1. if $a, b \in A$, then ab is defined if and only if $(a, b) \in A^{(2)}$, the set of consecutive edges of A;



2. if $(a, b) \in A^{(2)}$, then $ab \in A$ with $s(ab) = s(a)$ and $t(ab) = t(b)$;

3. if $(a, b, c) \in A^{(3)}$, then $(ab)c = a(bc)$.

As for any directed graph, we identify a semigroupoid A with its edge set $A = A^{(1)}$ and denote its vertex set by $V(A) = A^{(0)}$. We write $a \in A^{(1)} \Leftrightarrow a \in A$, $(a, b) \in A^{(2)} \Leftrightarrow ab \in A$, and $(a, b, c) \in A^{(3)} \Leftrightarrow abc \in A$.

An *identity* element in a semigroupoid A is an edge e with the properties:

1. $s(e) = t(e)$;

2. if $(a, e) \in A^{(2)}$, then $ae = a$;

3. if $(e, b) \in A^{(2)}$, then $eb = b$.

Note that a small category is precisely a semigroupoid C containing an identity 1_v with $s(1_v) = t(1_v) = v$, for each $v \in V(C)$. We usually identify 1_v with v for all $v \in V(C)$. In relation to groupoid, a groupoid G is a small category in which every element is invertible: for each $g \in G$, there exists $g^{-1} \in G$ such that

$$s(g^{-1}) = t(g), t(g^{-1}) = s(g), gg^{-1} = 1_{s(g)}, \text{ and } g^{-1}g = 1_{t(g)}.$$

A *subsemigroupoid* of a semigroupoid A is a sub-directed graph B which is closed under the partial binary operation on A; that is, $b_1 b_2 \in B$ whenever b_1 and b_2 are consecutive edges of B. Thus, B is itself a semigroupoid whose partial binary operation is the restriction of that on A. The intersection of any family of subsemigroupoids of A is also a subsemigroupoid of A. Hence, given a sub-directed graph Δ of A, the intersection of all subsemigroupoids of A containing Δ is the unique smallest subsemigroupoid of A containing Δ; it is called the *subsemigroupoid of A generated by* Δ and is denoted $\langle \Delta \rangle$. We can see that $\langle \Delta \rangle$ consists of all products $a_1 \cdots a_n \in A$, where $(a_1, \cdots, a_n)$ is a nontrivial path in Δ, and the vertex set of $\langle \Delta \rangle$ is equal to $V(\Delta)$.

We view every subset X of a semigroupoid $A = A^{(1)}$ as a sub-directed graph with vertex set $V(X) = s(X) \cup t(X)$. For instance, if u and v are vertices of A, we write the set $A(u, v)$ of all edges of A with source u and target v as a sub-directed graph with vertex set

$\{u, v\}$. Note that $A(v, v) = A_v$ is the set of edges of a subsemigroupoid of A with a single vertex v, and so A_v is a semigroup; it is called the *vertex semigroup* based at v.

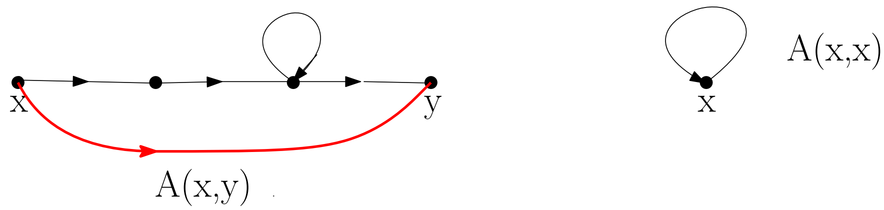


Figure 3.1: $A(x, y)$ is a subsemigroupoid with vertex set $\{x, y\}$ and $A(x, x) = A_x$ is a vertex semigroup based at x

In general, if $V_0 \subseteq V(A)$, then the subset A_{V_0} is a subsemigroupoid of A called the *full subsemigroupoid* of A with vertex set V_0.

The *product semigroupoid* of two semigroupoids A and B is the product directed graph $A \times B$ equipped with this partial binary operation: if $(a_1, b_1), (a_2, b_2) \in A \times B$ and $t(a_1, b_1) = s(a_2, b_2)$, then $(a_1, b_1)(a_2, b_2) = (a_1a_2, b_1b_2)$. Arbitrary products of semigroupoids are defined similarly.

Extending a notion for semigroups, we define a *homomorphism* $\phi \colon A \to B$ to be a map of directed graphs with the property: if $(a, b) \in A^{(2)}$, then $(\phi(a), \phi(b)) \in B^{(2)}$ and $\phi(ab) = \phi(a)\phi(b)$ where A and B are semigroupoids. If we extend $f : \Gamma \to \Delta$ to a map of directed graphs $f : \Gamma^+ \to \Delta^+$ by $f(x_1, \cdots, x_n) = (f(x_1), \cdots, f(x_n))$, we obtain the unique extension of f to a homomorphism of semigroupoids. That is, if p and q are paths in Γ such that $t(p) = s(q)$, then $f(pq) = f(p)f(q)$.

Equivalently, a homomorphism from A to B is a subsemigroupoid ϕ of the product semigroupoid $A \times B$ such that $\phi = \phi^{(1)} \subseteq A \times B$ and $\phi^{(0)} \subseteq V(A) \times V(B)$ are functions.

We will discuss congruences on semigroupoids in Section 3.2.

3.1.1 Free Semigroupoids

In this section, we define a special example of semigroupoids: the free ones. The following construction will be used in proving that free semigroupoids exist. Let Γ be a directed graph and Γ^+ be the set of all nontrivial paths. Then, Γ^+ is made into a semigroupoid, the *free semigroupoid* on Γ, with the partial binary operation of concatenation of consecutive paths:

$$\text{for } p = (x_1, \cdots, x_n) \text{ and } q = (y_1, \cdots, y_n) \text{ in } \Gamma, pq = (x_1, \cdots, x_n, y_1, \cdots, y_n)$$

is the free semigroupoid on Γ by the following theorem.

Theorem 3.1.1. *Let Γ^+ be the free semigroupoid on Γ, A be a semigroupoid, and $f : \Gamma \to A$ be a map of directed graphs. Then there exists a unique semigroupoid homomorphism $\bar{f}$: $\Gamma^+ \to A$ extending f.*

Proof. Every path $p = (x_1, \cdots, x_n)$ in Γ is the concatenation of its consecutive edges: $x_1, \cdots, x_n$. It follows that for a homomorphism $g : \Gamma^+ \to A$ extending f, we must have $g(p) = (f(x_1), \cdots, f(x_n))$, which proves the uniqueness. On the other hand, the map $\bar{f} : \Gamma^+ \to A$ given by

$$\bar{f}(x_1 \cdots x_n) = (f(x_1), \cdots, f(x_n))$$

is a homomorphism that extends f. □

We will write $f = \bar{f}$ throughout for convenience. Since every semigroupoid A is generated by some sub-directed graph $\langle \Delta \rangle$ of A, we have the universal property:

Corollary 3.1.2. *Every semigroupoid is the homomorphic image of a free one.*

3.2 Congruences of Semigroupoids

A *congruence* on a semigroupoid A is an equivalence relation $\rho = \rho^{(1)}$ on A and an equivalence relation $\rho^{(0)}$ on $V(A)$ satisfying the properties:

(1) if $(a, b) \in \rho$, then $(s(a), s(b)) \in \rho^{(0)}$ and $(t(a), t(b)) \in \rho^{(0)}$;

(2) if $(a_1, b_1), (a_2, b_2) \in \rho$ and $t(a_1, b_1) = s(a_2, b_2)$, then $(a_1a_2, b_1b_2) \in \rho$.

A congruence on A is a subsemigroupoid ρ of the product semigroupoid $A \times A$ such that $\rho = \rho^{(1)}$ is an equivalence relation on A and $\rho^{(0)}$ is an equivalence relation on $V(A)$.

Example 3.2.1. Let $\phi : A \to B$ be a homomorphism of semigroupoids. The *kernel* of ϕ is the subsemigroupoid of $A \times A$ given by

$$\ker \phi = \{(a, b) \in A \times A \mid \phi(a) = \phi(b)\}$$

with vertex set

$$V(\ker \phi) = \{(u, v) \in V(A) \times V(A) \mid \phi(u) = \phi(v)\}.$$

Note that $\ker(\phi)$ is an equivalence relation on Γ which is *compatible* with the source and target maps in the following sense: if $(a, b) \in \ker(\phi)$, then $(s(a), s(b)) \in \ker(\phi)$ and $(t(a), t(b)) \in \ker(\phi)$. We can see that the kernel of a homomorphism $\phi\colon A \to B$ of semigroupoids is a congruence on A.

The *quotient* of a semigroupoid A by a congruence ρ is the directed graph A/ρ with vertex set $V(A/\rho) = (A/\rho)^{(0)}$, and the natural map $\rho^\natural\colon A \to A/\rho$ is a map of directed graphs. However, in general it may be impossible to make A/ρ into a semigroupoid such that $\rho^\natural$ is a homomorphism.

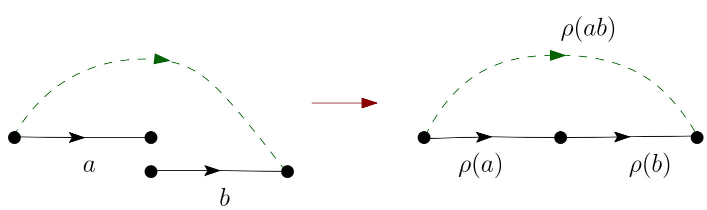


Figure 3.2: There is a pair of consecutive edges $(\rho(a), \rho(b))$ in A/ρ, but no edges at all from $s(a)$ to $t(b)$, and thus no way to define the product $\rho(ab)$.

To avoid this difficulty, we will only be concerned with congruences of the following special type. An *associative congruence* on a semigroupoid A is a congruence ρ on A with the two properties:

(3) $(\rho^\natural \times \rho^\natural)(A^{(2)}) = (A/\rho)^{(2)}$.

(4) The induced partial binary operation on the directed graph A/ρ is associative.

By the *induced partial binary operation* on A/ρ, where ρ is a congruence on A satisfying condition (3), the partial product is defined in the following obvious way. For $(x, y) \in (A/\rho)^{(2)}$, choose $(a, b) \in A^{(2)}$ such that $\rho^\natural(a) = x$ and $\rho^\natural(b) = y$ and define $xy = \rho^\natural(ab)$. This definition does not depend on the choice of $(a, b) \in A^{(2)}$ since ρ is a congruence on A, and so is a well-defined partial binary operation on A/ρ. However, this partial binary operation on A/ρ is not generally associative, and therefore we also need to assume condition (4) in order to guarantee that A/ρ is a semigroupoid. It is clear that, by an associative congruence ρ on A, the induced partial binary operation on A/ρ is the only way to make A/ρ into a semigroupoid such that the natural map $\rho^\natural \colon A \to A/\rho$ is a homomorphism of semigroupoids.

Homomorphisms of semigroupoids can also behave badly in general. For instance, the image of a homomorphism $\phi\colon A \to B$ is not always a subsemigroupoid of B because there may be edges $a_1, a_2 \in A$ such that $\phi(a_1)$ and $\phi(a_2)$ are consecutive edges of B, but no edge of A is mapped to the product $\phi(a_1)\phi(a_2)$. With homomorphisms that satisfy the analog of condition 3 above, we do not have this difficulty and we have the following first isomorphism theorem for semigroupoid homomorphisms of this special type.

Lemma 3.2.2. *Let $\phi\colon A \to B$ be a homomorphism of semigroupoids such that $\phi[A^{(2)}] = [\phi(A)]^{(2)}$. Then $\phi(A)$ is a subsemigroupoid of B, $\rho = \ker(\phi)$ is an associative congruence on A, and the unique map $\phi'\colon A/\rho \to B$ such that $\phi'\rho^\natural = \phi$ is an injective homomorphism of semigroupoids with image $\phi(A)$, and consequently $A/\rho \cong \phi(A)$.*

$$\begin{array}{ccc} A & \xrightarrow{\ \phi\ } & B \\ {\scriptstyle \rho^\natural}\downarrow & \nearrow{\scriptstyle \phi'} & \\ A/\rho & & \end{array}$$

Proof. Let $(b_1, b_2) \in [\phi(A)]^{(2)}$. By hypothesis, there exists $(a_1, a_2) \in A^{(2)}$ such that $\phi(a_1, a_2) = (b_1, b_2)$. Now $a_1a_2 \in A$ and $\phi(a_1a_2) = b_1b_2$, and so $b_1b_2 \in \phi(A)$. It follows that $\phi(A)$ is a subsemigroupoid of B.

Next let $(y_1, y_2) \in (A/\rho)^{(2)}$. Then $(\phi'(y_1), \phi'(y_2)) \in [\phi(A)]^{(2)}$, and so there exists $(a_1, a_2) \in A^{(2)}$ such that $\phi(a_1, a_2) = (\phi'(y_1), \phi'(y_2))$ by the hypothesis on ϕ. Since $\rho^{\natural}(a_i) = y_i$, it follows that $\rho^{\natural}[A^{(2)}] = [A/\rho]^{(2)}$. Furthermore, $\phi'(y_1 y_2) = \phi(a_1 a_2) = \phi(a_1)\phi(a_2) = \phi'(y_1)\phi'(y_2)$. Thus, ϕ' preserves the induced partial product on A/ρ and is an isomorphism of directed graphs from A/ρ onto $\phi(A)$. Since the partial product on $\phi(A)$ is associative, it follows that the induced partial product on A/ρ is also associative. □

3.3 Inverse Semigroupoids

An inverse semigroupoid can be thought of as an inverse semigroup in which multiplication is only required to be partially defined. Let A be a semigroupoid. An element $a \in A$ is called a *regular element* if there exists an edge $b \in A$ such that $s(b) = t(a)$, $t(b) = s(a)$, and $aba = a$. In this case, the element $c = bab$ has the stronger property that $aca = a$ and $cac = c$. Note that $c \in A$ with this latter property is called an *inverse* of a. Thus, an element $a \in A$ is regular if and only if it has an inverse. The set of regular elements of A is denoted $\mathrm{Reg}(A)$. We say that A is a *regular semigroupoid* if every edge of A is regular.

An *idempotent* in A is an element e such that $s(e) = t(e)$ and $e^2 = e$. Note that an idempotent is regular, and it is an inverse of itself. The set of all idempotents of A is denoted $E(A)$.

If $\phi\colon A \to B$ is a homomorphism of semigroupoids, then $\phi(\mathrm{Reg}(A)) \subseteq \mathrm{Reg}(B)$ and $\phi(E(A)) \subseteq E(B)$.

A classical result of [5] and [11] for semigroups can be easily extended to semigroupoids:

Lemma 3.3.1. *If A is a regular semigroupoid, then each pair of idempotents in $A^{(2)}$ commutes if and only if every edge of A has a unique inverse.*

Proof. Suppose each pair of idempotents in $A^{(2)}$ commutes and there exist two distinct inverses of $a \in A$, say $b, c \in A$, such that $s(b) = s(c) = t(a)$, $t(b) = t(c) = s(a)$. Then

ac, ab, ba, ca are idempotents in A, and so

$$ab = (aca)b = (ac)(ab) = (ab)(ac) = (aba)c = ac, \text{ and}$$

$$ba = b(aca) = (ba)(ca) = (ca)(ba) = c(aba) = ca.$$

Thus, $c = c(ac) = c(ab) = (ca)b = (ba)b = b$.

Now assume that every element of A has a unique inverse and $e_1, e_2 \in E(A)$ with $s(e_1) = t(e_2)$. We first show that $e_1e_2, e_2e_1 \in E(A)$. Let a be an inverse of e_1e_2 where

$$e_1e_2ae_1e_2 = e_1e_2 \text{ and } ae_1e_2a = a.$$

Then $(e_2ae_1)(e_2ae_1) = e_2(ae_1e_2a)e_1 = e_2ae_1$ and so $e_2ae_1 \in E(A)$. However,

$$e_1e_2(e_2ae_1)e_1e_2 = e_1e_2ae_1e_2 = e_1e_2 \text{ and}$$

$$e_2ae_1(e_1e_2)e_2ae_1 = e_2ae_1e_2ae_1 = e_2ae_1.$$

Since inverses are unique, $e_1e_2 = (e_1e_2)^{-1} = e_2ae_1 \in E(A)$. Similarly, $e_2e_1 \in E(A)$. Thus, $e_1e_2 = e_1^{-1}e_2^{-1} = (e_2e_1)^{-1} = e_2e_1$. Therefore, we conclude that each pair of idempotents in $A^{(2)}$ commutes. □

Consequently, an inverse semigroupoid is a semigroupoid A with the property: for each element $a \in A$, there is a unique element $a^{-1} \in A$ (called the *inverse* of a) with $s(a^{-1}) = t(a)$, $t(a^{-1}) = s(a)$ such that $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$. Due to the uniqueness of inverses, the following properties hold for an inverse semigroupoid A:
for all $a \in A^{(1)}$ and $(a, b) \in A^{(2)}$,

$$s(a^{-1}) = t(a),\ t(a^{-1}) = s(a),\ (a^{-1})^{-1} = a,\ (ab)^{-1} = b^{-1}a^{-1}.$$

It also follows that aa^{-1} and $a^{-1}a$ are idempotents and that $e^{-1} = e$ if e is an idempotent. We call A, with the above property, a *semigroupoid with involution.* Moreover, if A and B are inverse semigroupoids and $\phi : A \to B$ is a homomorphism of semigroupoids, then ϕ is involution-preserving: $\phi(a^{-1}) = \phi(a)^{-1}$ for $a \in A$. We will discuss this further in

Chapter 4.

Unfortunately, some important properties of inverse semigroups do not always hold for inverse semigroupoids. For instance, Lallement's lemma [3], stated in Section 2.1, does not generalize completely, and homomorphic images of inverse semigroupoids may fail to be inverse semigroupoids. Congruences on inverse semigroupoids also require more special attention. This is a simple counterexample.

Example 3.3.2. The following (Figure 3.3) is a surjective homomorphism $\phi\colon A \to B$ of semigroupoids such that A is regular (in fact, an inverse semigroupoid), however $\phi[E(A)] \neq E(B)$. Define $\phi(a) = c$ and $\phi(a^{-1}) = d$, and the multiplication in each semigroupoid is explained in the table.

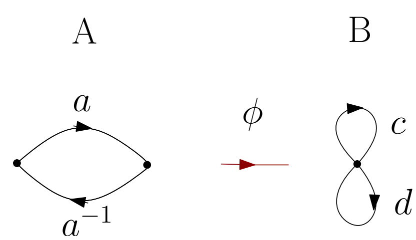


Figure 3.3: A surjective homomorphism $\phi\colon A \to B$ of semigroupoids such that A is regular, however $\phi[E(A)] \neq E(B)$

Table 3.1: Relations of elements of A and B

$*$	a	a^{-1}
a	Undefined	aa^{-1}
a^{-1}	$a^{-1}a$	Undefined

$*$	c	d	cd	dc
c	c	cd	cd	c
d	dc	d	d	dc
cd	c	cd	cd	c
dc	dc	d	d	dc

We obtain $A = \{a, a^{-1}, aa^{-1}, a^{-1}a\}$ and $B = \{c, d, cd, dc\}$. However, $E(A) = \{aa^{-1}, a^{-1}a\}$ and $E(B) = \{c, d, cd, dc\}$; therefore, $\phi[E(A)] \neq E(B)$.

A partial generalization for semigroupoids of Lallement's lemma is the following.

Lemma 3.3.3. *Let $\phi\colon A \twoheadrightarrow B$ be a surjective homomorphism of regular semigroupoids such that $\phi[A^{(2)}] = B^{(2)}$. Then $\phi[E(A)] = E(B)$.*

Proof. Let $e \in E(B)$. Since $(e, e) \in B^{(2)}$, there exists $(a_1, a_2) \in A^{(2)}$ such that $\phi(a_1) = \phi(a_2) = e$ (by the assumption on ϕ). Choose an inverse b of a_1a_2 in A. Then a_2ba_1 is an edge in A such that $(a_2ba_1)^2 = a_2b(a_1a_2)ba_1 = a_2ba_1$ and $\phi(a_2ba_1) = e\phi(b)e = e^2\phi(b)e^2 = \phi(a_1a_2ba_1a_2) = \phi(a_1a_2) = e^2 = e$. Thus $E(B) \subseteq \phi[E(A)]$ and the result follows, since clearly $\phi[E(A)] \subseteq E(B)$. □

Note that the additional assumption imposed here on ϕ in our extension of Lallement's lemma is a condition on the underlying map of directed graphs. In the case when A and B are semigroups, the condition automatically holds and the lemma reduces to Lallement's lemma for regular semigroups.

Recall that an immediate consequence of Lallement's lemma is that the homomorphic image of an inverse semigroup is an inverse semigroup (Proposition 2.1.3); equivalently, if ρ is a congruence on an inverse semigroup S, then S/ρ is an inverse semigroup. The following simple example shows that the corresponding statement for inverse semigroupoids is false, even under the hypothesis of Lemma 3.3.3.

Example 3.3.4. We define a surjective homomorphism $\phi\colon A \to B$ of semigroupoids such that A is an inverse semigroupoid as shown in Figure 3.4 with relations of elements that as described in Table 3.2, and $\phi[A^{(2)}] = B^{(2)}$ where $\phi(a) = c, \phi(a^{-1}) = c, \phi(b) = d, \phi(b^{-1}) = d$. By Lemma 3.3.3, $\phi[E(A)] = E(B)$. However B is not an inverse semigroupoid since the inverse of each element is not unique.

Next, we introduce some definitions. A *crossing* in a directed graph Γ is an element $(y_1, y_2) \times (z_1, z_2)$ of $\Gamma^{(2)} \times \Gamma^{(2)}$ such that $t(y_1) = t(z_1)$; i.e., a pair of paths of length two in Γ that cross at their middle vertices. Let $f\colon \Gamma \to \Delta$ be a map of directed graphs. If $c = (y_1, y_2) \times (z_1, z_2)$ is a crossing of Γ, then $(f(y_1), f(y_2)) \times (f(z_1), f(z_2))$ is a crossing of Δ,

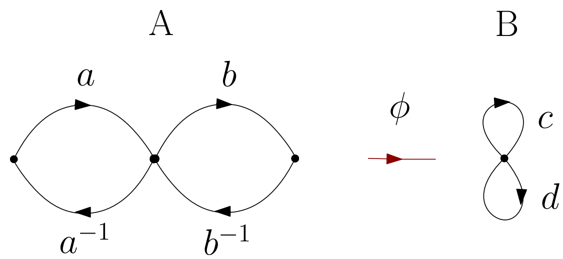


Figure 3.4: A surjective homomorphism $\phi\colon A \to B$ of semigroupoids such that A is an inverse semigroupoid and $\phi[A^{(2)}] = B^{(2)}$.

Table 3.2: Relations of elements of A and B

$*$	a	a^{-1}	b	b^{-1}
a	Undefined	aa^{-1}	ab	Undefined
a^{-1}	$a^{-1}a$	Undefined	Undefined	Undefined
b	Undefined	Undefined	Undefined	bb^{-1}
b^{-1}	Undefined	$b^{-1}a^{-1}$	bb^{-1}	Undefined

$*$	c	d	cd	dc
c	c	cd	cd	c
d	dc	d	d	dc
cd	c	cd	cd	c
dc	dc	d	d	dc

denoted by $f(c)$. We say that f is *crossings-surjective* if for each crossing d in Δ, there is a crossing c in Γ such that $f(c) = d$.

Theorem 3.3.5. *Let $\phi\colon A \twoheadrightarrow B$ be a surjective homomorphism of semigroupoids. If A is an inverse semigroupoid and ϕ is crossings-surjective, then B is an inverse semigroupoid.*

Proof. Since $B = \phi(A) = \phi[\mathrm{Reg}(A)] \subseteq \mathrm{Reg}(B)$, we see that B is regular. It remains to show that composible idempotents in B commute. Let $e, f \in E(B)$ such that $s(e) = s(f)$. Then $(e, e) \times (f, f)$ is a crossing in B, and so it is the image of a crossing $(a_1, a_2) \times (b_1, b_2)$ in A. As in the proof of Lemma 3.3.3, if c, d are inverses of a_1a_2, b_1b_2 in A, then $a_2ca_1, b_2db_1 \in E(A)$ with $\phi(a_2ca_1) = e$, $\phi(b_2db_1) = f$. Furthermore, the idempotents a_2ca_1, b_2db_1 are based at the same vertex of A, namely the common middle vertex of the paths (a_1, a_2) and (b_1, b_2). However, A is an inverse semigroupoid and so these idempotents in A commute. Therefore, their images e and f in B also commute. □

An inverse semigroupoid A has a natural partial order defined in the same way as for an inverse semigroup: for $a, b \in A^{(1)}$, we write $a \leq b$ if there exists an idempotent $e \in E(A)$ such that $a = eb$ with $s(a) = s(e) = s(b)$ and $t(a) = t(b)$.

Lemma 3.3.6. *Let A be an inverse semigroupoid. For $a, b \in A^{(1)}$, the following conditions are equivalent to $a \leq b$:*

(1) $a = aa^{-1}b$;

(2) $a = bf$ for some $f \in E(A)$;

(3) $a = ba^{-1}a$.

Proof. First note that if $a \leq b$, then $s(a) = s(b)$ and $t(a) = t(b)$.

To show (1)⇒(2): $a = aa^{-1}b = (aa^{-1})(bb^{-1})b = b(b^{-1}aa^{-1}b)$. However, it is clear that $b^{-1}aa^{-1}b \in E(A)$ and so $a = bf$ for some $f \in E(A)$. Then (2)⇒(3), since $a^{-1}a \in E(A)$, we can choose $f = a^{-1}a$. Thus, $a = ba^{-1}a$. Lastly, (3)⇒(1) $aa^{-1}b = (ba^{-1}a)(a^{-1}ab^{-1})b = b(a^{-1}a)(b^{-1}b) = bb^{-1}ba^{-1}a = ba^{-1}a = a$. □

Additionally, $\leq$ is a partial order on A, called the *natural partial order* on A with these additional properties:

(i) if $a \leq b$, then $a^{-1} \leq b^{-1}$;

(ii) if $a_1 \leq b_1$ and $a_2 \leq b_2$, and if $t(a_1) = s(a_2)$, then $a_1 a_2 \leq b_1 b_2$.

Lemma 3.3.7. *Let A be an inverse semigroupoid. If $a, b \in A$ and $e \in E(A)$ such that $t(a) = s(e) = s(b)$, then $ae \leq a$ and $aeb \leq ab$.*

Proof. Note that $ae = a(a^{-1}a)e = ae(a^{-1}a) = (aea^{-1})a$ and $aea^{-1} \in E(A)$. So $ae \leq a$. Then, $aeb = (aea^{-1})ab$ and so $aeb \leq ab$. □

We define E-unitary inverse semigroupoids analogously to the definition for inverse semigroups. We will show free inverse semigroupoid is an E-unitary inverse semigroupoid in Chapter 4.

3.4 Inverse Subsemigroupoids of Inverse Semigroupoids

Let A be an inverse semigroupoid. An *inverse subsemigroupoid* of A is a subsemigroupoid B that contains the unique inverse of each of its elements. If $\langle X \rangle$ is the inverse subsemigroupoid of A generated by X, then $\langle X \rangle$ consists of all products $x_1 \cdots x_n \in A$, where $(x_1, \cdots, x_n)$ is a path in $X^{\pm 1} = \{a \in A \mid a \in X \text{ or } a^{-1} \in X\}$. So the vertex set of X is equal to $V(X)$.

The *closure* of a subset X of an inverse semigroupoid A is the subset

$$X^{\uparrow} = \{a \in A \mid x \leq a \text{ for some } x \in X\}.$$

We say that X is closed if $X = X^{\uparrow}$. If B is an inverse subsemigroupoid of A, then so is $B^{\uparrow}$.

We see that if X is a subset of A, then $X^{\uparrow}$ is the smallest closed subset of A containing X in the sense:

1. $X \subseteq X^{\uparrow}$,
2. $X^{\uparrow}$ is a closed subset of A, and
3. if Y is closed subset of A such that $X \subseteq Y$, then $X^{\uparrow} \subseteq Y$.

Lemma 3.4.1. *Let A be an inverse semigroupoid with vertices v_1, v_2, and v_3. If $X \subseteq A(v_1, v_2)$ and $Y \subseteq A(v_2, v_3)$, then*

$$(X^{\uparrow}Y^{\uparrow})^{\uparrow} = (X^{\uparrow}Y)^{\uparrow} = (XY^{\uparrow})^{\uparrow} = (XY)^{\uparrow}.$$

Proof. It is clear that $(XY)^{\uparrow} \subseteq (X^{\uparrow}Y)^{\uparrow} \subseteq (X^{\uparrow}Y^{\uparrow})^{\uparrow}$ and $(XY)^{\uparrow} \subseteq (XY^{\uparrow})^{\uparrow} \subseteq (X^{\uparrow}Y^{\uparrow})^{\uparrow}$. Hence, we only need to show $(X^{\uparrow}Y^{\uparrow})^{\uparrow} \subseteq (XY)^{\uparrow}$. Let $x \in X^{\uparrow}$ and $y \in Y^{\uparrow}$. Then there exists $e_1, e_2 \in E(A)$ such that $e_1x \in X$ and $e_2y \in Y$. Thus, $e_1xe_2y \in XY$, but $e_1xe_2y \leq e_1xy \leq xy$ by Lemma 3.3.7, so $xy \in (XY)^{\uparrow}$. It follows that $X^{\uparrow}Y^{\uparrow} \subseteq (XY)^{\uparrow}$ and hence $(X^{\uparrow}Y^{\uparrow})^{\uparrow} \subseteq (XY)^{\uparrow}$. □

Lemma 3.4.2. *If B is an inverse subsemigroupoid of an inverse semigroupoid A, then $B^{\uparrow}$ is also an inverse subsemigroupoid of A.*

Proof. Let $a_1, a_2 \in B^{\uparrow}$ such that $t(a_1) = s(a_2)$. Then there exists $b_i \in B$ such that $b_i \leq a_i$, $i = 1, 2$. By properties of the natural partial order on A, $t(b_1) = s(b_2)$ and $b_1 b_2 \leq a_1 a_2$. Since $b_1 b_2 \in B$, it follows that $a_1 a_2 \in B^{\uparrow}$. Moreover, $b_1^{-1} \leq a_1^{-1}$ and $b_1^{-1} \in B$. So $a_1^{-1} \in B^{\uparrow}$. Hence $B^{\uparrow}$ is an inverse subsemigroupoid of A. □

Additionally, $V(B^{\uparrow}) = V(B)$. In particular, the closure of an inverse subsemigroupoid with a singleton vertex set is a closed inverse subsemigroup of A.

Lemma 3.4.3. *Let H and K be closed inverse subsemigroupoids of inverse semigroupoids A_{v_1} and A_{v_2} respectively, where A is an inverse semigroupoid and $v_1, v_2 \in V(A)$. If $a \in A(v_1, v_2)$, then the following are equivalent:*

1. *$a^{-1}Ha \subseteq K$ and $aKa^{-1} \subseteq H$;*

2. *$aa^{-1} \in H$ and $(a^{-1}Ha)^{\uparrow} = K$;*

3. *$a^{-1}a \in K$ and $(aKa^{-1})^{\uparrow} = H$.*

Proof. To show (1)⇒(2): Let $e \in E(K)$. Then $aea^{-1} \leq aa^{-1}$ and so $aa^{-1} \in H$ since H is closed. Now, let $k \in K$. Then $aka^{-1} \in H$ and $(a^{-1}a)k(a^{-1}a) = a^{-1}(aka^{-1})a \in a^{-1}Ha$. So $k \in (a^{-1}Ha)^{\uparrow}$.

To show (2)⇒(3): Let $e \in E(H)$. Then $a^{-1}ea \in K$. But K is closed and so $a^{-1}a \in K$. Next, since H is closed, $aKa^{-1} = a(a^{-1}Ha)^{\uparrow}a^{-1} \subseteq H^{\uparrow} = H$ and thus $(aKa^{-1})^{\uparrow} \subseteq H$. Now let $h \in H$. Then $a^{-1}ha \in K$ and $(aa^{-1})h(aa^{-1}) = a(a^{-1}ha)a^{-1} \in aKa^{-1}$ and so $h \in (aKa^{-1})^{\uparrow}$. Thus, $H \subseteq (aKa^{-1})^{\uparrow}$.

To show (3)⇒(1): Let $h \in H$. Then $h = aka^{-1}$ for some $k \in K$. Then, $a^{-1}ha = (a^{-1}a)k(a^{-1}a) \in K$ since $a^{-1}a \in K$. Hence $a^{-1}Ha \subseteq K$. It is straightforward to see that $aKa^{-1} \subseteq H$. □

We say that two closed inverse subsemigroups H and K of an inverse semigroupoid A are *conjugate* if there exists $a \in A$ such that $a^{-1}Ha \subseteq K$ and $aKa^{-1} \subseteq H$. Although it is easy to see that conjugation is an equivalence relation on the set of closed inverse subsemigroups of A, we will show the proof and write $H \approx K$ to indicate that H is conjugate to K as in [7].

Lemma 3.4.4. *$\approx$ is an equivalence relation on the set of closed inverse subsemigroups of an inverse semigroupoid A.*

Proof. The reflexive condition is obvious. To show that it is symmetric, let H and K be closed inverse subsemigroupoids of A. Suppose that $H \approx K$. Then there exists x such that $xKx^{-1} \subseteq H$ and $x^{-1}Hx \subseteq K$. Choose $y = x^{-1}$, so we have $yHy^{-1} \subseteq K$ and $y^{-1}Ky \subseteq H$. Hence, $H \approx K$. Now, let L also be a closed inverse semigroup and $yLy^{-1} \subseteq K$ and $y^{-1}Ky \subseteq L$ for $y \in A$. If there exists x such that $xKx^{-1} \subseteq H$ and $x^{-1}Hx \subseteq K$, we have $xy \in A$ such that $(xy)L(xy)^{-1} \subseteq H$ and $(xy)^{-1}H(xy) \subseteq L$. □

The next lemma shows that the closure operation behaves as expected with homomorphisms.

Lemma 3.4.5. *Let A, B be inverse semigroupoids and let $\phi : A \to B$ be a homomorphism. The following properties hold:*

1. *If $X \subseteq A$, then $\phi(X^{\uparrow}) \subseteq \phi(X)^{\uparrow}$.*

2. *If $Y \subseteq B$, then $\phi^{-1}(Y)^{\uparrow} \subseteq \phi^{-1}(Y^{\uparrow})$.*

3. *If Y is a closed subset of B, then $\phi^{-1}(Y)$ is closed in A.*

Proof. Let $a \in X^{\uparrow}$ and choose $e \in E(A)$ such that $ea \in X$. Then $\phi(e)\phi(a) = \phi(ea) \in \phi(X)$. Since $\phi(e) \in E(B), \phi(a) \in \phi(X)^{\uparrow}$. Thus, (1) holds. (2) follows by applying (1) to $X = \phi^{-1}(Y)$, and (3) follows immediately from (2). □

For any inverse semigroupoid A and a subset $V_0 \subseteq V(A)$, the *full semigroupoid* A_{V_0} with vertex set V_0 is an inverse subsemigroupoid of A. Moreover, assume that $b \in A_{V_0}^{\uparrow}$ and $a = eb \in A_{V_0}$ for $e \in E(A)$. Then $s(b) = t(e) = s(e) = s(a) \in V_0$, $t(b) = t(a) \in V_0$, and so $b \in A_{V_0}$. Hence $A_{V_0}^{\uparrow} = A_{V_0}$ and A_{V_0} is a closed inverse subsemigroupoid of A.

CHAPTER 4

FREE INVERSE SEMIGROUPOIDS

In Section 3.3, we saw that an inverse semigroupoid is an example of a semigroupoid with involution. In this chapter, we want to construct presentations of semigroupoids with involution and a free inverse semigroupoid from a graph Γ. Further, the idempotents in $\text{FIS}(\Gamma)$ are described. We revisit the concepts of coverings and immersions of graphs from [17] and show some of the results. We shall see in the next chapters how these results can be used to obtain the structure of closed inverse subsemigroupoids of a free inverse semigroupoid, and more generally, of an inverse semigroupoid.

4.1 Semigroupoids with Involution

Our goal in this section is to build a generating graph for any semigroupoid with involution, which is a semigroupoid A equipped with a function that assigns an edge a^{-1} to each edge $a \in A$ satisfying the properties:

$$s(a^{-1}) = t(a), t(a^{-1}) = s(a), (a^{-1})^{-1} = a, (bc)^{-1} = c^{-1}b^{-1}$$

for all $a \in A$ and all $(b, c) \in A^{(2)}$. We say that a homomorphism $\phi : A \to B$ is *involution-preserving* if $\phi(a^{-1}) = \phi(a)^{-1}$ for each edge $a \in A$ where A and B are semigroupoids with involution. A *congruence* ρ on A is *involution-preserving* if for all edges $a, b \in A$, $(a^{-1}, b^{-1}) \in \rho$ whenever $(a, b) \in \rho$.

4.1.1 Graphs

A *graph* is a directed graph Γ with an involution on its edge set, $x \mapsto x^{-1}$, with the properties:

$$s(x^{-1}) = t(x), t(x^{-1}) = s(x), (x^{-1})^{-1} = x, \text{ and } x^{-1} \neq x.$$

The edge x^{-1} is called the *inverse* of x. A *map of graphs* $f : \Gamma \to \Delta$ is an involution-preserving map of directed graphs. We extend the involution on a graph Γ to paths in Γ as follows: If $p = (x_1, \cdots, x_2)$ is a path in Γ, then $p^{-1} = (x_n^{-1}, \cdots, x_1^{-1})$. With this involution, Γ^+ becomes a semigroupoid with involution $p \mapsto p^{-1}$ and satisfies these properties:

$$s(p^{-1}) = t(p), t(p^{-1}) = s(p), (p^{-1})^{-1} = p, \text{ and } (pq)^{-1} = q^{-1}p^{-1}$$

whenever p, q are consecutive paths in Γ. Thus, a map of graphs $f : \Gamma \to \Delta$ induces an involution-preserving homomorphism $f : \Gamma^+ \to \Delta^+$.

4.1.2 Presentations of Semigroupoid with Involution

Let Γ be a graph (nonempty and without isolated vertices). For any binary relation R on Γ^+, we say that R is *rigid* if for all $(p, q) \in R$, we have $s(p) = s(q)$ and $t(p) = t(q)$. We say that R is *involution preserving* if $(p^{-1}, q^{-1}) \in R$ whenever $(p, q) \in R$.

Let ρ be a rigid congruence on Γ^+. We define the *quotient semigroupoid* Γ^+/ρ in the obvious way: The vertex set of Γ^+/ρ is $V(\Gamma^+) = V(\Gamma)$ and its edge set is the set of congruence classes of ρ. Denote the congruence class of path p by $\bar{p}$ and define $s(\bar{p}) = s(p)$, $t(\bar{p}) = t(p)$. Since ρ is rigid, it is clear that these operations are well-defined. Moreover, paths p and q in Γ are consecutive if and only if $\bar{p}$ and $\bar{q}$ are consecutive; and in this case we define $\bar{p}\bar{q} = \overline{pq}$. The natural map $\rho^\natural : \Gamma^+ \to \Gamma^+/\rho$, given by $p \mapsto \bar{p}$ on the edge sets and the identity map on the vertex sets, is clearly a homomorphism of semigroupoids. If in addition ρ is involution-preserving, then Γ^+/ρ is a semigroupoid with involution (given by $\bar{p}^{-1} = \overline{p^{-1}}$) and $\rho^\natural$ is involution-preserving.

Now let R be any rigid binary relation on Γ^+. We want to find a unique minimal congruence on Γ^+ containing R. The symmetric rewriting relation $\leftrightarrow_R$ induced by R on Γ^+ is defined as follows:

If $p, q \in \Gamma^+$, then $p \leftrightarrow_R q$ if and only if q is obtained from p by replacing a subpath p' of p by q' for some $(p', q') \in R \cup R^{-1}$.

Note that $\leftrightarrow_R$ is a rigid binary relation on Γ^+ and so is its reflexive transitive closure

$\leftrightarrow_R^*$. It is straightforward to check that $\leftrightarrow_R^* = R^\sharp$ is a congruence on Γ^+, and so we have the following.

Lemma 4.1.1. *Let Γ be a graph and let R be a rigid binary relation on Γ^+. Then the congruence $R^\sharp$ is also rigid and a pair of paths (p, q) is in $R^\sharp$ if and only if there is a finite sequence*

$$p = p_0 \leftrightarrow_R p_1 \leftrightarrow_R \cdots \leftrightarrow_R p_n = q$$

carrying p to q. Moreover, if in addition R is involution-preserving, then so is $R^\sharp$.

If Γ is a graph and R is an involution-preserving rigid binary relation on Γ^+, we say that the pair (Γ, R) is a *presentation* for the semigroupoid with involution $\Gamma^+/R^\sharp$. By a *generating graph* for a semigroupoid with involution A, we mean a graph Γ and an involution-preserving map of directed graphs $\alpha : \Gamma \to A$ such that:

- the map on the vertex sets $\alpha^{(0)} : \Gamma^{(0)} \to A^{(0)}$ is a bijection and
- the induced homomorphism $\alpha : \Gamma^+ \to A$ is surjective.

In this case, the kernel of $\alpha : \Gamma^+ \to A$ is an involution-preserving rigid congruence on Γ^+ given by

$$\ker \alpha = \{(p, q) \in \Gamma^+ \times \Gamma^+ \mid \alpha(p) = \alpha(q)\}.$$

So A is isomorphic to $\Gamma^+/\ker \alpha$. In this situation, if $\alpha : \Gamma \to A$ is a choice of generating graph for a semigroupoid A and R is any involution-preserving rigid binary relation on Γ^+ such that $R^\sharp = \ker \alpha$, then (Γ, R) is a presentation of A. We say that A is finitely generated if it has a finite generating graph, and that A is finitely presented if it has a presentation (Γ, R) where Γ and R are both finite.

Another observation regarding congruences is to consider the congruence μ on Γ given by

$$\mu = \{(p, q) \in \Gamma^+ \times \Gamma^+ \mid s(p) = s(q) \text{ and } t(p) = t(q)\}.$$

This gives us $\rho \subseteq \mu$ for any rigid congruence ρ on Γ^+ and the quotient Γ^+/μ is not only an inverse semigroupoid but also a groupoid.

We say that a rigid congruence ρ is an *inverse semigroupoid congruence* if the quotient Γ/ρ is an inverse semigroupoid. There is a unique minimal inverse semigroupoid congruence on Γ^+ where the congruence $\equiv$ is the intersection of all inverse semigroupoid congruences on Γ^+. Furthermore, the quotient of Γ^+ by $\equiv$ is the *free inverse semigroupoid* on Γ, denoted $\mathrm{FIS}(\Gamma)$ with the property: If A is an inverse semigroupoid and $f : \Gamma \to A$ is an involution-preserving map of directed graphs, then there is a unique extension of f to a homomorphism (with the same symbol) $f : \mathrm{FIS}(\Gamma) \to A$. This will be discussed further in the next chapter.

Likewise, ρ is a *groupoid congruence* if the quotient Γ/ρ is a groupoid, and there is a unique minimal groupoid congruence $\simeq$ on Γ^+ where $\Gamma^+/\simeq$ is a groupoid and $\simeq$ is contained in ρ, for all groupoid congruences ρ on Γ^+. The quotient of Γ by $\simeq$ is the *fundamental groupoid* of Γ, denoted $\pi_1(\Gamma)$. For example, let R be the binary relation on Γ^+ containing all pairs $(xx^{-1}x, x)$ and (xx^{-1}, yy^{-1}) for all $x, y \in \Gamma$ with $s(x) = s(y)$. Then (Γ, R) is a presentation for $\pi_1(\Gamma)$.

4.2 Construction of Free Inverse Semigroupoids

We construct free inverse semigroupoids from a graph Γ. Consider the subset $R \subseteq \Gamma^+ \times \Gamma^+$ consisting of all pairs $(xx^{-1}x,\ x)$ where $x \in \Gamma$, and all pairs $(pp^{-1}qq^{-1},\ qq^{-1}pp^{-1})$, where $p \in \Gamma^+$ with $s(p) = s(q)$. Thus if $(p, q) \in R$ then $s(p) = s(q)$, $t(p) = t(q)$, and $(p^{-1},\ q^{-1}) \in R$. Then $\equiv$, the congruence on Γ^+ generated by R, is rigid and involution-preserving. Furthermore, $\equiv$ is contained in $\simeq$, the homotopy congruence. That is, for all $p,\ q \in \Gamma^+$, if $p \equiv q$ then $p \simeq q$.

Lemma 4.2.1. *If $p \in \Gamma^+$, then $pp^{-1}p \equiv p$.*

Proof. If p has length 1, then $(pp^{-1}p,\ p) \in R$ and the property holds. Assume $p = x_1 \cdots x_n$ is a path of length $n \geq 2$.

By induction on n,

$$
\begin{aligned}
pp^{-1}p &\equiv (x_1 \cdots x_n)(x_1 \cdots x_n)^{-1}(x_1 \cdots x_n) \\
&\equiv x_1 \cdots x_n x_n^{-1} \cdots x_1^{-1} x_1 \cdots x_n \\
&\equiv x_1(x_2 \cdots x_n)(x_n^{-1} \cdots x_2^{-1})(x_1^{-1}x_1)x_2 \cdots x_n \\
&\equiv x_1(x_1^{-1}x_1)(x_2 \cdots x_n)(x_n^{-1} \cdots x_2^{-1})(x_2 \cdots x_n) \\
&\equiv x_1(x_2 \cdots x_n)(x_n^{-1} \cdots x_2^{-1})(x_2 \cdots x_n) \\
&\equiv \cdots \equiv x_1 \cdots x_n = p.
\end{aligned}
$$

□

Lemma 4.2.2. *Let $p \in \Gamma^+$. Then the following conditions are equivalent*

(1) p is null-homotopic

(2) $p \equiv pp^{-1}$

(3) $p^2 \equiv p$

Proof. To show (1) $\Rightarrow$ (2), suppose p is null-homotopic. Then $p = p_1 x x^{-1} p_2$, where $x \in \Gamma$ and p_1 and p_2 are paths in Γ such that $p_1 p_2$ is null-homotopic. If p_1 and p_2 are trivial paths, then $p = xx^{-1} \equiv (xx^{-1}x)x^{-1} = xx^{-1}(xx^{-1})^{-1} = pp^{-1}$. If p_1 and p_2 are not both trivial paths, then p_1^{-1} and p_2^{-1} are also not both trivial paths. By induction, we have $p_2^{-1}p_1^{-1} \equiv p_2^{-1}p_1^{-1}p_1p_2$.

By Lemma 4.2.1 and this fact, we obtain

$$
\begin{aligned}
p = p_1xx^{-1}p_2 &\equiv p_1xx^{-1}p_2(p_1xx^{-1}p_2)^{-1}p_1xx^{-1}p_2 \\
&= p_1(xx^{-1})(p_2p_2^{-1})(xx^{-1})(p_1^{-1}p_1)(xx^{-1})p_2 \\
&\equiv p_1xx^{-1}xx^{-1}p_2(p_2^{-1}p_1^{-1}p_1p_2) \\
&\equiv p_1xx^{-1}xx^{-1}p_2(p_2^{-1}p_1^{-1}) \\
&\equiv p_1xx^{-1}(xx^{-1})(p_2p_2^{-1})p_1^{-1} \\
&\equiv (p_1xx^{-1}p_2)(p_2^{-1}xx^{-1}p_1^{-1}) = pp^{-1}.
\end{aligned}
$$

(2) $\Rightarrow$ (3): By Lemma 4.2.1, $p^2 = pp \equiv pp^{-1}pp^{-1} \equiv pp^{-1} \equiv p$.

(3) $\Rightarrow$ (1): Suppose $p^2 \equiv p$. Then , $p^2 \simeq p$. Since $\Gamma^+/\simeq$ is a groupoid, p is null-homotopic.

□

Let $p = (x_1, \cdots, x_n) \in \Gamma^+$, we denote the smallest subgraph of Γ containing p by $\lceil p \rceil = \lceil x_1 \rceil \cup \cdots \cup \lceil x_n \rceil$. We will see this notation again in Chapter 5.

Lemma 4.2.3. *Let Γ be a graph and $p_1, p_2 \in \Gamma^+$. If $p_1 \equiv p_2$, then $\lceil p_1 \rceil = \lceil p_2 \rceil$.*

Proof. This is clear when $p_1 \leftrightarrow_R p_2$. The general case then follows from Lemma 4.2.1. Thus, $\lceil p_1 \rceil = \lceil p_2 \rceil$. □

Given a path $p \in \Gamma^+$, we denote the $\equiv$-class of p by $\bar{p}$ and we write FIS(Γ) for the quotient semigroupoid $\Gamma^+/\equiv$. As an immediate consequence of Lemma 4.2.3, we see that the natural map $\Gamma^+ \to$ FIS(Γ) restricts to an embedding of Γ into FIS(Γ). Identifying Γ with its image in FIS(Γ) under the natural map, $\Gamma \subseteq$ FIS(Γ) and the vertex set of FIS(Γ) is equal to $V(\Gamma) = V(\Gamma^+)$. The following result states that FIS(Γ) is the *free inverse semigroupoid* on Γ

Theorem 4.2.4. *Let Γ be a graph. Then* FIS(Γ) *is an inverse semigroupoid with the property: Every involution-preserving map of directed graphs $f : \Gamma \to A$, where A is an inverse semigroupoid, has a unique extension to a homomorphism $f :$* FIS(Γ) $\to A$.

Proof. By Lemma 4.2.1, FIS(Γ) is a regular semigroupoid. Then every idempotent of FIS(Γ) is the $\equiv$-class of some path in Γ of the form pp^{-1} by Lemma 4.2.2. Thus, by the defining relation, idempotents based at the same vertex of Γ commute. Hence FIS(Γ) is an inverse semigroupoid.

Now let A be an inverse semigroupoid and $f : \Gamma \to A$ be an involution-preserving map of directed graphs. Since A is an inverse semigroupoid, $R \subseteq \ker f$. So $\equiv\ \subseteq \ker f$. Therefore $f : \Gamma^+ \to A$ factors through the natural map $\Gamma^+ \to$ FIS(Γ) giving us the desired homomorphism (denoted by the same symbol) $f :$ FIS(Γ) $\to A$. Since the image of Γ in FIS(Γ) is a generating set, the homomorphism is unique. □

As a result, FIS(Γ) equipped with the natural map $\Gamma \to$ FIS(Γ) is the free inverse semigroupoid on Γ where the vertex set $V(\Gamma) = V(\Gamma^+)$. For any vertex v of Γ, FIS(Γ, v) is the vertex semigroup of FIS(Γ) based at v and we call it the *vertex inverse semigroup* of Γ based at v.

Let $f : \Gamma \to \Delta$ be a map of graphs. We also see by Theorem 4.2.4, f extends to a unique homomorphism f : FIS(Γ) $\to$ FIS(Δ) and it is given by $f(\bar{p}) = \overline{f(p)}$ for all $p \in \Gamma^+$. Writing FIS(f) for the induced homomorphism, we can see that FIS is a functor from the category of graphs to the category of inverse semigroupoids.

Let Σ be a subgraph of a graph Γ. If p and q are paths in Σ, then $p \equiv q$ in Γ if and only if $p \equiv q$ in Σ by definition of $\equiv$. Thus, the inclusion map $i : \Sigma \hookrightarrow \Gamma$ induces a monomorphism i : FIS(Σ) $\to$ FIS(Γ) by identifying FIS(Σ) with its image in FIS(Γ). We make the following observation for future reference.

Lemma 4.2.5. *Let Γ be a graph, Σ_1 and Σ_2 be connected subgraphs of Γ, and v is a vertex of $\Sigma_1 \cap \Sigma_2$. Then* FIS(Σ_1) = FIS(Σ_2) *if and only if $\Sigma_1 = \Sigma_2$.*

Proof. Suppose FIS(Σ_1) = FIS(Σ_2) and p_1 is a path in Σ_1 with source v. Then there is a path $p_2 \in \Sigma_2$ with source v such that $p_1 \equiv p_2$. By Lemma 4.2.3, $\lceil p_1 \rceil = \lceil p_2 \rceil \subseteq \Sigma_2$. Thus, $\Sigma_1 \subseteq \Sigma_2$. Similarly, $\Sigma_2 \subseteq \Sigma_1$. The converse is clear. □

Recall that an inverse semigroupoid A is called E-unitary if and only if given $e \leq a$ where $e \in E(A)$ implies $a \in E(A)$ with $s(e) = s(a)$ and $t(e) = t(a)$. A model of free inverse semigroups derived from [14] and the solution of the word problem, which will be discussed in Chapter 5, have been used to prove that a free inverse semigroup is E-unitary [4]. On the other hand, our description of idempotents in free inverse semigroupoids enables us to get the following straightforward result.

Proposition 4.2.6. *Let Γ be a graph. Then* FIS(Γ) *is E-unitary.*

Proof. Let $a \in$ FIS(Γ) and $e \leq a$ for $e \in E\big($FIS(Γ)$\big)$. By the Lemma 3.3.6, $ae \equiv e$. Since e is null-homotopic and $ae \simeq e$, a is also an idempotent of FIS(Γ). □

Consequently, $E(\mathrm{FIS}(\Gamma))$ is a closed inverse subsemigroupoid of $\mathrm{FIS}(\Gamma)$.

4.3 Coverings and Immersions of Graphs

Before we discuss coverings and immersion of graphs, we state a definition from Stallings [17] for any directed graph Γ. The *star* of a vertex $v \in V(\Gamma)$ is the set of edges of Γ with source v, denoted

$$\mathrm{St}(v,\Gamma) = \{x \in \Gamma \mid s(x) = v\}.$$

Note that if $x \in \mathrm{St}(v,\Gamma)$ then x^{-1} is not within the $\mathrm{St}(v,\Gamma)$ unless both source and target is v as seen in Figure 4.1.

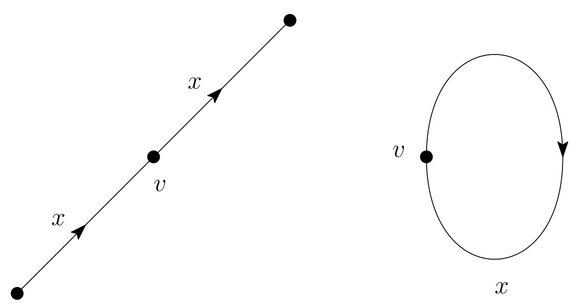


Figure 4.1: The inverse of an element in $\mathrm{St}(v,\Gamma)$.

Example 4.3.1. Here are examples of the elements of the star of a vertex in a graph Γ.

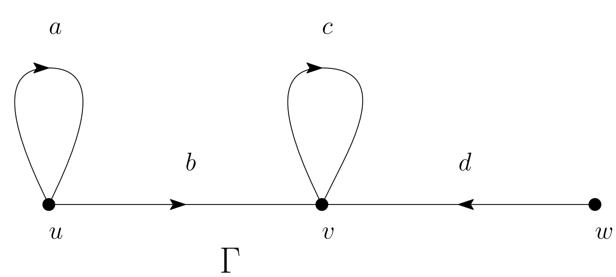


Figure 4.2: $\mathrm{St}(u,\Gamma) = \{a, a^{-1}, b\}, \mathrm{St}(v,\Gamma) = \{c, c^{-1}, b^{-1}, d^{-1}\}, \mathrm{St}(w,\Gamma) = \{d\}$.

If $f : \Gamma \to \Delta$ is a map of directed graphs, then f restricts to a function $f_v : \mathrm{St}(v,\Gamma) \to \mathrm{St}(f(v),\Delta)$. We say that f is *locally injective* if each f_v is injective for each vertex v of Γ; *locally surjective* if each f_v is surjective, and *locally bijective* if each f_v is bijective. In the case of graphs, a locally injective map of graphs is called an *immersion* and a locally bijective map is called a *covering*.

Example 4.3.2. A simple example is described as shown in Figure 4.3.

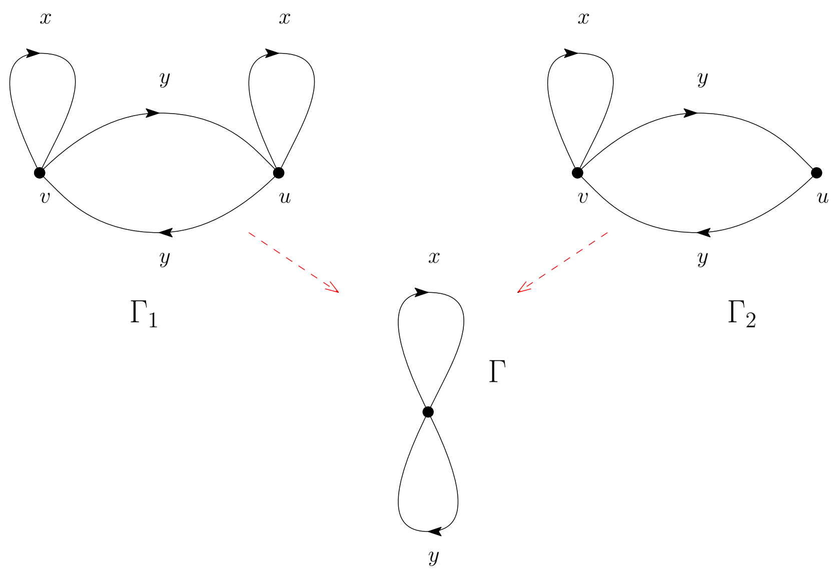


Figure 4.3: The map from the graph Γ_1 to the graph Γ is a covering and the map from the graph Γ_2 to the graph Γ is an immersion.

The following observations turn out to be useful later.

Proposition 4.3.3. *Let $f : \Gamma \to \Delta$ be map of graphs. If f is an immersion, then the induced homomorphisms*

$$f : \Gamma^+ \to \Delta^+ \text{ and } f : \pi_1(\Gamma) \to \pi_1(\Delta)$$

are also locally injective. Likewise, if f is locally surjective, then so are these two induced homomorphisms.

Proof. Suppose v is a vertex of Γ, $p = (x_1, \cdots, x_n), q = (y_1, \cdots, y_n) \in \Gamma^+$ with $s(p) = s(q) = v$, and $f(p) = f(q)$. Then, $s(x_1) = s(y_1)$ and $f(x_1) = f(y_1)$. Since f is an immersion, $x_1 = y_1$. It now follows that the paths $(x_2, \cdots, x_n)$ and $(y_2, \cdots, y_n)$ also have equal source in Γ and image in Δ. By induction on length of paths, $p = q$. Thus $f : \mathrm{St}(v, \Gamma^+) \to \mathrm{St}(f(v), \Delta^+)$ is injective.

By identifying each element of $\pi_1(\Gamma)$ with its unique reduced path, we can regard $\mathrm{St}(v, \pi_1(\Gamma)) \subseteq \mathrm{St}(v, \Gamma^+))$ and $\mathrm{St}(f(v), \pi_1(\Delta)) \subseteq \mathrm{St}(f(v), \Delta^+)$. Since an immersion maps reduced paths to reduced paths, it follows that $f_v : \mathrm{St}(v, \pi_1(\Gamma)) \to \mathrm{St}(f(v), \pi_1(\Delta))$ is the restriction map, and so it is injective as well.

Hence $f : \Gamma^+ \to \Delta^+$ and $f : \pi_1(\Gamma) \to \pi_1(\Delta)$ are locally injective. The second part also follows using induction on length on paths. □

Notice that the induced homomorphism of groups $f : \pi_1(\Gamma, v) \to \pi_1(\Delta, f(v))$ is a restriction of the map $f_v : \mathrm{St}(v, \pi_1(\Gamma)) \to \mathrm{St}(f(v), \pi_1(\Delta))$. Thus, it follows from Proposition 4.3.3 that $f : \pi_1(\Gamma, v) \to \pi_1(\Delta, f(v))$ is a monomorphism whenever $f : \Gamma \to \Delta$ is an immersion.

Corollary 4.3.4. *Let $f : \Gamma \to \Delta$ be an immersion of graphs. If Γ is connected and v is a vertex of Γ such that $f(\pi_1(\Gamma, v)) = \pi_1(\Delta, f(v))$, then $f : \Gamma \to \Delta$ is an embedding.*

Proof. Let v_1 and v_2 be vertices in Γ where $f(v_1) = f(v_2)$. Since Γ is connected, there exists p with source v_1 and target v_2. So $f(p)$ is a loop in Δ. Again, by connectedness of Γ, f is surjective on fundamental groups and so there exists $q \in \Gamma$ such that $s(q) = v_1$ and $f(q) \simeq f(p)$. But, $f_v : \mathrm{St}(v, \pi_1(\Gamma)) \to \mathrm{St}(f(v), \pi_1(\Delta))$ is injective. By Proposition 4.3.3, $p \simeq q$ and $v_1 = t(q) = t(p) = v_2$. Therefore, $f : \Gamma \to \Delta$ is injective on vertices.
Let x_1 and x_2 be edges in Γ and $f(x_1) = f(x_2)$. Then, $f(s(x_1)) = s(f(x_1)) = s(f(x_2)) = f(s(x_2))$ and similarly with the target, and thus $s(x_1) = s(x_2)$. Since f is an immersion, $x_1 = x_2$. Therefore, f is also injective on edges. □

When $f : \Gamma \to \Delta$ is a covering of graphs, it follows from Proposition 4.3.3 that $f : \Gamma^+ \to \Delta^+$ and $f : \pi_1(\Gamma) \to \pi_1(\Delta)$ are locally bijective. This is equivalent to the path-lifting and homotopy lifting properties of covering of graphs, which leads us to the next section.

4.3.1 Universal Covering of Connected Graphs

The universal covering of a connected graph Γ, denoted $\tilde{f} : \widetilde{\Gamma} \to \Gamma$, is the unique covering of Γ with $\widetilde{\Gamma}$ connected and $\pi_1(\widetilde{\Gamma})$ trivial. See [17] for other basic theory of coverings of graphs. Now let $f : \Gamma \to \Delta$ be a map of graphs. Then f lifts to a map of the covering graphs $\tilde{f} : \widetilde{\Gamma} \to \widetilde{\Delta}$. That is, there is a commutative diagram of maps of graphs

$$\begin{array}{ccc} \widetilde{\Gamma} & \xrightarrow{\tilde{f}} & \widetilde{\Delta} \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle q_2} \\ \Gamma & \xrightarrow{f} & \Delta \end{array}$$

where the vertical maps are universal coverings. Recall the construction of such map $\tilde{f}$: Choose base points x_0, y_0 in $V(\widetilde{\Gamma}), V(\widetilde{\Delta})$ respectively such that $f(q_1(x_0)) = q_2(y_0)$. For each $x \in V(\widetilde{\Gamma})$, choose a path α in $\widetilde{\Gamma}$ from x_0 to x. Then $f(q_1(\alpha))$ is a path in Δ with source $f(q_1(x_0)) = q_2(y_0)$. Hence there is a unique lift to a path β in $\widetilde{\Delta}$ starting at y_0 such that $q_2(\beta) = f(q_1(\alpha))$. Define $\tilde{f}(x) = t(\beta)$. Since any two paths in $\widetilde{\Gamma}$ from x_0 to x are homotopic, it follows that the definition of $\tilde{f}(x)$ does not depend on the choice of α and thus is well-defined. Extend this map of the vertex sets to a map of graphs $\tilde{f} : \widetilde{\Gamma} \to \widetilde{\Delta}$ as follows. If y is an edge of $\widetilde{\Gamma}$, then $f(q_1(y))$ is in the star of the vertex $f(q_1(x)) = q_2(\tilde{f}(x))$ of Δ, where $x = s(y)$. Since q_2 is a covering, there is a unique edge z in $\mathrm{St}(\tilde{f}(x), \widetilde{\Delta})$ such that $q_2(z) = f(q_1(y))$. Define $\tilde{f}(y) = z$. It is easy to check that $\tilde{f}$ is a map of graphs and that it is the unique lift of f such that $\tilde{f}(x_0) = y_0$.

Lemma 4.3.5. *let $f : \Gamma \to \Delta$ be a map of graphs.*

(1) If f is an immersion, then $\tilde{f}$ is an embedding.

(2) If f is locally surjective, then $\tilde{f}$ is surjective.

Proof. We will show the proof of (1) as the proof of (2) can be done with a similar argument.

Note that $\tilde{f}$ is also an immersion. Let $v_1 \neq v_2$ and let p be the reduced path from v_1 to V_2. Then $f(p)$ is a reduced path, so not a loop.

Let x_1 and x_2 be edges in $\widetilde{\Gamma}$ and $\tilde{f}(x_1) = \tilde{f}(x_2)$. Since $\tilde{f}$ is a map of graphs, $\tilde{f}(s(x_1)) = s(\tilde{f}(x_1)) = s(\tilde{f}(x_2)) = \tilde{f}(s(x_2))$. So $\tilde{f}(x_1), \tilde{f}(x_2)$ is in the star of the vertex $f(q_1(x)) = q_2(\tilde{f}(x))$ of Δ, where $x = s(x_1) = s(x_2)$. Thus, $f(q_1(x_1)) = q_2(\tilde{f}(x_1)) = q_2(\tilde{f}(x_2)) = f(q_1(x_2))$. Since $\tilde{f}$ is the unique lift of f and f is an immersion, it follows that $x_1 = x_2$ and f is also injective on edges. Therefore $\tilde{f}$ is an embedding. □

The converse of (1) is true, but the converse of (2) is false in general. However, we have the partial converse from both (1) and (2): If f is a covering, then $\tilde{f}$ is an isomorphism of graphs.

4.3.2 Immersion of Graphs

In this section, we assume $f : \Gamma \to \Delta$ is an immersion of graphs. The following is a correspondent for immersions of graphs of the covering homotopy theorem:

Lemma 4.3.6. *Let $p \in \Gamma^+$ and $q = f(p) \in \Delta^+$. If $q \equiv q'$, then there exists $p' \in \Gamma^+$ such that $p \equiv p'$ and $f(p') = q'$.*

Proof. It is enough to show this for one step $q \leftrightarrow q'$. We have three cases.

Case 1: $q = q_1 y q_2$ and $q' = q_1 y y^{-1} y q_2$ where y is an edge of Δ. Write $p = p_1 x p_2$ such that $f(p_1) = q_1, f(x) = y$, and $f(p_2) = q_2$. Choose $p' = p_1 x x^{-1} x p_2$. Then $p \equiv p'$ and $f(p') = q'$.

Case 2: $q = q_1 y y^{-1} y q_2$ and $q' = q_1 y q_2$ where y is an edge of Δ. Write $p = p_1 x_1 x_2 x_3 p_2$ such that $f(p_1) = q_1, f(x_1) = y, f(x_2) = y^{-1}, f(x_3) = y$ and $f(p_2) = q_2$. But f is an immersion, so $x_1 = x_3$. Also, $s(x^{-1}) = s(x_2)$ and $f(x_1^{-1}) = f(x_1)^{-1} = y^{-1} = f(x_2)$. Thus, $x_1^{-1} = x_2$. We can rewrite $p = p_1 x_1 x_1^{-1} x_1 p_2$. Choose $p' = p_1 x_1 p_2$. Then $p \equiv p'$ and $f(p') = q'$.

Case 3: $q = q_1 q_2 q_2^{-1} q_3 q_3^{-1} q_4$ and $q' = q_1 q_3 q_3^{-1} q_2 q_2^{-1} q_4$. Write $p = p_1 p_2 p_3 p_4 p_5 p_6$ such that $f(p_1) = q_1, f(p_2) = q_2, f(p_3) = q_2^{-1}, f(p_4) = q_3, f(p_5) = q_3^{-1}$ and $f(p_6) = q_4$. Note that $s(p_2^{-1}) = s(p_3)$, $s(p_4^{-1}) = s(p_5)$, $f(p_2^{-1}) = q_2^{-1} = f(p_3)$, and $f(p_4^{-1}) = q_3^{-1} = f(p_5)$. By Proposition 4.3.3, $p_2^{-1} = p_3$ and $p_4^{-1} = p_5$. We can rewrite $p = p_1 p_2 p_2^{-1} p_4 p_4^{-1} p_6$. Choose

$p' = p_1 p_4 p_4^{-1} p_2 p_2^{-1} p_6$, we have $p \equiv p'$ and $f(p') = q'$. $\square$

Theorem 4.3.7. *The induced homomorphism $f : \mathrm{FIS}(\Gamma) \to \mathrm{FIS}(\Delta)$ is locally injective and maps closed subsets of $\mathrm{FIS}(\Gamma)$ to closed subsets of $\mathrm{FIS}(\Delta)$.*

Proof. Let p_1 and p_2 be paths in Γ with $s(p_1) = s(p_2) = v \in V(\Gamma)$ and $f(p_1) \equiv f(p_2)$. By Lemma 4.3.6, there exists a path $p'_1 \in \Gamma$ such that $p_1 \equiv p'_1$ and $f(p'_1) = f(p_2)$. By Proposition 4.3.3, $p'_1 = p_2$. So $p_1 \equiv p_2$. It follows that $f : \mathrm{FIS}(\Gamma) \to \mathrm{FIS}(\Delta)$ is locally injective.

Let X be closed subset of $\mathrm{FIS}(\Gamma)$ and $a = eb \in f(X)$ for some idempotent e of $\mathrm{FIS}(\Delta)$. There exists $p \in \Gamma^+$ such that $\bar{p} \in X$ and $f(\bar{p}) = a$. Choose $q_1, q_2 \in \Delta$ where $\bar{q_1} = e$ and $\bar{q_2} = b$. Since q_1 is an idempotent, by Lemma 4.2.2, $q_1 \equiv q_1 q_1^{-1}$ and so $f(p) = q_1 q_2 \equiv q_1 q_1^{-1} q_2$. By Lemma 4.3.6, we can choose $p' \in \Gamma^+$ where $p \equiv p'$ and $f(p') = q_1 q_1^{-1} q_2$. Since f is an immersion, we can write $p' = p_1 p_1^{-1} p_2$ where $f(p_1) = q_1$ and $f(p_2) = q_2$. Now, $\bar{p_1}\bar{p_1}^{-1}\bar{p_2} = \bar{p} \in X$ and $\bar{p_1}\bar{p_1}^{-1}$ is an idempotent. Since X is closed, $\bar{p_2} \in X$. Thus, $b = \bar{q_2} = f(\bar{p_2}) \in f(X)$, and we see that $f(X)$ is closed. Therefore f maps closed subsets of $\mathrm{FIS}(\Gamma)$ to closed subsets of $\mathrm{FIS}(\Delta)$. $\square$

Corollary 4.3.8. *Let v be a vertex of Γ. Then $f : \mathrm{FIS}(\Gamma, v) \to \mathrm{FIS}(\Delta, f(v))$ is a monomorphism of semigroups and its image is a closed inverse semigroup of $\mathrm{FIS}(\Delta)$.*

Proof. Notice that the homomorphism $f : \mathrm{FIS}(\Gamma, v) \to \mathrm{FIS}(\Delta, f(v))$ is a restriction of the map $f_v : \mathrm{St}(v, \mathrm{FIS}(\Gamma)) \to \mathrm{St}(f(v), \mathrm{FIS}(\Delta))$. So it is a monomorphism of semigroups. Suppose $a = eb \in \mathrm{FIS}(\Gamma, v)$ where e is an idempotent of $\mathrm{FIS}(\Gamma)$. Then $s(a) = s(e) = s(b)$. Thus $b \in \mathrm{FIS}(\Gamma, v)$ and $\mathrm{FIS}(\Gamma, v)$ is a closed subset of $\mathrm{FIS}(\Gamma)$. $\square$

Corollary 4.3.9. *If Γ is connected. Then the set of all images $f(\mathrm{FIS}(\Gamma, v))$, as v ranges over all vertices of Γ, is a conjugacy class of closed inverse subsemigroups of $\mathrm{FIS}(\Delta)$.*

Proof. Let v_1 and v_2 be vertices of Γ. Since Γ is connected, there exists a path p in Γ from v_1 to v_2 such that

$$f(\bar{p})^{-1}f(\mathrm{FIS}(\Gamma, v_1))f(\bar{p}) = f(\bar{p}^{-1}\,\mathrm{FIS}(\Gamma, v_1)\bar{p}) \subseteq f(\mathrm{FIS}(\Gamma, v_2))$$

and

$$f(\bar{p})f(\mathrm{FIS}(\Gamma, v_2))f(\bar{p})^{-1} = f(\bar{p}\,\mathrm{FIS}(\Gamma, v_2)\bar{p}^{-1}) \subseteq f(\mathrm{FIS}(\Gamma, v_1)).$$

Since $\mathrm{FIS}(\Gamma, v_i)$ is closed for $i = 1, 2$, by Theorem 4.3.7, $f(\mathrm{FIS}(\Gamma, v_1))$ and $f(\mathrm{FIS}(\Gamma, v_2))$ are both closed inverse subsemigroupoids. Thus, by Lemma 3.4.3 (1), $f(\mathrm{FIS}(\Gamma, v_1))$ and $f(\mathrm{FIS}(\Gamma, v_2))$ are conjugate.

Suppose now that K is a closed inverse subsemigroup of $\mathrm{FIS}(\Delta)$ conjugate to $f(\mathrm{FIS}(\Gamma, v_1))$. We want to show that $K = \mathrm{FIS}(\Gamma, v)^{\uparrow}$. By Lemma 3.4.3 (3), there exists a path $q \in \Delta$ such that $\bar{q}\bar{q}^{-1} \in K$ and $(\bar{q}^{-1}f(\mathrm{FIS}(\Gamma, v_1)\bar{q})^{\uparrow} = K$. Since f is an immersion, we can choose a path p from v_1 to v such that $f(p) = q$. Hence

$$\begin{aligned} f(\mathrm{FIS}(\Gamma, v)) = f(\mathrm{FIS}(\Gamma, v))^{\uparrow} = f(\bar{p}^{-1}\,\mathrm{FIS}(\Gamma, v_1)\bar{p})^{\uparrow} = (f(\bar{p})^{-1}f(\mathrm{FIS}(\Gamma, v_1))f(\bar{p}))^{\uparrow} \\ = (\bar{q}^{-1}f(\mathrm{FIS}(\Gamma, v_1))\bar{q})^{\uparrow} = K. \end{aligned}$$

□

The following lemma can be deduced from [7, Theorem 4.5], however we also give an elementary proof.

Lemma 4.3.10. *Let $f_1 : \Gamma_1 \to \Delta$, $f_2 : \Gamma_2 \to \Delta$ be immersions of graphs, where Γ_1, Γ_2 are connected. If v_1, v_2 are vertices of Γ_1, Γ_2, respectively, such that $f_1(\mathrm{FIS}(\Gamma_1, v_1)) = f_2(\mathrm{FIS}(\Gamma_2, v_2))$, then f_1 and f_2 are equivalent maps into Δ.*

Proof. By replacing Δ with its component containing $f_1(v_1) = f_2(v_2)$, we may assume that Δ is connected. Let $j : \widetilde{\Delta} \to \Delta$ be the connected covering with a vertex $u \in V(\widetilde{\Delta})$ such that $j(\pi_1(\widetilde{\Delta}, u)) = f_i(\pi_1(\Gamma_i, v_i))$ for $i = 1, 2$. By the general lifting property of coverings, there exists a unique map of graphs $\tilde{f}_i : \Gamma_i \to \widetilde{\Delta}$ such that $\tilde{f}(v_i) = u$ and $j\tilde{f}_i = f_i$.

$$\begin{array}{ccccc} \Gamma_1 & \xrightarrow{\tilde{f}_1} & \widetilde{\Delta} & \xleftarrow{\tilde{f}_2} & \Gamma_2 \\ & {\scriptstyle f_1} \searrow & \downarrow {\scriptstyle j} & \swarrow {\scriptstyle f_2} & \\ & & \Delta & & \end{array}$$

Note that $j\tilde{f}_i(\pi_1(\Gamma_i, v_i)) = j(\pi_1(\widetilde{\Delta}, u))$. By Lemma 4.3.3, $j : \pi_1(\widetilde{\Delta}, u) \to \pi_1(\Delta, j(u))$ is injective. So $\tilde{f}_i(\pi_1(\Gamma_i, v_i)) = \pi_1(\widetilde{\Delta}, u)$. Also note that $\tilde{f}_i$ is an immersion since $f_i = j\tilde{f}_i$ and j are immersions. By Corollary 4.3.4, $\tilde{f}_i$ is an embedding and so $\text{FIS}(\tilde{f}_i(\Gamma_i), u) = \tilde{f}_i(\text{FIS}(\Gamma_i, v_i))$. Furthermore, by Corollary 4.3.8, $j : \text{FIS}(\widetilde{\Delta}, u) \to \text{FIS}(\Delta, j(u))$ is injective and $j\tilde{f}_1(\text{FIS}(\Gamma_1, v_1)) = j\tilde{f}_2(\text{FIS}(\Gamma_2, v_2))$. Thus, $\tilde{f}_1(\text{FIS}(\Gamma_1, v_1)) = \tilde{f}_2(\text{FIS}(\Gamma_2, v_2))$. Applying Lemma 4.2.5, $\tilde{f}_1(\Gamma_1) = \tilde{f}_2(\Gamma_2)$. Hence there is an isomorphism of graphs $h : \Gamma_1 \to \Gamma_2$ given by $h = \tilde{f}_2^{-1}\tilde{f}_1$ satisfying $f_2 h = f_1$. Consequently, $f_1 : \Gamma_1 \to \Delta$ and $f_2 : \Gamma_2 \to \Delta$ are equivalent maps into Δ. $\square$

CHAPTER 5

CANONICAL FORMS OF FREE INVERSE SEMIGROUPOIDS

We have shown that free inverse semigroupoids exist, but the construction of Section 4.2 does not tell us whether distinct words in a free inverse semigroupoid may represent the same element. This is what we mean by the word problem for the free inverse semigroupoids.

Example 5.0.1. Let a, b be edges in Γ with $s(a) = t(a) = s(b) = t(b)$. If $\bar{p} = \bar{a}^2\bar{a}^{-3}\bar{a}\bar{b}\bar{b}^{-1}\bar{a}\bar{b}^{-1}\bar{b}$ and $\bar{q} = \bar{a}^{-1}\bar{a}\bar{b}\bar{b}^{-1}\bar{a}\bar{b}^{-1}\bar{b}\bar{a}\bar{a}^{-1}$, we claim that they both represent the same element of FIS(Γ). We first separate the idempotents by the parentheses. Then, by the definition of free inverse semigroupoids,

$$\begin{aligned}
a^2a^{-3}abb^{-1}ab^{-1}b &\equiv aaa^{-1}a^{-1}a^{-1}abb^{-1}ab^{-1}b \\
&\equiv (\mathbf{aaa^{-1}a^{-1}})(\mathbf{a^{-1}a})(\mathbf{bb^{-1}})a(b^{-1}b) \\
&\equiv (\mathbf{bb^{-1}})(\mathbf{a^{-1}a})(\mathbf{aaa^{-1}a^{-1}})ab^{-1}b \\
&\equiv (bb^{-1})(a^{-1}a)a(\mathbf{aa^{-1}})(\mathbf{a^{-1}a})(b^{-1}b) \\
&\equiv (bb^{-1})(a^{-1}a)a(\mathbf{a^{-1}a})(\mathbf{aa^{-1}})(b^{-1}b) \\
&\equiv (bb^{-1})(a^{-1}a)\mathbf{aa^{-1}a}(aa^{-1})(b^{-1}b) \\
&\equiv (bb^{-1})(a^{-1}a)\mathbf{a}(aa^{-1})(b^{-1}b) \\
&\equiv (\mathbf{a^{-1}a})(\mathbf{bb^{-1}})a(\mathbf{b^{-1}b})(\mathbf{aa^{-1}}) \\
&\equiv a^{-1}abb^{-1}ab^{-1}baa^{-1}.
\end{aligned}$$

As we see from the example above, it can be time intensive to solve the word problem by using the definition. Fortunately, the ideas from free inverse semigroups can be extended to free inverse semigroupoids. Additionally, we show that a free inverse semigroupoid has a unique basis. Furthermore, we show some results of closed inverse subsemigroups of free inverse semigroups using the techniques that will be introduced in this chapter.

5.1 Inverse Semigroupoid of Finite Connected Subgraphs

Associated to a graph Γ, we construct an inverse semigroupoid $M(\Gamma)$ as follows. Both have vertex set $V(\Gamma)$. An edge of $M(\Gamma)$ consists of all triples (X, x_1, x_2), where X is a finite connected subgraph of Γ and $x_1, x_2 \in V(X)$; and the edges of $M^+(\Gamma)$ are those edges $(X, x_1, x_2) \in M(\Gamma)$ with X nontrivial (i.e., it has at least one edge). For $(X, x_1, x_2) \in M(\Gamma)$, we define

$$s(X, x_1, x_2) = x_1 \text{ and } t(X, x_1, x_2) = x_2.$$

If (X, x_1, x_2) and (Y, y_1, y_2) are consecutive edges in the directed graph $M(\Gamma)$, then we define $(X, x_1, x_2)(Y, y_1, y_2) = (X \cup Y, x_1, y_2)$. Note that in this situation, $x_2 = y_1$ and so $X \cup Y$ is connected—as required.

It is clear that this partial binary operation on $M(\Gamma)$ is associative, and so $M(\Gamma)$ a semigroupoid. Obviously, an element $(X, x_1, x_2) \in M(\Gamma)$ is an idempotent if and only if $x_1 = x_2$. So if (X, x, x) and (Y, y, y) are idempotents of $M(\Gamma)$ with $x = y$, then

$$(X, x, x)(Y, y, y) = (X \cup Y, x, x) = (Y, y, y)(X, x, x).$$

That is, pairs of idempotents in $M(\Gamma)^{(2)}$ commute. Furthermore, each element $(X, x_1, x_2) \in M(\Gamma)$ has a unique inverse given by $(X, x_1, x_2)^{-1} = (X, x_2, x_1)$. Therefore, $M^+(\Gamma)$ is an inverse semigroupoid.

In Section 4.2, we briefly introduced the smallest subgraph $\lceil p \rceil$ of a graph Γ containing p where p is a path in Γ^+. In this situation, we first have a natural map of graphs $\theta : \Gamma \to M(\Gamma)$ which is the identity map on $V(\Gamma)$ and maps each edge $x \in \Gamma$ to $\theta(x) = (\lceil x \rceil, s(x), t(x))$, where $\lceil x \rceil = \{x, x^{-1}, s(x), t(x)\}$ is the smallest subgraph of Γ containing x. Then, the induced homomorphisms $\theta : \Gamma^+ \to M^+(\Gamma)$ extends θ to nontrivial paths as follows. If $p = (x_1, \cdots, x_2)$

is a nontrivial path in Γ^+, then $\theta(p) = (\lceil p \rceil, s(p), t(p))$ where $\lceil p \rceil = \lceil x_1 \rceil \cup \cdots \cup \lceil x_n \rceil$ is the smallest subgraph of Γ containing the path p. Since every element of $M(\Gamma)$ is obviously equal to $(\lceil p \rceil, s(p), t(p))$ for some path $p \in \Gamma$, it follows that θ is an epimorphism.

Lemma 5.1.1. *If $p, q \in \Gamma^+$ and $p \equiv q$, then $\theta(p) = \theta(q)$.*

Proof. If x is an edge of Γ, then $x \equiv xx^{-1}x$. Additionally, $\lceil xx^{-1}x \rceil = \lceil x \rceil \cup \lceil x^{-1} \rceil = \lceil x \rceil$, $s(xx^{-1}x) = s(x)$, and $t(xx^{-1}x) = t(x)$. Thus $\theta(xx^{-1}x) = \theta(x)$.

If p and q are paths in Γ and $p \equiv q$, then $\lceil pp^{-1}qq^{-1} \rceil = \lceil p \rceil \cup \lceil q \rceil = \lceil qq^{-1}pp^{-1} \rceil$. Thus, $\theta(pp^{-1}qq^{-1}) = \theta(qq^{-1}pp^{-1})$. Since the congruence $\equiv$ on Γ^+ is generated by $(xx^{-1}x,\, x)$ where $x \in \Gamma$, and all pairs $(pp^{-1}qq^{-1},\, qq^{-1}pp^{-1})$, where $p, q \in \Gamma^+$ with $s(p) = s(q)$, the result follows. □

We can conclude from this lemma that there exists an induced homomorphism, denoted by the same symbol, $\theta : \mathrm{FIS}(\Gamma) \to M^+(\Gamma)$, and it is surjective. To understand this homomorphisms better, we extend the notion of Scheiblich normal forms of elements in free inverse semigroups [14].

5.2 Scheiblich Normal Form

Scheiblich normal forms are the suitable interpretation to free inverse semigroups of the reduced words used to solve the word problem in free groups. While words of the form xx^{-1} can be eliminated in free groups, these words represent idempotents and cannot be erased in free inverse semigroups. Nonetheless, they can be moved around since the idempotents commute. The idea behind the algorithm is: given a word w from a free inverse semigroup, certain "obvious" idempotents are moved to the front, and the resulting words, consisting of a product of idempotents followed by a non-idempotent element, will be equivalent to w. The following example shows the algorithm for computing a Scheiblich normal form.

Example 5.2.1. Let $w = bb^{-1}ab^{-1}bcaa^{-1}cc^{-1}$. Choose the reduced maximal prefix u_i then omit $u_iu_i^{-1}$ from w_{i-1} to get w_i. We perform this until w_i is a reduced word:

1. Choose $u_1 = b$, then $w = \underbrace{bb^{-1}}_{u_1u_1^{-1}}\underbrace{ab^{-1}bcaa^{-1}cc^{-1}}_{w_1}$,

2. Choose $u_2 = ab^{-1}$, then $w_1 = \underbrace{ab^{-1}ba^{-1}}_{u_2u_2^{-1}}\underbrace{acaa^{-1}cc^{-1}}_{w_2}$,

3. Choose $u_3 = aca$, then $w_2 = \underbrace{acaa^{-1}c^{-1}a^{-1}}_{u_3u_3^{-1}}\underbrace{accc^{-1}}_{w_3}$,

4. Choose $u_4 = acc$, then $w_3 = \underbrace{accc^{-1}c^{-1}a^{-1}}_{u_4u_4^{-1}}\underbrace{ac}_{w_4}$.

Thus, $w \equiv (u_1u_1^{-1})(u_2u_2^{-1})(u_3u_3^{-1})(u_4u_4^{-1})w_4$.

For free inverse semigroupoids, let $p = (x_1, \cdots, x_n) \in \Gamma^+$. Denote its prefix path of length $i \geq 1$ by $p_i = (x_1, \cdots, x_i)$.

Claim: $p \equiv (u_1u_1^{-1}) \cdots (u_nu_n^{-1})u_n$ where $u_i = \text{red}(p_i)$ is the unique reduced path in Γ^+ homotopic to p_i.

Proof. If $n = 1$, then $p = x_1 \equiv x_1x_1^{-1}x_1 = (u_1u_1^{-1})u_1$ as claimed. If $n > 1$, then $p = p_{n-1}x_n \equiv (u_1u_1^{-1}) \cdots (u_{n-1}u_{n-1}^{-1})u_{n-1}x_n$, by induction on n, and there are two cases to consider.

Case 1. If $u_{n-1}x_n$ is a reduced path, then $u_{n-1}x_n = u_n \equiv (u_nu_n^{-1})u_n$ and the claim follows for this case.

Case 2. If $u_{n-1}x_n$ is not reduced, then $u_{n-1} = u_nx_n^{-1}$ and

$$\begin{aligned} u_{n-1}x_n = u_nx_n^{-1}x_n &\equiv (u_nu_n^{-1}u_n)(x_n^{-1}x_n) \\ &\equiv u_n(u_n^{-1}u_n)(x_n^{-1}x_n) \\ &\equiv u_n(x_n^{-1}x_n)(u_n^{-1}u_n) \\ &\equiv (u_nx_n^{-1})(u_nx_n^{-1})^{-1}u_n \\ &\equiv (u_{n-1}u_{n-1}^{-1})u_n \\ &\equiv (u_{n-1}u_{n-1}^{-1})(u_nu_n^{-1})u_n \end{aligned}$$

Therefore, the claim also holds in this case.

□

To obtain the so-called *Scheiblich normal form* of an element $\bar{a} \in \mathrm{FIS}(\Gamma)$, choose a path $p \in \Gamma^+$ representing $\bar{a}$. Use the claim above to find reduced paths $u_1, \cdots, u_n$ such that $p \equiv (u_1 u_1^{-1}) \cdots (u_n u_n^{-1}) u_n$. Then delete the factors $(u_i u_i^{-1})$ that are redundant or where u_i is a proper prefix of some u_j. The resulting path is the Scheiblich normal form of a, which is unique up to the order of the idempotent factors $(u_i u_i^{-1})$.

Proposition 5.2.2. *If $\pi_1(\Gamma)$ is trivial, then $\theta : \mathrm{FIS}(\Gamma) \to M^+(\Gamma)$ is an isomorphism.*

Proof. Let $p = (x_1, \ldots, x_n) \in \Gamma^+$. Since $\pi_1(\Gamma)$ is trivial (i.e., Γ is a forest), the subgraph $\lceil p \rceil$ is a finite tree. The vertex set of $\lceil p \rceil$ consists of all the targets $t(p_i)$ of the prefixes $p_i = (x_1, \ldots, x_i)$ of p. Putting $u_i = \mathrm{red}(p_i)$, we see that $\{u_1, \ldots, u_n\}$ is exactly the set of all reduced paths in Γ that start at the vertex $s(p)$ and terminate at a vertex in $\lceil p \rceil$; and u_n is the unique reduced path in Γ from $s(p)$ to $t(p)$. By the claim above, $p \equiv (u_1 u_1^{-1}) \cdots (u_n u_n^{-1}) u_n$ and it follows that the element that p represents in $\mathrm{FIS}(\Gamma)$ is completely determined by $\lceil p \rceil$, $s(p)$, and $t(p)$. So $\theta\colon \mathrm{FIS}(\Gamma) \to M^+(\Gamma)$ is injective, and thus an isomorphism. □

Recall from Section 2.1 that for an associative congruence ρ on A, there is a natural way to make A/ρ into a semigroupoid. We see that a group acting freely on a semigroupoid by semigroupoid automorphisms is an associative congruence.

Theorem 5.2.3. *If A is a semigroupoid and G is a group acting freely on A by semigroupoid automorphisms, then the congruence determined by G is an associative congruence. Hence, A/G is a semigroupoid.*

Proof. Define ρ on A by $(a, b) \in \rho$ if there exists $g \in G$ such that $g \cdot a = b$. So $g \cdot s(a) = s(g \cdot a) = s(b)$ and $g \cdot t(a) = t(g \cdot a) = t(b)$. If $(a_1, a_2) \in A^{(2)}$ and $(b_1, b_2) \in A^{(2)}$ are such that $(a_1, b_1) \in \rho$ and $(a_2, b_2) \in \rho$, then there exists $g, h \in G$ such that $g \cdot a_1 = b_1, h \cdot a_2 = b_2$. Then, $t(a_1) = s(a_2)$ and $t(b_1) = s(b_2)$. So $g \cdot t(a_1) = t(g \cdot a_1) = t(b_1) = s(b_2) = s(h \cdot a_2) = h \cdot s(a_2) = h \cdot t(a_1)$. We have $g = h$ as G acts freely. Then $g \cdot a_1 a_2 = (g \cdot a_1)(g \cdot a_2) = (g \cdot a_1)(h \cdot a_2) = b_1 b_2$. Thus, $(a_1 b_1, a_2 b_2) \in \rho$.

To show ρ is an associative congruence, we let $\nu : A \to A/G$ and $a, b \in A$ such that $\nu(a)\nu(b)$ is defined. Then $\nu(t(a)) = \nu(s(b))$, so there exists $g \in G$ such that $g \cdot t(a) = s(b)$. Thus, $(g \cdot a)b$ is defined and $\nu(g \cdot a)\nu(b) = \nu(a)\nu(b)$. Hence $(\nu \times \nu)(A^{(2)}) = (A/\rho)^{(2)}$. Similarly, $(\nu \times \nu \times \nu)(A^{(3)}) = (A/\rho)^{(3)}$ and so it follows that A/G is associative. Therefore, ρ is an associative congruence and so A/G is a semigroupoid. $\square$

We have the following straightforward consequences.

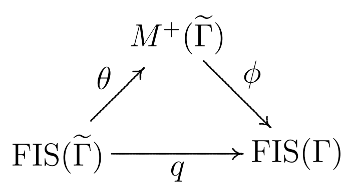
Corollary 5.2.4. *Let A be a semigroupoid and G be a group acting freely on A by semigroupoid automorphisms. If $\nu : A \longrightarrow A/G$ is the natural map, then $\nu(A^{(2)}) = (A/G)^{(2)}$.*

Corollary 5.2.5. *Let A be a semigroupoid and G be a group acting freely on A by semigroupoid automorphisms. If $\nu : A \longrightarrow A/G$ is the natural map, then $\nu(A^{(3)}) = (A/G)^{(3)}$.*

We conclude this section with an extension of Proposition 5.2.2 to arbitrary graphs Γ.

Theorem 5.2.6. *Let $q \colon \widetilde{\Gamma} \to \Gamma$ be the universal covering of a connected graph Γ. Then $\mathrm{FIS}(\Gamma) \cong M^+(\widetilde{\Gamma})/G$, where $G = \mathrm{Aut}(\widetilde{\Gamma}, q)$ is the group of covering transformations.*

Proof. Let $\theta : \mathrm{FIS}(\widetilde{\Gamma}) \to M^+(\widetilde{\Gamma})$ be the isomorphism of Proposition 5.2.2.



Consider the diagram above where $\phi = q\theta^{-1}$. First, we show that ϕ is an epimorphism. Let p be a path in Γ representing $\bar{p} \in \mathrm{FIS}(\Gamma)$. Then there exists a path $\tilde{p}$ in $\widetilde{\Gamma}$ representing $\bar{\tilde{p}} \in \mathrm{FIS}(\widetilde{\Gamma})$ such that $q(\tilde{p}) = p$. Since θ is an isomorphism, by Proposition 5.2.2, we have $(\lceil \tilde{p} \rceil, s(\tilde{p}), t(\tilde{p})) \in M^+(\widetilde{\Gamma})$ and $\theta^{-1}(\lceil \tilde{p} \rceil, s(\tilde{p}), t(\tilde{p})) = \tilde{p}$. Then, $\phi(\lceil \tilde{p} \rceil, s(\tilde{p}), t(\tilde{p})) = q\big(\theta^{-1}(\lceil \tilde{p} \rceil, s(\tilde{p}), t(\tilde{p}))\big) = q(\tilde{p}) = p$. It follows that ϕ is an epimorphism.

We define the G-action on $M^+(\widetilde{\Gamma})$ by: for all $g \in G$, let $g \cdot (X, x_1, x_2) = (g \cdot X, g \cdot x_1, g \cdot x_2)$. It is clear that G is acting by semigroupoid automorphisms on $M^+(\widetilde{\Gamma})$. Now, we want to show that $\ker\phi$ is the congruence determined by G. Let $(X, x_1, x_2), (Y, y_1, y_2) \in M^+(\widetilde{\Gamma})$ and suppose $\phi(X, x_1, x_2) = \phi(Y, y_1, y_2)$. Choose paths p_1, p_2 in $\widetilde{\Gamma}$ such that

$$\theta(\bar{p_1}) = (\lceil p_1 \rceil, s(p_1), t(p_1)) = (X, x_1, x_2)$$

and

$$\theta(\bar{p_2}) = (\lceil p_2 \rceil, s(p_2), t(p_2)) = (Y, y_1, y_2).$$

Then, $q(p_1) \equiv q(p_2)$. By lemma 4.3.6, there exists a path p_1' in $\widetilde{\Gamma}$ such that $p_1 \equiv p_1'$ and $q(p_1') = q(p_2)$. Note that p_1' and p_2 are two lifts of the path $q(p_1')$ to the universal cover $\widetilde{\Gamma}$. So there exists a covering transformation $g \in G$ such that $g \cdot p_1' = p_2$. Therefore, $g \cdot (X, x_1, x_2) = g \cdot (\lceil p_1 \rceil, s(p_1), t(p_1)) = (g \cdot \lceil p_1 \rceil, g \cdot s(p_1), g \cdot s(p_1)) = (\lceil p_2 \rceil, s(p_2), t(p_2)) = (Y, y_1, y_2)$. It follows that $\ker\phi \subseteq$ the congruence determined by G.

For the reverse inclusion, we suppose $(X, x_1, x_2) \subseteq M^+(\widetilde{\Gamma})$. Choose a path p in $\widetilde{\Gamma}$ such that $(\lceil p \rceil, s(p), t(p)) = (X, x_1, x_2)$. Then

$$g \cdot (X, x_1, x_2) = g \cdot (\lceil p \rceil, s(p), t(p)) = (\lceil g \cdot p \rceil, s(g \cdot p), t(g \cdot p)).$$

Thus, $\phi(X, x_1, x_2) = q(\bar{p}) = q(\overline{g \cdot p}) = \phi(g \cdot (\lceil p \rceil, s(p), t(p)) = \phi(g \cdot (X, x_1, x_2))$. Therefore, the congruence determined by $G \subseteq \ker\phi$. Consequently, the congruence determined by G is $\ker\phi$.

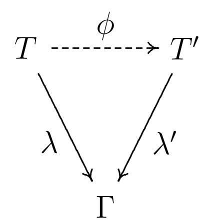
Next, suppose $g \cdot (X, x_1, x_2) = (X, x_1, x_2)$. Choose a path p in $\widetilde{\Gamma}$ such that $\lceil p \rceil = X$, $s(p) = x_1$, and $t(p) = x_2$. Then $g \cdot p$ is a path in $\widetilde{\Gamma}$ with source $g \cdot s(p) = s(g \cdot p) = s(p)$ and similarly $g \cdot t(p) = t(p)$. So $g = 1$ as G acts freely on $\widetilde{\Gamma}$. Hence, G acts freely on $M^+(\widetilde{\Gamma})$. By Theorem 5.2.3, G is an associative congruence. It follows from Lemma 3.2.2 that $M^+(\widetilde{\Gamma})/G \cong \text{FIS}(\Gamma)$.

□

5.3 Munn Trees

A different approach to the solution of the word problem, suggested by Douglas Munn [10], is important in combinatorial inverse semigroup theory. Munn trees provide a convenient canonical form for elements of free inverse semigroups. We show that this idea easily generalizes to free inverse semigroupoids.

Let Γ be a graph. A *Munn Tree* over Γ is a quadruple (T, λ, t_1, t_2), where T is a finite tree with at least one edge, $\lambda : T \to \Gamma$ is an immersion of graphs (called the labeling function), and $t_1, t_2 \in V(T)$. We say that two Munn trees, (T, λ, t_1, t_2) and (T, λ, t_1, t_2), are *equivalent* if there exists a label-preserving isomorphism of graphs $\phi : T \to T'$ such that $\phi(t_1) = t'_1$ and $\phi(t_2) = t'_2$.



If (T, λ, t_1, t_2) is a Munn tree over Γ and p is a path in T from t_1 to t_2 such that $\lceil p \rceil = T$ and $\lambda(p) = q \in \Gamma$, then we say that (T, λ, t_1, t_2) is a Munn tree over Γ for the path q.

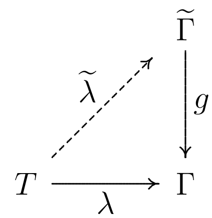
We first show that every nontrivial path p in Γ^+ has a unique Munn tree over Γ.

Lemma 5.3.1. *Let Γ be a graph and $p \in \Gamma^+$. Then p has a Munn tree (T, λ, t_1, t_2) over Γ and it is unique up to the equivalence of Munn trees.*

Proof. Since we are only interested in the component of Γ containing the path p, we may assume that Γ is connected. Let $q \colon \widetilde{\Gamma} \to \Gamma$ be the universal covering of Γ and choose a path $\widetilde{p}$ such that $q(\widetilde{p}) = p$ (a "lift" of p to $\widetilde{\Gamma}$). Then $(\lceil \widetilde{p} \rceil, q_0, s(\widetilde{p}), t(\widetilde{p}))$, where $q_0 \colon \lceil \widetilde{p} \rceil \to \Gamma$ is the restriction of q, is a Munn tree over Γ. Since $\widetilde{p}$ is a path in the tree $\lceil \widetilde{p} \rceil$ from $s(\widetilde{p})$ to $t(\widetilde{p})$ and $q_0(\widetilde{p}) = p$, we see that p has a Munn tree over Γ.

Now let (T, λ, t_1, t_2) be any Munn tree over Γ for the path p. Since $\pi_1(T, t_1)$ is the trivial group, there exists a unique map of graphs $\widetilde{\lambda} \colon T \to \widetilde{\Gamma}$ such that $\widetilde{\lambda}(t_1) = s(\widetilde{p})$ and

$q\widetilde{\lambda} = \lambda$ (by the general lifting property of coverings).



Since λ is an immersion, it follows from Lemma 4.3.5 that $\widetilde{\lambda}$ is an embedding. That is, $\widetilde{\lambda}$ is an isomorphism of graphs from T onto its image in $\widetilde{\Gamma}$. Let p' be the path in T from t_1 to t_2 with $\lceil p' \rceil = T$ and $\lambda(p') = p$. Note that $\widetilde{\lambda}(p')$ is a path in $\widetilde{\Gamma}$ from $\widetilde{\lambda}(t_1) = s(\widetilde{p})$ with $q(\widetilde{\lambda}(p')) = \lambda(p') = p$, and so $\widetilde{\lambda}(p') = \widetilde{p}$ by the unique path lifting property of coverings. It follows that the image of $\widetilde{\lambda}$ is the subgraph $\lceil \widetilde{p} \rceil$ and that the Munn trees (T, λ, t_1, t_2) and $(\lceil \widetilde{p} \rceil, q_0, s(\widetilde{p}), t(\widetilde{p}))$ are equivalent. $\square$

In light of the previous lemma, we can now speak of the Munn Tree of a nontrivial path p of Γ; it is denoted by $\mathrm{MT}(p)$. Furthermore, we next show that the Munn tree of p is a canonical form for the element in $\mathrm{FIS}(\Gamma)$ represented by p.

Lemma 5.3.2. *If Γ is a graph and $p_1, p_2 \in \Gamma^+$, then $p_1 \equiv p_2$ if and only if $\mathrm{MT}(p_1) = \mathrm{MT}(p_2)$.*

Proof. Suppose first that $p_1 \equiv p_2$. We may assume that Γ is connected and let $q\colon \widetilde{\Gamma} \to \Gamma$ be its universal covering. Choose a vertex $\tilde{x}$ of $\widetilde{\Gamma}$ such that $q(\tilde{x}) = s(p_1) = s(p_2)$ and let $\widetilde{p}_1$ and $\widetilde{p}_2$ be the unique paths in $\widetilde{\Gamma}$ with source $\tilde{x}$ such that $q(\widetilde{p}_1) = p_1$ and $q(\widetilde{p}_2) = p_2$. By Theorem 4.3.7, $\widetilde{p}_1 \equiv \widetilde{p}_2$ and thus, by Lemma 5.1.1, $(\lceil \widetilde{p}_1 \rceil, s(\widetilde{p}_1), t(\widetilde{p}_1)) = (\lceil \widetilde{p}_2 \rceil, s(\widetilde{p}_2), t(\widetilde{p}_2))$. Consequently, the Munn trees $\mathrm{MT}(p_1) = (\lceil \widetilde{p}_1 \rceil, q|_{\lceil \widetilde{p}_1 \rceil}, s(\widetilde{p}_1), t(\widetilde{p}_1))$ and $\mathrm{MT}(p_2) = (\lceil \widetilde{p}_2 \rceil, q|_{\lceil \widetilde{p}_2 \rceil}, s(\widetilde{p}_2), t(\widetilde{p}_2))$ are equal.

Conversely, assume that $\mathrm{MT}(p_1) = \mathrm{MT}(p_2) = (T, \lambda, t_1, t_2)$. Then there exist paths p_1', p_2' in T from t_1 to t_2 such that $\lceil p_1' \rceil = T$, $\lceil p_2' \rceil = T$ and $\lambda(p_1') = p_1$, $\lambda(p_2') = p_2$. Recall the isomorphism in Proposition 5.2.2 of the semigroupoids $\mathrm{FIS}(T)$ and $M^+(T)$ induced by the homomorphism $\theta_T\colon T^+ \to M^+(T)$. We see that $\theta_T(p_1') = (\lceil p_1' \rceil, t_1, t_2)$ and $\theta_T(p_2') =$

$([p'_2], t_1, t_2)$ are equal, and so $p'_1 \equiv p'_2$. Therefore, by Lemma 5.1.1, $p_1 = \lambda(p'_1) \equiv \lambda(p'_2) = p_2$. □

Moreover, the Munn tree over a graph Γ of a nontrivial path p can be effectively constructed using Stallings foldings.

5.4 Stallings Foldings

We start by recalling some ideas from Stallings foldings. For a finite graph, [17] introduces Stallings foldings to obtain graph immersions. Let $f : \Gamma \to \Delta$ be a map of graphs which is not an immersion, then there exists $x_1, x_2 \in \Gamma$ such that $f(x_1) = f(x_2)$ and $s(x_1) = s(x_2)$. Hence, f factors through the quotient $\Gamma \longrightarrow \Gamma/[x_1 = x_2]$ obtained by identifying $t(x_1)$ to $t(x_2)$, x_1 to x_2, and x_1^{-1} to x_2^{-1}. This quotient map is a so called *folding prescribed by f*.

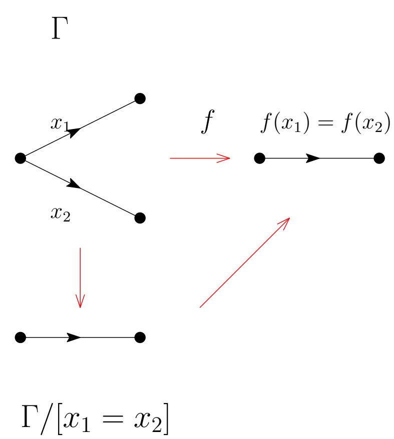


Figure 5.1: Folding prescribed by f.

When Γ is a finite graph and $f\colon \Gamma \to \Delta$ is a map of graphs, it follows that there is a finite sequence of foldings $\Gamma \to \Gamma_1 \to \cdots \to \Gamma_n$ and an immersion $\Gamma_n \to \Delta$ whose composition is the map f.

To obtain the Munn tree over a graph Γ of a nontrivial path p, let's represent the path $p = (x_1, \cdots, x_n)$ by a map of graphs $q\colon [0, n] \to \Gamma$, where $[0, n]$ is the combinatorial arc of length n, and q is defined so that the unique reduced path from 0 to n is mapped to the path

p. Then as recalled above, there is a finite sequence of foldings $[0,n] = T_0 \to T_1 \to \cdots \to T_k$ and an immersion $\lambda\colon T_k \to \Gamma$ whose composition is f. It is clear that a folding of a tree is again a tree. Thus (T_k, λ, t_1, t_2), where t_1, t_2 are the images of 0, n respectively under the composition of the foldings, is the Munn tree of p.

Example 5.4.1. A simple way to obtain the Munn tree using Stallings folding is to fold the paths that have the same source or the same target as in Figure 5.2.

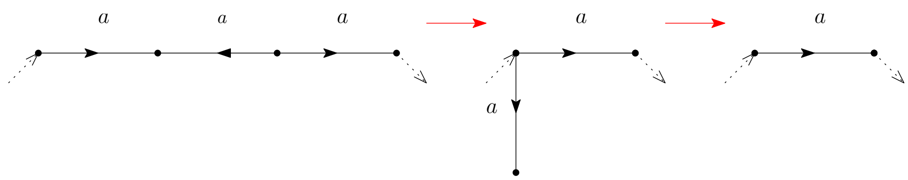


Figure 5.2: $aa^{-1}a \equiv a$

Example 5.4.2. In this example, we see that using Munn Trees is a more straightforward way to decide whether two words represent the same element compared to Example 5.0.1. Let Γ be a graph. If $\bar{p} = \bar{a}^2\bar{a}^{-3}\bar{a}\bar{b}\bar{b}^{-1}\bar{a}\bar{b}^{-1}\bar{b} \in \text{FIS}(\Gamma)$, then there is a path p that represents $\bar{p}$ with edges labeled without the bars, and $\text{MT}(p)$ is the Figure 5.3. To see whether $\bar{p} \equiv \bar{q} \in \text{FIS}(\Gamma)$, we trace the path that represents $\bar{q}$ from the starting vertex. If it passes through all the edges at least once and stops at the ending vertex, then $\text{MT}(p) = \text{MT}(q)$ and so $p \equiv q$ as stated in Lemma 5.3.2. Therefore, $\bar{p} = \bar{q}$.

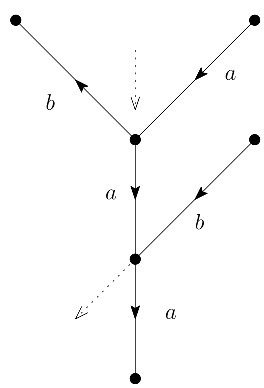


Figure 5.3: $\text{MT}(p) = \text{MT}(q)$ if and only if $p \equiv q$

Consequently, the word problem for finitely generated free inverse semigroupoids is solvable. That is, there is an algorithm that takes as input a finite graph Γ and two nontrivial paths p, q of Γ and decides whether or not $p \equiv q$.

5.5 Uniqueness of Basis Graph

A free inverse semigroupoid has a (symmetric) basis, which is a graph; and it turns out to be unique. Recall from Chapter 4, we identify Γ with its image under the natural map $\Gamma \to \mathrm{FIS}(\Gamma)$ where the vertex set $V(\Gamma) = V(\Gamma^+)$. It is clear that this natural map is injective. To prove the following Lemma, we note that if Γ is connected then $\mathrm{MT}(p) \subseteq \mathrm{MT}(q)$ if and only if $p \geq q$ for all $p, q \in \Gamma^+$.

Lemma 5.5.1. *Let Γ be a connected graph. Then,*

$$\Gamma = \{a \in \mathrm{FIS}(\Gamma) \mid a \notin E(\mathrm{FIS}(\Gamma)) \text{ and } aa^{-1} \text{ is a maximal idempotent } \}.$$

Proof. We denote $S = \{a \in \mathrm{FIS}(\Gamma) \mid a \notin E(\mathrm{FIS}(\Gamma))$ and aa^{-1} is a maximal idempotent$\}$. Let $x \in \Gamma$. Then $x \in \mathrm{FIS}(\Gamma)$ by the natural inclusion map. Since x is not null-homotopic, x is not an idempotent by Lemma 4.2.2. Thus, $\mathrm{MT}(xx^{-1})$ is a minimal Munn Tree on Γ, i.e. $\mathrm{MT}(xx^{-1}) \subseteq \mathrm{MT}(e)$ for all $e \in E(\Gamma^+)$ with $s(e) = s(x)$ and so $e \leq xx^{-1}$. Consequently, xx^{-1} is a maximal idempotent. Therefore, $\Gamma \subseteq S$.

For the reverse inclusion, we let $a \in S$. Choose a path $p = (x_1, x_2, \cdots, x_n) \in \Gamma^+$ such that $\bar{p} = a$. Then, $\mathrm{MT}(x_1x_1^{-1}) \subseteq \mathrm{MT}(pp^{-1})$. So $pp^{-1} \leq x_1x_1^{-1}$. However, pp^{-1} is a maximal idempotent. Thus, $pp^{-1} \equiv x_1x_1^{-1}$ and so $\mathrm{MT}(pp^{-1}) = MT(xx^{-1})$ by Lemma 5.3.2. Since p is not an idempotent, $\mathrm{MT}(p) = \mathrm{MT}(x_1)$. Again, by Lemma 5.3.2, $p \equiv x_1$. Therefore, $a = \bar{p} \in \Gamma$ and so $S \subseteq \Gamma$. Consequently, $\Gamma = S$. □

Suppose Γ and Δ are connected graphs. It is clear that if $\Gamma \cong \Delta$ then $\mathrm{FIS}(\Gamma) \cong \mathrm{FIS}(\Delta)$. We want to show that the converse is true.

Theorem 5.5.2. *Let Γ and Δ be connected graphs. If $\mathrm{FIS}(\Gamma) \cong \mathrm{FIS}(\Delta)$ then $\Gamma \cong \Delta$.*

Proof. Let $\theta : \mathrm{FIS}(\Gamma) \to \mathrm{FIS}(\Delta)$ be an isomorphism. By previous Lemma, $\theta(\Gamma) \subseteq \Delta$ and $\theta^{-1}(\Delta) \subseteq \Gamma$. Additionally, $\theta(a^{-1}) = \theta(a)^{-1}$ and $\theta^{(0)}(s(a)) = s(\theta(a))$. Thus, $\theta|_{\Gamma} : \Gamma \to \Delta$ is a map of graphs and $\theta^{-1}|_{\Delta} : \Delta \to \Gamma$ is its inverse. Therefore, the result follows. □

5.6 Closed Inverse Subsemigroups of Free Inverse Semigroups

This section contains some results for closed inverse subsemigroups of free inverse semigroups using the techniques of coverings, Munn trees, and Stallings foldings; see also [7]. First, we have an alternate proof of Corollary 4.3.8.

Theorem 5.6.1. *If* $f : \Gamma \to \Delta$ *is an immersion of graphs and* $v \in V(\Gamma)$*, then* $f\colon \mathrm{FIS}(\Gamma, v) \to \mathrm{FIS}(\Delta, f(v))$ *is injective and its image is a closed inverse subsemigroup of* $\mathrm{FIS}(\Delta, f(v))$.

Proof. Consider the commutative diagram below where q_1 and q_2 are the universal coverings of Γ and Δ respectively.

$$\begin{array}{ccc} \widetilde{\Gamma} & \xrightarrow{\tilde{f}} & \widetilde{\Delta} \\ {\scriptstyle q_1}\big\downarrow & & \big\downarrow{\scriptstyle q_2} \\ \Gamma & \xrightarrow{f} & \Delta \end{array}$$

Let p_1 and p_2 be paths of Γ with source v representing $\bar{p_1}$ and $\bar{p_2}$ of $\mathrm{FIS}(\Gamma, v)$ respectively. Suppose that $f(p_1) \equiv f(p_2)$. Since f is an immersion, $s(f(p_1)) = s(f(p_2)) = f(v)$. Choose a vertex $\widetilde{v}$ of $\widetilde{\Gamma}$ such that $q_1(\widetilde{v}) = v$, paths $\widetilde{p}_1$ and $\widetilde{p}_2$ of $\widetilde{\Gamma}$ such that $q_1(\widetilde{p}_1) = p$, $q_1(\widetilde{p}_2) = p_2$, and such that $\widetilde{v} = s(\widetilde{p}_1) = s(\widetilde{p}_2)$. By the $\equiv$ lifting property of coverings, $\widetilde{f}(\widetilde{p}_1) \equiv \widetilde{f}(\widetilde{p}_2)$. By Lemma 4.3.5, $\widetilde{f}$ is an embedding. Thus, $\widetilde{p}_1 \equiv \widetilde{p}_2$ and so $p_1 = q_1(\widetilde{p}_1) \equiv q_1(\widetilde{p}_2) = p_2$. Hence, $\bar{p_1} = \bar{p_2}$. Therefore, $f\colon \mathrm{FIS}(\Gamma, v) \to \mathrm{FIS}(\Delta, f(v))$ is injective.

To prove that the image is a closed inverse semigroup of $\mathrm{FIS}(\Delta, f(v))$, we let $a \in \mathrm{FIS}(\Delta, f(v))$ such that $ea \in f(\mathrm{FIS}(\Gamma, v))$ for some $e \in E\big(\mathrm{FIS}(\Delta, f(v))\big)$. Choose paths r and p in Δ such that $\bar{r} = e$ and $\bar{p} = a$ with $f(v)$ as their source and target. We also choose $b \in \mathrm{FIS}(\Gamma, v)$ such that $f(b) = ea$. Then there exists a path p_0 in Γ with $s(p_0) = t(p_0) = v$

such that $\bar{p}_0 = b$, $q_1(\widetilde{p}_0) = p_0$ for $\widetilde{p}_0$ in $\widetilde{\Gamma}$, and $q_1(\tilde{v}) = v$ for $\tilde{v} \in V(\widetilde{\Gamma})$. Then $q_2(\tilde{f}(\widetilde{p}_0)) = f(p_0) \equiv rp$. So $\mathrm{MT}(f(p_0)) = \mathrm{MT}(rp)$ and thus $\lceil \tilde{f}(\widetilde{p}_0) \rceil = \lceil \tilde{r}\tilde{p} \rceil$ where $q_2(\tilde{r}) = r$ and $q_2(\tilde{p}) = p$. Since $\tilde{f}$ is an embedding, there exists a path $\widetilde{p}_1$ in $\lceil \widetilde{p}_0 \rceil$ with $s(\widetilde{p}_1) = \tilde{v}$ and $\tilde{f}(\widetilde{p}_1) = \tilde{p}$. Put $p_1 = q_1(\widetilde{p}_1)$. Since $f(t(p_1)) = q_2(t(\tilde{p})) = f(v)$ and f is an immersion, $t(p_1) = v$. Hence, p_1 is a loop in Γ based at v such that $f(p_1) \equiv p$. Therefore, $a = \bar{p} \in f(\mathrm{FIS}(\Gamma, v))$. $\square$

According to Section 5.4, Stallings showed how immersions between finite graphs may be used to study finitely generated subgroups of free groups as described by the following example.

Example 5.6.2. *Constructing the graph using foldings associated to a finitely generated subgroup*
$\langle a^2, ba^{-1}a^{-1}, a^{-1}ab^{-1} \rangle$ *of a free group* $F(a, b)$*. The circle vertices represent the images of the original base vertex in the intermediate graphs.*

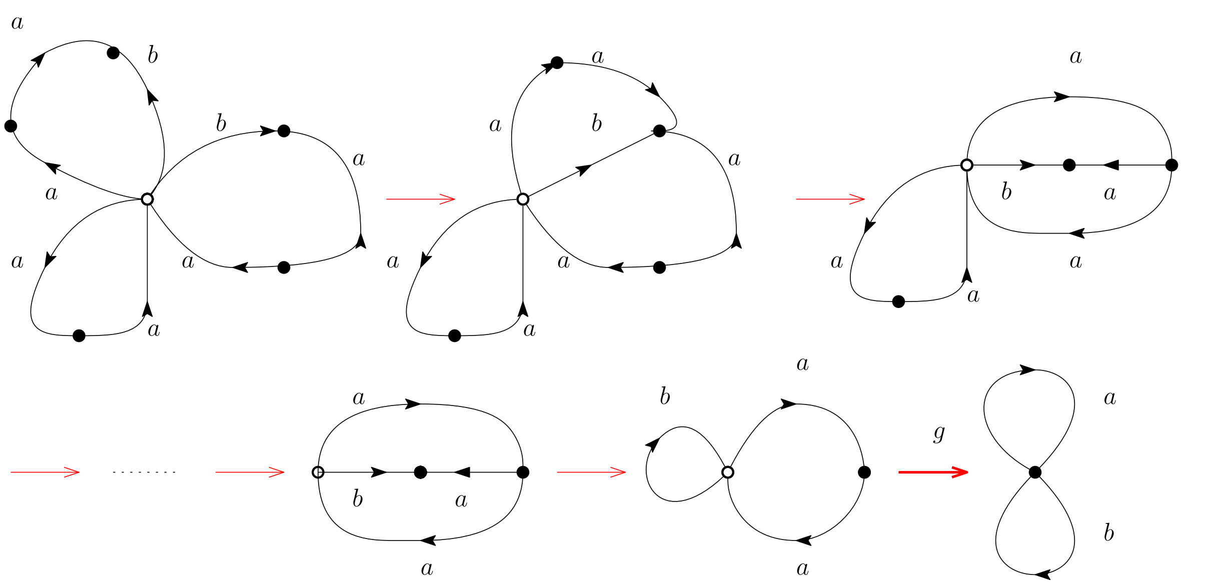


Figure 5.4: A sequence of folded graphs and an immersion whose composition is f.

The following proposition shows that closed inverse subsemigroups of free inverse semigroups play the same role as subgroups of free groups.

Proposition 5.6.3. *Let* Δ *be a graph and* $v \in V(\Delta)$*. If* A *is a finitely generated closed* *inverse subsemigroup of* FIS(Δ, v)*, then there exists a graph* Γ *and an immersion* $g\colon \Gamma \to \Delta$ *such that* $g(\text{FIS}(\Gamma, u)) = A$ *for some* $u \in V(\Gamma)$*.*

Proof. Suppose $A = \langle X \rangle^{\uparrow}$ with a finite set $X = \{x_1, x_2, x_3, \cdots, x_k\}$. Then Γ consists of a single vertex v with n loops. Each loop is subdivided into the number of edges of x_i which are oriented and labelled by the word x_i. Since Γ is a finite graph and $f : \Gamma \to \Delta$ is a map of graphs, by Stalling's foldings, there exists a graph Γ_n such that $g : \Gamma_n \to \Delta$ is an immersion. So there exists a vertex $u \in V(\Gamma)$ such that $g(u) = v$. By Corollary 4.3.8, $g : \text{FIS}(\Gamma_n, u) \to \text{FIS}(\Delta, v)$ is a monomorphism and its image is a closed inverse semigroup of FIS(Δ, v); therefore, $g(\text{FIS}(\Gamma_n, u)) = \langle X \rangle^{\uparrow} = A$. □

Note that a closed inverse subsemigroup of a free inverse semigroup is not necessarily free [7], and it is only free when it has only one vertex. In Corollary 7.1.4, we show that the converse of the lemma above also holds. The general results for the structure of closed inverse subsemigroupoids of free inverse semigroupoids are proven in Chapter 7.

CHAPTER 6

GRAPHS AND FREE INVERSE SEMIGROUPOIDS

To obtain our results for the next chapter, we establish two useful concepts in this chapter: the Stallings kernel and subdivisions of a graph.

6.1 Stallings Kernel

While the concept of Stallings foldings works for finite graphs; in this section, we introduce a new concept that will generalize this idea for any graphs. By an *equivalence relation* R on a graph Γ, we mean a pair of equivalence relations, $R^{(0)}$ on $V(\Gamma)$ and $R = R^{(1)}$ on $\Gamma = \Gamma^{(1)}$, that are compatible with the graph structure: if $(x, y) \in R$, then $(s(x), s(y)) \in R^{(0)}, (t(x), t(y)) \in R^{(0)}$, and $(x^{-1}, y^{-1}) \in R$. In this case, the *quotient graph* Γ/R is the graph with vertex set $V(\Gamma)/R^{(0)}$ and edge set $\Gamma^{(1)}/R^{(1)}$ equipped with the unique source, target, and edge inversion maps such that the natural map $\nu : \Gamma \longrightarrow \Gamma/R$ is a map of graphs.

The *kernel* of a map of graphs $f : \Gamma \to \Delta$ is the equivalence relation $\ker f$ on Γ defined on the vertices and edges by:

$$(\ker f)^{(i)} = \{(x, y) \in \Gamma^{(i)} \times \Gamma^{(i)} \mid f(x) = f(y)\}, i = 0, 1.$$

As a result, we have the first isomorphism theorem for graphs. In other words, f factors through the quotient map $\nu : \Gamma \to \Gamma/\ker f$ via a unique map of graphs $g : \Gamma/\ker f \to \Delta$, and g is injective.

To construct the Stallings kernel, we start with this observation. Let R_1 and R_2 be kernels of compositions of finite sequences of foldings prescribed by $f : \Gamma \to \Delta$. Suppose $\Gamma \to \Gamma_1 \to \Gamma_2 \to \cdots \Gamma_n = \Gamma/R_2$. Then we can build a sequence of foldings $\Gamma/R_1 = \Gamma' \to \Gamma'_1 \to \Gamma'_2 \cdots \to \Gamma'_n$ prescribed by $f_1 : \Gamma' \to \Delta$ forming a commutative diagram:

$$\begin{array}{ccccccccc}
\Gamma & \longrightarrow & \Gamma_1 & \longrightarrow & \Gamma_2 & \longrightarrow & \cdots & \longrightarrow & \Gamma_n \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
\Gamma' & \longrightarrow & \Gamma'_1 & \longrightarrow & \Gamma'_2 & \longrightarrow & \cdots & \longrightarrow & \Gamma'_n
\end{array}$$

Let R_3 be the kernel of $\Gamma \to \Gamma'_n$, the composition of a finite sequence of foldings prescribed by f. Then the quotient map $\Gamma \to \Gamma/R_3 = \Gamma'_n$ factors through both $\Gamma/R_1 = \Gamma'$ and $\Gamma/R_2 = \Gamma_n$, and hence $R_1, R_2 \subseteq R_3$. Consequently, if $\{R_i\}$ is the set of all of the kernels of compositions of finite sequences of foldings prescribed by a map of graphs $f : \Gamma \to \Delta$, then $R = \bigcup R_i$, the *Stallings kernel* of f, is an equivalence relation on Γ.

Theorem 6.1.1. *Let $f : \Gamma \to \Delta$ be a map of graphs and let R be its Stallings kernel. Then f factors through the natural quotient map $\nu : \Gamma \to \Gamma/R$ via a unique map of graphs $g : \Gamma/R \to \Delta$, and g is an immersion of graphs.*

$$\begin{array}{ccc}
\Gamma & \xrightarrow{f} & \Delta \\
{\scriptstyle \nu}\downarrow & \nearrow_{g} & \\
\Gamma/R & &
\end{array}$$

Moreover, if $X \subseteq \mathrm{FIS}(\Gamma)$ is any subset, then $g(\nu(X)^{\uparrow}) = f(X)^{\uparrow}$.

Proof. Let $\{R_i\}$ be the set of all the kernels of compositions of finite sequences of foldings prescribed by f. Since each $R_i \subset \ker f$, $R = \bigcup R_i \subset \ker f$. So f factors through $\nu : \Gamma \to \Gamma/R$. Since ν is surjective, the map of graphs $g : \Gamma/R \to \Delta$ such that $g\nu = f$ is unique. Now let y_1, y_2 be edges of Γ/R with $g(y_1) = g(y_2)$ and $s(y_1) = s(y_2)$. There exists $x_1, x_2 \in \Gamma$ such that $\nu(x_1) = y_1$ and $\nu(x_2) = y_2$. Then $f(x_1) = g(\nu(x_1)) = g(y_1) = g(y_2) = g(\nu(x_2)) = f(x_2)$, and $(s(x_1), s(x_2)) \in R^{(0)}$. It follows that there exist some i such that $(s(x_1), s(x_2)) \in R_i^{(0)}$. Now $\Gamma \to \Gamma/R_i \to (\Gamma/R_i)/[\nu(x_1) = \nu(x_2)]$ is the composition of a finite sequence of foldings prescribed by f. We denote the kernel of this composition by $R_{i'} \subseteq R$. However $(x_1, x_2) \in R_{i'}^{(0)}$, and hence $y_1 = \nu(x_1) = \nu(x_2) = y_2$. Therefore, g is an immersion.

For the second part, note that $g(\nu(X)^{\uparrow}) \subseteq g(\nu(X))^{\uparrow} = f(X)^{\uparrow}$ by Lemma 3.4.5. For the reverse inclusion, $f(X) = g(\nu(X)) \subseteq g(\nu(X)^{\uparrow})$. Since g is an immersion, $g(\nu(X)^{\uparrow})$ is a closed subset of FIS(Δ) by Theorem 4.3.7. Hence $f(X)^{\uparrow} \subseteq g(\nu(X)^{\uparrow})$. □

We recall the basic properties of fundamental groupoids of graphs. Let p be a path in a graph Γ. If p' is a (possibly trivial) path obtained from p by deleting a subpath of the form (x, x^{-1}), then we write $p \searrow p'$ and say that p' is obtained from p by an *elementary reduction*. A path is *reduced* if it contains no subpath of the form (x, x^{-1}). The reflexive and transitive closure of $\searrow$ is denoted by $\searrow^*$. It turns out that given any path p, there is a reduced path r such that $p \searrow^* r$ and r is unique, we write $\mathrm{red}(p) = r$.

It is noted in [17] that a folding map $\alpha : \Gamma \to \Gamma/[x_1 = x_2]$ is surjective on fundamental groups. More generally, if $v_1, v_2 \in V(\Gamma)$ and q is a path in $\Gamma/[x_1 = x_2]$ from $\alpha(v_1)$ to $\alpha(v_2)$ then there exists a path p in Γ from v_1 to v_2 such that $\alpha(p) \searrow^* q$. By iteration, this property holds also when α is the composition of any finite sequence of foldings. Furthermore, we extend this to any quotient of Γ by the Stallings kernel of a map of graphs.

Lemma 6.1.2. *Let $f : \Gamma \to \Delta$ be a map of graphs, let R be its Stallings kernel, and let $\nu : \Gamma \to \Gamma/R$ be the natural map. If $v_1, v_2 \in V(\Gamma)$ and q is a path in Γ/R from $\nu(v_1)$ to $\nu(v_2)$, then there exists a path p in Γ from v_1 to v_2 such that $\nu(p) \searrow^* q$.*

Proof. Let $q = (y_1, y_2, \cdots, y_n)$ be a path in Γ/R from v_1 to v_2. Then there exists $x_i \in \Gamma$ such that $\nu(x_i) = y_i$ for $i = 1, \cdots, n$. Note that $(s(x_1), v_1) \in R^{(0)}$, $(t(x_n), v_2) \in R^{(0)}$, and $(t(x_i), s(x_{i+1})) \in R^{(0)}$. Hence, there exists $R_0 \subseteq R$, the kernel of a finite sequence of foldings prescribed by f, such that $(s(x_1), v_1) \in R_0$, $(t(x_n), v_2) \in R_0$ and $(t(x_i), s(x_{i+1})) \in R_0$ for all $i = 1, \cdots, n-1$. Let $\nu_0 : \Gamma \to \Gamma/R_0$ be the natural map. Then, $p' = (\nu_0(x_1), \cdots, \nu_0(x_n))$ is a path in Γ/R_0 from $\nu_0(v_1)$ to $\nu_0(v_2)$. Hence, there exists a path p in Γ from v_1 to v_2 such that $\nu_0(p) \searrow^* p'$. Since $\nu : \Gamma \to \Gamma/R$ factors through $\nu_0 : \Gamma \to \Gamma/R_0$, it follows that $\nu(p) \searrow^* q$. □

Consequently, we obtain a couple of applications of Theorem 6.1.1 that we use later. Notice that elementary reduction means deleting round-trip paths, and round-trip paths represent idempotents. Hence, if $p \searrow^* q$, it follows that $\bar{p} \leq \bar{q}$.

Theorem 6.1.3. *In the situation of Theorem 6.1.1, for any non-empty $V_0 \subseteq V(\Gamma)$,*

$$g(\mathrm{FIS}(\Gamma/R, \nu(V_0))) = f(\mathrm{FIS}(\Gamma, V_0))^{\uparrow}.$$

Proof. First, let q be a path in Γ/R where $s(q)$ and $t(q)$ are in $\nu(V_0)$. By the Lemma above, there exists a path p in Γ such that $s(p), t(p) \in V_0$ and $\nu(p) \searrow^* q$. Then $\nu(\bar{p}) \leq \bar{q}$. Thus, $f(\bar{p}) = g(\nu(\bar{p})) \leq g(\bar{q})$. Therefore, $g(\bar{q}) \in f(\mathrm{FIS}(\Gamma, V_0))^{\uparrow}$.

Conversely, $f(\mathrm{FIS}(\Gamma, V_0)) = g(\nu(\mathrm{FIS}(\Gamma, V_0))) \subseteq g(\mathrm{FIS}(\Gamma/R, \nu(V_0)))$. Since g is an immersion and $\mathrm{FIS}(\Gamma/R, \nu(V_0))$ is a closed subset of $\mathrm{FIS}(\Gamma/R)$, by Theorem 4.3.7, $g(\mathrm{FIS}(\Gamma/R, \nu(V_0)))$ is closed in $\mathrm{FIS}(\Delta)$. Therefore $f(\mathrm{FIS}(\Gamma, V_0))^{\uparrow} \subseteq g(\mathrm{FIS}(\Gamma/R, \nu(V_0)))$.

□

6.2 Subdivisions

Suppose we have a graph Γ and x is an edge of Γ. A *subdivision* of Γ is a graph obtained from Γ by subdividing each of its edges; roughly speaking, we replace x with some combinatorial arc subdividing the edge x, as shown in Figure 6.1. Formally, a graph obtained from Γ by *subdividing the edge* x is a graph of the form $\Gamma' = (\Gamma \setminus \{x^{\pm 1}\}) \cup \{x_1^{\pm 1}, \cdots, x_n^{\pm 1}\}$ with $V(\Gamma') = V(\Gamma) \cup \{v_1, \cdots, v_{n-1}\}$, where $v_1, \cdots, v_{n-1}$ are the new vertices (not in $V(\Gamma)$) and $x_1^{\pm 1}, \cdots, x_n^{\pm 1}$ are new edges (not in Γ). The source, target, and edge inversion maps on $\Gamma \setminus \{x^{\pm 1}\}$ are extended to Γ' so that: $s(x_1) = s(x), t(x_i) = s(x_{i+1}) = v_i$ for $1 \leq i \leq n-1$, and $t(x_n) = t(x)$.

In this situation, there is an involution-preserving map of directed graphs $\mathrm{sd} : \Gamma \to (\Gamma')^+$, called the corresponding *subdivision map*, such that $\mathrm{sd}(x) = (x_1, \cdots, x_n)$, where $x_1, \cdots, x_n$ are the new edges in Γ' subdividing the edge x (for each edge x of Γ).

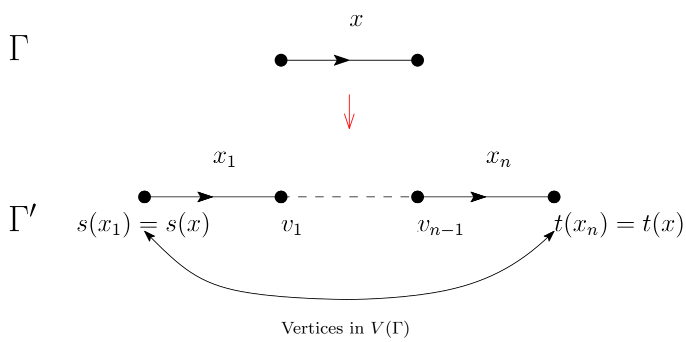


Figure 6.1: Γ' is subdividing the edge $x \in \Gamma$.

Lemma 6.2.1. *Let Γ be a graph and Γ' be a subdivision of Γ with subdivision map $\mathrm{sd} : \Gamma \to (\Gamma')^+$. If q is a path in Γ' with $s(q), t(q) \in V(\Gamma)$, then there exists a path p in Γ such that $\mathrm{sd}(p) \searrow^* q$.*

Proof. Let $q = (q_1, \cdots, q_n)$ be a path in Γ' with $s(q_i), t(q_i) \in V(\Gamma)$ but all of its intermediate vertices are in $V(\Gamma') \backslash V(\Gamma)$. For each i, there is an edge $x \in \Gamma$ such that:

Case 1: if q is not null-homotopic, set $p_i = xx^{-1} \cdots xx^{-1}x$;

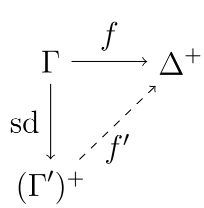
Case 2: if q is null-homotopic, set $p_i = xx^{-1} \cdots xx^{-1}$;

then it is easy to see that $\mathrm{sd}(p_i) \searrow^* q_i$. Put $p = (p_1, \cdots, p_n)$, we have p is a path in Γ and $\mathrm{sd}(p) \searrow^* q_1 \cdots q_n = q$. □

For the next theorem, as usual, we denote the homomorphisms of free inverse semigroupoids induced by f, f', and sd by the same symbols.

Theorem 6.2.2. *Let $f : \Gamma \to \Delta^+$ be an involution-preserving map of directed graphs and let Γ' be the subdivision of Γ that admits a map of graphs $f' : \Gamma' \to \Delta$ such that $f'\,\mathrm{sd} = f$. If $V_0 = V(\Gamma) \subseteq V(\Gamma')$, then*

$$f(\mathrm{FIS}(\Gamma)) \subseteq f'(\mathrm{FIS}(\Gamma', V_0)) \subseteq f(\mathrm{FIS}(\Gamma))^{\uparrow}.$$



Proof. It is clear that the above graph is commutative, so $f(\mathrm{FIS}(\Gamma)) = f'(\mathrm{sd}(\mathrm{FIS}(\Gamma))) \subseteq f'(\mathrm{FIS}(\Gamma', V_0))$. For the second inclusion, we let q be a path in Γ' where $s(q), t(q) \in V_0$. By the previous Lemma, there exists a path $p \in \Gamma$ such that $\mathrm{sd}(p) \searrow^* q$. Then $f(p) = f'(\mathrm{sd}(p)) \searrow^* f'(q)$ and so $f(\bar{p}) \leq f'(\bar{q})$. Hence $f'(\bar{q}) \in f(\mathrm{FIS}(\Gamma))^{\uparrow}$. $\square$

CHAPTER 7

CLOSED INVERSE SUBSEMIGROUPOIDS OF INVERSE SEMIGROUPOIDS

We have introduced basic definitions and properties of inverse subsemigroupoids in Section 3.4. The aim of this chapter is to extend these well-known properties to closed inverse subsemigroupoids of free inverse semigroupoids and closed inverse subsemigroupoids of inverse semigroupoids. We are able to do this by focusing on the full inverse subsemigroupoid. Moreover, with motivation from [7], we prove that every finitely generated closed inverse semigroupoid of a free inverse semigroupoid F has finite index, whether or not F is finitely generated.

7.1 Closed Inverse Subsemigroupoids of Free Inverse Semigroupoids

In this section, we characterize the closed inverse subsemigroupoids of a free inverse semigroupoid using immersion of graphs. Consider the full inverse subsemigroupoid of $\mathrm{FIS}(\Gamma)$ with vertex set $V_0 \subseteq V(\Gamma)$ consisting of all $a \in \mathrm{FIS}(\Gamma)$ with $s(a), t(a) \in V_0$. We know $\mathrm{FIS}(\Gamma, V_0)$ is a closed inverse subsemigroupoid of $\mathrm{FIS}(\Gamma)$. However, for any map of graphs $f : \Gamma \to \Delta$, if A is an inverse subsemigroupoid of $\mathrm{FIS}(\Gamma)$, then $f(A)$ need not be a subsemigroupoid of $\mathrm{FIS}(\Delta)$ as noted in Section 2.1. This difficulty is avoided by a full subsemigroupoid $\mathrm{FIS}(\Gamma, V_0)$ with vertex set $V_0 \subseteq V(\Gamma)$ for which the restriction of the vertex map $f\,|_{V_0}\colon V_0 \to V(\Delta)$ is injective. Hence, if $a_1, a_2 \in \mathrm{FIS}(\Gamma, V_0)$ and $f(a_1)f(a_2)$ is defined, then also a_1a_2 is defined and thus $f(a_1)f(a_2) = f(a_1a_2) \in f(\mathrm{FIS}(\Gamma, V_0))$.

We begin with a slight extension of Corollary 4.3.8.

Theorem 7.1.1. *Let $g : \Gamma \to \Delta$ be an immersion of graphs and let V_0 be a non-empty subset of $V(\Gamma)$ such that $g^{(0)}|_{V_0} : V_0 \to V(\Delta)$ is injective. Then the induced homomor-*

phism $g : \mathrm{FIS}(\Gamma, V_0) \to \mathrm{FIS}(\Delta)$ *is injective and its image* $g(\mathrm{FIS}(\Gamma, V_0))$ *is a closed inverse subsemigroupoid of* $\mathrm{FIS}(\Delta)$.

Proof. Let $a, b \in \mathrm{FIS}(\Gamma, V_0)$ and suppose that $g(a) = g(b)$. Since $g^{(0)}|_{V_0}$ is injective, $s(a) = s(b)$. By Theorem 4.3.7, g is locally injective and so $a = b$. Since $\mathrm{FIS}(\Gamma, V_0)$ is a closed subset of $\mathrm{FIS}(\Gamma, V_0)$ and g is an immersion, it also follows from Theorem 4.3.7 that $g(\mathrm{FIS}(\Gamma, V_0))$ is a closed subset of $\mathrm{FIS}(\Delta)$. Therefore, $g(\mathrm{FIS}(\Gamma, V_0))$ is a closed inverse subsemigroupoid of $\mathrm{FIS}(\Delta)$ from the condition on the restriction of vertex map. □

The following is one of our main results of this section. We show that the converse of Theorem 7.1.1 holds.

Theorem 7.1.2. *Let* Δ *be a graph and let* A *be a closed inverse subsemigroupoid of* $\mathrm{FIS}(\Delta)$. *Then there exists a graph* Γ*, an immersion* $g : \Gamma \to \Delta$*, and a non-empty subset* $V_0 \subseteq V(\Gamma)$ *such that* $g^{(0)} |_{V_0}: V_0 \to V(\Delta)$ *is injective and* $g(\mathrm{FIS}(\Gamma, V_0)) = A$.

Proof. Let X be any subset of $\mathrm{FIS}(\Delta)$ such that $\langle X \rangle^{\uparrow} = A$. We view X as a sub-directed graph of A with $V(X) = s(X) \cup t(X)$. Let X^{-1} be a disjoint copy of X and in one-to-one correspondence with X by $x \mapsto x^{-1}$. We construct a graph Γ by letting $\Gamma = X \cup X^{-1}$ where $V(\Gamma) = V(X)$ and every edge $x \in \Gamma$ has $s(x) = t(x^{-1})$ and $t(x) = s(x^{-1})$. Note that $V(\Gamma) = V(X)$ and that there is a unique involution-preserving map of directed graphs $\Gamma \to \mathrm{FIS}(\Delta)$ which restricts to the identity map on X.

Construct a map of directed graphs $f : \Gamma \to \Delta^+$ as follows. Let $f^{(0)} : V(\Gamma) \to V(\Delta^+)$ be the inclusion map of $V(\Gamma) = V(X)$ into $V(\Delta^+) = V(\Delta)$. To define $f = f^{(1)}$, for each $x \in X$, choose a path $p_x \in \Delta$ such that $\overline{p_x} = x$; define $f(x) = p_x$ and $f(x^{-1}) = p_x^{-1}$. Thus, $f : \Gamma \to \Delta^+$ is a map of directed graphs and involution preserving. Furthermore, the image of the induced homomorphism $f : \mathrm{FIS}(\Gamma) \to \mathrm{FIS}(\Delta)$ is $\langle X \rangle$.

Next, let Γ' be the subdivision of Γ that admits a map of graphs $f' : \Gamma' \to \Delta$ such that $f' \mathrm{sd} = f$ where $\mathrm{sd} : \Gamma \to (\Gamma')^+$. Choose $V_0 = V(\Gamma) \subseteq V(\Gamma')$. By Theorem 6.2.2, $f'(\mathrm{FIS}(\Gamma', V_0))^{\uparrow} = f(\mathrm{FIS}(\Gamma))^{\uparrow} = \langle X \rangle^{\uparrow} = A$.

Now let R be the Stallings kernel of f'. By Theorem 6.1.1, f' factors through the quotient map $\nu : \Gamma' \to \Gamma'/R$ via a unique map of graphs $g : \Gamma'/R \to \Delta$. Then g is an immersion of graphs (by Theorem 6.1.1), and $g(\text{FIS}(\Gamma'/R, \nu(V_0))) = f'(\text{FIS}(\Gamma', V_0))^{\uparrow} = A$ (by Theorem6.1.3).

Finally, note that $(gv)^{(0)} \mid_{V_0} = (f')^{(0)} \mid_{V_0} = f^{(0)} \mid_{V_0}: V_0 \hookrightarrow V(\Delta)$, and so $g^{(0)} \mid_{\nu(V_0)}: \nu(V_0) \to V(\Delta)$ is injective. Therefore, $g : \Gamma'/R \to \Delta$ and $\nu(V_0)$ are the desired immersion and subset of vertices. □

Corollary 7.1.3. *Every closed inverse subsemigroupoid A of a free inverse semigroupoid* $\text{FIS}(\Delta)$ *is isomorphic to a full subsemigroupoid* $\text{FIS}(\Gamma, V_0)$ *of the free inverse semigroupoid on* $\text{FIS}(\Gamma)$ *for some graph Γ with some non-empty vertex set $V_0 \subseteq V(\Gamma)$.*

Proof. By Theorem 7.1.2, there exists a graph Γ, an immersion $g : \Gamma \to \Delta$, and a subset $V_0 \subseteq V(\Gamma)$ such that $g^{(0)} \mid_{V_0}: V_0 \to V(\Delta)$ is injective and $g(\text{FIS}(\Gamma, V_0)) = A$. It follows from Theorem 7.1.1, the induced homomorphism $g : \text{FIS}(\Gamma, V_0) \to \text{FIS}(\Delta)$ is a monomorphism. Hence, g is an isomorphism of $\text{FIS}(\Gamma, V_0)$ onto A. □

Thus, we may view a closed inverse subsemigroupoid A of a free inverse semigroupoid $\text{FIS}(\Delta)$ as the full subsemigroupoid with vertex set V_0 of some free inverse semigroupoid $\text{FIS}(\Gamma)$. We can construct a set of generators for A by the following familiar method. We are only interested in the components of Γ that contain a vertex in V_0, and so all other components can be deleted. In other words, we may assume that the quotient graph Γ/V_0, obtained by identifying the vertices in V_0 as a single vertex, is connected.

Choose a maximal forest Γ_0 of Γ; i.e., the union of choices of a maximal tree in each component of Γ. Also choose a subset $V_0' \subseteq V_0$ consisting of one vertex from each component of Γ. Then for each $x \in \Gamma$, there are unique reduced paths p and q in Γ_0 such that $s(p) = s(q) \in V_0'$, $t(p) = s(x)$, and $t(q) = t(x)$. Put $a(x) = \bar{p}\bar{x}\bar{q}^{-1} \in \text{FIS}(\Gamma)$, and note that $a(x) \in \text{FIS}(\Gamma, V_0) = A$. Let X be the subset of $\text{FIS}(\Gamma)$ containing the elements $a(x)$, for all $x \in \Gamma$, and the elements of the form $\bar{p}$, for each reduced path $p \in \Gamma_0^+$ with $s(p) \in V_0'$

and $t(p) \in V_0$:

$$X = \{a(x) \mid x \in \Gamma\} \cup \{\bar{p} \mid p \in \Gamma_0^+ \text{ is reduced, } s(p) \in V_0',\, t(p) \in V_0\}.$$

Note that $X \subseteq A$, and so $\langle X \rangle^{\uparrow} \subseteq A$. We claim that $\langle X \rangle^{\uparrow} = A$. The example below focusing on a vertex v is to give an idea to prove the claim.

Consider the graph Γ of Figure 7.1. Let $V_0 = \{v, v_1, v_2\} \subseteq V(\Gamma)$. Choose the maximal tree as the highlighted path in Γ and $V_0' = \{v\}$. Then $X = \{aa^{-1}, ab^{-1}ba^{-1}, ab^{-1}c^{-1}d^{-1}a^{-1}, ade, e^{-1}e, ab^{-1}, ad, add^{-1}a^{-1}, e^{-1}\}$.

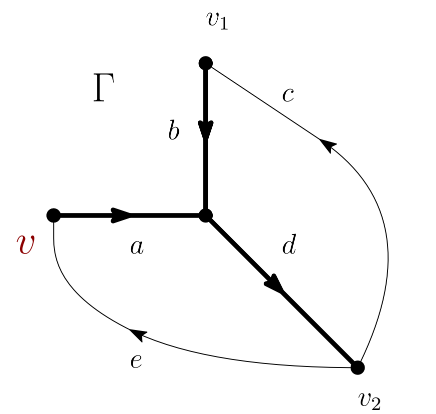


Figure 7.1: The maximal tree is the highlighted path in Γ

If $p = c^{-1}e$, then $s(p), t(p) \in V_0$ and we have $p_0 = ab^{-1}, p_1 = ad, p_2 =$ trivial . So $q = p_0^{-1}p_0c^{-1}p_1^{-1}p_1e$ and $\bar{q} \in\in \langle X \rangle$. Furthermore, $q = (p_0^{-1}p_0)c^{-1}(p_1^{-1}p_1)e$; by Lemma 3.3.7, $p \geq q$ and so $\bar{p} \in \langle X \rangle^{\uparrow}$.

In general, let $p = (x_1, \cdots, x_n)$ be a path in Γ with $s(p)$, $t(p) \in V_0$. Let $p_0, \cdots, p_n$ be the unique reduced paths in the forest Γ_0 such that $s(p_0) = \cdots = s(p_n) \in V_0'$, $t(p_0) = s(x_1)$, and $t(p_i) = t(x_i)$ for $1 \leq i \leq n$, some of the p_i may be trivial paths. Then

$$q = p_0^{-1}(p_0x_1p_1^{-1})(p_1x_2p_2^{-1}) \cdots (p_{n-1}x_np_n^{-1})p_n$$

is a path in Γ and $\bar{q} \in \langle X \rangle$. Moreover, $\bar{p} \geq \bar{q}$ and so $\bar{p} \in \langle X \rangle^{\uparrow}$. It follows that $A \subseteq \langle X \rangle^{\uparrow}$, establishing our claim.

Corollary 7.1.4. *A closed inverse subsemigroupoid A of a free inverse semigroupoid* FIS(Δ) *is finitely generated if and only if there exists an immersion of graphs $g : \Gamma \to \Delta$, where Γ is a finite graph with a subset $V_0 \subseteq V(\Gamma)$, such that $g^{(0)} |_{V_0}: V_0 \to V(\Delta)$ is injective and $g(\mathrm{FIS}(\Gamma, V_0)) = A$.*

Proof. Suppose $A = \langle X \rangle^{\uparrow}$ where X is a finite subset of FIS(Δ). Then using the construction from Theorem 7.1.2, Γ is finite and the conclusion follows with the desired subset $V_0 \subseteq V(\Gamma)$ and immersion $g : \Gamma \to \Delta$. Conversely, assume that there exists an immersion of graphs $g : \Gamma \to \Delta$, where Γ is a finite graph with a subset $V_0 \subseteq V(\Gamma)$, such that $g^{(0)} |_{V_0}: V_0 \to V(\Delta)$ is injective and $g(\text{FIS}(\Gamma, V_0)) = A$. By the previous Corollary, A is isomorphic to a full semigroupoid $\text{FIS}(\Gamma, V_0)$. Thus, the generating set X constructed by the method described by choosing a maximal forest is a finite set. □

We turn now to the uniqueness of the immersion representing a closed inverse subsemigroupoid of a free inverse semigroupoid. As noted above, if $g : \Gamma \to \Delta$ is an immersion and $V_0 \subseteq V(\Gamma)$ is a set of vertices representing a closed inverse subsemigroupoid A of $\text{FIS}(\Delta)$ as in Theorem 7.1.2, then, by deleting the components of Γ that do not contain any element of V_0, we may assume that the quotient graph Γ/V_0 is connected. We then obtain the following result.

Theorem 7.1.5. *Let $g_1 : \Gamma_1 \to \Delta$ and $g_2 : \Gamma_2 \to \Delta$ be two immersions of graphs into a graph Δ and let V_1, V_2 be subsets of the vertex sets of Γ_1, Γ_2 respectively such that $g_1^{(0)} |_{V_1}: V_1 \to V(\Delta)$ and $g_2^{(0)} |_{V_2}: V_2 \to V(\Delta)$ are injective. If Γ_1/V_1, Γ_2/V_2 are connected and $g_1(\text{FIS}(\Gamma_1, V_1) = g_2(\text{FIS}(\Gamma_2, V_2))$, then $g_1 : \Gamma_1 \to \Delta$ and $g_2 : \Gamma_2 \to \Delta$ are equivalent maps into Δ.*

Proof. Let $A = g_1(\text{FIS}(\Gamma_1, V_1) = g_2(\text{FIS}(\Gamma_2, V_2))$ and $v \in V(A)$. Then there exists $v_1 \in V(\Gamma_1)$ and $v_2 \in V(\Gamma_2)$ such that $g_i(v_i) = v$ for $i = 1, 2$. Since Γ_1/V_1, Γ/V_2 are connected and $g_1(\text{FIS}(\Gamma_1, v_1) = g_2(\text{FIS}(\Gamma_2, v_2)) = A_v$, the restrictions of g_1 and g_2 to the components containing v_1 and v_2 are equivalent maps into Δ by Theorem 4.3.10. But, $g_1^{(0)} |_{V_1}$ and $g_2^{(0)} |_{V_2}$ are injective, so there is a one-to-one correspondence between the components of Γ_1 and Γ_2 such that the restrictions of g_1 and g_2 to corresponding components are equivalent maps into Δ. Putting all these equivalences together yields an equivalence of the maps $g_1 : \Gamma_1 \to \Delta$ and $g_2 : \Gamma_2 \to \Delta$. □

Recall that the Nielsen-Schreier theorem states that every subgroup of a free group is itself a free group. In other words, every subgroup of the fundamental group of a graph is itself the fundamental group of some graph. In our situation, we find the analogue that holds for vertex inverse semigroups of graphs.

Corollary 7.1.6. *Let Δ be a graph and let w be a vertex of Δ. If A is a closed inverse subsemigroup of* $\mathrm{FIS}(\Delta, w)$*, then there exists a graph Γ with a vertex v such that $A \cong$* $\mathrm{FIS}(\Gamma, v)$.

Proof. Since A is also closed in $\mathrm{FIS}(\Delta)$, by Theorem 7.1.2, there exists a graph Γ, an immersion $g : \Gamma \to \Delta$, and a subset $V_0 \subseteq V(\Gamma)$ such that $g^{(0)}|_{V_0} : V_0 \to V(\Delta)$ is injective and $g(\mathrm{FIS}(\Gamma, V_0)) = A$. Since $w \in V(\Delta)$, we can find $v \in V_0$ such that $g(v) = w$. It follows from the fact that g is an immersion and Theorem 7.1.1 (set $V_0 = \{v\}$), we have $g : \mathrm{FIS}(\Gamma, v) \to \mathrm{FIS}(\Delta, w)$ is injective. Additionally, A is isomorphic to a vertex subsemigroupoid $\mathrm{FIS}(\Gamma, v)$ (Corollary 7.1.3). Therefore, $A \cong \mathrm{FIS}(\Gamma, v)$. □

Moreover, the immersion is unique by Theorem 7.1.5. Applying the results of this section, we observe that vertex inverse semigroupoids play the same role in the theory of immersions that fundamental groups play in the theory of coverings; see [7] for similar results using free inverse monoids.

7.2 Intersections of Closed Inverse Subsemigroupoids

Let Γ_1, Γ_2, and Δ be any graphs. Given maps of graphs $f_1 \colon \Gamma_1 \to \Delta$ and $f_2 \colon \Gamma_2 \to \Delta$. Then the *product graph* $\Gamma_1 \times \Gamma_2$ is defined as follows. The vertex set of $\Gamma_1 \times \Gamma_2$ is the set $V(\Gamma_1) \times V(\Gamma_2)$. For a pair of vertices (v_1, v_2), $(u_1, u_2) \in V(\Gamma_1 \times \Gamma_2)$, $v_1, u_1 \in V(\Gamma_1)$ and $v_2, u_2 \in V(\Gamma_2)$. So if there is an edge, labeled p, from v_1 to u_1 in Γ_1 and there is an edge, also labeled p, from v_2 to u_2 in Γ_2, then p labels an edge with source (v_1, v_2) and target (u_1, u_2).

Let Γ be the subgraph of the product graph $\Gamma_1 \times \Gamma_2$ with vertex set

$$V(\Gamma) = \{(v_1, v_2) \in V(\Gamma_1) \times V(\Gamma_2) \mid f_1(v_1) = f_2(v_2)\}$$

and edge set

$$\Gamma = \Gamma^{(1)} = \{(y_1, y_2) \in \Gamma_1 \times \Gamma_2 \mid f_1(y_1) = f_2(y_2)\}.$$

The projection maps of Γ to Γ_1 and Γ_2 form a commutative diagram

$$\begin{array}{ccc} \Gamma & \longrightarrow & \Gamma_1 \\ \big\downarrow & & \big\downarrow f_1 \\ \Gamma_2 & \xrightarrow[f_2]{} & \Delta \end{array}$$

called a *pullback diagram* of graphs and the composition $f : \Gamma \to \Delta$ is the *pullback* of f_1 and f_2. It is easy to see that if f_1 and f_2 are immersions, then so is their pullback f. Furthermore, pullbacks of immersions represent intersections of closed inverse subsemigroupoids of a free semigroupoid in the following way. See [17, Theorem 5.5] for the group-theoretic analogue.

Theorem 7.2.1. *For $i = 1, 2$, let $g_i \colon \Gamma_i \to \Delta$ be an immersion of graphs with a subset $V_i \subseteq V(\Gamma_i)$ such that $g_i^{(0)}|_{V_i} \colon V_i \to V(\Delta)$ is injective, and put $A_i = g_i\big(\operatorname{FIS}(\Gamma_i, V_i)\big)$. Let $g \colon \Gamma \to \Delta$ be the pullback of g_1 and g_2 and let $V_0 = \{(v_1, v_2) \in V_1 \times V_2 \mid g_1^{(0)}(v_1) = g_2^{(0)}(v_2)\} \subseteq V(\Gamma)$. Then $g^{(0)}|_{V_0} \colon V_0 \to V(\Delta)$ is injective and $g\big(\operatorname{FIS}(\Gamma, V_0)\big) = A_1 \cap A_2$.*

Proof. It is clear that $g^{(0)}|_{V_0} \colon V_0 \to V(\Delta)$ is injective. For the second part, we let $w \in A_1 \cap A_2$ and choose a path $q \in \Delta^+$ such that $\bar{q} = w$. Then, there exists a path $p_i \in \Gamma_1$ with $s(p_i)$ and $t(p_i)$ are in $V(\Gamma_i)$ such that $g_i(p_i) = q$ for $i = 1, 2$. It follows that if $p = (p_1, p_2)$ then $s(p), t(p) \in V_0$ and $g(p) = q$. Thus $w = \bar{q} = g(\bar{p}) \in g(\operatorname{FIS}(\Gamma, V_0))$. Hence $A_1 \cap A_2 \subseteq g\big(\operatorname{FIS}(\Gamma, V_0)\big)$. To establish the reverse, note that g factors through the projection onto Γ_i, which maps V_0 to V_i. This implies $g\big(\operatorname{FIS}(\Gamma, V_0)\big) \subseteq g_i\big(\operatorname{FIS}(\Gamma_i, V_i)\big) = A_i$ for each $i = 1, 2$, and thus $g\big(\operatorname{FIS}(\Gamma, V_0)\big) = A_1 \cap A_2$. □

Recall that Howson's theorem [1] states that the intersection of two finitely generated subgroups of a free group is itself finitely generated. The technique used in [17] to prove Howson's theorem can be adapted to our situation yielding the following result.

Corollary 7.2.2. *If A_1 and A_2 are finitely generated closed inverse subsemigroupoids of a free inverse semigroupoid* $\mathrm{FIS}(\Delta)$ *and $A_1 \cap A_2$ is nonempty, then the closed inverse subsemigroupoid $A_1 \cap A_2$ is also finitely generated.*

Proof. For $i = 1, 2$, by Theorem 7.1.2, we can construct Γ_i with a subset $V_i \subseteq V(\Gamma_i)$ and an immersion $g_i : \Gamma_i \to \Delta$ such that $g_i^{(0)}|_{V_i} : V_i \to V(\Delta)$ is injective and $g_i\big(\mathrm{FIS}(\Gamma_i, V_i)\big) = A_i$. But, the pullback $g : \Gamma \to \Delta$ of g_1 and g_2 is also an immersion, and Γ is finite since Γ_i are finite. Moreover, by Theorem 7.2.1, there is a subset $V_0 \subseteq V(\Gamma)$ such that $g^{(0)}|_{V_0} : V_0 \to V(\Delta)$ is injective and $g\big(\mathrm{FIS}(\Gamma, V_0)\big) = A_1 \cap A_2$. Clearly, $A_1 \cap A_2$ is a closed inverse subsemigroup of $\mathrm{FIS}(\Delta)$. By Corollary 7.1.4, $A_1 \cap A_2$ is finitely generated. □

7.3 Index of Closed Inverse Subsemigroupoids

Now, we instigate the index of a closed inverse subsemigroupoid A' of an inverse semigroupoid A. Define an equivalence relation on the set

$$A_0 = \{a \in A \mid aa^{-1} \in A'\}$$

as follows: for all $a, b \in A_0$, write $a \sim b$ if and only if $ab^{-1} \in A'$. Note in particular that if $a \sim b$, then $t(a) = t(b)$. It is clear that $\sim$ is reflexive and symmetric. Transitivity uses the assumption that A' is closed: if $ab^{-1} \in A'$ and $bc^{-1} \in A'$, then $ab^{-1}bc^{-1} \in A'$ and $b^{-1}b$ is an idempotent, and thus $ac^{-1} \in (A')^{\uparrow} = A'$, by Lemma 3.3.7.

Lemma 7.3.1. *For all $a \in A_0$, the equivalence class of a is the set $(A'a)^{\uparrow}$, where $A'a$ is the set of all products $a'a$ such that $a' \in A'$ and $t(a') = s(a)$.*

Proof. Suppose $b \in A_0$ and $b \sim a$. Then $ba^{-1} \in A'$ and so $ba^{-1}a \in A'a$. But $a^{-1}a$ is an idempotent in A. Thus $b \in (A'a)^{\uparrow}$.

Conversely, assume that $b \in (A'a)^{\uparrow}$. Then there exists $e \in E(A)$ and $a' \in A'$ such that $eb = a'a$. Note that $ebb^{-1}e = eb(eb)^{-1} = a'(aa^{-1})(a')^{-1} \in A'$ and $a'aa^{-1} = eba^{-1} \in A'$ Hence, $ba^{-1} \in A'$ and $bb^{-1} \in A'$ since A' is closed (by Lemma 3.3.7). That is, $b \in A_0$ and $b \sim a$. □

The equivalence classes $(A'a)^{\uparrow} = \{b \mid a'a \leq b, a' \in A\}$, where $a \in A_0$, are called the (right) *cosets of* A' *in* A. Being the equivalence classes of $\sim$ on A_0, these cosets have the following elementary properties:

Lemma 7.3.2. *Let A' be a closed inverse subsemigroupoid of an inverse semigroupoid A. For all $a, b \in A_0$, the following statements are equivalent:*

1. $(A'a)^{\uparrow} = (A'b)^{\uparrow}$
2. $ba^{-1} \in A'$
3. $b \in (A'a)^{\uparrow}$
4. $a \in (A'b)^{\uparrow}$

We also have additional properties: For all $a \in A_0$,

1. $a \in (A'a)^{\uparrow}$;
2. $(A'a)^{\uparrow} = A'$ if and only if $a \in A'$.

The cardinality of the set of all cosets of A' is called the *index of* A' *in* A and is denoted $|A : A'|$. That is, $|A : A'| = |A_0/{\sim}|$. This is an obvious generalization of the notion of the index of a subgroup of a group. Indeed when $A = G$ is a group, every subgroup H of G is closed and $A_0 = \{g \in G \mid gg^{-1} \in H\} = G$. Also the right cosets of H in G are the equivalence classes of the equivalence relation on G given by $g \sim h$ if and only if $gh^{-1} \in H$; and the cardinality of $G/{\sim}$ is the index of H in G.

We first observe that the correspondence theorem for groups generalizes to some extent for inverse semigroupoids as follows.

Theorem 7.3.3. *Let $\phi\colon A \twoheadrightarrow B$ be an epimorphism of inverse semigroupoids such that $\phi^{(0)}\colon V(A) \to V(B)$ is a bijection. Let B' be a closed inverse subsemigroupoid of B and let $A' = \phi^{-1}(B')$. Then A' is a closed inverse subsemigroupoid of A and $|A : A'| = |B : B'|$.*

Proof. To show that A' is closed, we let $ea \in A'$ for some $e \in E(A)$. Then $\phi(ea) \in B'$ and so $\phi(a) \in B'$ since $\phi(e) \in E(B)$ and B' is closed. Thus, $a \in \phi^{-1}(B') = A'$. Next, let $A_0 = \{a \in A \mid aa^{-1} \in A'\}$ and $B_0 = \{b \in B \mid bb^{-1} \in B'\}$. Since ϕ is onto, $\phi(A_0) = B_0$. Furthermore, $a_1 a_2^{-1} \in A'$ if and only if $\phi(a_1)\phi(a_2)^{-1} \in B'$. Hence ϕ determines a one-to-one correspondence $\phi' : A_0/\sim \to B/\sim$ between the sets of cosets, as required. □

In the situation of Theorem 7.3.3 when $A = \text{FIS}(\Delta)$ is a free inverse semigroupoid, ϕ is equivalent to what we previously called a generating graph for B. That is, such a homomorphism is determined by its restriction to Δ, which is an involution-preserving map of directed graphs $\phi\colon \Delta \to B$ such that $\phi^{(0)}\colon V(\Delta) \to V(B)$ is a bijection and $\phi\colon \Delta^+ \to B$ is surjective. Then, applying Theorem 7.3.3, we can reduce finding the index of a closed inverse subsemigroupoid of B to finding the index of a closed inverse subsemigroupoid of a free inverse semigroupoid. For this purpose, we make the following observation.

Theorem 7.3.4. *Let $A = \text{FIS}(\Delta)$ and let A' be a closed inverse subsemigroupoid of A. Let $g\colon \Gamma \to \Delta$ be an immersion and $V_0 \subseteq V(\Gamma)$ such that Γ/V_0 is connected, $g^{(0)}|_{V_0}\colon V_0 \to V(\Delta)$ is injective, and $g\big(\text{FIS}(\Gamma, V_0)\big) = A'$. Then $|V(\Gamma)| = |A : A'|$.*

Proof. Define $f : V(\Gamma) \to A/\sim$ by $f(v) = [A' g(\bar{p})]^{\uparrow}$ as follows. Let $v \in V(\Gamma)$ and choose a path $p \in \Gamma$ with $s(p) \in V_0$ and $t(p) = v$. Then $\bar{p}\bar{p}^{-1} \in \text{FIS}(\Gamma, V_0)$ and $g(\bar{p})g(\bar{p})^{-1} \in A'$. To see that f is a well-defined function, suppose q is another path in Γ with $s(q) \in V_0$ and $t(q) = v$. Then $\bar{p}\bar{q}^{-1} \in \text{FIS}(\Gamma, V_0)$ and $g(\bar{p})g(\bar{q})^{-1} \in A'$. Thus, $[A' g(\bar{p})]^{\uparrow} = [A' g(\bar{q})]^{\uparrow}$. To complete the proof, we will show that f is a bijection.

Suppose $v_1, v_2 \in V(\Gamma)$ and $f(v_1) = f(v_2)$. Then there exist paths $p_1, p_2 \in \Gamma^+$ such that $s(p_1), s(p_2) \in V_0$, $t(p_1) = v_1$, and $t(p_2) = v_2$,. Then $[A' g(\bar{p_1})]^{\uparrow} = [A' g(\bar{p_2})]^{\uparrow}$ and so $g(\bar{p_1})g(\bar{p_2})^{-1} \in A'$. Since g is an immersion, we can find a path $p \in \Gamma$ with $s(p), t(p) \in V_0$ such that $g(p) \equiv g(p_1)g(p_2)^{-1}$. By Lemma 4.3.6, there exists a path $q \in \Gamma$ such that $p \equiv q$ and $g(q) = g(p_1)g(p_2)^{-1}$. Write $q = q_1 q_2$ where $g(q_1) = g(p_1)$ and $g(q_2) = g(p_2)^{-1}$. Note that $g^{(0)}(s(q_1)) = g^{(0)}(s(p_1))$ and since $g^{(0)}$ is injective on V_0, $s(q_1) = s(p_1)$. So $q_1 = p_1$ by

Proposition 4.3.3. Similarly, $q_2 = p_2^{-1}$. Hence $q = p_1 p_2^{-1}$. In particular, $v_1 = t(p_1) = t(p_2) = v_2$. Thus f is injective.

Let $a \in A$ such that $aa^{-1} \in A'$ and choose a path $q \in \Delta^+$ such that $\bar{q} = a$. As in the previous paragraph, there exists a path $p \in \Gamma^+$ such that $s(p) \in V_0$ and $g(p) = q$. Thus if $v = t(p)$, then $f(v) = (A'\bar{q})^{\uparrow} = (A'a)^{\uparrow}$. Hence f is also surjective. □

Combining the results of this section and previous observations, we obtain the following corollaries.

Corollary 7.3.5. *In the situation of Theorem 7.3.4, if $W = V(A')$, or more generally any subset of $V(\Delta)$ containing $V(A')$, then the index of A' in $\mathrm{FIS}(\Delta, W)$ is equal to the cardinality of $g^{-1}(W)$.*

Proof. For the bijection $f : V(\Gamma) \to A_0/\sim$ defined in the proof of Theorem 7.3.4, we note that $f(v)$ is a coset of A' in $\mathrm{FIS}(\Delta, W)$ if and only if $v \in g^{-1}(W)$. □

Consider the case of a free inverse semigroup $F = \mathrm{FIS}(\Delta)$ for some graph Δ with a single vertex. In [7], among other things, the authors prove the following result: If H is a finitely generated closed inverse subsemigroup of F, then H has finite index in F. The following is a full generalization of this result for free inverse semigroupoids:

Corollary 7.3.6. *Let Δ be a graph. Let $A = \mathrm{FIS}(\Delta)$ and let A' be a closed inverse subsemigroupoid of A. If A' is finitely generated, then $|A : A'|$ is finite.*

Proof. By Corollary 7.1.4, there exists an immersion $g : \Gamma \to \Delta$ where Γ is a finite graph with $V_0 \subseteq V(\Gamma)$ such that $g^{(0)}|_{V_0} : V_0 \to V(\Delta)$ is injective and $g\big(\mathrm{FIS}(\Gamma, V_0)\big) = A'$. It follows from Theorem 7.3.4 that $|A : A'|$ is finite. □

In [7], the authors prove that the converse of Corollary 7.3.6 also holds when F is finitely generated for the case of a free inverse semigroup $F = \mathrm{FIS}(\Delta)$, where Δ has a single vertex. Note that the converse is obviously false if F is not finitely generated; for example, F has finite index (one) in itself, however F is not finitely generated when Δ has finitely many edges. Indeed the converse of Corollary 7.3.6 is true in general when A is any finitely generated inverse semigroupoid, regardless of whether A is free or not. That is, we have the following generalization of a well-known property of groups.

Corollary 7.3.7. *Let A be any inverse semigroupoid and let A' be a closed inverse subsemigroupoid of finite index in A. If A is finitely generated, then so is A'.*

Proof. Let $\phi : \Delta \to A$ be a finite generating graph for A. By Theorem 7.3.3, $\phi^{-1}(A')$ is a closed inverse subsemigroupoid of finite index in $\mathrm{FIS}(\Delta)$. We obtain a graph Γ with an immersion $g : \Gamma \to \Delta$ and a subset $V_0 \subseteq V(\Gamma)$ where $g^{(0)}|_{V_0} : V_0 \to V(\Delta)$ is injective and $g(\mathrm{FIS}(\Gamma, V_0)) = \phi^{-1}(A')$ by Theorem 7.1.2. It follows from Theorem 7.3.4 that $V(\Gamma)$ is finite. Since g is an immersion from a graph with finite vertex set to a finite graph, Γ must be a finite graph. Applying Corollary 7.1.4, $\phi^{-1}(A')$ is finitely generated. Now, choose a finite subset $X \subseteq \mathrm{FIS}(\Gamma)$ such that $\langle X\rangle^{\uparrow} = \phi^{-1}(A')$. Lastly, we show that $\phi(X)$ generates A' to complete the proof. Note first that $\phi(X) \subseteq A'$ and A' is a closed inverse subsemigroupoid of A. So $\langle\phi(X)\rangle^{\uparrow} \subseteq A'$. On the other hand, by Lemma 3.4.5, $A' = \phi\big(\langle X\rangle^{\uparrow}\big) \subseteq \phi\big(\langle X\rangle\big)^{\uparrow} = \langle\phi(X)\rangle^{\uparrow}$. □

Recall that in the group case, it is well known that if a subgroup of a group has finite index and is finitely generated, then the group itself is finitely generated. However, in our case, this need not hold - even for inverse semigroups. For example, let Δ be an infinite graph and let Σ be a finite subgraph of Δ. Then $A' = \mathrm{FIS}(\Sigma)$ is a closed inverse subsemigroupoid of $A = \mathrm{FIS}(\Delta)$ which is finitely generated (by Corollary 7.1.4) and of finite index in A (by Corollary 7.3.6). But A is not finitely generated.

The following are additional generalizations of a well-known results for groups.

Corollary 7.3.8. *If A_1 and A_2 are closed inverse subsemigroupoids of finite index in an inverse semigroupoid A and $A_1 \cap A_2 \neq \emptyset$, then $A_1 \cap A_2$ is also a closed inverse subsemigroupoid of finite index in A. Moreover, $|A : A_1 \cap A_2| \leq |A : A_1||A : A_2|$.*

Proof. Let $\phi : \Delta \to A$ be a choice of generating graph for A. By Theorem 7.3.3, $\phi^{-1}(A_1)$, $\phi^{-1}(A_2)$, and $\phi^{-1}(A_1) \cap \phi^{-1}(A_2) = \phi^{-1}(A_1 \cap A_2)$ are closed inverse subsemigroupoids of $\mathrm{FIS}(\Delta)$ with indices $|A : A_1|$, $|A : A_2|$, and $|A : A_1 \cap A_2|$ respectively. For each $i = 1, 2$, by Theorem 7.1.2, there exists a graph Γ_i, an immersion $g_i : \Gamma_i \to \Delta$, and a subset $V_i \subseteq V(\Gamma_i)$ such that $g_i^{(0)} |_{V_i}: V_i \to V(\Delta)$ is injective and $g(\mathrm{FIS}(\Gamma, V_i)) = \phi^{-1}(A_i)$. We may assume that Γ/V_i is connected; by Theorem 7.3.4, since $|A : A_i|$ is finite, $|V(\Gamma_i)|$ is finite. Then, the pullback $g : \Gamma \to \Delta$ of g_1 and g_2 represents $\phi^{-1}(A_1 \cap A_2)$ (Theorem 7.2.1) and so $V(\Gamma)$ is finite. Therefore, by Theorem 7.3.4, $|A : A_1 \cap A_2| = |V(\Gamma)|$ is finite.

For the last statement, note that $V(\Gamma) \subseteq V(\Gamma_1) \times V(\Gamma_2)$. Thus, $|A : A_1 \cap A_2| = |V(\Gamma)| \leq |V(\Gamma_1)||V(\Gamma_2)| = |A : A_1||A : A_2|$. □

The following is an alternate proof for the second statement of Corollary 7.3.8.

Corollary 7.3.9. *If A_1 and A_2 are closed inverse subsemigroupoids of finite index in an inverse semigroupoid A and $A_1 \cap A_2 \neq \emptyset$, then $|A : A_1 \cap A_2| \leq |A : A_1||A : A_2|$.*

Proof. Let $a, b \in A$ such that $aa^{-1}, bb^{-1} \in A_1 \cap A_2$. Since $A_1 \cap A_2$ is a closed inverse subsemigroupoid, $[(A_1 \cap A_2)a]^\uparrow = [(A_1 \cap A_2)b]^\uparrow$ if and only if $ab^{-1} \in A_1 \cap A_2$ if and only if $ab^{-1} \in A_1$ and $ab^{-1} \in A_2$ if and only if $[A_1 a]^\uparrow = [A_1 b]^\uparrow$ and $[A_2 a]^\uparrow = [A_2 b]^\uparrow$ if and only if $([A_1 a]^\uparrow, [A_2 b]^\uparrow) = ([A_1 b]^\uparrow, [A_2 a]^\uparrow)$. Thus, the mapping

$$[(A_1 \cap A_2)a]^\uparrow \mapsto ([A_1 a]^\uparrow, [A_1 a]^\uparrow)$$

is well defined and an injective map from the set of cosets of $A_1 \cap A_2$ to the product of the sets of cosets of A_1 and A_2. The result follows. □

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