TWISTING BORDERED KHOVANOV HOMOLOGY

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ABSTRACT

We describe a bordered version of totally twisted Khovanov homology. We first twist Roberts's type D structure by adding a "vertical" type D structure which generalizes the vertical map in twisted tangle homology. One of the distinct advantages of our type D structure is that it is homotopy equivalent to a type D structure supported on "spanning tree" generators. We also describe how to twist Roberts's type A structure for a left tangle in such a way that pairing our type A and type D structures will result in the totally twisted Khovanov homology. Analogous to the type D structure, there is a spanning-tree-like model for the type A structure.

LIST OF ABBREVIATIONS AND SYMBOLS

$\mathbb{R}^3$	The three dimensional real vector space
$\otimes$	Tensor product
$\oplus$	Direct sum
S^3	The three dimensional sphere
$\mathbb{Z}$	The set of integers
$\mathbb{Z}_2$	The set of 2-adic integers
Σ	Summation
$\circ$	Function composition
$\boxtimes$	Box tensor product of a type A and a type D structures
APS	M. Asaeda, J. Przytycki, and A. Sikora
DGA	Differential graded algebra

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CHAPTER 1

INTRODUCTION

Khovanov Homology, defined by M. Khovanov in the late 1990s, is an invariant of oriented knots and links that arises as the homology of a chain complex, and whose graded Euler characteristics is the Jones polynomial (see [7]). More precisely, let L be a link diagram for a link $\mathcal{L}$ embedded in $\mathbb{R}^3$ and let $(C(L), \partial_{\mathcal{KH}})$ denote the Khovanov chain complex associated to L. $(C(L), \partial_{\mathcal{KH}})$ is a bigraded complex of free abelian groups with a $(1,0)$ differential $\partial_{\mathcal{KH}}$. In [7], M. Khovanov proved that the chain complex $(C(L), \partial_{\mathcal{KH}})$ is invariant up to homotopy equivalence under Reidemeister moves, and thus provides an invariant for $\mathcal{L}$. Analogous to the Kauffman's state summation approach to Jones polynomial, the generators of Khovanov complex are obtained by exponentially many ways to resolve the link diagram. By checkerboard coloring a link diagram, M. Thistlethwaite showed that the Jones polynomial can be expressed in terms of spanning trees obtained from the Tait graph ([24]). Motivated by this idea, A. Champanerkar and I. Kofman in [5], and S. Wehrli in [25] proved that there exists a chain complex, generated by spanning trees, whose homology is the reduced Khovanov homology. Unfortunately, there was no explicit formula for the differential of this complex and thus, the calculation based on this spanning tree complex was still vague.

In an attempt to find a better description for the spanning tree complex, in [19], L. Roberts defined a totally twisted version of Khovanov homology by using an idea from Heegaard Floer homology. The construction is based on the extra decorations on the diagram L: a marked point and labels x_f for each arc of L. With this extra data, $C(L)$ also has a Koszul $(0,-2)$ vertical differential $\partial_{\mathcal{V}}$ which commutes with $\partial_{\mathcal{KH}}$. $(C(L), \partial_{\mathcal{KH}} + \partial_{\mathcal{V}})$ is the totally twisted Khovanov chain complex and its homology is an invariant of $\mathcal{L}$. The distinct advantage of this complex is that it also admits a description for the deformation retraction chain subcomplex whose generators are the spanning trees and the differential can be computed explicitly from the labels $\{x_f\}$. In particular, for the case of knots, in [23], T. Jaeger proved that the totally twisted Khovanov homology is isomorphic to the reduced Khovanov homology. Therefore, we get a spanning tree complex description with an explicit differential for the reduced Khovanov homology.

Modeled from the construction of the bordered Heegaard-Floer homology given by R. Lipshitz, P. Ozsvath, D. Thurston [11], L. Roberts, in [17] and [18], described how to obtain the Khovanov homology of a link from a diagram divided into two parts, left and right tangles. To those tangles, he associated different types of tangle invariants in such a way that gluing those two invariants recovers the Khovanov homology. There are several ways to extend Khovanov homology to tangles by M. Asaeda, J. Przytycki and A. Sikora in [2], A. Lauda and H. Pfeiffer in [10], D. Bar-Natan in [4] or M. Khovanov in [8]. In [20], by using the idea of the totally twisted Khovanov homology in [19], the author and L. Roberts took the construction of M. Asaeda, J. Przytycki and A. Sikora and twisted them to obtain a tangle invariant (called twisted tangle homology).

In an effort of combining those two theories to study the reduced Khovanov homology, we will twist the Roberts' tangle invariance to get the twisted invariances for each tangle-component of a link, which recovers the reduced Khovanov homology by gluing the tangle-components. Since the cores of the underlying complexes in "bordered" Khovanov homology construction was based on the extension of Khovanov homology for tangles given by M. Asaeda, J. Przytycki, and A. Sikora [1], [2], we will take the approach of twisted skein homology, described by the author and L. Roberts.

Before we describe the chain complex, let's recall the setting of [17], [18]. Let T be a link diagram of a link $\mathcal{T}$ in $\mathbb{R}^2$, which is transverse to the y-axis. The y-axis divides T into two parts: a left tangle $\overleftarrow{T}$ and a right one $\overrightarrow{T}$. Two left (right) tangle diagrams will be equivalent if they are related by an ambient isotopy of $\mathbb{R}^2$ preserving the y-axis pointwise and a sequences of the three Reidemeister moves.

Labeling each arc f between two crossings, or the boundary, of the tangle diagram $\overrightarrow{T}$ with a formal variable x_f, we form the polynomial ring $\mathbb{P}_{\overrightarrow{T}} = \mathbb{Z}_2[x_f | f \in \text{ARC}(\overrightarrow{T})]$ where $\text{ARC}(\overrightarrow{T})$ denotes the set of arcs of $\overrightarrow{T}$. The field of fractions of $\mathbb{P}_{\overrightarrow{T}}$ will be denoted $\mathbb{F}_{\overrightarrow{T}}$. The same thing can be done for $\overleftarrow{T}$ to get $\mathbb{F}_{\overleftarrow{T}}$.

We then associate to each right (left) tangle diagram $\overrightarrow{T}$ (respectively $\overleftarrow{T}$) a bigraded vector space $[\![\overrightarrow{T} \rangle\!\rangle$ over $\mathbb{F}_{\overrightarrow{T}}$ (respectively $\langle\!\langle \overleftarrow{T}]\!]$ over $\mathbb{F}_{\overleftarrow{T}}$), generated by a collection of states (r, s), described as follows:

1. r is a pair $(\overleftarrow{m}, \rho)$ (respectively $(\rho, \overrightarrow{m})$) where $\overleftarrow{m}$ ($\overrightarrow{m}$) is a specific representative of its equivalence as a left (right) planar matching (a left (right) planar matching is a collection of n embedded arcs in the left (right) half plane whose boundaries are the intersection points of $\overrightarrow{T}$ ($\overleftarrow{T}$) and the y-axis) and ρ is a resolution of $\overrightarrow{T}$ ($\overleftarrow{T}$).

2. s is a decoration on each circle of $\overleftarrow{m} \cup \rho$ (respectively $\rho \cup \overrightarrow{m}$) by either $+$ or $-$.

In [17], L. Roberts defines a type D structure $\overrightarrow{\delta_T}$ on $[\![\overrightarrow{T} \rangle\!\rangle$ over a differential graded algebra $(\mathcal{B}\Gamma_n, \mathcal{I}_n)$ such that the homotopy class as a type D structure of $([\![\overrightarrow{T} \rangle\!\rangle, \overrightarrow{\delta_T})$ is an invariant of $\overrightarrow{\mathcal{T}}$. Inspired by the idea of the twisted tangle homology [20], we will define a $(0, -2)$ "vertical" type D structure $\overrightarrow{\delta_{\mathcal{V}}}$ on $[\![\overrightarrow{T} \rangle\!\rangle$. With respect to the bigrading of $[\![\overrightarrow{T} \rangle\!\rangle$ and $\mathcal{B}\Gamma_n$, $\overrightarrow{\delta_T}$ is a degree $(1, 0)$ map while $\overrightarrow{\delta_{\mathcal{V}}}$ is a degree $(0, -2)$ map. By collapsing the bigrading using the formula $\zeta = i - j/2$, both $\overrightarrow{\delta_T}$ and $\overrightarrow{\delta_{\mathcal{V}}}$ become degree 1 maps. Note that this is the usual δ-grading on Khovanov homology but we will not use δ here since it overlaps the notation of the type D structure $\overrightarrow{\delta_T}$. We will also prove that $\overrightarrow{\delta_\mathcal{V}}$ commutes with $\overrightarrow{\delta_T}$ in a sense that $\overrightarrow{\delta_{T,\bullet}} := \overrightarrow{\delta_T} + \overrightarrow{\delta_\mathcal{V}}$ is a type D structure on $[\![\overrightarrow{T} \rangle\!\rangle$ (see Proposition 29). Furthermore, using a trick to move weights (the formal variables labeled to the arcs) and definitions of stable A_∞-homotopy equivalence and stable D-homotopy equivalence (see Chapter 9), we will prove the following theorem in Chapter 10 :

Theorem 1. *The stable homotopy class of the type D structure* $([\![\overrightarrow{T} \rangle\!\rangle, \overrightarrow{\delta_{T,\bullet}})$ *is an isotopy invariant of the tangle defined by* $\overrightarrow{T}$

Additionally, in Chapter 7, using the cancellation lemma, we will get a type D structure homotopy equivalent to $([\![\overrightarrow{T} \rangle\!\rangle, \overrightarrow{\delta_{T,\bullet}})$ supported on states which do not contain any free circles (a free circle is the one which does not intersect with the y-axis). The collection of such states will be denoted $\textsc{st}_n(\overrightarrow{T})$. We also denote:

$$[\![\overrightarrow{CT} \rangle\!\rangle := span_{\mathbb{F}_{\overrightarrow{T}}}\{(r,s) \in \textsc{st}_n(\overrightarrow{T})\}.$$

We will describe a type D structure $\overrightarrow{\delta_{T,n}}$ on $[\![\overrightarrow{CT} \rangle\!\rangle$, which is a deformation retraction of $([\![\overrightarrow{T} \rangle\!\rangle, \overrightarrow{\delta_{T,\bullet}})$. The map

$$\overrightarrow{\delta_{T,n}} : [\![\overrightarrow{CT} \rangle\!\rangle \longrightarrow \mathcal{B}\Gamma_n \otimes_{\mathcal{I}_n} [\![\overrightarrow{CT} \rangle\!\rangle[-1]$$

is defined by specifying its image on each generator $(r,s) \in \textsc{st}_n(\overrightarrow{T})$ as

$$\overrightarrow{\delta_{T,n}}(r,s) = \sum_{(r',s')} \langle (r,s),(r',s') \rangle I_{\partial(r,s)} \otimes (r',s') + \sum_{\gamma \in \textsc{Bridge}(r)} B(\gamma)$$
$$+ \sum_{C \in \textsc{cir}(\partial(r,s)), s(C)=+} (\overleftarrow{e_C} + \overrightarrow{w_C}\overrightarrow{e_C}) \otimes (r, s_C).$$

These summands will be defined in Chapter 7.

As the consequence of Theorem 1, we have the following corollary:

Corollary 2. *The stable homotopy class of the type D structure* $([\![\overrightarrow{CT} \rangle\!\rangle, \overrightarrow{\delta_{T,n}})$ *is an invariant of* $\overrightarrow{\mathcal{T}}$.

In [18], associated to a left tangle diagram $\overleftarrow{T}$ of a tangle $\overleftarrow{\mathcal{T}}$, there exists a differential bigraded module $(\langle\!\langle \overleftarrow{T}]\!], m_1, m_2)$ over $(\mathcal{B}\Gamma_n, \mathcal{I}_n)$ (also called a type A structure in the language of bordered Heegaard Floer homology [11]) where m_1 is a modified version of the differential defined by M. Asaeda, J. Przytycki and A. Sikora and m_2 is a right action $\langle\!\langle \overleftarrow{T}]\!] \otimes_{\mathcal{I}_n} \mathcal{B}\Gamma_n \longrightarrow \langle\!\langle \overleftarrow{T}]\!]$. In Chapter 11, we will define a new type A structure $(\langle\!\langle \overleftarrow{T}]\!], m_{1,\bullet}, m_{2,\bullet})$ where $m_{1,\bullet}$ is obtained from m_1 by adding a modified version of the vertical differential in the twisted tangle homology. The only difference between the actions m_2 and $m_{2,\bullet}$ is the action of left decoration elements on $\langle\!\langle \overleftarrow{T}]\!]$. In Chapter 13, we will sketch a proof which shows that $(\langle\!\langle \overleftarrow{T}]\!], m_{1,\bullet}, m_{2,\bullet})$ is an invariant of $\overleftarrow{\mathcal{T}}$ in the category of A_∞ modules:

Theorem 3. *Let $\overleftarrow{\mathcal{T}}$ be a left tangle with a diagram $\overleftarrow{T}$. The stable homotopy class of $(\langle\!\langle \overleftarrow{T}]\!], m_{1,\bullet}, m_{2,\bullet})$ as an A_∞ module is an invariant of the tangle $\overleftarrow{\mathcal{T}}$.*

Analogous to the type D structure, in Chapter 12, we will use a simplification process to obtain a type A structure $(\langle\!\langle \overleftarrow{CT}]\!], m_{1,T}, m_{2,T})$ homotopy equivalent to $(\langle\!\langle \overleftarrow{T}]\!], m_{1,\bullet}, m_{2,\bullet})$ supported on states which do not contain any free circles. As the consequence of Theorem 3, we have the following corollary:

Corollary 4. *The stable homotopy class of $(\langle\!\langle \overleftarrow{CT}]\!], m_{1,T}, m_{2,T})$ as an A_∞ module is an invariant of $\overleftarrow{\mathcal{T}}$.*

Using the gluing theory described in [11, Section 2.4], we can pair the type D structure $([\![\overrightarrow{T} \rangle\!\rangle, \overrightarrow{\delta_{T,\bullet}})$ and the type A structure $(\langle\!\langle \overleftarrow{T}]\!], m_{1,\bullet}, m_{2,\bullet})$ to form a chain complex whose underlying module is $\langle\!\langle \overleftarrow{T}]\!] \otimes_{\mathcal{I}_n} [\![\overrightarrow{T} \rangle\!\rangle$ and differential is defined by the following formula:

$$\partial_\bullet^{\boxtimes}(x \otimes y) = m_{1,\bullet}(x) \otimes y + (m_{2,\bullet} \otimes \mathbb{I})(x \otimes \overrightarrow{\delta_{T,\bullet}}(y)).$$

Let T be the link diagram obtained by gluing $\overleftarrow{T}$ and $\overrightarrow{T}$ along their end points. In Chapter 14, we will prove one of the main theorems of this paper:

Theorem 5. $(\langle\!\langle \overleftarrow{T}]\!] \otimes_{\mathcal{I}_n} [\![\overrightarrow{T} \rangle\!\rangle, \partial_\bullet^{\boxtimes})$ *is chain isomorphic to* $([\![T]\!], \widetilde{\partial})$ *where* $([\![T]\!], \widetilde{\partial})$ *denotes the totally twisted Khovanov chain complex associated to T.*

1.1 The structure of this thesis

In Chapter 2, we recall the definitions of Kauffman bracket, Jones polynomial, and Khovanov homology. We also sketch the ideas of a spanning tree complex in Khovanov homology and bordered Khovanov homology. In Chapter 3, we recall the definition of the cleaved algebra $\mathcal{B}\Gamma_n$ in [17] with two minor modifications: the ground ring $\mathbb{Z}_2$ instead of $\mathbb{Z}$ and the requirement on decorations of marked circles. In Chapter 4, we construct the expanded complex $[\![\overrightarrow{T}\rangle\!\rangle$ and recall our main result for the twisted tangle homology. It will be followed by the construction of the "vertical" type D structure $([\![\overrightarrow{T}\rangle\!\rangle, \overrightarrow{\delta_{\mathcal{V}}})$ in Chapter 5. In Chapter 6, we will prove that $\overrightarrow{\delta_{T,\bullet}} := \overrightarrow{\delta_T} + \overrightarrow{\delta_{\mathcal{V}}}$ is a type D structure on $[\![\overrightarrow{T}\rangle\!\rangle$. We, next, will define the map $\overrightarrow{\delta_{n,T}}$ and prove that $([\![\overrightarrow{CT}\rangle\!\rangle, \overrightarrow{\delta_{n,T}})$ is a deformation retraction of $([\![\overrightarrow{T}\rangle\!\rangle, \overrightarrow{\delta_{T,\bullet}})$ in Chapter 7. The proof of Theorem 1 will be described in Chapter 10 by using a trick to move weights, described in Chapter 8. The whole Chapter 9 is devoted to establishing the definitions of the stable homotopy equivalence of type A and type D structures. In Chapter 11, we define the type A structure $(\langle\!\langle\overleftarrow{T}]\!], m_{1,\bullet}, m_{2,\bullet})$. After that, we will define the type A structure $(\langle\!\langle\overleftarrow{CT}]\!], m_{1,T}, m_{2,T})$ and prove that it is A_∞ homotopy equivalent to $(\langle\!\langle\overleftarrow{T}]\!], m_{1,\bullet}, m_{2,\bullet})$. We will sketch the proof of theorem 3 in Chapter 13. The relationship between the totally twisted Khovanov homology and the chain complex obtained by pairing our twisted type A and type D structures will be described in Chapter 14. In Chapter 15, we will give examples calculating our type D and type A structures for several knots and links.

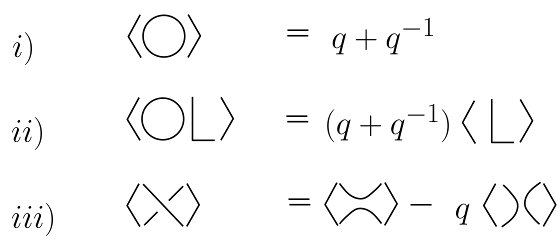
CHAPTER 2

PRELIMINARIES

2.1 Kauffman bracket and Jones Polynomial

Consider a link diagram L of an oriented link $\mathcal{L}$, embedded in S^3. The set of crossings of L is denoted by $\textsc{cr}(L)$. In Figure 2.1, the picture on the left (respectively right) represents a $-$ (respectively $+$) crossing. The numbers of positive and negative crossings will be denoted n_+ and n_- respectively. First of all, we recall the definition of Kauffman brackets associated to link diagrams:

Definition 6. *The Kauffman bracket is a function from link diagrams to Laurent polynomials with integer coefficients in an indeterminate q. It maps a link diagram L to $\langle L\rangle \in \mathbb{Z}[q^{-1}, q]$, characterised by three following formulas:*



Using the numbers of positive and negative crossings of L, the Jones polynomial of L is

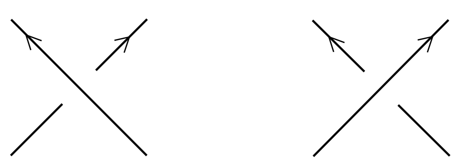


Figure 2.1: Negative and positive crossings.

defined to be $J(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L \rangle$. The Jones polynomial can be proved easily to be invariant under Reidemeister moves I, II, and III. Therefore, we obtain the following result:

Theorem 7. *The Jones polynomial $J(L)$ is an invariant of the oriented link $\mathcal{L}$.*

To compute Jones polynomial of a link diagram L, one can start at one arbitrary crossing c. Using the third Kauffman bracket relation, we then can express $\langle L \rangle$ in terms of $\langle L_1 \rangle$ and $\langle L_2 \rangle$ where L_1 and L_2 are obtained from L by resolving c. By induction, we repeat this process until there is no crossing to resolve. We then use the first and second Kauffman bracket relations to compute $\langle L \rangle$ and thus $J(L)$. We denote $n = n_+ + n_-$ the number of crossings of L and index the set of crossings in an arbitrary order. A complete resolution of L is a collection of circles on a plane, obtained by a choice to resolve each crossing by either 0 or 1 resolution (in the third Kauffman bracket relation, the left and right pictures in the right hand side describe 0 and 1 resolutions respectively). Equivalently, a resolution r is a vertex of the cube $\{0,1\}^n$. Thus, r will stand both for the resolution diagram and for the indicator function for the set of crossings defining the resolution. We denote $h(r)$, called the height of r, the number of 1 resolutions. If r has $k(r)$ disjoint circles in its resolution, the summand of Kauffman bracket corresponding to this resolution will be $(-1)^{h(r)} q^{h(r)} (q + q^{-1})^{k(r)}$. Therefore, if we denote $\textsc{res}(L)$ the set of resolution of L, then we have the following formula:

$$J(L) = (-1)^{n_-} q^{n_+ - 2n_-} \sum_{r \in \textsc{res}(L)} (-1)^{h(r)} q^{h(r)} (q + q^{-1})^{k(r)}. \tag{2.1}$$

2.2 Khovanov homology

On the other hand, the motivation of the construction of Khovanov homology is to try to interpret the Jones polynomial as Euler characteristics of some homology theories. To describe Khovanov's construction, we first recall the notations for the graded dimension and the degree shifts:

Definition 8. *Given a bigraded R-module $V = \bigoplus V_{i,j}$ where R is a ring, the graded dimen-*

sion of V is a Laurent polynomial $P(V) = \sum q^j dimV_{i,j}$.

Definition 9. *Let $M = \oplus_{\vec{v}\in\mathbb{Z}^k} M_{\vec{v}}$ be a $\mathbb{Z}^k$-graded R-module where R is a ring, then $M[\vec{w}]$ is the $\mathbb{Z}^k$-graded module with $(M[\vec{w}])_{\vec{v}} \cong M_{\vec{v}-\vec{w}}$.*

By a simple calculation, we see that $q+q^{-1}$ is the graded dimension of a bigraded two dimensional vector space $\mathcal{V}$ with two basis elements $v_{\pm}$ whose degrees are $(0, \pm 1)$ respectively. We then associate to each resolution $r \in \text{RES}(L)$ a bigraded vector space $\mathcal{V}_r = \mathcal{V}^{\otimes k}[h(r), h(r)]$ where k is the number of disjoint circles in r and $h(r)$ is the height of r. Note that, a generator of $\mathcal{V}$ can be thought of as being the set of circles in r provided with a way to decorate each circle in r by either $+$ or $-$. Define $[\![L]\!]^m := \bigoplus_{r\in\text{RES}(L),h(r)=m} V_r$ to be the m-chain group of complex $[\![L]\!]$ and let $C(L) := [\![L]\!]\{-n\}[n_+ - 2n_-]$. Before recalling the differential d of this complex, we have a remark from the relationship between this complex and Jones polynomial. From the construction, it is straightforward to see that ignoring the sign, the graded dimension of $\mathcal{V}_r$ is exactly its portion in Kauffman bracket of L. Therefore, we have the following theorem which can be found in [3, Section 3]:

Theorem 10. *If the differential d of $(C(L), d)$ preserves the second degree, the graded Euler characteristics $\chi(C(L))$ of $C(L)$, defined to be the alternating sum of the graded dimensions of its homology groups, is the Jones polynomial of L:*

$$\chi(C(L)) = J(L).$$

As the next step we recall the construction of the differential d. It suffices to define the image of d on each generator (r, s) of $\mathcal{V}_r$ where $r \in \text{RES}(L)$ and s is a decoration on the circles of r. We define:

$$d(r, s) = \sum_{c\in\text{CR}(L)} (-1)^{\sum_{c_1<c} r(c_1)} d_c(r, s)$$

where 1) $d_c(r, s) = 0$ if $r(c) = 1$ or 2) if $r(c) = 0$ then $d_c(r, s) = \sum_{\alpha}(r_c, s_\alpha)$ where $r_c \in \text{RES}(L)$ obtained from r by changing the resolution at c from 0 to 1. s_α is calculated in terms of s

by defining $s_\alpha(C) = s(C)$ if C is not abutting c and the signs on other circles is defined as following:

1. Resolve c from 0 to 1 merges two circles of r then we use:

$$\mu = \begin{cases} v_+ \otimes v_+ \longrightarrow v_+ \\ v_+ \otimes v_- \longrightarrow v_- \\ v_- \otimes v_+ \longrightarrow v_- \\ v_- \otimes v_- \longrightarrow 0. \end{cases}$$

2. Resolving c from 0 to 1 splits a circle of r into two circles of r_c then we use:

$$\Delta = \begin{cases} v_+ \longrightarrow v_+ \otimes v_- + v_- \otimes v_+ \\ v_- \longrightarrow v_- \otimes v_-. \end{cases}$$

It is not hard to check that d is a differential, preserving the second degree and thus, it turns out that $(C(L), d)$ encodes the information of the Jones polynomial of L. Additionally, we even have stronger result:

Theorem 11. *[7] The homology group $\mathcal{H}^m(L)$ of $(C(L), d)$ is an invariant of $\mathcal{L}$ that encodes the information of the Jones polynomial.*

Remark 12. *If $\mathcal{L}$ is equipped with a marked point $p \in \mathcal{L}$ and L is a generic projection of $\mathcal{L}$, taking p to a non-crossing point, we will have the reduced version of Khovanov homology supported on generators whose decorations on the marked circle (the one which contain the marked point) are $-$. Without an abuse of notation, we will still use $(C(L), d)$ and $\mathcal{H}^*$ to stand for the reduced Khovanov chain complex and its homologies.*

2.3 Tait graph and the totally twisted Khovanov homology

We first recall the definition of a Tait graph associated to a link diagram L.

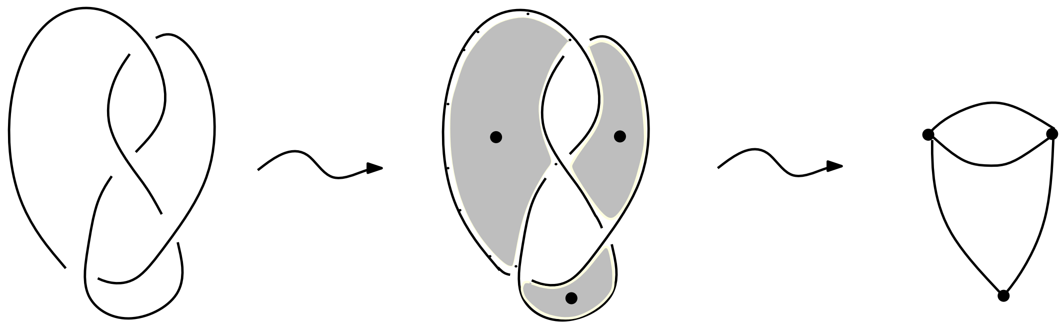


Figure 2.2: The Tait graph of 8-figure knot

Definition 13. *Given a link diagram L, its Tait graph is obtained by first checkerboard coloring the faces of L by white and black colors, then taking the black faces as vertices. Each crossing in* $\textsc{cr}(L)$ *will provides an edge between two vertices since it abuts one or two black faces.*

Example. We illustrate the Tait graph of 8-figure knot as in Figure 2.2.

Remark 14. *A set of resolutions of L will be $1-1$ corresponding to a set of subgraphs of the Tait graph associated to L. Indeed, each crossing is corresponding to an edge of the graph and resolving a crossing by either 0 or 1 resolution will locally split or merge black regions. Therefore, each resolution is determined by a specific choice to resolve diagram's crossings and this choice will determine the subgraph. Note: resolutions which contain only one circle will be corresponding to a spanning tree subgraph of the Tait graph.*

As we can see from the construction of the (reduced) Khovanov homology, its generators are found by using exponentially many ways a link diagram can be resolved. Therefore, to have a better understanding of this complex, it is reasonable to find a deformation retraction of it. In fact, A. Champanerkar, I. Kofman in [5] and S. Wehrli in [25] proved that there exists a chain complex, generated by spanning trees, whose homology is the reduced Khovanov homology. However, they did not fully describe the formula of the differential. Following the work of L. Roberts ([19], we briefly recall the construction of the totally twisted Khovanov homology (the detail is similar to the twisted tangle case, described in Section 4.1):

1. Label each arc, not contains marked point p, of L with a distinct formal variable x_f. Let $\mathbb{F}_L$ be the field of rational functions $\mathbb{Z}_2[x_f|f$ is an arc of $L]$. Denote $(\overline{KT}(L), \partial_{KH}) = (C(L) \otimes \mathbb{F}_L, d \otimes \mathbb{I}_d)$. Note that ∂_{KH} is $(+1, 0)$ differential.

2. There is a vertical Koszul $(0, -2)$ differential $\partial_{\mathcal{V}}$ on $\overline{KT}(L)$, determined by its image on each generator (r, s) as following:

$$\partial_{\mathcal{V}}(r, s) = \sum_{C \in r, s(C) = +} w_C(r, s_C)$$

   where w_C is the sum of weights on C and s_C is obtained from s by only changing the decoration on C from $+$ to $-$.

We state the main theorems of the totally twisted Khovanov homology:

Theorem 15. *[19] Let $\partial_{KH} : \overline{KT}^{*,*}(L) \to \overline{KT}^{*+1,*}(L)$ be the Khovanov differential and let $\partial_{\mathcal{V}} : \overline{KT}^{*,*}(L) \to \overline{KT}^{*,*-2}(L)$ be the Koszul differential. Then $\partial = \partial_{KH} + \partial_{\mathcal{V}}$ is a differential on $\overline{KT}(L)$. Furthermore, if we collapse the bigrading to singly grading by using the formula $\delta = 2i - j$, then δ will provides a grading to the complex $(\overline{KT}(L), \partial)$.*

The complex $(\overline{KT}(L)[-n_+(L)], \partial)$ is called the totally twisted Khovanov complex and we denote the homology of this complex (with respect to δ-grading) by $\underline{HT}_*(L)$.

Theorem 16. *[19] The homologies $\underline{HT}_*(L)$ is a stably invariant of isotopy class of $\mathcal{L}$.*

One of the distinct advantage of this type of chain complex is that by using the Gauss elimination method, we can find a deformation retraction $(CT(L), \partial_L)$ of $(\overline{KT}(L)[-n_+(L)], \partial)$ supported on the generators which corresponds to the spanning trees. Here, we only briefly describe the formula of differential ∂_L without further comments. More properties of this spanning tree complex can be found in [19]. It suffices to define the image of ∂_L on the spanning tree generators. Let r be such generator, then:

$$\partial_L(r) = \sum_{r_1} \langle r, r_1 \rangle r_1$$

where r_1 is any spanning tree generator, obtained from r by changing two crossings c_1, c_2 from 0 to 1. Additionally, let r_{c_i} be a resolution obtained from r by changing the resolution at c_i from 0 to 1. r_{c_i} has two disjoint circles and we denote the non-marked circle by C_i $(i = 1, 2)$. Then, we have:

$$\langle r, r_1 \rangle = \frac{1}{w_{C_1}} + \frac{1}{w_{C_2}}.$$

Despite the fact that the relationship between the reduced Khovanov homology and its twisted version is not well understood for the link case, in the case of knot, thank to the work of T. Jaeger, we have the following theorem:

Theorem 17. *[23] If K is a knot diagram of knot $\mathcal{K}$ with a marked point p, as δ-grading complex, the reduced Khovanov homology is chain isomorphic to its totally twisted Khovanov homology:*

$$(\overline{KT}(L), \partial_{KH}) \cong (\overline{KT}(L), \partial_{KH} + \partial_{\mathcal{V}}).$$

2.4 Bordered Khovanov homology

From a different perspective, in [2], M. Asaeda, J. Przytycki, and A. Sikora extented the Khovanov constrution to associte to each tangle $\mathcal{T}$ embedded in I-bundle over orientable surfaces a chain complex $(C(T), d_{APS})$ whose homologies are isotopy invariant of $\mathcal{T}$. Therefore, there is a natural question about whether or not we can recover Khovanov homology of links from the homology of its tangle components. This question is answered by L. Roberts by modeling off the ideas from bordered Heegaard Floer homology. Despite the fact that the M. Asaeda, J. Przytycki, and A. Sikora's construction does not allow one to recover the original Khovanov homology of $\mathcal{T}$, its module will play an essential role in Roberts construction. Let's consider a link diagram L, obtained by gluing two tangles $\overleftarrow{T}$ and $\overrightarrow{T}$, we will see in Chapter 5 that $C(\overleftarrow{T})$ and $C(\overrightarrow{T})$ will be the "core" of its expansion $\langle\!\langle \overleftarrow{T}]\!]$ and $[\![\overrightarrow{T} \rangle\!\rangle$ (the arrows here indicate the left and right tangle components of L. We now will give an example to illustrate the main idea of bordered Khovanov homology: the gluing process.

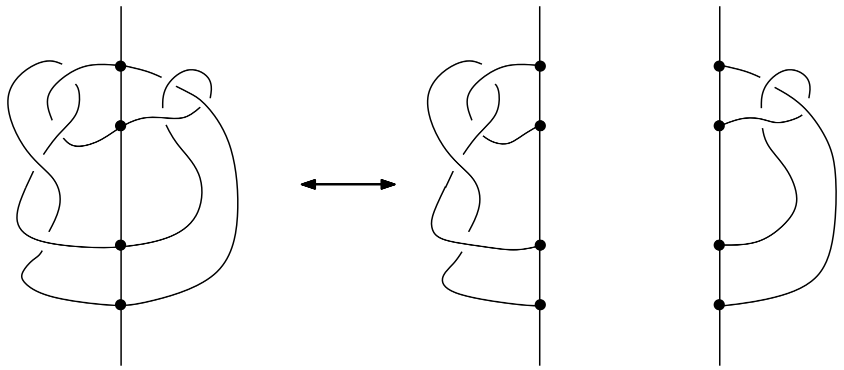


Figure 2.3: A link obtained by gluing two tangles.

Let's consider a link L and think of it as being glued by two tangles (see Figure 2.3 for an example). The generators of complex $C(L)$ are obtained by smoothing each crossing of L by either 0 or 1 and then, decorating the planar circles by $\pm$. Such generator ξ might look like the left top corner in Figure 2.4. If we strip out the free circles, which do not intersect the y-axis, on one side and still keep track of what is going on another side, we will get the pictures at the right top and left bottom corners. These pictures will correspond to a generator $\overrightarrow{\xi}$ of $\llbracket \overrightarrow{T} \rrangle$ (the module associated to the right tangle) and a generator $\overleftarrow{\xi}$ of $\llangle \overleftarrow{T} \rrbracket$ (the module associated to the left tangle). As we can see, ξ is a result of gluing $\overleftarrow{\xi}$ and $\overrightarrow{\xi}$ along their "boundary"-the right bottom corner picture obtained by stripping off all free circles of ξ. Therefore, if we let the latter diagram to be the idempotent in $\mathcal{B}\Gamma_n$ (described in the next chapter) which acts on two states as the identity and let $\mathcal{I}$ be the idempotent subalgebra, $\llangle \overleftarrow{T} \rrbracket \otimes_{\mathcal{I}} \llbracket \overrightarrow{T} \rrangle \cong C(L)$.

Now, we will recall why the idea of bordered Heegaard Floer plays a role here. We note that the differential of Khovanov chain complex is contributed by changing types of resolution at right or left crossings. However, these contributions might change the idempotent and thats why there should be an appropriate map on the right tangle: $\overrightarrow{\delta} : \llbracket \overrightarrow{T} \rrangle \longrightarrow \mathcal{B}\Gamma_n \otimes_{\mathcal{I}} \llbracket \overrightarrow{T} \rrangle$ which records the change of idempotents and satisfies type D structure equations as in [11]. On the other hand, on the left tangle, the maps which reflect the action of the change

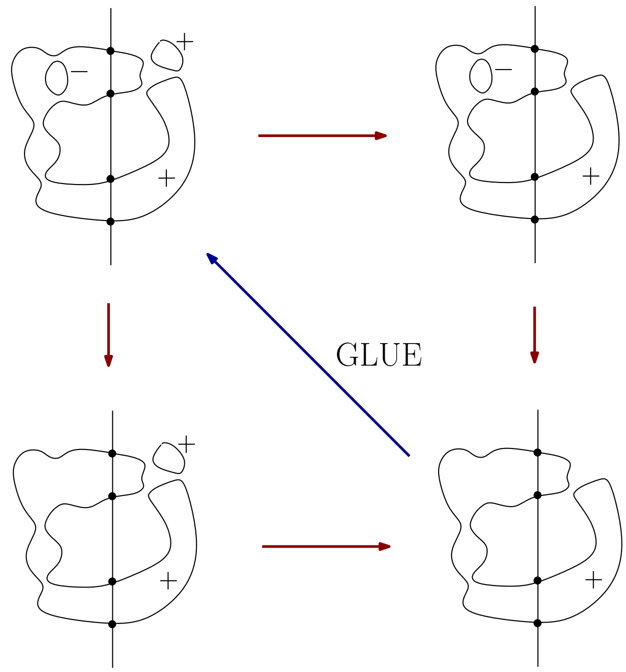


Figure 2.4: This figure illustrates how to obtain a generator from the left and right tangles.

of idempotents are defined to satisfies type A structure equations. Using a gluing process in [11], we can pair the type D structure of $\overrightarrow{T}$ and the type A structure of $\overleftarrow{T}$ to recover Khovanov chain complex of L (see [18] for detail).

CHAPTER 3

RECALL THE DEFINITION OF THE ALGEBRA FROM CLEAVED LINKS IN [17]

Let P_n be the set of $2n$ points $p_1 = (0,1)$, $p_2 = (0,2)$,..., $p_{2n} = (0,2n)$ on the y-axis. In [17, Section 2], Roberts associates P_n with a bigraded differential algebra $\mathcal{B}\Gamma_n$ (over $\mathbb{Z}$) equipped with a $(1,0)$-differential d_{Γ_n} which satisfies a Leibniz identity.

In this chapter, we will recall the definitions and properties of $(\mathcal{B}\Gamma_n, d_{\Gamma_n})$. As we mentioned earlier, we will replace the ground ring $\mathbb{Z}$ by $\mathbb{Z}_2$ in our definitions and thus, we do not need to worry about the sign issues.

We denote the closed half-planes $\mathbb{R}\times(-\infty,0]$ and $\mathbb{R}\times[0,\infty)$ by $\overleftarrow{\mathbb{H}}$ and $\overrightarrow{\mathbb{H}}$ respectively. Since $\mathcal{B}\Gamma_n$ is described by generators and relations, we first recall the definitions of its generators and then rewrite the set of the relations, ignoring the signs.

Definition 18. *[17, Section 2] An n-decorated, cleaved link (L,σ) is an embedding of disjoint circles in $\mathbb{R}^2$ such that:*

1. *Each circle contains at least 2 points in P_n,*
2. *$L \cap \{0\times(-\infty,\infty)\} = P_n$,*
3. *σ is a function which assigns either $+$ or $-$ to each circle of L, called decoration.*

Let $\textsc{cir}(L)$ be the set of circle components of L. We call p_{2n} the marked point and the circle of L which passes through the marked point is called the marked circle. We denote the marked circle by $L(*)$. We then denote $\mathcal{CL}_n^*$ the set of equivalence classes of n-decorated, cleaved links whose decorations on the marked circle is $-$.

Definition 19. *[17, Section 2] A right (left) planar matching M of P_n in the right (left) half plane $\overrightarrow{\mathbb{H}}$ ($\overleftarrow{\mathbb{H}}$) is a proper embedding of n-arcs $\alpha_i : [0,1] \to \overrightarrow{\mathbb{H}}$ ($\overleftarrow{\mathbb{H}}$) such that $\alpha_i(0)$ and $\alpha_i(1)$ belong to P_n.*

L can be obtained canonically by gluing a right planar matching $\overrightarrow{L}$ and a left one $\overleftarrow{L}$ along P_n. We denote the equivalence classes of right and left planar matchings by $\overrightarrow{\text{MATCH}(n)}$ and $\overleftarrow{\text{MATCH}(n)}$ respectively.

We next describe the definition of a bridge of a cleaved link.

Definition 20. *[17, Section 2] A bridge of a cleaved link L is an embedding of $\gamma : [0,1] \to \mathbb{R}^2 \backslash \{0 \times (-\infty, \infty)\}$ such that:*

1. *$\gamma(0)$ and $\gamma(1)$ are on distinct arcs of $\overrightarrow{L}$ (or $\overleftarrow{L}$),*
2. *$\gamma(0,1) \cap L = \emptyset$.*

Depending upon the location of γ, γ is called either left bridge or right bridge. Two left (right) bridges γ_1 of L_1 and γ_2 of L_2 will be equivalent if there is a planar isotopy of $\overleftarrow{\mathbb{H}}$ ($\overrightarrow{\mathbb{H}}$), fixing $\partial\mathbb{H}$, which takes γ_1 into γ_2 and $\overleftarrow{L_1}$ ($\overrightarrow{L_1}$) into $\overleftarrow{L_2}$ ($\overrightarrow{L_1}$). The equivalence classes of bridges, left bridges and right bridges of L are denoted by $\text{BRIDGE}(L)$, $\overleftarrow{\text{BR}}(L)$ and $\overrightarrow{\text{BR}}(L)$ respectively.

A generator e of $\mathcal{B}\Gamma_n$ can be thought as an oriented edge in a directed graph whose source and target vertices, denoted by $s(e)$ and $t(e)$, are decorated cleaved links (see [17, Section 2] for more detail). Corresponding to each $(L, \sigma) \in \mathcal{CL}_n^*$, there is an idempotent $I_{(L,\sigma)} \in \mathcal{B}\Gamma_n$. Let $\mathcal{I}_n$ denote the sub-algebra generated by the idempotents $I_{(L,\sigma)}$. $\mathcal{B}\Gamma_n$ is freely generated over $\mathbb{Z}_2$ by the idempotents and the following elements, subject to the relations described below:

1. For each circle $C \in \text{CIR}(L)$ where $\sigma(C) = +$, we have a "dual" decorated cleaved link (L, σ_C) where $\sigma_C(C) = -$ and $\sigma_C(D) = \sigma(D)$ for each $D \in \text{CIR}(L) \backslash \{C\}$. There are two elements $\overrightarrow{e_C}$ and $\overleftarrow{e_C}$, called *right and left decoration elements*, whose sources are (L, σ) and targets are (L, σ_C). C is called the support of $\overrightarrow{e_C}$ and $\overleftarrow{e_C}$.

2. Let $\gamma \in \textsc{Bridge}(L)$, then there is a *bridge element* $e_{(\gamma;\sigma,\sigma_\gamma)}$ whose source is (L,σ) and target is $(L_\gamma, \sigma_\gamma)$ where L_γ is obtained from L by surgering along γ and σ_γ is any decoration compatible with σ and computed from the Khovanov Frobenius algebra. Additionally, $e_{(\gamma;\sigma,\sigma_\gamma)}$ is called a left (right) bridge element if $\gamma \in \overleftarrow{\textsc{Br}}(L)$ $(\overrightarrow{\textsc{Br}}(L))$ and will be denoted by $\overleftarrow{e_\gamma}$ $(\overrightarrow{e_\gamma})$ if the context is clear. Note that L_γ has a special bridge $\gamma^\dagger$ which is the image of the co-core of the surgery.

With these generators and idempotents, we have:

Proposition 21. *[17] $\mathcal{B}\Gamma_n$ is finite dimensional.*

Furthermore, $\mathcal{B}\Gamma_n$ can be bigraded as in [17]. In this paper, we collapse the bigrading by using $\zeta(i,j) = i - j/2$ to give a new grading on $\mathcal{B}\Gamma_n$. On the generating elements, the new grading is specified by setting:

$$\begin{array}{lll}
\mathrm{I}_{(L,\sigma)} & \longrightarrow (0,0) & \longrightarrow 0. \\
\overrightarrow{e_C} & \longrightarrow (0,-1) & \longrightarrow 1/2. \\
\overleftarrow{e_C} & \longrightarrow (1,1) & \longrightarrow 1/2. \\
\overrightarrow{e_\gamma} & \longrightarrow (0,-1/2) & \longrightarrow 1/4. \\
\overleftarrow{e_\gamma} & \longrightarrow (1,1/2) & \longrightarrow 3/4.
\end{array}$$

This assignment provides the grading to every other element by extending the grading on generators homomorphically.

Based on the above set of generators, there is a set of commutativity relation of generators, divided into the following groups:

Group I$-$Disjoint support and squared bridge relations:

We describe the set of relations in this case by using the following model:

$$e_\alpha e_{\beta'} = e_\beta e_{\alpha'} \tag{3.1}$$

We require that e_α and $e_{\alpha'}$ are the same type of elements (decoration or bridge) and they also have same locations (left or right). The same requirements are applied for the pair e_β and $e_{\beta'}$.

Let $(L,\sigma) \in \mathcal{CL}_n^*$ such that $I_{(L,\sigma)}$ is the source of both e_α and e_β, we have the following cases:

1. If C and D are two distinct $+$ circles of (L,σ), there are two ways to obtain $(L,\sigma_{C,D})$ from (L,σ) by changing the decoration on either C or D from $+$ to $-$ first and then changing the decoration on the remaining $+$ circle. The recorded algebra elements for two paths will form a relation.

2. If $e_\alpha = e_{(\gamma,\sigma,\sigma')}$ for a bridge γ in (L,σ) and e_β is a decoration element for $C \in \textsc{cir}(L)$, with C not in the support of γ, due to the disjoint support, there will exist $e_{\alpha'} = e_{(\gamma,\sigma_C,\sigma'_C)}$ and $e_{\beta'}$ which is a decoration element whose source is (L_γ, s') and target is (L_γ, s'_C).

3. Given any bridge γ of L. Let $B_d(L,\gamma)$ denote the set of bridges of (L,γ) neither of whose ends is on an arc with γ. If $e_\alpha = e_{(\gamma,\sigma,\sigma')}$ and $e_\beta = e_{(\eta,\sigma,\sigma'')}$ are bridge elements for distinct bridges γ and η in (L,σ), with $\eta \in B_d(L,\gamma)$ and $e_{\beta'} = e_{(\eta,\sigma',\sigma''')}$ and $e_{\alpha'} = e_{(\gamma,\sigma'',\sigma''')}$ for some decoration σ''' on $L_{\gamma,\eta}$, we obtain a commutativity relation.

4. If $e_\alpha = e_{(\overrightarrow{\gamma},\sigma,\sigma')}$ and $e_\beta = e_{(\overrightarrow{\eta},\sigma,\sigma'')}$ are bridge elements for distinct *right* bridges γ and η in (L,σ), and $e_{\beta'} = e_{(\overrightarrow{\delta},\sigma',\sigma''')}$ and $e_{\alpha'} = e_{(\overrightarrow{\omega},\sigma'',\sigma''')}$, such that $L_{\gamma,\delta} = L_{\eta,\omega}$, and some compatible decoration σ''', it will form a commutativity relation.

5. Let $\overleftarrow{\gamma}$ be a left bridge of L. Let $B_o(L,\overleftarrow{\gamma})$ denote the set of bridges of $(L,\overleftarrow{\gamma})$ one of whose ends is on the same arc as $\overleftarrow{\gamma}$ and lying on opposite side of the arc as $\overleftarrow{\gamma}$. If $e_\alpha = e_{(\overleftarrow{\gamma},\sigma,\sigma')}$ and $e_\beta = e_{(\overleftarrow{\eta},\sigma,\sigma'')}$ are bridge elements for distinct *left* bridges in (L,σ), with $\overleftarrow{\eta} \in B_o(L,\overleftarrow{\gamma})$, and $e_{\beta'} = e_{(\overleftarrow{\delta},\sigma',\sigma''')}$, $e_{\alpha'} = e_{(\overleftarrow{\omega},\sigma'',\sigma''')}$ with $L_{\gamma,\delta} = L_{\eta,\omega}$, and some compatible decoration σ''', we form a commutativity relation.

Group II−Other bridge relations:

1. Suppose $\gamma \in \overleftarrow{\text{BR}}(L)$ and $\eta \in B_{\pitchfork}(L_\gamma, \gamma^\dagger)$ where $B_{\pitchfork}(L_\gamma, \gamma^\dagger)$ consists of the classes of bridges all of whose representatives intersect $\gamma^\dagger$, then

$$e_{(\gamma,\sigma,\sigma')}e_{(\eta,\sigma',\sigma'')} = 0$$

whenever σ' and σ'' are compatible decorations.

2. The second possibility is that $\overleftarrow{\alpha} \in B_s(L, \overleftarrow{\beta})$ where $B_s(L, \overleftarrow{\gamma})$ stands for the set of bridges one of whose ends is on the same arc as $\overleftarrow{\gamma}$ and lying on same side of the arc as $\overleftarrow{\gamma}$. There is a natural left bridge $\overleftarrow{\gamma}$ by sliding $\overleftarrow{\alpha}$ over $\overleftarrow{\beta}$. In this case there are three paths from $s(\overleftarrow{e_\gamma}\overleftarrow{e_\beta})$ to $t(\overleftarrow{e_\gamma}\overleftarrow{e_\beta})$ and they will form a relation whenever the decorations are compatible:

$$\overleftarrow{e_\alpha}\overleftarrow{e_\beta} + \overleftarrow{e_\beta}\overleftarrow{e_\delta} + \overleftarrow{e_\gamma}\overleftarrow{e_\eta} = 0$$

where $\overleftarrow{\delta}$ and $\overleftarrow{\eta}$ are the images of $\overleftarrow{\alpha}$ in $L_{\overleftarrow{\beta}}$ and in $L_{\overleftarrow{\gamma}}$ respectively.

3. If there is a circle $C \in \text{CIR}(L)$ with $\sigma(C) = +$, and there are elements $\overrightarrow{e}_{(\gamma,\sigma,\sigma')}$ and $\overrightarrow{e}_{(\gamma^\dagger,\sigma',\sigma_C)}$ for a bridge $\gamma \in \overrightarrow{\text{BR}}(L)$ then

$$\overrightarrow{e}_{(\gamma,\sigma,\sigma')}\overrightarrow{e}_{(\gamma^\dagger,\sigma',\sigma_C)} = \overrightarrow{e_C}.$$

Such a circle C is unique for the choice of γ and σ' and is called the active circle for γ.

Group III−Relations for decoration edges:

When the support of e_C is not disjoint from the bridge γ of $e_{(\gamma,\sigma,\sigma_\gamma)}$, the relations are different depending upon the location of e_C.

1. **The relations for $\overrightarrow{e_C}$:** If C_1 and C_2 are two + circles in L and γ is a bridge which merges C_1 and C_2 to form a new circle C, we then obtain the following relation:

$$\overrightarrow{e_{C_1}} m_{(\gamma,\sigma_{C_1},\sigma_C)} = \overrightarrow{e_{C_2}} m_{(\gamma,\sigma_{C_2},\sigma_C)} = m_{(\gamma,\sigma,\sigma_\gamma)} \overrightarrow{e_C}$$

   Similarly, if C is a + circle in L and γ is a bridge which divides C into C_1 and C_2 in $\textsc{cir}(L_\gamma)$, then we impose the relation:

$$\overrightarrow{e_C} f_{(\gamma,\sigma_C,\sigma_{C,\gamma})} = f_{(\gamma,\sigma,\sigma^1_\gamma)} \overrightarrow{e_{C_1}} = f_{(\gamma,\sigma,\sigma^2_\gamma)} \overrightarrow{e_{C_2}}$$

   where σ^i_γ assigns + to C_i and − to C_{3-i}.

2. **The relations for $\overleftarrow{e_C}$:** Since there are two types of decoration elements, in the above relations, if we replace the right decoration elements by the left ones, we obtain the following relations:

$$\overleftarrow{e_{C_1}} m_{(\gamma,\sigma_{C_1},\sigma_C)} + \overleftarrow{e_{C_2}} m_{(\gamma,\sigma_{C_2},\sigma_C)} + m_{(\gamma,\sigma,\sigma_\gamma)} \overleftarrow{e_C} = 0$$

$$\overleftarrow{e_C} f_{(\gamma,\sigma_C,\sigma_{C,\gamma})} + f_{(\gamma,\sigma,\sigma^1_\gamma)} \overleftarrow{e_{C_1}} + f_{(\gamma,\sigma,\sigma^2_\gamma)} \overleftarrow{e_{C_2}} = 0.$$

The main result about $\mathcal{B}\Gamma_n$ in [17] is:

Proposition 22. *Let $(L,\sigma) \in \mathcal{CL}^*_n$ such that there is a circle $C \in \textsc{cir}(L)$ with $\sigma(C) = +$. Let $\overleftarrow{e_C}$ be the decoration element corresponding to C. Let*

$$d_{\Gamma_n}(\overleftarrow{e_C}) = \sum e_{(\gamma,\sigma,\sigma_\gamma)} e_{(\gamma^\dagger,\sigma_\gamma,\sigma_C)} \tag{3.2}$$

where the sum is over all $\gamma \in \overleftarrow{\mathrm{BR}}(L)$ with C as active circle, and all decorations σ_γ which define compatible elements. Let $d_{\Gamma_n}(e) = 0$ for every other generator e (including idempotents). Then d_{Γ_n} can be extended to an order 1 differential on the graded algebra $\mathcal{B}\Gamma_n$ which *satisfies the following Leibniz identity:*

$$d_{\Gamma_n}(\alpha\beta) = \big(d_{\Gamma_n}(\alpha)\big)\beta + \alpha\big(d_{\Gamma_n}(\beta)\big). \tag{3.3}$$

$(\mathcal{B}\Gamma_n, d_{\Gamma_n})$ denotes this differential, graded algebra over $\mathbb{Z}_2$.

Note. Because we require the decoration on a marked circle is $-$, our $\mathcal{B}\Gamma_n$, described above, is actually a subalgebra of $\mathcal{B}\Gamma_n$ defined in [17].

CHAPTER 4

TWISTED TANGLE HOMOLOGY AND ITS EXPANSION

4.1 Twisted tangle homology

In [20], the author and L. Roberts define a twisted Khovanov homology version for tangles embedded in thickened surfaces by twisting the reduced Khovanov tangle chain complex, defined by M. Asaeda, J. Przytycki and A. Sikora in [2].

Let $\overrightarrow{T} \subset \overrightarrow{\mathbb{H}} = [0,\infty)\times\mathbb{R}\times\{0\}$ be a tangle diagram for an oriented tangle $\overrightarrow{\mathcal{T}} \subset [0,\infty)\times\mathbb{R}^2 \subset \mathbb{R}^3$. The set of crossings in $\overrightarrow{T}$ will be denoted $\text{CR}(\overrightarrow{T})$. An arc is a segment between two crossings, or the boundary, of the tangle diagram $\overrightarrow{T}$ and the set of arcs of the tangle diagram $\overrightarrow{T}$ will be denoted $\text{ARC}(\overrightarrow{T})$. The number of positive crossings will be denoted $n_+(\overrightarrow{T})$ and the number of negative crossings will be denoted $n_-(\overrightarrow{T})$. We will often omit the reference to $\overrightarrow{T}$ when the choice of the diagram is clear.

Following [19], [20], we label each arc $f \in \text{ARC}(\overrightarrow{T})$ with a formal variable x_f and form the polynomial ring $\mathbb{P}_{\overrightarrow{T}} = \mathbb{Z}_2[x_f | f \in \text{ARC}(\overrightarrow{T})]$. The field of fractions of $\mathbb{P}_{\overrightarrow{T}}$ will be denoted $\mathbb{F}_{\overrightarrow{T}}$.

Definition 23. *For each subset $S \subset \text{CR}(\overrightarrow{T})$, the resolution ρ_S of $\overrightarrow{T}$ is a collection of arcs and circles in $\overrightarrow{\mathbb{H}}$, considered up to isotopy, found by locally replacing each crossing $s \in \text{CR}(\overrightarrow{T})$ according to the rule as in Figure 4.1*

The set of resolutions for $\overrightarrow{T}$ will be denoted $\text{RES}(\overrightarrow{T})$. For each resolution ρ, denote by $h(\rho)$ the number of elements in the corresponding subset $S \subset \text{CR}(\overrightarrow{T})$.

Given a crossing $c \in \text{CR}(\overrightarrow{T})$ and a resolution $\rho = \rho_S$, we will also use the notation $\rho(c) = 0$ for $c \notin S$ and $\rho(c) = 1$ for $c \in S$. Thus, ρ will stand both for the resolution diagram and for the indicator function for the set of crossings defining the resolution. The local arc β which

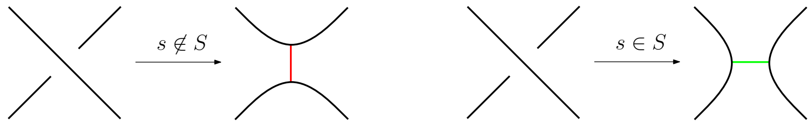


Figure 4.1: A rule to resolve each crossing locally

shows up when we resolve the crossing c is called a resolution bridge. Depending on the value of ρ at c, which is either 0 or 1, β is active (colored red) or inactive (colored green) respectively.

We denote $\rho_\beta = \rho \cup \{c\}$ if β is active and the notation is just not used when β is inactive. Furthermore, let $\textsc{bc}(\rho)$ be the set of circles and arcs in ρ while $\overrightarrow{m_\rho}$ stands for the planar matching obtained from ρ by deleting all of circles in $\textsc{bc}(\rho)$. The set of circles of ρ will be denoted by $\textsc{fcir}(\rho)$.

We next assign a weight to each circle (or arc) in a resolution by adding the formal variables along each circle (or arc) (See Figure 4.2 for an example).

Definition 24. *Let ρ be a resolution for $\overrightarrow{T}$, then for each $C \in \textsc{bc}(\rho)$, we define:*

$$\overrightarrow{w_C} = \sum_{f \in \textsc{arc}(C)} x_f.$$

For each resolution $\rho(\overrightarrow{T})$, let $\textsc{fcir}(\rho) = \{C_1, ..., C_k\}$. To each C_i, we associate the complex $\mathcal{K}[C_i]$:

$$0 \longrightarrow \mathbb{F}_{\overrightarrow{T}}\, v_+ \xrightarrow{\cdot \overrightarrow{w_{C_i}}} \mathbb{F}_{\overrightarrow{T}}\, v_- \longrightarrow 0$$

where $v_\pm$ occur in bigradings $(0, \pm 1)$. The differential in $\mathcal{K}[C_i]$ will be denoted ∂_{C_i}.

Then we associate ρ with the bigraded Koszul chain complex defined as

$$(\mathcal{K}(\rho), \partial_{\mathcal{K}(\rho)}) = (\mathcal{K}[C_1], \partial_{C_1}) \otimes_{\mathbb{F}_{\overrightarrow{T}}} (\mathcal{K}[C_2], \partial_{C_2}) \otimes_{\mathbb{F}_{\overrightarrow{T}}} \cdots \otimes_{\mathbb{F}_{\overrightarrow{T}}} (\mathcal{K}[C_k], \partial_{C_k}).$$

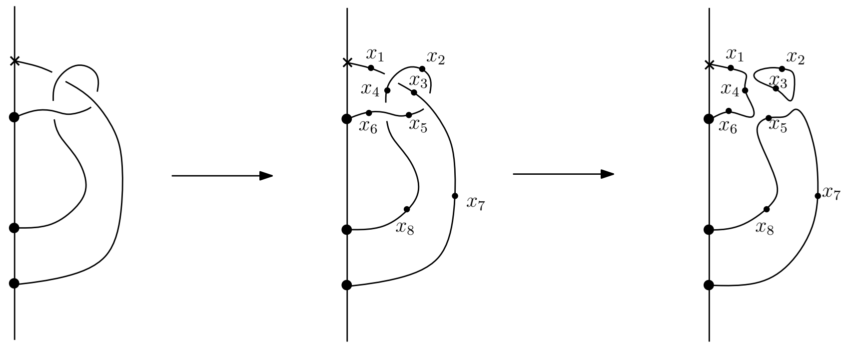


Figure 4.2: This figure illustrates how to assign weights to circles and arcs of a resolution. In this example, the far right picture represents a resolution of the tangle diagram. The weight of the free circle is $x_2 + x_3$ while the weights assigned to the top and bottom arcs are $x_3 + x_4 + x_6$ and $x_5 + x_7 + x_8$ respectively.

Because we need to shift the gradings, we use the following notation:

Definition 25. *Let $M = \oplus_{\vec{v} \in \mathbb{Z}^k} M_{\vec{v}}$ be a $\mathbb{Z}^k$-graded R-module where R is a ring, then $M\{\vec{w}\}$ is the $\mathbb{Z}^k$-graded module with $(M\{\vec{w}\})_{\vec{v}} \cong M_{\vec{v}-\vec{w}}$.*

The vertical complex for the resolution ρ of the tangle diagram $\overrightarrow{T}$ is defined as:

$$\mathcal{V}(\rho) = \mathcal{K}(\rho)\{[h(\rho), h(\rho)]\}$$

where the differential $\partial_{\mathcal{V}(\rho)}$ will change the bigrading by $(0, -2)$.

We now define a bigraded chain group:

$$C_{\text{APS}}(\overrightarrow{T}) = \bigoplus_{\rho \in \text{RES}(\overrightarrow{T})} \mathcal{V}(\rho).$$

$C_{\text{APS}}(\overrightarrow{T})$ is a bigraded chain complex with a $(0, -2)$ differential:

$$\partial'_{\mathcal{V}} = \bigoplus_{\rho \in \text{RES}(\overrightarrow{T})} \partial_{\mathcal{V}(\rho)}.$$

Additionally, in [2], M. Asaeda, J. Przytycki and A. Sikora define a $(1, 0)$ differential ∂_{APS}

on this bigraded module, satisfying:

$$\partial_{APS}\partial'_{\mathcal{V}} + \partial'_{\mathcal{V}}\partial_{APS} = 0.$$

Therefore, by collapsing the bigrading using $\zeta(i,j) = i - j/2$, we can make $\partial_{APS} + \partial'_{\mathcal{V}}$ a differential on $C_{\mathrm{APS}}(\overrightarrow{T})$. Furthermore, since both ∂_{APS} and $\partial'_{\mathcal{V}}$ are constructed to preserve the right planar matching, we can decompose:

$$(C_{\mathrm{APS}}, \partial_{APS} + \partial'_{\mathcal{V}}) = \bigoplus_{m \in \overrightarrow{\textsc{Match}(n)}} (C_{\mathrm{APS}}(\overrightarrow{T}, m), \partial_{APS,m} + \partial'_{\mathcal{V},m}).$$

We now can describe the main theorem of [20]:

Theorem 26. *For each $m \in \overrightarrow{\textsc{Match}(n)}$, the homology $H_*(C_{\mathrm{APS}}, \partial_{APS,m} + \partial'_{\mathcal{V},m})$, as a relative ζ-graded module, is an invariant of the isotopy class of $\overrightarrow{T}$.*

Following [17, Section 3.2], we first expand the chain group $[\![\overrightarrow{T} \rangle\!\rangle$:

Definition 27. *Given $(L,\sigma) \in \mathcal{CL}^*_n$, let*

$$C_{\mathrm{APS}}(\overrightarrow{T}, \overrightarrow{L}, \sigma) = C_{\mathrm{APS}}(\overrightarrow{T}, \overrightarrow{L})\{[0, \frac{i(L,\sigma)}{2}]\}$$

and

$$[\![\overrightarrow{T} \rangle\!\rangle = \bigoplus_{(L,\sigma) \in \mathcal{CL}^*_n} C_{\mathrm{APS}}(\overrightarrow{T}, \overrightarrow{L}, \sigma)$$

where $i(L,\sigma)$ is computed by subtracting the number of $-$ non-marked circles in L from the number of $+$ circles in L.

We also denote:

$$d_{APS} = \bigoplus_{(L,\sigma) \in \mathcal{CL}^*_n} \partial_{APS,\overrightarrow{L}}$$

$$\partial_{\mathcal{V}} = \bigoplus_{(L,\sigma) \in \mathcal{CL}^*_n} \partial'_{\mathcal{V},\overrightarrow{L}}.$$

As we can see, a generator of $[\![\overrightarrow{T} \rangle\!\rangle$ corresponds 1-1 with a triple $(\overleftarrow{m}, \rho, s)$ where $\overleftarrow{m}$ is a specific representative of its equivalence as a left planar matching, ρ is a resolution of $\overrightarrow{T}$ and $s : \textsc{cir}(\overleftarrow{m}\#\rho) \longrightarrow \{+,-\}$ such that $s(\overleftarrow{m}\#\overrightarrow{m_\rho}(*)) = -$. Here $\textsc{cir}(\overleftarrow{m}\#\rho)$ is the collection of all cleaved and free circles of $\overleftarrow{m}\#\rho$.

We denote $r = \overleftarrow{m}\#\rho$ and we call (r,s) a state. The collection of states of $\overrightarrow{T}$ will be denoted $\textsc{State}(\overrightarrow{T})$. $\partial(r,s)$ denotes a decorated cleaved link, obtained by deleting all of the free circles of $\overleftarrow{m}\#\rho$ and the decoration of this cleaved link is induced from the decoration s on $\textsc{cir}(\overleftarrow{m}\#\rho)$. Therefore:

$$[\![\overrightarrow{T} \rangle\!\rangle = \bigoplus_{(L,\sigma)\in\mathcal{CL}_n^*} \bigoplus_{\partial(r,s)=(L,\sigma)} \mathbb{F}_{\overrightarrow{T}}(r,s).$$

Additionally, each state $(r,s) = (\overleftarrow{m}\#\rho, s)$ has a ζ-grading, computed from a bigrading $(h(r,s), q(r,s))$ where $h(r,s) = h(\rho) - n_-$ and $q(r,s) = h(\rho) + \frac{i(L,\sigma)}{2} + \#(+ \text{ free circles}) - \#(- \text{ free circles}) + n_+ - 2n_-$.

Note. The whole process can be applied exactly the same to give the construction of the expanded complex $\langle\!\langle \overleftarrow{T}]\!]$ associated to a left tangle $\overleftarrow{T}$.

In the next chapter, we will describe a "vertical" type D structure on this underlying module $[\![\overrightarrow{T} \rangle\!\rangle$ with respect to the ζ-grading.

CHAPTER 5

A TYPE D STRUCTURE IN THE TWISTED TANGLE HOMOLOGY

Using the idea of the twisted tangle homology, we will describe a way of twisting Roberts's type D structure on $[\![\overrightarrow{T} \rangle\!\rangle$ [17].

Before defining a "vertical" type D structure $\overrightarrow{\delta_{\mathcal{V}}}$ on $[\![\overrightarrow{T} \rangle\!\rangle$, we have the following remarks:

1. $[\![\overrightarrow{T} \rangle\!\rangle$ is a left $\mathcal{I}_n$-module, defined by the trivial action: $I_{(L,\sigma)}(r,s) = (r,s)$ if $\partial(r,s) = (L,\sigma)$ or 0 else.

2. In the below definition of $\overrightarrow{\delta_{\mathcal{V}}}$ and elsewhere in this paper, $\mathcal{B}\Gamma_n$ stands for $\mathcal{B}\Gamma_n \otimes_{\mathbb{Z}_2} \mathbb{F}_{\overrightarrow{T}}$. Note that $\mathcal{B}\Gamma_n \otimes_{\mathbb{Z}_2} \mathbb{F}_{\overrightarrow{T}}$ has a differential graded algebra structure over $\mathbb{F}_{\overrightarrow{T}}$, induced by the differential graded algebra structure over $\mathbb{Z}_2$ of $\mathcal{B}\Gamma_n$ (this fact will be mentioned in Chapter 9).

3. If C is a $+$ cleaved circle of a state (r,s), (r,s_C) denotes a state obtained from (r,s) by changing the decoration on C from $+$ to $-$ and using the same decorations for other circles. Therefore, if $\partial(r,s) = (L,\sigma)$ then $\partial(r,s_C) = (L,\sigma_C)$.

Given $[\![\overrightarrow{T} \rangle\!\rangle$ equipped with the ζ-grading, we define a left $\mathcal{I}_n$-module map:

$$\overrightarrow{\delta_{\mathcal{V}}} : [\![\overrightarrow{T} \rangle\!\rangle \longrightarrow \mathcal{B}\Gamma_n \otimes_{\mathcal{I}_n} [\![\overrightarrow{T} \rangle\!\rangle[-1]$$

by specifying the image of $\overrightarrow{\delta_{\mathcal{V}}}$ on each generator $\xi = (r,s)$ of $[\![\overrightarrow{T} \rangle\!\rangle$:

$$\overrightarrow{\delta_{\mathcal{V}}}(r,s) = I_{\partial(r,s)} \otimes \partial_{\mathcal{V}}(r,s) + \sum_{C \in \text{CIR}(\partial(r,s)), s(C)=+} \overrightarrow{w_C}\overrightarrow{e_C} \otimes (r,s_C). \tag{5.1}$$

The bigradings of the terms $I_{\partial(r,s)} \otimes \partial_{\mathcal{V}}$ are decreased by $(0,2)$ because the bigradings of idempotents are $(0,0)$ and $\partial_{\mathcal{V}}$ is a $(0,-2)$ differential. Therefore, the ζ-grading is increased by 1. Similarly, since the ζ-grading of $\overrightarrow{e_C}$ is $1/2$ and $\zeta(r,s) - \zeta(r,s_C) = 1/2[i(L,\sigma)/2 - i(L,\sigma_C)/2] = 1/2$, the ζ-gradings of the terms $\overrightarrow{e_C} \otimes (r,s_C)$ are larger by 1. As a result, $\overrightarrow{\delta_{\mathcal{V}}}$ is ζ-grading preserving.

Next, we prove the following proposition:

Proposition 28. *$\overrightarrow{\delta_{\mathcal{V}}}$ is a type D-structure on $[\![\overrightarrow{T}]\!\rangle$:*

$$(\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \overrightarrow{\delta_{\mathcal{V}}})\overrightarrow{\delta_{\mathcal{V}}} + (d_{\Gamma_n} \otimes \mathbb{I}_d)\overrightarrow{\delta_{\mathcal{V}}} = 0.$$

Proof. Since the image of d_{Γ_n} on idempotents or right decoration elements equals 0, it suffices to verify that:

$$(\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \overrightarrow{\delta_{\mathcal{V}}})\overrightarrow{\delta_{\mathcal{V}}}(r,s) = 0$$

for each generator $\xi = (r,s)$ of $[\![\overrightarrow{T}]\!\rangle$.

Since the image of $\overrightarrow{\delta_{\mathcal{V}}}(r,s)$ contains states (r,s_C) (with coefficients in $\mathcal{B}\Gamma_n$), the image of $(\mathbb{I} \otimes \overrightarrow{\delta_{\mathcal{V}}})\overrightarrow{\delta_{\mathcal{V}}}(r,s)$ are the states $(r,s_{C_{1,2}})$ where the decoration $s_{C_{1,2}}$ of r is obtained from s by changing the decoration on $C_1, C_2 \in \textsc{cir}(r)$ from $+$ to $-$. Therefore, we have:

$$(\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \overrightarrow{\delta_{\mathcal{V}}})\overrightarrow{\delta_{\mathcal{V}}}(r,s) = \sum_{\substack{C_1,C_2 \in \textsc{cir}(r) \\ C_1 \neq C_2}} A_{(r,s_{C_{1,2}})}(r,s_{C_{1,2}}).$$

We will prove that each $A_{(r,s_{C_{1,2}})} = 0$. Since each circle in $\textsc{cir}(r)$ is either cleaved or free, we have the following cases:

1. Both C_1 and C_2 are free circles, there are two ways to obtain $(r,s_{C_{1,2}})$ from (r,s):

$$(r,s) \xrightarrow{\overrightarrow{w_{C_1}} I_{\partial(r,s)}} (r,s_{C_1}) \xrightarrow{\overrightarrow{w_{C_2}} I_{\partial(r,s)}} (r,s_{C_{1,2}})$$

$$(r,s) \xrightarrow{\overrightarrow{w_{C_2}} I_{\partial(r,s)}} (r,s_{C_2}) \xrightarrow{\overrightarrow{w_{C_1}} I_{\partial(r,s)}} (r,s_{C_{1,2}}).$$

Therefore, $A_{(r,s_{C_{1,2}})} = (\overrightarrow{w_{C_1}}\overrightarrow{w_{C_2}} + \overrightarrow{w_{C_2}}\overrightarrow{w_{C_1}})I_{\partial(r,s)} = 0$. Note that the elements above the arrows indicate the algebra elements in $\mathcal{B}\Gamma_n$.

2. C_1 is free and C_2 is cleaved. We have two following paths:

$$(r,s) \xrightarrow{\overrightarrow{w_{C_1}}I_{\partial(r,s)}} (r,s_{C_1}) \xrightarrow{\overrightarrow{w_{C_2}}\overrightarrow{e_{C_2}}} (r,s_{C_{1,2}})$$

$$(r,s) \xrightarrow{\overrightarrow{w_{C_2}}\overrightarrow{e_{C_2}}} (r,s_{C_2}) \xrightarrow{\overrightarrow{w_{C_1}}I_{\partial(r,s_{C_2})}} (r,s_{C_{1,2}})$$

and therefore,

$$A_{(r,s_{C_{1,2}})} = \overrightarrow{w_{C_2}}\overrightarrow{w_{C_1}}\overrightarrow{e_{C_2}}I_{\partial(r,s_{C_{1,2}})} + \overrightarrow{w_{C_1}}\overrightarrow{w_{C_2}}I_{\partial(r,s_{C_1})}\overrightarrow{e_{C_2}} = 0.$$

3. Both C_1 and C_2 are cleaved circles. We have:

$$(r,s) \xrightarrow{\overrightarrow{w_{C_1}}\overrightarrow{e_{C_1}}} (r,s_{C_1}) \xrightarrow{\overrightarrow{w_{C_2}}\overrightarrow{e_{C_2}}} (r,s_{C_{1,2}})$$

$$(r,s) \xrightarrow{\overrightarrow{w_{C_2}}\overrightarrow{e_{C_2}}} (r,s_{C_1}) \xrightarrow{\overrightarrow{w_{C_1}}\overrightarrow{e_{C_1}}} (r,s_{C_{1,2}}).$$

By Relation (1) in the **Group I**, we have $\overrightarrow{e_{C_2}}\overrightarrow{e_{C_1}}+\overrightarrow{e_{C_1}}\overrightarrow{e_{C_2}} = 0$. Therefore, $A_{(r,s_{C_{1,2}})} = 0$.
◇

In [17, Section 5], L. Roberts defined a type D structure $\overrightarrow{\delta_T}$ on the bigraded module $[\![\overrightarrow{T} \rangle\!\rangle$, which is bigrading preserving into $\mathcal{B}\Gamma_n \otimes_{\mathcal{I}_n} [\![\overrightarrow{T} \rangle\!\rangle[(-1,0)]$ and thus, is also ζ-grading preserving into $\mathcal{B}\Gamma_n \otimes_{\mathcal{I}_n} [\![\overrightarrow{T} \rangle\!\rangle[-1]$.
As the next step, we show that $\overrightarrow{\delta_{\mathcal{V}}}$ commutes with $\overrightarrow{\delta_T}$ in the following sense:

Proposition 29. *The type D structures $\overrightarrow{\delta_{\mathcal{V}}}$ and $\overrightarrow{\delta_T}$ on $[\![\overrightarrow{T} \rangle\!\rangle$ satisfy:*

$$(\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \overrightarrow{\delta_{\mathcal{V}}})\overrightarrow{\delta_T} + (\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \overrightarrow{\delta_T})\overrightarrow{\delta_{\mathcal{V}}} = 0.$$

The proof of this proposition will be presented in the next chapter. Combined with the fact that $\overrightarrow{\delta_{\mathcal{V}}}$ and $\overrightarrow{\delta_T}$ are type D structures on $\llbracket \overrightarrow{T} \rangle\!\rangle$, Proposition 29 ensures that $\overrightarrow{\delta_{T,\bullet}} = \overrightarrow{\delta_{\mathcal{V}}} + \overrightarrow{\delta_T}$ is a type D structure on $\llbracket \overrightarrow{T} \rangle\!\rangle$. In Chapter 10, we will prove one of the main theorems of this paper:

Theorem 30. *The stable homotopy class of the type D structure $(\llbracket \overrightarrow{T} \rangle\!\rangle, \overrightarrow{\delta_{T,\bullet}})$ is an isotopy invariant of the tangle defined by $\overrightarrow{T}$.*

CHAPTER 6

PROOF OF PROPOSITION 29

Before proving the proposition, let us briefly recall the definition of $\overrightarrow{\delta_T}$ defined in [17, Section 5]. For each generator (r, s) of $[\![\overrightarrow{T} \rangle\!\rangle$, let

$$\overrightarrow{\delta_T}(r,s) := I_{\partial(r,s)} \otimes d_{APS} + \sum_{\gamma \in \textsc{Bridge}(r)} \mathcal{B}(\gamma)$$

$$+ \sum_{\gamma \in \textsc{dec}(r,s)} \overrightarrow{e_{C(\gamma)}} \otimes (r_\gamma, s_\gamma) + \sum_{C \in \textsc{cir}(\partial(r,s)), s(C)=+} \overleftarrow{e_C} \otimes (r, s_C)$$

where:

1. $\textsc{Bridge}(r)$ is the union of left bridges of $\partial(r,s)$ and the active resolution bridges γ of r so that the right planar matching of r_γ is different from the right planar matching of r. Furthermore, $\mathcal{B}(\gamma)$ is the sum of (r_γ, s^i_γ) where s^i_γ is computed from the Khovanov Frobenius algebra, with the recorded coefficient in $\mathcal{B}\Gamma_n$ corresponding to the bridge element whose source is $\partial(r,s)$ and target is $\partial(r_\gamma, s^i_\gamma)$. We note that the first row of the below figure illustrates a term in $\mathcal{B}(\gamma)$ obtained by surgering along an active resolution bridge γ.

2. $\textsc{dec}(r,s)$ is a collection of active resolution bridges γ of r such that either both feet of γ belong to the same component of $C \cap \overrightarrow{\mathbb{H}}$ where C is a $+$ cleaved circle of r, or one foot of γ belongs to a $+$ cleaved circle C of r and another one belongs to a $-$ free circle of r. In both cases, s_γ is computed from the Khovanov Frobenius algebra with a condition that $s_\gamma(C_\gamma) = -$ where C_γ is a cleaved circle of r_γ, obtained from C by surgering along γ. The second row of the below figure illustrates the case when both feet of γ belong to the same component of $C \cap \overrightarrow{\mathbb{H}}$.

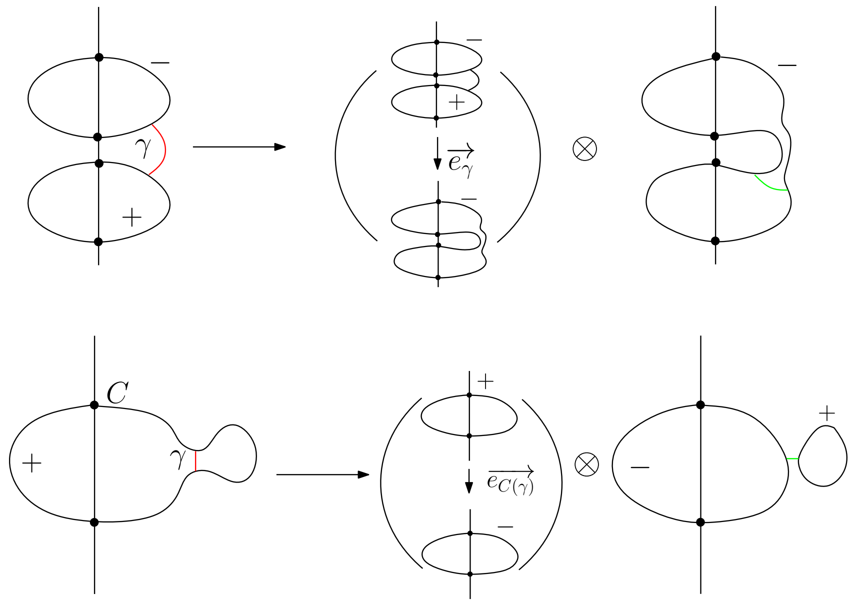


Figure 6.1: Examples about terms in the type D structure

Proof of Proposition 29. It suffices to prove that for each generator $\xi = (r, s)$ of $[\![\overrightarrow{T} \rangle\!\rangle$,

$$(\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \overrightarrow{\delta_{\mathcal{V}}})\overrightarrow{\delta_T}(r, s) + (\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \overrightarrow{\delta_T})\overrightarrow{\delta_{\mathcal{V}}}(r, s) = 0.$$

We rewrite the left hand side as

$$(\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \overrightarrow{\delta_{\mathcal{V}}})\overrightarrow{\delta_T}(r, s) + (\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \overrightarrow{\delta_T})\overrightarrow{\delta_{\mathcal{V}}}(r, s) = \sum_{(r', s')} A(r', s') \otimes (r', s')$$

where $A(r', s')$ is computed by taking the sum of products $e_\alpha e_\beta$, modeled by:

$$(r, s) \xrightarrow{e_\alpha} (r_1, s_1) \xrightarrow{e_\beta} (r', s').$$

Note that e_α and e_β are the elements in $\mathcal{B}\Gamma_n$ corresponding to $\partial(r, s) \to \partial(r_1, s_1)$ and $\partial(r_1, s_1) \to \partial(r', s')$, respectively. Additionally, one of them is either an idempotent or a right decoration element (this term comes from $\overrightarrow{\delta_{\mathcal{V}}}$) and the other is either an idempotent, or a bridge element, or a right decoration element, or a left decoration element (this term comes from $\overrightarrow{\delta_T}$).

Our goal is to prove that $A(r', s') = 0$. We have the following two cases:

Case I. The term coming from $\overrightarrow{\delta_\mathcal{V}}$ is a right decoration element $\overrightarrow{e_C}$. In this case, we have the following subcases:

1. The term coming from $\overrightarrow{\delta_T}$ is a left decoration element $\overleftarrow{e_D}$. Then (r', s') is obtained from (r, s) by changing the decorations on two cleaved circles C, D of r from $+$ to $-$. Therefore, there are exactly two paths from (r, s) to (r', s'):

$$(r, s) \xrightarrow{\overrightarrow{w_C}\overrightarrow{e_C}} (r, s_C) \xrightarrow{\overleftarrow{e_D}} (r', s')$$

$$(r, s) \xrightarrow{\overleftarrow{e_D}} (r, s_D) \xrightarrow{\overrightarrow{w_C}\overrightarrow{e_C}} (r', s')$$

   We have: $A(r', s') = \overrightarrow{w_C}[\overrightarrow{e_C}\overleftarrow{e_D} + \overleftarrow{e_D}\overrightarrow{e_C}] = 0$ by Relation (1) in the **Group I**.

2. The term coming from $\overrightarrow{\delta_T}$ is either an idempotent, or a right decoration element, or a bridge element e_γ where $\gamma \in \textsc{Bridge}(r)$, such that the support of γ is disjoint from C. In this case, there are again exactly two paths from (r, s) to (r', s'). Similar to the previous case and the proof of Proposition 28, using Relation (1) or (2) in the **Group I**, we can compute that $A(r', s') = 0$.

3. The term from $\overrightarrow{\delta_T}$ is a bridge element e_γ where $\gamma \in \textsc{Bridge}(r)$ and C is in the support of γ. We have the two following subcases:

   (a) The merging case: let C_1 be a cleaved circle in $\partial(r, s)$, merged with C by merging on γ to form another cleaved circle C_2. For the path from (r, s) to (r', s') to exist, $s(C_1) = +$. Then, there are three paths in $A(r', s')$:

$$(r, s) \xrightarrow{\overrightarrow{w_C}\overrightarrow{e_C}} (r, s_C) \xrightarrow{e_\gamma} (r', s')$$

$$(r,s) \xrightarrow{\overrightarrow{w_{C_1}}\overrightarrow{e_{C_1}}} (r, s_{C_1}) \xrightarrow{e_\gamma} (r', s')$$

$$(r,s) \xrightarrow{e_\gamma} (r_\gamma, s_\gamma) \xrightarrow{\overrightarrow{w_{C_2}}\overrightarrow{e_{C_2}}} (r', s').$$

As the result, $A(r',s') = \overrightarrow{w_C}\overrightarrow{e_C}e_\gamma + \overrightarrow{w_{C_1}}\overrightarrow{e_{C_1}}e_\gamma + e_\gamma\overrightarrow{w_{C_2}}\overrightarrow{e_{C_2}}$. Since $\overrightarrow{w_{C_2}} = \overrightarrow{w_{C_1}} + \overrightarrow{w_C}$, we can rewrite: $A(r',s') = \overrightarrow{w_C}(\overrightarrow{e_C}e_\gamma + e_\gamma\overrightarrow{e_{C_2}}) + \overrightarrow{w_{C_1}}(\overrightarrow{e_{C_1}}e_\gamma + e_\gamma\overrightarrow{e_{C_2}}) = 0$, by Relation (1) for the merging case in the **Group III**.

(b) The case of division: let C_1 be a cleaved circle in $\partial(r,s)$, divided into C and C_2 by a surgery along γ. For the path from (r,s) to (r',s') to exist, $s(C_1) = +$. Then, there are three paths in $A(r',s')$:

$$(r,s) \xrightarrow{\overrightarrow{w_{C_1}}\overrightarrow{e_{C_1}}} (r, s_{C_1}) \xrightarrow{e_\gamma} (r', s')$$

$$(r,s) \xrightarrow{e_\gamma} (r_\gamma, s^2_\gamma) \xrightarrow{\overrightarrow{w_{C_2}}\overrightarrow{e_{C_2}}} (r', s')$$

$$(r,s) \xrightarrow{e_\gamma} (r_\gamma, s_\gamma) \xrightarrow{\overrightarrow{w_C}\overrightarrow{e_C}} (r', s')$$

where s^2_γ (respectively s_γ) assigns $+$ $(-)$ to C_2 and $-$ $(+)$ to C.
As the result, $A(r',s') = \overrightarrow{w_{C_1}}\overrightarrow{e_{C_1}}e_\gamma + e_\gamma\overrightarrow{w_{C_2}}\overrightarrow{e_{C_2}} + e_\gamma\overrightarrow{w_C}\overrightarrow{e_C}$. Since $\overrightarrow{w_{C_1}} = \overrightarrow{w_{C_2}} + \overrightarrow{w_C}$, we can rewrite: $A(r',s') = \overrightarrow{w_C}(\overrightarrow{e_{C_1}}e_\gamma + e_\gamma\overrightarrow{e_C}) + \overrightarrow{w_{C_2}}(\overrightarrow{e_{C_1}}e_\gamma + e_\gamma\overrightarrow{e_{C_2}}) = 0$, by Relation (1) for the case of division in the **Group III**.

Case II. The term coming from $\overrightarrow{\delta_\mathcal{V}}$ is an idempotent obtained by changing the decoration on a free circle C from $+$ to $-$. We have the following subcases:

1. The term from $\overrightarrow{\delta_T}$ is an idempotent. In this case, we know that the product of the weights of two paths from (r,s) to (r',s') will be canceled out by [20, Proposition 3.6]. Therefore, $A(r',s') = 0$.

2. The term from $\overrightarrow{\delta_T}$ is a bridge element e_γ where $\gamma \in \textsc{Bridge}(r)$. Since the support of γ is disjoint from C, we only have two paths from (r,s) to (r',s') and $A(r',s') = \overrightarrow{w_C}I_{\partial(r,s)}e_\gamma + e_\gamma\overrightarrow{w_C}I_{\partial(r,s)} = 0$.

3. The term from $\overrightarrow{\delta_T}$ is a right decoration element, obtained by surgery of (r, s) along $\gamma \in \text{DEC}(r, s)$. It is possible that the support of γ is disjoint or not disjoint from C. When the support of γ is disjoint from C, the proof of $A(r, s) = 0$ is similar to the case II.2. When the support of γ is not disjoint from C, the situation is more interesting. Since the case of division can be handled similarly, we only present the proof of the case when γ merges a $+$ cleaved circle $D \in \partial(r, s_C)$ with C to give a cleaved circle $C_1 \in \partial(r', s')$. In this case, there are exactly three paths from (r, s) to (r', s'):

$$(r, s) \xrightarrow{I_{\partial(r,s)}} (r_2, s_2) \xrightarrow{\overrightarrow{w_{C_1}}\overrightarrow{e_{C_1}}} (r', s')$$

$$(r, s) \xrightarrow{\overrightarrow{w_C} I_{\partial(r,s)}} (r, s_C) \xrightarrow{\overrightarrow{e_D}} (r', s')$$

$$(r, s) \xrightarrow{\overrightarrow{w_D}\overrightarrow{e_D}} (r, s_D) \xrightarrow{I_{\partial(r,s_D)}} (r', s').$$

Therefore, $A(r', s') = (\overrightarrow{w_{C_1}} + \overrightarrow{w_D} + \overrightarrow{w_C})\overrightarrow{e_D} = 0$ because $\overrightarrow{e_D} = \overrightarrow{e_{C_1}}$ and $\overrightarrow{w_{C_1}} = \overrightarrow{w_C} + \overrightarrow{w_D}$.

$\diamond$

CHAPTER 7

THE DEFORMATION RETRACTION OF THE TYPE D STRUCTURE

In this chapter, we will define the type D structure $(\llbracket \overrightarrow{CT} \rrbracket, \overrightarrow{\delta_{n,T}})$ described in Chapter 1 and prove that it is homotopy equivalent to $(\llbracket \overrightarrow{T} \rrbracket, \overrightarrow{\delta_{T,\bullet}})$ as type D structures.

Let $\mathrm{ST}_n(\overrightarrow{T})$ be the collection of states of $\overrightarrow{T}$ that do not have any free circles in their resolutions. Recall that $\llbracket \overrightarrow{CT} \rrbracket$ is a vector space over $\mathbb{F}_{\overrightarrow{T}}$ generated by $\mathrm{ST}_n(\overrightarrow{T})$.

We define the left $\mathcal{I}_n$-module map:

$$\overrightarrow{\delta_{T,n}} : \llbracket \overrightarrow{CT} \rrbracket \longrightarrow \mathcal{B}\Gamma_n \otimes_{\mathcal{I}_n} \llbracket \overrightarrow{CT} \rrbracket[-1]$$

by specifying the image of $\overrightarrow{\delta_{\mathcal{V}}}$ on each generator $\xi = (r, s)$ of $\llbracket \overrightarrow{CT} \rrbracket$:

$$\overrightarrow{\delta_{T,n}}(r,s) = \sum_{(r',s') \in \mathrm{ST}_n(\overrightarrow{T})} \big\langle (r,s),(r',s') \big\rangle I_{\partial(r,s)} \otimes (r',s') + \sum_{\gamma \in \mathrm{BRIDGE}(r)} B(\gamma)$$

$$+ \sum_{C \in \mathrm{CIR}(\partial(r,s)), s(C)=+} (\overleftarrow{e_C} + \overrightarrow{w_C}\overrightarrow{e_C}) \otimes (r, s_C)$$

where

1. The coefficient $\big\langle\, (r,s)\,,\,(r',s')\,\big\rangle$ in the first summand is calculated from the weights assigned to the arcs. First, $\big\langle\, (r,s)\,,\,(r',s')\,\big\rangle = 0$ unless they satisfy the two following conditions:

    (a) r is different from r' at two crossings c_1 and c_2 which are resolved with the 0-resolution in r and the 1-resolution in r'.

    (b) $\partial(r,s) = \partial(r',s')$ where $\partial(r,s)$ and $\partial(r',s')$ are the cleaved links associated to (r,s) and (r',s') respectively.

If (r,s) and (r',s') satisfy the above conditions, let r_{01} (r_{10}) be the resolution where c_1 is resolved with the 0-resolution (respectively 1-resolution) and c_2 resolved with the 1-resolution (respectively 0-resolution). Then:

$$\big\langle\, (r,s)\,,\,(r',s')\,\big\rangle = \begin{cases} 1/\overrightarrow{w_{C_{10}}} + 1/\overrightarrow{w_{C_{01}}} & r_{10} \text{ and } r_{01} \text{ have free circles } C_{10} \text{ and } C_{01} \text{ respectively} \\ 1/\overrightarrow{w_{C_{10}}} & r_{10} \text{ has a free circle } C_{10} \text{ but } r_{01} \text{ does not} \\ 1/\overrightarrow{w_{C_{01}}} & r_{01} \text{ has a free circle } C_{01} \text{ but } r_{10} \text{ does not} \\ 0 & \text{ neither } r_{10} \text{ nor } r_{01} \text{ has a free circle.} \end{cases} \tag{7.1}$$

2. The second summand is defined exactly the same as the second summand in the definition of $\overrightarrow{\delta_T}$ described at the beginning of Chapter 6. We note that if (r',s') is obtained from (r,s) by surgering along either a left bridge of $\partial(r,s)$ or an active resolution bridge γ of r so that the right planar matching of r_γ is different from the right planar matching of r, $\langle\overrightarrow{\delta_{T,n}}(r,s),(r',s')\rangle = \langle\overrightarrow{\delta_T}(r,s),(r',s')\rangle = \langle\overrightarrow{\delta_{T,\bullet}}(r,s),(r',s')\rangle$.

3. A state (r,s_C) in the third summand is obtained from (r,s) by changing the decoration on a cleaved circle C of r from $+$ to $-$. As we can see, $\langle\overrightarrow{\delta_{T,n}}(r,s),(r,s_C)\rangle = \langle\overrightarrow{\delta_{T,\bullet}}(r,s),(r,s_C)\rangle = \overleftarrow{e_C} + \overrightarrow{w_C}\overrightarrow{e_C}$.

We, next, will recall the type D cancellation lemma whose proof can be found in [17, Appendix A].

Let (A,d) be a differential graded algebra over a ground ring R (characteristic 2). Let N be a graded module over R. Suppose over R, N can be generated by a basis $\{x_1,...,x_n\}$. Suppose $a_{ij}\in A$ so that:

$$d(a_{ik}) + \sum_{j=1}^{n} a_{ij}\,a_{jk} = 0 \qquad\qquad i,k\in\{1,\ldots,n\} \tag{7.2}$$

and $gr(a_{ij}) = |x_i| - |x_j| + 1$. Then the a_{ij} can be used as structure coefficients in the definition of a type D structure $\delta : N \to (A \otimes_R N)[-1]$ defined by:

$$\delta(x_i) = \sum_{j=1}^{n} a_{ij} \otimes x_j.$$

Lemma 31. *[17, Proposition 39] Let δ be a D structure on N. Suppose there is a basis B for N whose structure coefficients satisfy $a_{ii} = 0$ and a_{12} is invertible. Let $\overline{N} = \operatorname{span}_R\{\overline{x}_3, \dots, \overline{x}_n\}$. Then*

$$\overline{\delta}(\overline{x}_i) = \sum_{j \geq 3} (a_{ij} - a_{i2}\, a_{1j}) \otimes \overline{x}_j$$

is a D structure on $\overline{N}$. Furthermore, the maps

$$\iota : \overline{N} \to A \otimes N \qquad \iota(\overline{x}_i) = 1_A \otimes x_i - a_{i2} \otimes x_1$$

$$\pi : N \to A \otimes \overline{N} \qquad \pi(x_i) = \begin{cases} 0 & i = 1 \\ \sum_{j \geq 3} a_{1j} \otimes \overline{x}_j & i = 2 \\ 1_A \otimes \overline{x}_i & i \geq 3 \end{cases}$$

realize $\overline{N}$ as a deformation retraction of N with $\iota \circ \pi \simeq_H \mathbb{I}_N$ using the homotopy $H : N \to A \otimes N[-1]$

$$H(x_i) = \begin{cases} 1_A \otimes x_1 & i = 2 \\ 0 & i \neq 2. \end{cases}$$

Proposition 32. *$(\llbracket \overrightarrow{CT} \rrbracket, \overrightarrow{\delta_{n,T}})$ is a type D structure over $(\mathcal{B}\Gamma_n, \mathcal{I}_n)$. Additionally, $(\llbracket \overrightarrow{T} \rrbracket, \overrightarrow{\delta_{T,\bullet}})$, defined in Chapter 5, is homotopy equivalent to $(\llbracket \overrightarrow{CT} \rrbracket, \overrightarrow{\delta_{n,T}})$.*

Proof. Let (r, s) be a state of $\llbracket \overrightarrow{T} \rrbracket$ containing a free circle C. Then the decoration $s(C)$ is either $+$ or $-$. Corresponding to (r, s), there is a state (r, s') of $\llbracket \overrightarrow{T} \rrbracket$ obtained by changing the decoration on C from $\pm$ to $\mp$. We call this pair of states a mutual pair. We will use the cancellation lemma 31 to cancel the mutual pairs (r, s) and (r, s'). Without loss of generality, we suppose $s(C) = +$. Therefore, $\overrightarrow{\delta_{T,\bullet}}(r, s) = \overrightarrow{w_C} I_{\partial(r,s)} \otimes (r, s') + Q = \overrightarrow{w_C} 1_{\mathcal{B}\Gamma_n} \otimes (r, s') + Q$

where Q is a linear combination supported on states not equal to either (r,s) or (r,s'). Since $\overrightarrow{w_C}1_{\mathcal{B}\Gamma_n}$ is invertible, we can cancel this pair to get a deformation retraction (N,δ) of $(\llbracket \overrightarrow{T} \rrbracket, \overrightarrow{\delta_{T,\bullet}})$ supported on $\textsc{State}(\overrightarrow{T}) \setminus \{(r,s),(r,s')\}$. In particular, if (r_1,s_1) and (r_1,s_1') are another mutual pair, $\langle \delta(r_1,s_1),(r_1,s_1')\rangle = \langle \overrightarrow{\delta_{T,\bullet}}(r_1,s_1),(r_1,s_1')\rangle$. As a result, we can cancel all of the mutual pairs of states and what we have left is a homotopy equivalent type D structure δ_n supported on $\textsc{st}_n(\overrightarrow{T})$. We also need to verify that δ_n is the same as $\overrightarrow{\delta_{n,T}}$. Let $(r,s),(r',s')$ be two states in $\textsc{st}_n(\overrightarrow{T})$ such that $\langle \delta_n(r,s),(r',s')\rangle \neq 0$. Therefore, under the action of $\overrightarrow{\delta_{T,\bullet}}$, we have the following sequence of transitions of states in $\textsc{State}(\overrightarrow{T})$:

$$(r,s) = (r_0,s_0^+) \to (r_1,s_1^-) \to (r_1,s_1^+) \to .... \to (r_k,s_k^+) \to (r_{k+1},s_{k+1}^-) = (r',s')$$

where each transition $(r_i,s_i^-) \to (r_i,s_i^+)$ comes from a mutual pair and each transition $(r_i,s_i^+) \to (r_{i+1},s_{i+1}^-)$ corresponds to a term in $\overrightarrow{\delta_{T,\bullet}}$. We also let C_i be the free circle where $s_i^-(C_i) = -$ and $s_i^+(C_i) = +$. By denoting the number of $+$ and $-$ free circles for each state (r,s) of $\overrightarrow{T}$ by $J(r,s) = (J_+(r,s), J_-(r,s))$, we see that $J(r_i,s_i^+) - J(r_i,s_i^-) = (1,-1)$ for each $i \in \{1,...,k\}$. Additionally, we evaluate $J_i = J(r_{i+1},s_{i+1}^-) - J(r_i,s_i^+)$ as the following cases:

1. If the corresponding coefficient from $\partial(r_i,s_i^+) \to \partial(r_{i+1},s_{i+1}^-)$ is either $\overleftarrow{e_C}$ or bridge element then $J_i = (0,0)$.

2. If the corresponding coefficient is $\overrightarrow{e_C}$ then J_i belongs to $\{(1,0),(0,-1),(0,0)\}$.

3. If this transition comes from $I \otimes d_{APS}$, J_i belongs to $\{(-1,0),(0,1)\}$.

Since $(r,s),(r',s') \in \textsc{st}_n(\overrightarrow{T})$, we have $J(r,s) = J(r',s') = (0,0)$. Furthermore, we have:

$$J(r',s') - J(r,s) = \sum_{i=1}^{k} \left[J(r_i,s_i^+) - J(r_i,s_i^-) \right] + \sum_{i=0}^{k} J_i = (0,0).$$

Therefore, $\sum\limits_{i=0}^{k} J_i = (-k,k)$. Looking through all of possible cases of J_i, k has to be either 0 or 1. If $k=0$, (r',s') is obtained from (r,s) by either changing a decoration on a cleaved circle from + to − or performing surgery along $\gamma \in \textsc{Bridge}(r)$. In this case:

$$\langle \delta_n(r,s), (r',s') \rangle = \langle \overrightarrow{\delta_{T,\bullet}}(r,s), (r',s') \rangle.$$

If $k = 1$, we need to have $\partial(r,s) = \partial(r_1, s_1^-) = \partial(r_1, s_1^+) = \partial(r', s')$. As a result, we know how to calculate δ_n. We call a sequence of transitions an $\mathcal{I}$-transition if it is of the following form: $(r,s) \to (r_1, s_1^-) \to (r_1, s_1^+) \to (r', s')$ where the states in this sequence of transitions have the same "cleaved link" boundaries. By using the formula in Lemma 31, we have:

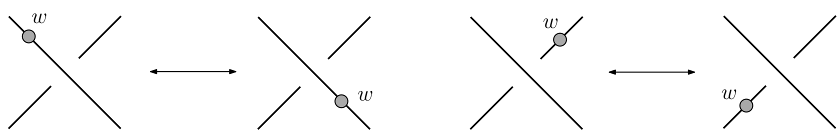
$$\langle \delta_n(r,s), (r',s') \rangle = \sum_{\mathcal{I}-transition} (1/\overrightarrow{w_C}) I_{\partial(r,s)}.$$

Also, we see that (r_1, s_1^-) is obtained from (r,s) by surgery along an active resolution bridge γ_1 of r. Since $(r,s) \in \mathrm{ST}_n(\overrightarrow{T})$ and a new free circle C is created in (r_1, s_1^-), γ_1 has to divide a cleaved circle C_1 of r into C and another cleaved circle C_2 of r_1. Furthermore, (r', s') is obtained from (r_1, s_1^+) by surgering along an active resolution bridge γ_2 of r_1. Since $(r', s') \in \mathrm{ST}_n(\overrightarrow{T})$, γ_2 merges C to a cleaved circle D of r_1. If $C_2 = D$, there are two $\mathcal{I}$-transitions from (r,s) to (r',s'). If $C_2 \neq D$, there is only one $\mathcal{I}$-transition from (r,s) to (r',s'). Therefore, δ_n is $\overrightarrow{\delta_{T,n}}$. $\diamond$

CHAPTER 8

INVARIANCE OF THE TYPE D STRUCTURE UNDER THE WEIGHT MOVES

Following [20] and [23], we will prove that the homotopy type of $(\llbracket \overrightarrow{T} \rrbracket, \overrightarrow{\delta_{T,\bullet}})$ is invariant under the following weight moves. Let $\overrightarrow{T}$ and $\overrightarrow{T}'$ be weighted tangle diagrams of a tangle $\overrightarrow{\mathcal{T}}$ with weighted arcs before and after the weight w is moved along the crossing c as the following figures:



We need to show that $(\llbracket \overrightarrow{T} \rrbracket, \overrightarrow{\delta_{T,\bullet}})$ is homotopy equivalent to $(\llbracket \overrightarrow{T}' \rrbracket, \overrightarrow{\delta_{T',\bullet}})$.
Let $\overline{D_c} : \llbracket \overrightarrow{T} \rrbracket \longrightarrow \mathcal{B}\Gamma_n \otimes_{\mathcal{I}_n} \llbracket \overrightarrow{T}' \rrbracket$ be the $\mathbb{F}_{\overrightarrow{T}}$-linear map defined as follows. Let $\xi = (r, s)$ be a generator of $\llbracket \overrightarrow{T} \rrbracket$. $\overline{D_c}(r, s)$ is defined to be 0 if $r(c) = 0$. If $r(c) = 1$, define $\overline{D_c}(r, s)$ to be the sum of element(s) $e \otimes (r', s')$ such that:

1. r' is the resolution obtained from r by surgering along the inactive bridge γ at c of r.

2. $s'(D) = s(D)$ for all circles D not abutting c and the signs on circles abutting c are computed by using the Khovanov Frobenius algebra.

3. e is the element in $\mathcal{B}\Gamma_n$ whose source is $\partial(r, s)$ and target is $\partial(r', s')$. We note that since r' is obtained from r by surgering a bridge on the right side, e is either a right bridge element, or a right decoration element, or an idempotent.

Additionally, if e is a bridge element in $\mathcal{B}\Gamma_n$, we have:

$$\zeta(r', s') = h(r', s') - q(r', s')/2 = [h(r, s) - 1] - [q(r, s) - 3/2]/2 = \zeta(r, s) - 1/4.$$

Therefore, in this case, $\zeta(r', s') + \zeta(e) = \zeta(r, s)$. By a similar computation, if e is either an idempotent or a right decoration element, then $\zeta(r', s') + \zeta(e) = \zeta(r, s)$. As a result, $\overline{D_c}$ is a ζ-grading preserving map.

Proposition 33. *The map* $\Psi : [\![\overrightarrow{T} \rangle\!\rangle \longrightarrow \mathcal{B}\Gamma_n \otimes_{\mathcal{I}_n} [\![\overrightarrow{T}' \rangle\!\rangle$ *where* $\Psi(r, s) = I_{\partial(r,s)} \otimes (r, s) + w \cdot \overline{D_c}(r, s)$ *is a type D homomorphism.*

Proof. To ease the notation, we let δ, δ' stand for $\overrightarrow{\delta_{T,\bullet}}$, $\overrightarrow{\delta_{T',\bullet}}$ respectively.
It suffices to prove:

$$(\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \delta')\Psi + (\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \Psi)\delta + (d_{\Gamma_n} \otimes \mathbb{I}_d)\Psi = 0 \tag{8.1}$$

when applied to each $(r, s) \in \text{STATE}(\overrightarrow{T})$.
Since the image of Ψ does not have any term of the form $\overleftarrow{e_C} \otimes (r_1, s_1)$, the last term of (8.1) will be 0. Also, the map δ can be written as the sum: $\delta = D_T + D_{T,c} + E_T + E_{T,c}$ where D_T and $D_{T,c}$ are the terms in the image of δ obtained by surgering either one left bridge or one active bridge at $c_1 \neq c$, and c respectively. E_T ($E_{T,c}$ respectively) is the term in the image of δ obtained by changing the decoration on a circle not abutting c (abutting c) from $+$ to $-$. We can write down a similar sum for δ': $\delta' = D'_T + D_{T',c} + E_{T'} + E_{T',c}$.
As we can canonically identify $[\![\overrightarrow{T} \rangle\!\rangle$ as $[\![\overrightarrow{T'} \rangle\!\rangle$, let δ_1 be a D structure on $[\![\overrightarrow{T} \rangle\!\rangle$, which is precisely the same as δ' on $[\![\overrightarrow{T'} \rangle\!\rangle$. Note that: $\delta + \delta_1 = E_{T,c} + E_{T',c}$.
Let $\mathbb{I}_T : [\![\overrightarrow{T} \rangle\!\rangle \longrightarrow \mathcal{B}\Gamma_n \otimes_{\mathcal{I}_n} [\![\overrightarrow{T}' \rangle\!\rangle$ be the map defined by $\mathbb{I}_T(r, s) = I_{\partial(r,s)} \otimes (r, s)$. The left hand side of (8.1) can be written as the sum:

$$(\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \delta')\mathbb{I}_T + (\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \mathbb{I}_T)\delta_1 + (\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \delta')w \cdot \overline{D_c}$$

$$+(\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes w \cdot \overline{D_c})\delta + (\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \mathbb{I}_T)(E_{T,c} + E_{T',c}).$$

The sum of the first two terms will be 0 since both equal δ'. Rearranging the other terms, we need to prove:

$$(\mu_{\mathcal{B}\Gamma_n}\otimes\mathbb{I}_d)(\mathbb{I}\otimes\delta')w\cdot\overline{D_c}+(\mu_{\mathcal{B}\Gamma_n}\otimes\mathbb{I}_d)(\mathbb{I}\otimes w\cdot\overline{D_c})\delta=(\mu_{\mathcal{B}\Gamma_n}\otimes\mathbb{I}_d)(\mathbb{I}\otimes\mathbb{I}_T)(E_{T,c}+E_{T',c}). \quad (8.2)$$

By using the decomposition of δ and δ', we will in turn prove the following equations:

$$(\mu_{\mathcal{B}\Gamma_n}\otimes\mathbb{I}_d)(\mathbb{I}\otimes D_{T'})w\cdot\overline{D_c}+(\mu_{\mathcal{B}\Gamma_n}\otimes\mathbb{I}_d)(\mathbb{I}\otimes w\cdot\overline{D_c})D_T=0 \quad (8.3)$$

$$(\mu_{\mathcal{B}\Gamma_n}\otimes\mathbb{I}_d)(\mathbb{I}\otimes E_{T'})w\cdot\overline{D_c}+(\mu_{\mathcal{B}\Gamma_n}\otimes\mathbb{I}_d)(\mathbb{I}\otimes w\cdot\overline{D_c})E_T=0 \quad (8.4)$$

$$(\mu_{\mathcal{B}\Gamma_n}\otimes\mathbb{I}_d)(\mathbb{I}\otimes E_{T',c})w\cdot\overline{D_c}+(\mu_{\mathcal{B}\Gamma_n}\otimes\mathbb{I}_d)(\mathbb{I}\otimes w\cdot\overline{D_c})E_{T,c}=0 \quad (8.5)$$

$$(\mu_{\mathcal{B}\Gamma_n}\otimes\mathbb{I}_d)(\mathbb{I}\otimes D_{T',c})w\cdot\overline{D_c}+(\mu_{\mathcal{B}\Gamma_n}\otimes\mathbb{I}_d)(\mathbb{I}\otimes w\cdot\overline{D_c})D_{T,c}=(\mu_{\mathcal{B}\Gamma_n}\otimes\mathbb{I}_d)(\mathbb{I}\otimes\mathbb{I}_T)(E_{T,c}+E_{T',c}). \quad (8.6)$$

Let T_c be a tangle diagram obtained from T by switching the crossing at c (left-handed $\leftrightarrow$ right-handed), but with the same locations for all the weights. There is a natural identification of $[\![\overrightarrow{T}\rangle\!\rangle$ with $[\![\overrightarrow{T_c}\rangle\!\rangle$. Using this identification and the fact that $\overline{D_c}$ goes backwards from the Khovanov differential, the proof of Equation (8.3) comes from the proof that $\overrightarrow{\delta_{T_c}}$ is a type D structure on $[\![\overrightarrow{T_c}\rangle\!\rangle$. Using a similar argument, we also have Equation (8.4) is one of the cases of Proposition 29 (for the tangle diagram $\overrightarrow{T_c}$). For Equation (8.5), we remark that both terms will be 0 if $r(c)=0$ because $E_{T,c}$ does not change the value of $r(c)$. $(\mu_{\mathcal{B}\Gamma_n}\otimes\mathbb{I}_d)(\mathbb{I}\otimes E_{T',c})w\cdot\overline{D_c}$ will be 0 unless there is a + circle abutting c in the image of $\overline{D_c}$. As a result, $(\mu_{\mathcal{B}\Gamma_n}\otimes\mathbb{I}_d)(\mathbb{I}\otimes E_{T',c})w\cdot\overline{D_c}$ will be 0 if one of the circle(s) abutting c contains the marked point. In this case, it is also true that $(\mu_{\mathcal{B}\Gamma_n}\otimes\mathbb{I}_d)(\mathbb{I}\otimes w\cdot\overline{D_c})E_{T,c}=0$ because either $E_{T,c}$ is 0 or its image contains two $-$ circles abutting c. Since $\overline{D_c}$, $E_{T,c}$ and $E_{T',c}$ only change the decorations on the circles abutting c, it suffices to prove Equation (8.5) by checking all of the possibilities for the circles abutting c. We can also assume that the circles abutting c do not contain the marked point. Therefore, we have the following cases to check:

1. All of circles abutting c are free circles.

2. There is at least one free circle and one cleaved circle.

3. All of circles abutting c are cleaved circles.

Case (1) can be done similarly to case (2) and has already been described in [20, Proposition 7.1].

For case (2), we only address the merging case (the argument for division is similar). Let $\pm_c$ and $\pm_f$ denote the decoration on the cleaved and free circles, respectively. Then:

$$(\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes E_{T',c})\overline{D_c}(+_c+_f) + (\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \overline{D_c})E_{T,c}(+_c+_f)$$

$$= (\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes E_{T',c})(I_{\partial(r,s)} \otimes +_c) + (\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \overline{D_c})\big[(\overleftarrow{e_C} + \overrightarrow{w_C}\overrightarrow{e_C}) \otimes -_c+_f$$

$$+(\overrightarrow{w_f} I_{\partial(r,s)} \otimes +_c-_f)\big]$$

$$= (\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{I}_d)\big[I_{\partial(r,s)} \otimes (\overleftarrow{e_C} + (\overrightarrow{w_C} + \overrightarrow{w_f})\overrightarrow{e_C}) \otimes -_c\big] + (\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{I}_d)\big[(\overleftarrow{e_C} + \overrightarrow{w_C}\overrightarrow{e_C})\otimes I_{\partial(r,s)} \otimes -_c + \overrightarrow{w_f} I_{\partial(r,s)} \otimes \overrightarrow{e_C} \otimes -_c\big]$$

$$= (\overleftarrow{e_C} + (\overrightarrow{w_C} + \overrightarrow{w_f})\overrightarrow{e_C}) \otimes -_c + (\overleftarrow{e_C} + \overrightarrow{w_C}\overrightarrow{e_C}) \otimes -_c + \overrightarrow{w_f}\overrightarrow{e_C} \otimes -_c$$

$$= 0.$$

If either $s_c = -$ or $s_f = -$, then both terms will be 0 and, thus, Equation (8.5) is true.

For case (3), we will again only prove the identity for the merging case. Let $\pm_1$, $\pm_2$ be the decorations on cleaved circles C_1, C_2 respectively and $\pm_c$ be the decoration on a merged cleaved circle C. Let e_γ, e_{γ_1} and e_{γ_2} be the bridge elements representing the changes of the cleaved links: $+_1+_2 \longrightarrow +_c$, $-_1+_2 \longrightarrow -_c$ and $+_1-_2 \longrightarrow -_c$ respectively. Then

$$(\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes E_{T',c})\overline{D_c}(+_1+_2) + (\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \overline{D_c})E_{T,c}(+_1+_2)$$

$$= (\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes E_{T',c})(e_\gamma \otimes +_c) + (\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \overline{D_c})\big[\big((\overleftarrow{e_{C_1}} + \overrightarrow{w_{C_1}}\overrightarrow{e_{C_1}}) \otimes -_1 +_2\big)$$

$$+\big((\overleftarrow{e_{C_2}} + \overrightarrow{w_{C_2}}\overrightarrow{e_{C_2}}) \otimes +_1 -_2\big)\big]$$

$$= e_\gamma(\overleftarrow{e_C} + \overrightarrow{w_C}\overrightarrow{e_C}) \otimes -_c + (\overleftarrow{e_{C_1}} + \overrightarrow{w_{C_1}}\overrightarrow{e_{C_1}})e_{\gamma_1} \otimes -_c + (\overleftarrow{e_{C_2}} + \overrightarrow{w_{C_2}}\overrightarrow{e_{C_2}})e_{\gamma_2} \otimes -_c$$

$$= \big(e_\gamma\overleftarrow{e_C} + \overleftarrow{e_{C_1}}e_{\gamma_1} + \overleftarrow{e_{C_2}}e_{\gamma_2}\big) \otimes -_c + \overrightarrow{w_{C_1}}(e_\gamma\overrightarrow{e_C} + \overrightarrow{e_{C_1}}e_{\gamma_1}) \otimes -_c + \overrightarrow{w_{C_2}}(e_\gamma\overrightarrow{e_C} + \overrightarrow{e_{C_2}}e_{\gamma_2}) \otimes -_c$$

$$= 0.$$

since the first, second and third terms equal 0 by Relations 1 and 2 in the **Group III**.

Finally, we prove that Equation (8.6) is true. Again, since the proof for divisions is similar, we only present the proof for the merging case. We have the following subcases:

1. All of circles are free: The proof of Equation (8.6) is then similar to case (2) below and already described in [20, Proposition 7.1].

2. There is at least one free circle F and one cleaved circle C. Since what we are about to prove includes the case when the decoration on C is $-$, we can assume C does not contain the marked point. Let $\pm_c$ and $\pm_f$ be the decorations on the cleaved and free circles respectively. We rewrite the map on the RHS of Equation (8.6) as

$$+_c+_f \quad \longrightarrow \quad w \cdot \big[(\overrightarrow{e_C} \otimes -_c+_f) + (I_{\partial(r,s_C)} \otimes +_c-_f)\big]$$

$$-_c+_f \quad \longrightarrow \quad w \cdot I_{\partial(r,s_C)} \otimes -_c-_f$$

$$+_c-_f \quad \longrightarrow \quad w \cdot \overrightarrow{e_C} \otimes -_c-_f$$

if $r(c) = 1$ or $+_c \xrightarrow{0} -_c$ if $r(c) = 0$.

If $r(c) = 0$, the first term of the LHS of Equation (8.6) equals 0 because $\overline{D_c}$ is supported on states (r', s') with $r'(c) = 1$. The second term of the LHS maps $+_c$ to

$(\overrightarrow{e_C} + \overrightarrow{e_C}) \otimes -_c = 0$. Therefore, Equation (8.6) is also true in this case.

If $r(c) = 1$, the second term of the LHS of Equation (8.6) is 0 and the first term can be described as

$$+_c+_f \longrightarrow (w \cdot I_{\partial(r,s)}\overrightarrow{e_C} \otimes -_c+_f) + (w \cdot I_{\partial(r,s)} I_{\partial(r,s)} \otimes +_c-_f)$$

$$-_c+_f \longrightarrow w \cdot I_{\partial(r,s_C)} I_{\partial(r,s_C)} \otimes -_c-_f$$

$$+_c-_f \longrightarrow w \cdot \overrightarrow{e_C} I_{\partial(r,s_C)} \otimes -_c -_f .$$

By comparing the coefficients on each generator, the images of the LHS and the RHS agree. As a result, Equation (8.6) is true in this case.

3. All of the circles are cleaved circles: The proof will be similar to the above cases when we can prove that two sides of Equation (8.6) agree on every generator $\xi = (r, s)$ of $[\![\overrightarrow{T} \rangle\!\rangle$ by using Relation $\overrightarrow{e}_{(\gamma,\sigma,\sigma')}\overrightarrow{e}_{(\gamma^\dagger,\sigma',\sigma_C)} = \overrightarrow{e_C}$. Figure 8.1 illustrates the proof for this case. In this figure, C is the cleaved circle of the 0-resolution while C_1 and C_2 correspond to the top and the bottom cleaved circles of the 1-resolution. In addition, $x_1 = \langle \overrightarrow{\delta_{T',\bullet}}(+_{C_1}+_{C_2}), -_{C_1}+_{C_2} \rangle = \overleftarrow{e_{C_1}} + (w+b)\overrightarrow{e_{C_1}}$ and $x_2 = \langle \overrightarrow{\delta_{T',\bullet}}(+_{C_1}+_{C_2}), +_{C_1}-_{C_2} \rangle = \overleftarrow{e_{C_2}} + (c+a)\overrightarrow{e_{C_2}}$ while $x_1' = \langle \overrightarrow{\delta_{T,\bullet}}(+_{C_1}+_{C_2}), -_{C_1}+_{C_2} \rangle = \overleftarrow{e_{C_1}} + b\overrightarrow{e_{C_1}}$ and $x_2' = \langle \overrightarrow{\delta_{T,\bullet}}(+_{C_1}+_{C_2}), +_{C_1}-_{C_2} \rangle = \overleftarrow{e_{C_2}} + (w + a + c)\overrightarrow{e_{C_2}}$ (x, x', x_3, x_3', x_4, and x_4' can be computed similarly from the weights). Also, in the figure, e_{γ_1}, e_{γ_2} and e_{γ_3} are the bridge elements representing the changes of the cleaved links: $+_{C_1}+_{C_2} \longrightarrow +_C$, $-_{C_1}+_{C_2} \longrightarrow -_C$ and $+_{C_1}-_{C_2} \longrightarrow -_C$ respectively ($e_{\gamma_1^\dagger}$, $e_{\gamma_1^\dagger}$, and $e_{\gamma_3^\dagger}$ can be defined similarly). Therefore, $\langle \overline{D_c}(+_{C_1}+_{C_2}), +_C \rangle = w \cdot e_{\gamma_1}$ and so on. In Equation (8.6):

$$\text{LHS}(+_{C_1}+_{C_2}) = (w \cdot e_{\gamma_1} e_{\gamma_1^\dagger} \otimes -_{C_1}+_{C_2}) + (w \cdot e_{\gamma_1} e_{\gamma_2^\dagger} \otimes +_{C_1}-_{C_2})$$

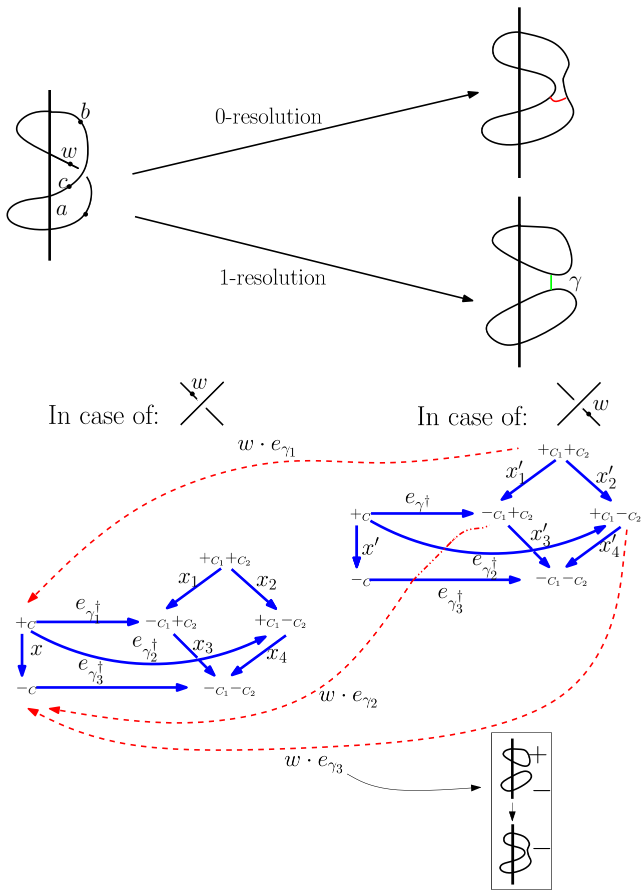


Figure 8.1: In this figure, the right and left columns contain the generators of the complexes associated to the two tangles respectively. The dashed red and thick blue arrows illustrate the definitions of the type D homomorphism and the type D structures respectively. The recorded algebra elements above the arrows correspond to the change in the cleaved links (see e_{γ_3} for an example) and will be explained in more details as in case (3) below. Note that the diagram does not cover all the terms of the type D structures or the homomorphism. It only illustrates the last case of Proposition 33. However, this picture can be modified to give the pictures for the other cases by changing e_{γ_i} and $e^{\dagger}_{\gamma_i}$ to suitable algebra elements.

and:

$$\mathrm{RHS}(+_{C_1}+_{C_2}) = ((x_1 + x_1') \otimes -_{C_1}+_{C_2}) + ((x_2 + x_2') \otimes -_{C_2}+_{C_1}).$$

Using Relation (3) in the **Group II**, we see that $\mathrm{LHS}(+_{C_1}+_{C_2}) = \mathrm{RHS}(+_{C_1}+_{C_2})$. Similarly, we can prove that Equation (8.6) is true when applied for $-_{C_1}+_{C_2}$ or $+_{C_1}-_{C_2}$. Therefore, Ψ is a type D homomorphism. $\diamond$

Proposition 34. *$(\llbracket \overrightarrow{T} \rrbracket, \overrightarrow{\delta_{T,\bullet}})$ is isomorphic to $(\llbracket \overrightarrow{T'} \rrbracket, \overrightarrow{\delta_{T',\bullet}})$ as type D structures.*

Proof. Let $\Phi : \llbracket \overrightarrow{T}' \rrbracket \longrightarrow \mathcal{B}\Gamma_n \otimes_{\mathcal{I}_n} \llbracket \overrightarrow{T} \rrbracket$ where $\Phi(r,s) = I_{\partial(r,s)} \otimes (r,s) + w.\underline{D_c}(r,s)$ where $\underline{D_c}$ is defined identically as $\overline{D_c}$ but from $\overrightarrow{T'}$ to $\overrightarrow{T}$. By Proposition 33, Φ is a D structure homomorphism. We will prove: $\Phi \circ \Psi = I_{\llbracket \overrightarrow{T} \rrbracket}$ and $\Psi \circ \Phi = I_{\llbracket \overrightarrow{T'} \rrbracket}$ where $\circ$ stands for the composition of two type D structure homomorphisms (described in Chapter 9). After that, we can conclude $(\llbracket \overrightarrow{T} \rrbracket, \overrightarrow{\delta_{T,\bullet}})$ is isomorphic to $(\llbracket \overrightarrow{T'} \rrbracket, \overrightarrow{\delta_{T',\bullet}})$. Indeed,

$$\begin{aligned}
&\Phi \circ \Psi \\
&\quad = (\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \Phi)\Psi \\
&\quad = (\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \mathbb{I}_{T'})\mathbb{I}_T + w.\big[(\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \underline{D_c})\mathbb{I}_T + (\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \mathbb{I}_{T'})\overline{D_c}\big] \\
&\qquad + (\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \underline{D_c})\overline{D_c}.
\end{aligned}$$

We note that the last term is 0 since $\underline{D_c}$ is supported on states (r,s) where $r(c) = 1$ while the image of $\overline{D_c}$ contains only the states (r_1, s_1) where $r_1(c) = 0$. The sum of the two middle terms is also 0 since the second term is equal to the third term. Moreover, $(\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{I}_d)(\mathbb{I} \otimes \mathbb{I}_{T'})\mathbb{I}_T = I_{\llbracket \overrightarrow{T} \rrbracket}$. Combining all these facts, we obtain: $\Phi \circ \Psi = I_{\llbracket \overrightarrow{T} \rrbracket}$. Similarly, $\Psi \circ \Phi = I_{\llbracket \overrightarrow{T'} \rrbracket}$. $\diamond$

CHAPTER 9

GRADED DIFFERENTIAL ALGEBRA AND STABLE HOMOTOPY

9.1 Preliminary

Different projections of the right tangle $\overrightarrow{\mathcal{T}}$ can have a different number of arcs and thus, the corresponding type D structures will be vector spaces over different base fields. To relate these structures, we will need an appropriate algebraic tool: the stable homotopy, whose construction is based on the idea in [19, Section 4]. Let $\mathbb{F}$ be a field and W be a vector space over $\mathbb{F}$. Let $\mathbb{F}_W$ be the field of rational function of P_W where P_W is the symmetric algebra of W. Recall: let W, W' be two vector spaces over $\mathbb{F}$ and let M, M' be two vector spaces over $\mathbb{F}_W$, $\mathbb{F}_{W'}$ respectively. A pair (M, W) is stably isomorphic to (M', W') if there is a triple (W'', i, i') where W'' is a vector space over $\mathbb{F}$, and $W \xrightarrow{i} W''$ and $W \xrightarrow{i'} W''$ are injective linear maps, such that $M \otimes_{\mathbb{F}_W} \mathbb{F}_{W''} \cong M' \otimes_{\mathbb{F}_{W'}} \mathbb{F}_{W''}$ as vector spaces over $\mathbb{F}_{W''}$. This relation is proved to be an equivalence relation in [19, Lemma 4.3]. We always assume that $\mathbb{F}$ is $\mathbb{Z}_2$ and W is a vector space over $\mathbb{Z}_2$, unless otherwise stated. In this section, we give a modification of this definition which will allow us to relate two type D (or A) structures over two different fields. Furthermore, the complexes obtained by gluing stable homotopy equivalences of type A structures to stable homotopy equivalences of type D structures will be stably homotopic in the sense of [19].

9.2 $A_{W,\mathcal{I}}$ category and stable A_∞ homotopy equivalence

We will describe a way to construct A_∞ modules over the ground field $\mathbb{F}_W$ from A_∞ over $\mathbb{Z}_2$ by first upgrading our unital DGA $(\mathcal{B}\Gamma_n, d_{\Gamma_n})$ (over $\mathbb{Z}_2$) to one over $\mathbb{F}_W$.

We will let $(A, \mathcal{I})$, μ_1 and μ_2 stand for $(\mathcal{B}\Gamma_n, \mathcal{I}_n)$, d_{Γ_n} and the product on $\mathcal{B}\Gamma_n$ respectively.

Definition 35. *For each vector space W over $\mathbb{Z}_2$, let $A_W := A \otimes_{\mathbb{Z}_2} \mathbb{F}_W$, equipped with the following $\mathbb{F}_W-$linear maps*

$$\mu_{W,1} : A_W \longrightarrow A_W[-1]$$

$$\mu_{W,2} : A_W \otimes_{\mathbb{F}_W} A_W \longrightarrow A_W$$

where $\mu_{W,1} = \mu_1 \otimes \mathbb{I}_{\mathbb{F}_W}$ and $\mu_{W,2} = \mu_2 \otimes \mathbb{I}_{\mathbb{F}_W}$, under the canonical isomorphism: $A_W \otimes_{\mathbb{F}_W} A_W \cong A^{\otimes 2} \otimes_{\mathbb{Z}_2} \mathbb{F}_W$. We also let $\mathcal{I}_W$ be $\mathcal{I} \otimes_{\mathbb{Z}_2} \mathbb{F}_W$

Using the fact that tensor product is a functor, it is straightforward to verify the following proposition:

Proposition 36. *If (A, μ_1, μ_2) is unital DGA over $\mathbb{Z}_2$ then $(A_W, \mu_{W,1}, \mu_{W,2})$ is unital DGA over $\mathbb{F}_W$.*

The proof of this proposition is left to the reader. We recall the definition of an A_∞ module which can be found in [11, Chapter 2].

Definition 37. *$(M, \{m_i\}_{i \in \mathbb{N}})$ is a right A_∞ module over $(A_W, \mathcal{I}_W)$ if M is a graded $\mathcal{I}_W$-module and for each $i \in \mathbb{N}$, $m_i : M \otimes_{\mathcal{I}_W} A_W^{\otimes(i-1)} \longrightarrow M[i-2]$ is an $\mathcal{I}_W$-linear map, which satisfies the following compatibility condition:*

$$0 = \sum_{i+j=n+1} m_i(m_j \otimes \mathbb{I}^{\otimes(i-1)}) + \sum_{i+j=n+1, j<3, k>0} m_i(\mathbb{I}^{\otimes k} \otimes \mu_{W,j} \otimes \mathbb{I}^{\otimes(i-k-1)}).$$

$(M, \{m_i\})$ is said to be strictly unital if for any $\xi \in M$, $m_2(\xi \otimes 1_{A_W}) = \xi$, but for $n > 2$, $m_n(\xi \otimes a_1 \otimes ... \otimes a_{n-1}) = 0$ if any $a_i = 1_{A_W}$. $(M, \{m_i\})$ is said to be bounded if $m_i = 0$ for all sufficient large i.

Definition 38. *Let $(M, \{m_i\})$ and $(M', \{m'_i\})$ be two A_∞ modules over $(A_W, \mathcal{I}_W)$. An A_∞ homomorphism Ψ is a collection of maps:*

$$\psi_i : M \otimes_{\mathcal{I}_W} A_W^{\otimes(i-1)} \longrightarrow M'[i-1]$$

indexed by $i \in \mathbb{N}$, satisfying the compatibility conditions:

$$0 = \sum_{i+j=n+1} m'_i(\psi_j \otimes \mathbb{I}^{\otimes(i-1)}) + \sum_{i+j=n+1} \psi_i(m_j \otimes \mathbb{I}^{\otimes(i-1}) + \sum_{i+j=n+1,k>0,j<3} \psi_i(\mathbb{I}^{\otimes k} \otimes \mu_{W,j} \otimes \mathbb{I}^{\otimes(i-k-1)}).$$

Additionally, a homotopy H between two A_∞ morphisms Ψ and Φ is a set of maps $\{h_i\}$ with $h_i : M \otimes_{\mathcal{I}_W} A_W^{\otimes(i-1)} \longrightarrow M'[i]$ such that:

$$\psi_n + \phi_n = \sum_{i+j=n+1} m'_i(h_j \otimes \mathbb{I}^{\otimes(i-1)}) + \sum_{i+j=n+1} h_i(m_j \otimes \mathbb{I}^{\otimes(i-1)}) + \sum_{i+j=n+1,k>0,j<3} h_i(\mathbb{I}^{\otimes k} \otimes \mu_{W,j} \otimes \mathbb{I}^{\otimes(i-k-1)}). \tag{9.1}$$

For ease of notation, we sometimes let M stand for $(M, \{m_i\})$ when it is clear from the context.

Proposition 39. *Let $A_{W,\mathcal{I}}$ be a collection of A_∞ modules over $(A_W, \mathcal{I}_W)$. $Hom(M, M')$ is a collection of A_∞ homomorphisms $(\Psi, \{\psi_i\})$ and the composition of $\Phi \circ \Psi$ is the set of maps:*

$$(\Phi \circ \Psi)_n = \sum_{i+j=n+1} \phi_i(\psi_j \otimes \mathbb{I}^{\otimes(i-1)})$$

for each $n \in \mathbb{N}$. Then $A_{W,\mathcal{I}}$ forms a category.

Definition 40. *Let $\varphi : W \hookrightarrow W'$ be a linear injection. Let $(M, \{m_i\}_{i\in\mathbb{N}})$ be an object of $A_{W,\mathcal{I}}$ and $(\Psi, \{\psi\}_{i\in\mathbb{N}}) \in Hom((M, \{m_i\}), (M', \{m'_i\}))$ be a homomorphism of $A_{W,\mathcal{I}}$. We define:*

1. $$\mathcal{F}_\varphi(M) = M \otimes_\varphi \mathbb{F}_{W'}.$$

2. *For each $i \in \mathbb{N}$, $\mathcal{F}_\varphi(m_i) : \mathcal{F}_\varphi(M) \otimes_{\mathcal{I}_{W'}} A_{W'}^{\otimes(i-1)} \longrightarrow \mathcal{F}_\varphi(M)[i-2]$ is defined by:*

   $$\mathcal{F}_\varphi(m_i)\big((x \otimes r) \otimes (a_1 \otimes r_1) \otimes .... \otimes (a_{i-1} \otimes r_{i-1})\big) = m_i(x \otimes a_1 \otimes ... \otimes a_{i-1}) \otimes r r_1 ... r_{i-1}$$

   where $x \in M$, $a_j \in A_W$ and r, $r_j \in \mathbb{F}_{W'}$ for $j = 1, ..., i-1$.

3. *Similarly, for each* $i \in \mathbb{N}$, $\mathcal{F}_\varphi(\psi_i) : \mathcal{F}_\varphi(M) \otimes_{\mathcal{I}_{W'}} A_{W'}^{\otimes(i-1)} \longrightarrow \mathcal{F}_\varphi(M')[i-1]$ *is defined in the same manner as* $\mathcal{F}_\varphi(m_i)$.

Combining the facts that tensor product is a functor and there is the following canonical isomorphism:

$$(M \otimes_\varphi \mathbb{F}_{W'}) \otimes_{\mathcal{I}_{W'}} (A_W \otimes_\varphi \mathbb{F}_{W'})^{\otimes(i-1)} \cong (M \otimes_{\mathcal{I}_W} A_W^{\otimes(i-1)}) \otimes_\varphi \mathbb{F}_{W'}$$

for each $i \in \mathbb{N}$, it is straightforward to prove the two following propositions:

Proposition 41. $\mathcal{F}_\varphi(M, \{m_i\}) = (\mathcal{F}_\varphi(M), \{\mathcal{F}_\varphi(m_i)\})$ *is well-defined and it is an object of* $A_{W',\mathcal{I}}$. *Likewise,* $\mathcal{F}_\varphi(\Psi)$ *is well-defined and belongs to* $Hom(\mathcal{F}_\varphi(M, \{m_i\}), \mathcal{F}_\varphi(N, \{n_i\}))$.

Proposition 42. *For each injection* $\varphi : W \hookrightarrow W'$, *there is a functor* $\mathcal{F}_\varphi : A_{W,\mathcal{I}} \longrightarrow A_{W',\mathcal{I}}$ *defined by*

$$\mathcal{F}_\varphi(M, \{m_i\}) = (\mathcal{F}_\varphi(M), \{\mathcal{F}_\varphi(m_i)\})$$

$$\mathcal{F}_\varphi(\Psi, \{\psi_i\}) = (\mathcal{F}_\varphi(\Psi), \{\mathcal{F}_\varphi(\psi_i)\}).$$

Furthermore, if Ψ *and* Φ *are* A_∞ *homotopic in* $A_{W,\mathcal{I}}$ *then* $\mathcal{F}_\varphi(\Psi)$ *is* A_∞ *homotopic to* $\mathcal{F}_\varphi(\Phi)$ *in* $A_{W',\mathcal{I}}$.

We now have enough tools to relate two A_∞ structures over different fields by defining the stable homotopy equivalence of A_∞ modules:

Definition 43. *Let* W *and* W' *be two vector spaces over* $\mathbb{Z}_2$. *Let* $(M, \{m_i\})$, $(M', \{m'_i\})$ *be objects of* $A_{W,\mathcal{I}}$ *and* $A_{W',\mathcal{I}}$ *respectively. Then* $(M, \{m_i\})$ *is stably homotopy equivalent to* $(M', \{m'_i\})$ *if there is a triple* (φ, φ', W'') *where* W'' *is a vector space over* $\mathbb{Z}_2$, *and* $\varphi : W \hookrightarrow W''$ *and* $\varphi' : W' \hookrightarrow W''$ *are linear injections, such that* $(\mathcal{F}_\varphi(M), \{\mathcal{F}_\varphi(m_i)\})$ *is homotopy equivalent to* $(\mathcal{F}_{\varphi'}(M'), \{\mathcal{F}_\varphi(m'_i)\})$ *in the category* $A_{W'',\mathcal{I}}$.

Proposition 44. *Stable homotopy equivalence of* A_∞ *modules is an equivalence relation.*

The proof of this proposition follows directly from [19, Lemma 4.3], the fact that homotopy equivalence of A_∞ modules is an equivalence relation, and the following lemma:

Lemma 45. *Let $\widetilde{W}$ be a vector space over $\mathbb{Z}_2$ and $\widetilde{\varphi} : W'' \hookrightarrow \widetilde{W}$ be a linear injection. If $(M, \{m_i\})$ is stably homotopy equivalent to $(M', \{m'_i\})$ via (φ, φ', W''), then $(M, \{m_i\})$ is stably homotopy equivalent to $(M', \{m'_i\})$ via $(\widetilde{\varphi} \circ \varphi, \widetilde{\varphi} \circ \varphi', \widetilde{W})$*

Proof. Since $(\mathcal{F}_\varphi(M), \{\mathcal{F}_\varphi(m_i)\})$ is homotopy equivalent to $(\mathcal{F}_{\varphi'}(M'), \{\mathcal{F}_{\varphi'}(m'_i)\})$, $(\mathcal{F}_{\widetilde{\varphi}}(\mathcal{F}_\varphi(M),$ $\{\mathcal{F}_{\widetilde{\varphi}}(\mathcal{F}_\varphi(m_i))\})$ is homotopy equivalent to $(\mathcal{F}_{\widetilde{\varphi}}(\mathcal{F}_{\varphi'}(M')), \{\mathcal{F}_{\widetilde{\varphi}}(\mathcal{F}_{\varphi'}(m'_i))\})$ by Proposition 42. Thus, we will finish the proof of this lemma if we can prove

$$(\mathcal{F}_{\widetilde{\varphi}\circ\varphi}(M), \{\mathcal{F}_{\widetilde{\varphi}\circ\varphi}(m_i)\}) \cong_{A_{\widetilde{W},\mathcal{I}}} (\mathcal{F}_{\widetilde{\varphi}} \circ \mathcal{F}_\varphi(M), \{\mathcal{F}_{\widetilde{\varphi}}(\mathcal{F}_\varphi(m_i))\})$$

and

$$(\mathcal{F}_{\widetilde{\varphi}\circ\varphi'}(M'), \{\mathcal{F}_{\widetilde{\varphi}\circ\varphi'}(m'_i)\}) \cong_{A_{\widetilde{W},\mathcal{I}}} (\mathcal{F}_{\widetilde{\varphi}} \circ \mathcal{F}_{\varphi'}(M'), \{\mathcal{F}_{\widetilde{\varphi}}(\mathcal{F}_{\varphi'}(m'_i))\}).$$

First of all, we have:

$$\mathcal{F}_{\widetilde{\varphi}\circ\varphi}(M) = M \otimes_{\widetilde{\varphi}\circ\varphi} \mathbb{F}_{\widetilde{W}} \cong (M \otimes_\varphi \mathbb{F}_{W''}) \otimes_{\widetilde{\varphi}} \mathbb{F}_{\widetilde{W}} = \mathcal{F}_{\widetilde{\varphi}} \circ \mathcal{F}_\varphi(M).$$

Secondly, under this identification of the underlying modules, we need to prove $\mathcal{F}_{\widetilde{\varphi}\circ\varphi}(m_i) = \mathcal{F}_{\widetilde{\varphi}} \circ \mathcal{F}_\varphi(m_i)$. Using Definition 40, we have:

$$\mathcal{F}_{\widetilde{\varphi}\circ\varphi}(m_i)\big((x \otimes r) \otimes (a_1 \otimes r_1) \otimes ... \otimes (a_{i-1} \otimes r_{i-1})\big) = m_i(x \otimes a_1 \otimes ... \otimes a_{i-1}) \otimes_{\widetilde{\varphi}\circ\varphi} rr_1...r_{i-1}.$$

On the other hand,

$$\mathcal{F}_{\widetilde{\varphi}} \circ \mathcal{F}_\varphi(m_i)\big(((x \otimes_\varphi 1) \otimes_{\widetilde{\varphi}} r) \otimes ((a_1 \otimes_\varphi 1) \otimes_{\widetilde{\varphi}} r_1) \otimes ... \otimes ((a_{i-1} \otimes_\varphi 1) \otimes_{\widetilde{\varphi}} r_{i-1})\big)$$

$$= \mathcal{F}_\varphi(m_i)\big((x \otimes_\varphi 1) \otimes (a_1 \otimes_\varphi 1) \otimes ... \otimes (a_{i-1} \otimes 1)\big) \otimes_{\widetilde{\varphi}} rr_1...r_{i-1}$$

$$= \big(m_i(x \otimes a_1 \otimes ... \otimes a_{i-1}) \otimes_{\varphi} 1\big) \otimes_{\widetilde{\varphi}} rr_1...r_{i-1}.$$

Therefore, $\mathcal{F}_{\widetilde{\varphi}\circ\varphi}(m_i) = \mathcal{F}_{\widetilde{\varphi}} \circ \mathcal{F}_{\varphi}(m_i)$. $\diamond$

9.3 $D_{W,\mathcal{I}}$ Category and Stable D homotopy equivalence

We first review the definition of the type D structure and the D_W category as in [11, Chapter 2.3].

Definition 46. *[11, Chapter 2.3] Let N be a graded $\mathcal{I}_W$-module. A (left) D structure on N is a linear map:*

$$\delta : N \longrightarrow (A_W \otimes_{\mathcal{I}_W} N)[-1] \tag{9.2}$$

satisfying:

$$(\mu_{W,2} \otimes \mathbb{I}_N)(\mathbb{I}_{A_W} \otimes \delta)\delta + (\mu_{W,1} \otimes \mathbb{I}_N)\delta = 0.$$

$\delta^k : N \longrightarrow (A_W^{\otimes k} \otimes_{\mathcal{I}_W} N)[-k]$ is defined by induction $\delta^0 = \mathbb{I}_N, \delta^1 = \delta$ and the relation $\delta^n = (\mathbb{I}^{\otimes(n-1)} \otimes \delta)\delta^{n-1}$. We call the type D structure (N, δ) bounded if for all $x \in N$, there is an integer n so that for all $i \geq n$, $\delta^i = 0$.

A homomorphism of type D structures $(N, \delta) \longrightarrow (N', \delta')$ is a map $\psi : N \longrightarrow A_W \otimes_{\mathcal{I}_W} N'$ such that:

$$(\mu_{W,2} \otimes \mathbb{I}_N)(\mathbb{I}_{A_W} \otimes \delta')\psi + (\mu_{W,2} \otimes \mathbb{I}_N)(\mathbb{I}_{A_W} \otimes \psi)\delta + (\mu_{W,1} \otimes \mathbb{I}_N)\psi = 0.$$

$H : N \longrightarrow (A_W \otimes_{\mathcal{I}_W} N')[1]$ is a homotopy of two type D structure homomorphisms ψ and ϕ if:

$$\psi + \phi = (\mu_{W,2} \otimes \mathbb{I}_N)(\mathbb{I}_{A_W} \otimes \delta')H + (\mu_{W,2} \otimes \mathbb{I}_N)(\mathbb{I}_{A_W} \otimes H)\delta + (\mu_{W,1} \otimes \mathbb{I}_N)H.$$

Proposition 47. *[11, Chapter 2.3] Let $D_{W,\mathcal{I}}$ be the collection of type D structures (N, δ) over $(A_W, \mathcal{I}_W)$. Let $Hom((N, \delta), (N', \delta'))$ be the set of type D homomorphisms and the composition $\phi \circ \psi$ of $\psi : N \longrightarrow A_W \otimes N'$ and $\phi : N' \longrightarrow A_W \otimes_{\mathcal{I}_W} N''$ is defined to be*

$(\mu_{W,2} \otimes \mathbb{I}_{N''})(\mathbb{I}_{A_W} \otimes \phi)\psi$. *Additionally, the identity homomorphism at* (N, δ) *is* $I_N : N \longrightarrow A_W \otimes_{\mathcal{I}_W} N$ *given by* $x \longrightarrow 1_{A_W} \otimes x$. *Then* $D_{W,\mathcal{I}}$ *forms a category.*

Using the same technique as in Section 9.2, for each $\varphi : W \hookrightarrow W'$ a linear injection, we will construct a functor from $D_{W,\mathcal{I}}$ to $D_{W',\mathcal{I}}$. Before doing that, we need the following definition:

Definition 48. *Let* $\varphi : W \hookrightarrow W'$ *be a linear injection. Let* (N, δ), (N', δ') *be two objects of* $D_{W,\mathcal{I}}$ *and* $\psi \in Mor((N, \delta), (N', \delta'))$. *We define:*

1.

$$\mathcal{G}_\varphi(N) := N \otimes_\varphi \mathbb{F}_{W'}$$

2. *Furthermore,*

$$\mathcal{G}_\varphi(\delta) : N \otimes_\varphi \mathbb{F}_{W'} \longrightarrow A_{W'} \otimes_{\mathcal{I}_{W'}} (N \otimes_\varphi \mathbb{F}_{W'})[-1]$$

is defined to be $\delta \otimes \mathbb{I}_{\mathbb{F}_{W'}}$ *under the canonical isomorphism* $A_{W'} \otimes_{\mathcal{I}_{W'}} (N \otimes_\varphi \mathbb{F}_{W'}) \cong (A_W \otimes_{\mathcal{I}_W} N) \otimes_\varphi \mathbb{F}_{W'}$, *and*

3.

$$\mathcal{G}_\varphi(\psi) : N \otimes_\varphi \mathbb{F}_{W'} \longrightarrow A_{W'} \otimes_{\mathcal{I}_{W'}} (N' \otimes_\varphi \mathbb{F}_{W'})$$

is defined to be $\psi \otimes \mathbb{I}_{\mathbb{F}_{W'}}$ *under the canonical isomorphism* $A_{W'} \otimes_{\mathcal{I}_{W'}} (N' \otimes_\varphi \mathbb{F}_{W'}) \cong (A_W \otimes_{\mathcal{I}_W} N') \otimes_\varphi \mathbb{F}_{W'}$.

Proposition 49. *Using the same notation as in the above definition, there exists a functor* $\mathcal{G}_\varphi : D_{W,\mathcal{I}} \longrightarrow D_{W',\mathcal{I}}$ *defined by:*

$$\mathcal{G}_\varphi(N, \delta) = (\mathcal{G}_\varphi(N), \mathcal{G}_\varphi(\delta))$$

and $\mathcal{G}_\varphi(\psi)$ *is defined as above. Furthermore, let* ψ *and* ϕ *belong to* $Hom((N, \delta), (N', \delta'))$ *in* $D_{W,\mathcal{I}}$ *and if* H *is a homotopy from* ψ *to* ϕ, *then* $\mathcal{G}_\varphi(H)$ *is a homotopy from* $\mathcal{G}_\varphi(\psi)$ *to* $\mathcal{G}_\varphi(\phi)$ *in* $D_{W',\mathcal{I}}$ *where:*

$$\mathcal{G}_\varphi(H) : N \otimes_\varphi \mathbb{F}_{W'} \longrightarrow A_{W'} \otimes_{\mathbb{F}_{W'}} (N' \otimes_\varphi \mathbb{F}_{W'})[1]$$

is defined to be $H \otimes \mathbb{I}_{F_{W'}}$ *under the isomorphism* $A_{W'} \otimes_{\mathcal{I}_{W'}} (N' \otimes_\varphi \mathbb{F}_{W'}) \cong (A_W \otimes_{\mathcal{I}_W} N') \otimes_\varphi \mathbb{F}_{W'}$.

Therefore, $\mathcal{G}_\varphi$ *induces a functor from the homotopy category of* $D_{W,\mathcal{I}}$ *to the homotopy category of* $D_{W',\mathcal{I}}$.

We now can give a definition of stable D homotopy.

Definition 50. *Let* W *and* W' *be vector spaces over* $\mathbb{Z}_2$. *Let* (N, δ), (N', δ') *be objects of* $D_{W,\mathcal{I}}$ *and* $D_{W',\mathcal{I}}$, *respectively. Then* (N, δ) *is stably homotopy equivalent to* (N, δ') *if there is a triple* (φ, φ', W'') *where* W'' *is a vector space over* $\mathbb{Z}_2$, $\varphi : W \hookrightarrow W''$ *and* $\varphi : W' \hookrightarrow W''$ *are linear injections, such that* $\mathcal{G}_\varphi(N, \delta)$ *is homotopy equivalent to* $\mathcal{G}_{\varphi'}(N', \delta')$ *in the category* $D_{W'',\mathcal{I}}$

Proposition 51. *Stable homotopy of type* D *structures is an equivalence relation.*

Since the proofs of Propositions 49 and 51 are similar to the proofs of Propositions 42 and 44 respectively, we leave them to the readers. We have the following remark about the property of the composition of two functors, which is useful for the next section.

Remark 52. *Let* $\varphi_1 : W \hookrightarrow W_1$ *and* $\varphi_2 : W_1 \hookrightarrow W_2$ *be injective linear maps. Let* (N, δ) *be an object of* A_W. *Then* $\mathcal{G}_{\varphi_2 \circ \varphi_1}(N, \delta) \cong_D \mathcal{G}_{\varphi_2}(\mathcal{G}_{\varphi_1})(N, \delta)$.

9.4 Pairing an A_∞ module and a type D structure over different DGAs

In [11, Chapter 2.4], there is the result that we can pair an object $(M, \{m_i\})$ of $A_{W,\mathcal{I}}$ and an object (N, δ) of $D_{W,\mathcal{I}}$ to form a chain complex $(M \boxtimes N, \partial^\boxtimes)$ over $\mathbb{F}_W$. Additionally, if $(M, \{m_i\}) \simeq_{A_\infty} (M', \{m_i'\})$ in $A_{W,\mathcal{I}}$ category and $(N, \delta) \simeq_D (N', \delta')$ in $D_{W,\mathcal{I}}$ category, then $(M \boxtimes N, \partial^\boxtimes)$ is chain homotopic to $(M' \boxtimes N', \partial^\boxtimes)$.

For our purpose, since we need to pair a type A and a type D structures over distinct differential graded algebras, we will modify the way to pair them to get a chain complex. Furthermore, we will prove that under the change of either type A or type D by a stable homotopy equivalence, the glued chain complexes are stably chain homotopic.

Let W and W' be two vector spaces over $\mathbb{Z}_2$. Let $(M,\{m_i\})$ and (N,δ) be objects of $A_{W,\mathcal{I}}$ and $D_{W',\mathcal{I}}$ respectively. Suppose moreover that either $(M,\{m_i\})$ is a bounded A_∞ module or (N,δ) is a bounded type D structure. Let $W_1 := W\oplus W'$ and let $\varphi : W \hookrightarrow W_1$, $\varphi' : W' \hookrightarrow W_1$ be two canonical linear injections.

Definition 53. *Define $M\boxtimes_\bullet N$ to be the graded vector space $\mathcal{F}_\varphi(M)\otimes_{\mathcal{I}_{W_1}}\mathcal{G}_{\varphi'}(N)$ over $\mathbb{F}_{W_1}$ and $\partial^{\boxtimes}_\bullet : M\boxtimes_\bullet N \longrightarrow (M\boxtimes_\bullet N)[-1]$ to be the map :*

$$\partial^{\boxtimes}_\bullet = \sum_{k=0}^{\infty}\left(\mathcal{F}_\varphi(m_{k+1})\otimes \mathbb{I}^{\otimes(k+1)}_{\mathcal{G}_{\varphi'}(N)}\right)\circ\left(\mathbb{I}_{\mathcal{F}_\varphi(M)}\otimes\Delta_k\right)$$

where $\Delta_k : \mathcal{G}_{\varphi'}(N)\longrightarrow A^{\otimes k}_{W_1}\otimes_{\mathcal{I}_{W_1}}\mathcal{G}_{\varphi'}(N)[-k]$ is defined by induction $\Delta_0=\mathbb{I}_{\mathcal{G}_{\varphi'}(N)}$, $\Delta_1 = \mathcal{G}_{\varphi'}(\delta)$ and the relation: $\Delta_n = (\mathbb{I}^{\otimes(n-1)}\otimes\mathcal{G}_{\varphi'}(\delta))\Delta_{n-1}$.

Note. $(M\boxtimes_\bullet N,\partial^{\boxtimes}_\bullet)$ is defined exactly the same as the definition of $(\mathcal{F}_\varphi(M)\boxtimes\mathcal{G}_{\varphi'}(N),\partial^{\boxtimes})$ in [11, Definition 2.26] and thus, it is a chain complex.

Proposition 54. *Let W, W' and $\widetilde{W}$ be three vector spaces over $\mathbb{Z}_2$. Let (N,δ) be an object of $D_{\widetilde{W},\mathcal{I}}$. Let $(M,\{m_i\})$, $(M',\{m_i'\})$ be objects of $A_{W,\mathcal{I}}$ and $A_{W',\mathcal{I}}$ respectively, such that $(M,\{m_i\})$ is stably homotopy equivalent to $(M',\{m_i'\})$ via the triple (φ,φ',W''). Then $(M\boxtimes_\bullet N,\partial^{\boxtimes}_\bullet)$ is stably chain homotopic to $(M'\boxtimes_\bullet N,\partial^{\boxtimes}_\bullet)$.*

Before proving this proposition, we state the following lemma which can be proved in a similar manner as Lemma 45.

Lemma 55. *Let $(M,\{m_i\})$ and (N,δ) be objects of the categories $A_{W,\mathcal{I}}$ and $D_{W,\mathcal{I}}$ respectively. Let $\varphi : W\hookrightarrow W'$ be a linear injection. Then $((M\boxtimes N)\otimes_\varphi \mathbb{F}_{W'},\partial^{\boxtimes}\otimes\mathbb{I}_{\mathbb{F}_{W'}})$ is chain isomorphic to $(\mathcal{F}_\varphi(M)\boxtimes\mathcal{G}_\varphi(N),\partial^{\boxtimes})$.*

Proof of Proposition 54 .

Define $V := W \oplus \widetilde{W}$, $V' := W' \oplus \widetilde{W}$ and $V'' := W'' \oplus \widetilde{W}$. Let $\widetilde{\varphi} : V \longrightarrow V''$ and $\widetilde{\varphi}' : V' \longrightarrow V''$ denote the maps $\varphi \oplus \mathbb{I}_{\widetilde{W}}$ and $\varphi' \oplus \mathbb{I}_{\widetilde{W}}$. Let $p : \widetilde{W} \longrightarrow V$, $p' : \widetilde{W} \longrightarrow V'$, $\pi : W \longrightarrow V$, $\pi' : W' \longrightarrow V'$ and $\pi'' : W'' \longrightarrow V''$ be the natural injections. We immediately have the following relations:

1. $\widetilde{\varphi} \circ p = \widetilde{\varphi}' \circ p'$.

2. $\widetilde{\varphi} \circ \pi = \pi'' \circ \varphi$.

3. $\widetilde{\varphi}' \circ \pi' = \pi'' \circ \varphi'$.

We will prove that $(M \boxtimes_\bullet N, \partial_\bullet^\boxtimes)$ is stably chain homotopic to $(M' \boxtimes_\bullet N, \partial_\bullet^\boxtimes)$ via the triple $(\widetilde{\varphi}, \widetilde{\varphi}', V'')$. Indeed, using Lemma 55 and the fact that the underlying complex of $(M \boxtimes_\bullet N, \partial_\bullet^\boxtimes)$ is $\mathcal{F}_\pi(M) \otimes_{\mathcal{I}_V} \mathcal{G}_p(N)$, we have:

$$((M \boxtimes_\bullet N) \otimes_{\widetilde{\varphi}} F_{V''}, \partial_\bullet^\boxtimes \otimes \mathbb{I}_{F_{V''}}) \simeq (\mathcal{F}_{\widetilde{\varphi}}(\mathcal{F}_\pi(M)) \boxtimes \mathcal{G}_{\widetilde{\varphi}}(\mathcal{G}_p(N)), \partial^\boxtimes).$$

Additionally, by the proof of Lemma 45, we have:

$$\mathcal{F}_{\widetilde{\varphi}}(\mathcal{F}_\pi(M)) \simeq_A \mathcal{F}_{\widetilde{\varphi} \circ \pi}(M) = \mathcal{F}_{\pi'' \circ \varphi}(M) \simeq_A \mathcal{F}_{\pi''}(\mathcal{F}_\varphi(M)).$$

Note that the second identity comes from the relation $\widetilde{\varphi} \circ \pi = \pi'' \circ \varphi$. Similarly, $\mathcal{G}_{\widetilde{\varphi}}(\mathcal{G}_p(N))$ is D-homotopy equivalent to $\mathcal{G}_{\widetilde{\varphi} \circ p}(N)$. Therefore, following [11, Lemma 2.32], $((M \boxtimes_\bullet N) \otimes_{\widetilde{\varphi}} F_{V''}, \partial_\bullet^\boxtimes \otimes \mathbb{I}_{F_{V''}})$ is chain homotopic to $(\mathcal{F}_{\pi''}(\mathcal{F}_\varphi(M)) \boxtimes \mathcal{G}_{\widetilde{\varphi} \circ p}(N), \partial^\boxtimes)$. Likewise,

$$((M' \boxtimes_\bullet N) \otimes_{\widetilde{\varphi}'} F_{V''}, \partial_\bullet^\boxtimes \otimes \mathbb{I}_{F_{V''}}) \simeq (\mathcal{F}_{\pi''}(\mathcal{F}_{\varphi'}(M')) \boxtimes \mathcal{G}_{\widetilde{\varphi}' \circ p'}(N), \partial^\boxtimes).$$

Since $(M, \{m_i\})$ is stable A-homotopy equivalent to $(M', \{m_i'\})$ via the triple (φ, φ', W''), we have $\mathcal{F}_{\varphi}(M) \simeq_A \mathcal{F}_{\varphi'}(M')$ in the category $A_{W'',\mathcal{I}}$. By Proposition 42,

$$\mathcal{F}_{\pi''}(\mathcal{F}_{\varphi}(M)) \simeq_A \mathcal{F}_{\pi''}(\mathcal{F}_{\varphi'}(M')).$$

Furthermore, since $\widetilde{\varphi} \circ p = \widetilde{\varphi}' \circ p'$, we have $\mathcal{G}_{\widetilde{\varphi} \circ p}(N) = \mathcal{G}_{\widetilde{\varphi}' \circ p'}(N)$. Therefore,

$$(\mathcal{F}_{\pi''}(\mathcal{F}_{\varphi}(M)) \boxtimes \mathcal{G}_{\widetilde{\varphi} \circ p}(N), \partial^{\boxtimes}) \simeq (\mathcal{F}_{\pi''}(\mathcal{F}_{\varphi'}(M')) \boxtimes \mathcal{G}_{\widetilde{\varphi}' \circ p'}(N), \partial^{\boxtimes}).$$

Thus,

$$((M \boxtimes_{\bullet} N) \otimes_{\widetilde{\varphi}} F_{V''}, \partial_{\bullet}^{\boxtimes} \otimes \mathbb{I}_{F_{V''}}) \simeq ((M' \boxtimes_{\bullet} N) \otimes_{\widetilde{\varphi}'} F_{V''}, \partial_{\bullet}^{\boxtimes} \otimes \mathbb{I}_{F_{V''}}).$$

As a result, $(M \boxtimes_{\bullet} N, \partial_{\bullet}^{\boxtimes})$ is stably chain homotopic to $(M' \boxtimes_{\bullet} N, \partial_{\bullet}^{\boxtimes})$ via the triple $(\widetilde{\varphi}, \widetilde{\varphi}', V'')$.
$\diamond$

The same method can be applied to prove the following theorem:

Proposition 56. *Let W, W' and $\widetilde{W}$ be vector spaces over $\mathbb{Z}_2$. Let $(M, \{m_i\})$ be an object of $A_{\widetilde{W},\mathcal{I}}$. Let (N, δ) and (N', δ') be objects of $D_{W,\mathcal{I}}$ and $D_{W',\mathcal{I}}$ respectively such that (N, δ) is stably homotopy equivalent to (N', δ') via the triple (φ, φ', W''). Then $(M \boxtimes_{\bullet} N, \partial_{\bullet}^{\boxtimes})$ is stably chain homotopic to $(M \boxtimes_{\bullet} N', \partial_{\bullet}^{\boxtimes})$.*

CHAPTER 10

INVARIANCE UNDER REIDEMEISTER MOVES

In this chapter, we will prove that the stable homotopy type of $(\llbracket \overrightarrow{T} \rrangle, \overrightarrow{\delta_{T,\bullet}})$ is invariant under the Reidemeister moves. We will first use the "weight moves" trick, described in Chapter 8, to move the weights to the bottoms of the local diagrams, see Figures 10.1 and 10.2. After that, the result follows from a modification of the proof of the invariance in the untwisted case. As we notice before, the type D structures before and after a Reidemeister move will be defined over different fields. Therefore, to relate these structures, we will specify how the formal variables for the two diagrams need to be related to show invariance.

Attention: In the proof of invariance under Reidemeister moves, we sometimes use the index T in $\mathcal{B}\Gamma_{T,n}$ to emphasize the dependence of the ground field $\mathbb{F}_{\overrightarrow{T}}$ on the diagram $\overrightarrow{T}$.

10.1 Invariance under the first Reidemeister move

Figure 3 shows the complex for a diagram prior to and after an Reidemeister I move applied to a right-handed crossing. Let $\overrightarrow{T}$ be the tangle diagram before the Reidemeister I move, and let $\overrightarrow{T'}$ be the diagram afterwards.

As usual, we can decompose $\llbracket \overrightarrow{T'} \rrangle = V_0 \oplus V_1$ corresponding to states (r, s) where $r(c) = 0$ or $r(c) = 1$. Furthermore, since each state generating V_0 always has a free circle C as in the local diagram, we can continue decomposing $V_0 = (V' \otimes +_c) \oplus (V' \otimes -_c)$ where $\pm_c$ are the decorations on C.

Let (r, s) be a state in $V' \otimes +_c$. As we can see, the only state (r', s'), which is a generator of V_1 and is in the image of $\overrightarrow{\delta_{T',\bullet}}(r, s)$, is the one obtained by applying d_{APS}. In this case, the coefficient of (r', s') in $\mathcal{B}\Gamma_n$ is $I_{\partial(r,s)} = I_{\partial(r',s')}$. Using this fact, we can use Lemma 31

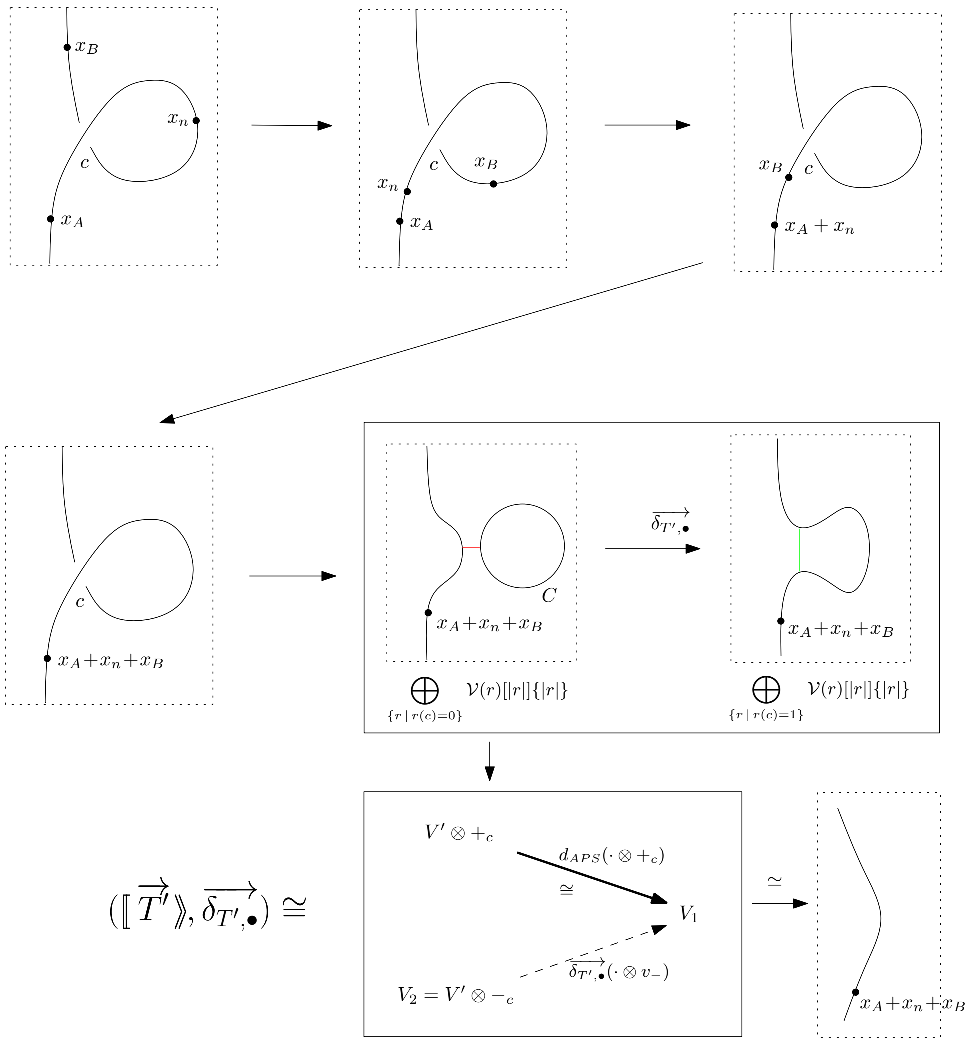


Figure 10.1: In the top row, we use the weight shift isomorphisms to move all the weights to the bottom of the diagram. Surgering the crossing c in both ways gives a finer view into the complex. Regardless of whether the local arc is on a free circle or a cleaved circle, the recorded algebra element of the thickened arrow is always an invertible element of $\mathcal{B}\Gamma_n$ (however, the recorded algebra element of the dashed arrow depends upon the type of the local arc). When the complex is reduced along the thickened arrow, we obtain the complex for the diagram before the Reidemeister I move with the weight $x_A + x_n + x_B$ on the local arc.

to cancel out $V' \otimes +_c$ and V_1. What we have left is $V_2 = V' \otimes -_c$ with the new type D structure $\overrightarrow{\delta_1} : V_2 \to \mathcal{B}\Gamma_n \otimes V_2$ where $\overrightarrow{\delta_1}$ is the sum of $\overrightarrow{\delta'} = \overrightarrow{\delta_{T,\bullet}}|_{V_2}$ and the perturbation term.

Taking a deeper look into the cancellation process, we see that the perturbation term arises from the following diagram: $\xi \to \xi_1 \to \xi_2 \to \xi_3$ where $\xi \in V_2$, ξ_1 is a generator of V_1 which is in the image of $\overrightarrow{\delta_{T',\bullet}}(\xi)$, $\xi_2 \in V' \otimes +_c$ such that $\langle \overrightarrow{\delta_{T',\bullet}}(\xi_2), \xi_1 \rangle = I_{\partial(\xi_1)} = I_{\partial(\xi_2)}$, and $\langle \overrightarrow{\delta_{T',\bullet}}(\xi_2), \xi_3 \rangle \neq 0$. We note that ξ_3 has to be a generator of $(V' \otimes +_c) \oplus V_1$. Indeed, the only possibility for $\xi_3 \in V' \otimes -_c = V_2$ is that ξ_3 is obtained from ξ_2 by applying the vertical map $\overrightarrow{\delta_{\mathcal{V}}}$ to change the decoration on C. But since we already moved the weight out of C as described at the beginning of this chapter, $\overrightarrow{w_C} = 0$. Therefore, $\xi_3 \in (V' \otimes +_c) \oplus V_1$. As a result, at the end of the cancellation process, ξ_3 is canceled out. Consequently, the perturbation term will be 0 and $(V_2, \overrightarrow{\delta'}) \simeq (\llbracket \overrightarrow{T'} \rrbracket, \overrightarrow{\delta_{T',\bullet}})$.

Although there is one-to-one corresponding between the generators of $\llbracket \overrightarrow{T} \rrbracket$ and of V_2, we are still working over different fields $\mathbb{F}_{\overrightarrow{T}}$ and $\mathbb{F}_{\overrightarrow{T'}}$. Additionally, the local arc in $\overrightarrow{T}$ is labeled by y_i while the one of V_2 is labeled by $x_A + x_B + x_n$. To relate them, let $\widetilde{\varphi} : \mathcal{B}\Gamma_{T,n} \to \mathcal{B}\Gamma_{T',n}$ be the map induced by the inclusion $\varphi : \mathbb{F}_{\overrightarrow{T}} \to \mathbb{F}_{\overrightarrow{T'}}$ defined by: $y_i \to x_A + x_B + x_n$ and $y_j \to x_j$ for $i \neq j$. Then, we define:

$$\overrightarrow{\delta_{T_\varphi,\bullet}} : \llbracket \overrightarrow{T} \rrbracket \otimes_\varphi \mathbb{F}_{\overrightarrow{T'}} \longrightarrow \mathcal{B}\Gamma_{T',n} \otimes_{\mathcal{I}_{T',n}} \left[\llbracket \overrightarrow{T} \rrbracket \otimes_\varphi \mathbb{F}_{\overrightarrow{T'}} \right]$$

by specifying $\left\langle \overrightarrow{\delta_{T_\varphi,\bullet}}\big((r,s) \otimes 1\big)\,,\,(r',s') \otimes 1 \right\rangle = \widetilde{\varphi} \left\langle \overrightarrow{\delta_{T,\bullet}}(r,s)\,,\,(r',s') \right\rangle$. By Proposition 49, $\overrightarrow{\delta_{T_\varphi,\bullet}}$ is a type D structure on $\llbracket \overrightarrow{T} \rrbracket \otimes_\varphi \mathbb{F}_{\overrightarrow{T'}}$ over $(\mathcal{B}\Gamma_{T',n}, \mathcal{I}_{T',n})$.

If we further identify the generators (r,s) of V_2 with $(r,s) \otimes 1$ of $\llbracket \overrightarrow{T} \rrbracket \otimes_\varphi \mathbb{F}_{\overrightarrow{T'}}$, we have $\left\langle \overrightarrow{\delta'}(r,s)\,,\,(r',s') \right\rangle = \left\langle \overrightarrow{\delta_{T_\varphi,\bullet}}\big((r,s) \otimes 1\big)\,,\,(r',s') \otimes 1 \right\rangle$. Therefore,

$$(\llbracket \overrightarrow{T} \rrbracket \otimes_\varphi \mathbb{F}_{\overrightarrow{T'}}, \overrightarrow{\delta_{T_\varphi,\bullet}}) \cong (V_2, \overrightarrow{\delta'}) \cong (\llbracket \overrightarrow{T'} \rrbracket, \overrightarrow{\delta_{T',\bullet}})$$

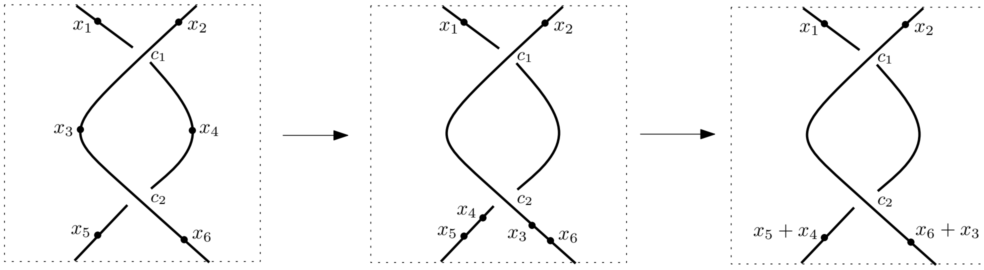


Figure 10.2: The weights moved in the second Reidemeister move

over $(\mathcal{B}\Gamma_{T',n}, \mathcal{I}_{T',n})$. Consequently, $(\llbracket \overrightarrow{T} \rrbracket, \overrightarrow{\delta_{T,\bullet}})$ is stably homotopy equivalent to $(\llbracket \overrightarrow{T'} \rrbracket, \overrightarrow{\delta_{T',\bullet}})$. $\diamond$

The invariance under a left-handed Reidemeister I move can be obtained by combining the invariance of the type D structure under the right-handed Reidemeister I move and under the Reidemeister II move.

10.2 Invariance under the second Reidemeister move

As before $\overrightarrow{T}$ will be the tangle before the move and $\overrightarrow{T'}$ will be the tangle after using a Reidemeister II move. Once again we shift all the weights to the bottom of the diagram (see above figure). Since the homotopy class of the type D structure associated to the tangle is invariant under the weight moves by Proposition 34, we still denote $\overrightarrow{T'}$ the weighted tangle after shifting the weights.

Let c_1, c_2 be the two crossings in the local diagrams. For $i = 0, 1$, $j = 0, 1$, we let V_{ij} be the set of states where c_0 is resolved by i and c_1 is resolved by j. We can decompose $\llbracket \overrightarrow{T'} \rrbracket = V_{00} \oplus V_{01} \oplus V_{10} \oplus V_{11}$. We further decompose $V_{10} = (V \otimes +_c) \oplus (V \otimes -_c)$ where $\pm_c$ are the decorations on the free circle C. Basically, the same sort of argument as in the proof of the Reidemeister I invariance can be used to explain why we can cancel out $V \otimes +_c$ and V_{11} without creating any new perturbation. The reason again comes from the fact that $\overrightarrow{w_C} = 0$, and thus, we do not have any maps from $V \otimes +_c$ to $V \otimes -_c$. This is illustrated in

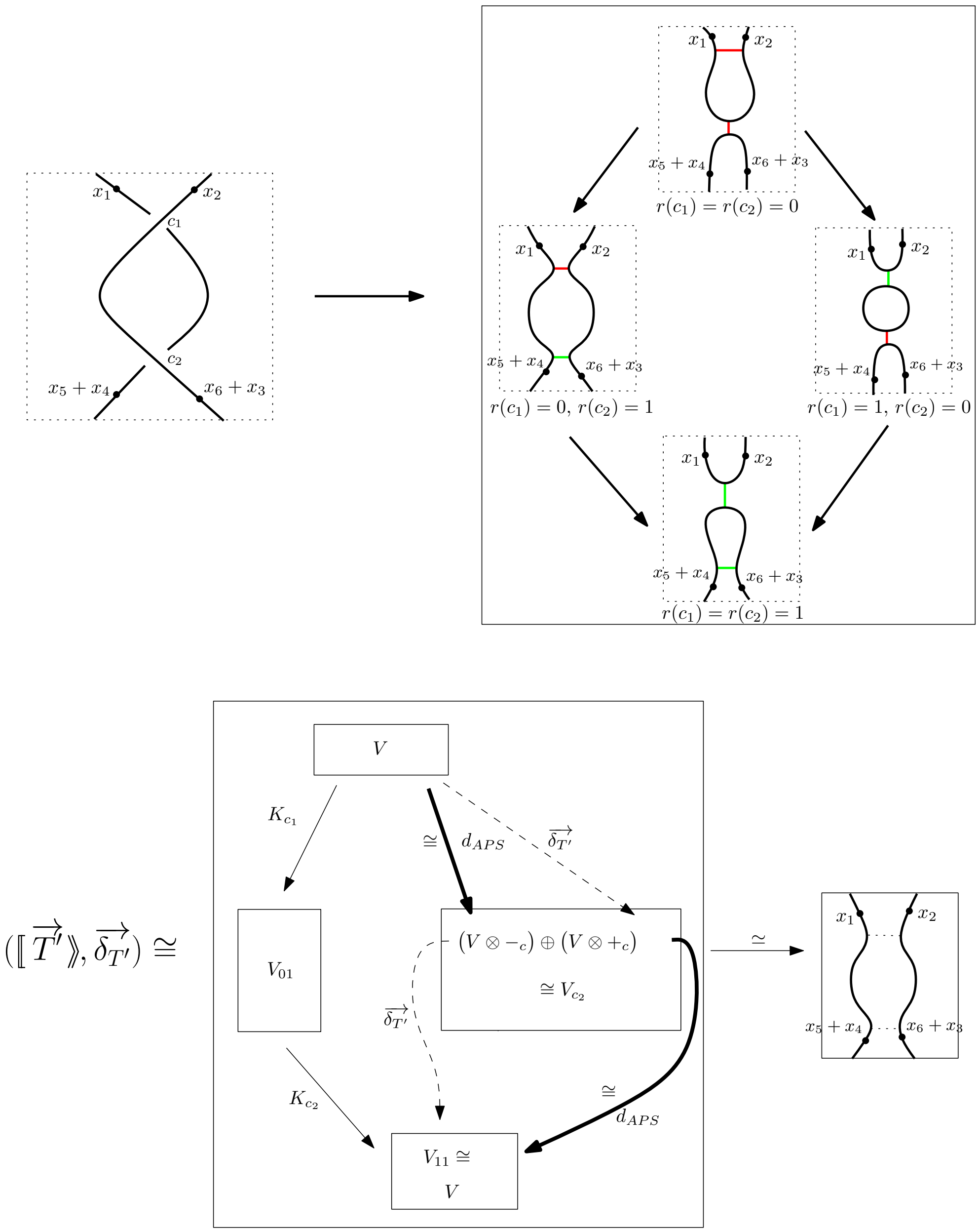


Figure 10.3: This figure illustrates the proof of invariance under the second Reidemeister move. As we can see, regardless of whether the local arcs lie on free circles or cleaved circles, the recorded algebra elements of the thicker arrows are always idempotents. Additionally, if we cancel the bottom thicker arrow first, and then the top thicker one, we introduce no new perturbation terms since the weight on C is 0 . These cancellations produce the deformation retraction of the type D structure.

Figure 10.3.

The next step is to cancel out V_{00} and $V \otimes -_c$ by applying an isomorphism $I \otimes d_{APS} : V_{00} \to V \otimes -_c$. Again, no perturbation term appears since there is no map from V_{01} to $V_{00} \oplus (V \otimes -_c)$. At the end, we will be left solely with V_{01} and a new type D structure $\delta' : V_{01} \to \mathcal{B}\Gamma_{T',n} \otimes V_{01}$ where $\delta' = \overrightarrow{\delta_{T',\bullet}}|_{V_{01}}$. Note that the generators of V_{01} corresponds 1-1 with the generators of $[\![\overrightarrow{T}]\!\rangle$, but they are two possibly distinct type D structures over the different ground fields. Let the weights on two local arcs of $\overrightarrow{T}$ be y_L and y_R. To relate $\mathcal{B}\Gamma_{T,n}$ and $\mathcal{B}\Gamma_{T',n}$, we construct the $\widetilde{\varphi} : \mathcal{B}\Gamma_{T,n} \longrightarrow \mathcal{B}\Gamma_{T',n}$ which is induced by the inclusion $\varphi : \mathbb{F}_{\overrightarrow{T}} \longrightarrow \mathbb{F}_{\overrightarrow{T'}}$, defined by: $y_L \to x_1 + x_4 + x_5$, $y_R \to x_2 + x_3 + x_6$ and $y_i \to x_i$ for $i \neq L$ or R. Using $\widetilde{\varphi}$, we can upgrade the type D structure $([\![\overrightarrow{T}]\!\rangle, \overrightarrow{\delta_{T,\bullet}})$ over $\mathcal{B}\Gamma_{T,n}$ to a type D structure $([\![\overrightarrow{T}]\!\rangle \otimes_{\varphi} \mathbb{F}_{\overrightarrow{T'}}, \overrightarrow{\delta_{T_\varphi,\bullet}})$ over $\mathcal{B}\Gamma_{T',n}$ and the latter is homotopy equivalent to (V_{01}, δ'). These can be proven exactly as in the proof of the invariance under the Reidemeister I. Consequently, $([\![\overrightarrow{T}]\!\rangle, \overrightarrow{\delta_{T,\bullet}})$ is stably homotopy equivalent to $([\![\overrightarrow{T'}]\!\rangle, \overrightarrow{\delta_{T',\bullet}})$. $\diamond$

10.3 Invariance under the third Reidemeister move

Since the proof of invariance under the third Reidemeister move is similar to the proof of invariance under the first and the second Reidemeister moves, we only mention the strategy of the proof and the figures to illustrate it. However, we will construct a homomorphism to relate $\mathcal{B}\Gamma_{T,n}$ and $\mathcal{B}\Gamma_{T',n}$ since it is somewhat different from the proof of invariance under the Reidemeister I and II moves. The strategy is to follow these steps: 1) shift the weights to the bottom as in Figures 10.4 and 10.5. 2) using the cancellation method exactly as in the Reidemeister II move for the top faces of the top diagrams in Figures 10.4 and 10.5, we will get the lower diagrams with the two new perturbation maps. The important point is that there is one-to-one corresponding between the generators of the lower diagram in Figure 10.4 and the generators of the lower diagram in Figure 10.5. Also, the maps from the tops to the bottoms of the lower diagrams in these two figures will be the same (see [17, Lemma 46] for more details).

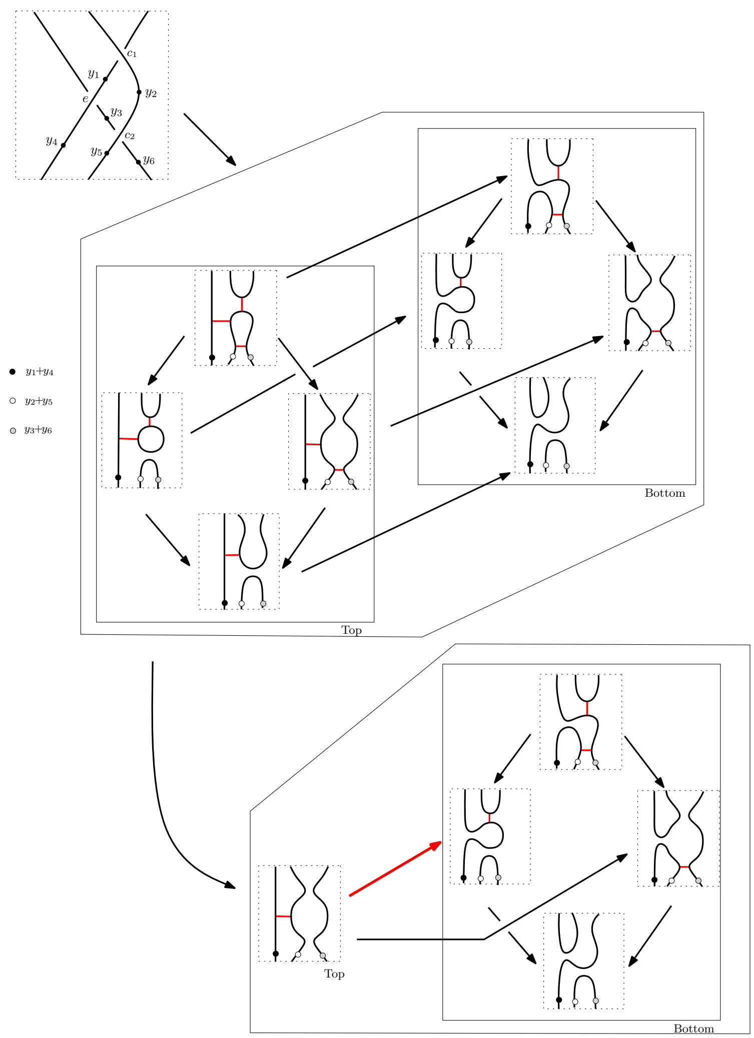


Figure 10.4: The local picture for a diagram before the Reidemeister III move. We decompose the module along the eight possible ways of resolving the local crossings. The four resolutions with the crossing c resolved by a 0-resolution replicate the diagrams in the proof of Reidemeister II invariance. Using the cancellation process in the top of the higher diagram (as in the case of the Reidemeister II move) gives the lower diagram. A new perturbation map may occur from the thicker red arrow in the bottom figure; however, under the identification of the generators of the lower diagrams in Figures 10.4 and 10.5, it will be the same as the map of the lower diagram of Figure 10.4 which is obtained by surgering a bridge at the crossing d .

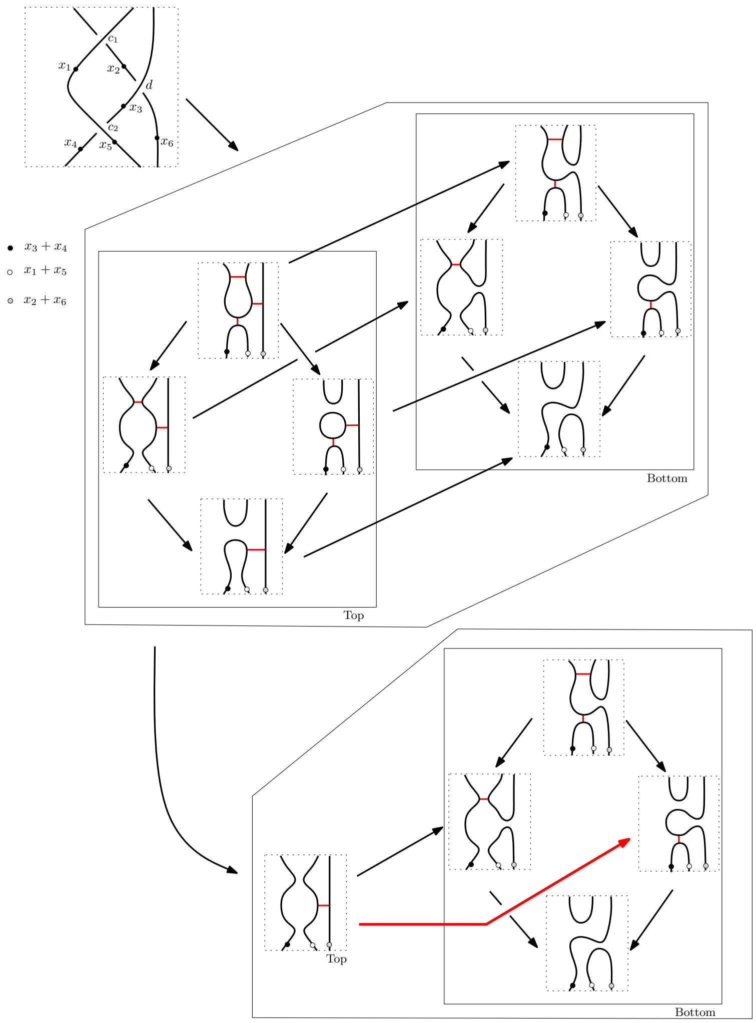


Figure 10.5: The local picture for a diagram after the Reidemeister III move. Once again there is a new perturbation map, shown by the thicker red arrow in the bottom figure.

We assume that the weights for the arcs not shown in the figures are $z_7, ..., z_t$. The type D structures of the lower diagrams in Figures 10.4 and 10.5 will be denoted by (R_1, δ_1) and (R_2, δ_2) respectively. Let (S, δ_S) be a type D structure with the identical maps as in the lower diagrams but with the black circle representing u_1, the white circle representing u_2 and the gray circle representing u_3. We note that (S, δ_S) is a type D structure over $\mathcal{B}\Gamma_{S,n} = \mathcal{B}\Gamma_n \otimes_{\mathbb{Z}_2} \mathbb{F}_S$ where $\mathbb{F}_S$ is the field of fractions of $\mathbb{Z}_2[u_1, u_2, u_3, z_7, ..., z_t]$. We define $\varphi_1 : \mathcal{B}\Gamma_{S,n} \to \mathcal{B}\Gamma_{T,n}$ which is induced by the inclusion $\widetilde{\varphi_1} : \mathbb{F}_S \to \mathbb{F}_T$: $u_1 \to y_1 + y_4$, $u_2 \to y_2 + y_5$ and $u_3 \to y_3 + y_6$. As in the proof of the Reidemeister I invariance, this map will give us a way to relate (S, δ_S) and (R_1, δ_1) and as a result, (S, δ_S) is stably homotopy equivalent to (R_1, δ_1).

Similarly, $\widetilde{\varphi_2} : \mathcal{B}\Gamma_{S,n} \to \mathcal{B}\Gamma_{T',n}$, which is induced by the inclusion $\varphi_2 : \mathbb{F}_S \to \mathbb{F}_{T'}$: $u_1 \to x_3 + y_4$, $u_2 \to y_1 + y_5$ and $u_3 \to y_2 + y_6$, will result in (S, δ_S) being stably homotopy equivalent to (R_2, δ_2). Then by Proposition 51 , (R_1, δ_1) is stably homotopy equivalent to (R_2, δ_2). As a result, $(\llbracket \overrightarrow{T} \rrbracket, \overrightarrow{\delta_{T,\bullet}})$ is stably homotopy equivalent to $(\llbracket \overrightarrow{T'} \rrbracket, \overrightarrow{\delta_{T',\bullet}})$. $\diamond$

CHAPTER 11

A TYPE A STRUCTURE IN TOTALLY TWISTED KHOVANOV HOMOLOGY

In [18], L. Roberts describes a type A structure on the underlying module $\llbracket \overleftarrow{T} \rrbracket$ (see the note at the end of Chapter 4) over $\mathcal{B}\Gamma_n$. It is characterized by two maps: 1) the differential m_1 which increases the bigrading by $(1,0)$ and thus is a degree 1 map relative to the $\zeta-$grading and 2) the action m_2 which preserves bigrading and thus is $\zeta-$grading preserving. For the twisted bordered homology, we will describe another type A structure on $\llbracket \overleftarrow{T} \rrbracket$ over $\mathcal{B}\Gamma_n$ by the following maps:

$$m_{1,\bullet} : \llbracket \overleftarrow{T} \rrbracket \to \llbracket \overleftarrow{T} \rrbracket[-1].$$

$$m_{2,\bullet} : \llbracket \overleftarrow{T} \rrbracket \otimes_{\mathcal{I}_n} \mathcal{B}\Gamma_n \longrightarrow \llbracket \overleftarrow{T} \rrbracket.$$

Let $\xi = (r,s)$ be a generator of $\llbracket \overleftarrow{T} \rrbracket$ and e be a generator of $\mathcal{B}\Gamma_n$.

For $m_{1,\bullet}$: we denote $m_{1,\bullet}(\xi) := d_{APS}(\xi) + \partial_{\mathcal{V}}(\xi) = m_1(\xi) + \partial_{\mathcal{V}}(\xi)$. Since the vertical map $\partial_{\mathcal{V}}$ decreases the bigrading by (0,2), $m_{1,\bullet}$ is $\zeta-$grading preserving into $\llbracket \overleftarrow{T} \rrbracket[-1]$.

For $m_{2,\bullet}$: we first define the action of a generator of $\mathcal{B}\Gamma_n$ on $\llbracket \overleftarrow{T} \rrbracket$.

- If $e \neq \overleftarrow{e_C}$, then $m_{2,\bullet}(\xi \otimes e) := m_2(\xi \otimes e)$ where e is either an idempotent, or a bridge element, or a right decoration element and m_2 is defined in [18, Section 4].

- If $e = \overleftarrow{e_C}$, then $m_{2,\bullet}(\xi \otimes \overleftarrow{e_C}) := m_2(\xi \otimes \overleftarrow{e_C}) + \overleftarrow{w_C} m_2(\xi \otimes \overrightarrow{e_C}) = m_2(\xi \otimes \overleftarrow{e_C}) + \overleftarrow{w_C}(r, s_C)$.

The fact that m_2 preserves the $\zeta-$grading immediately results in the fact that $m_{2,\bullet}$ also preserves the $\zeta-$grading.

To define the action of a general element of $\mathcal{B}\Gamma_n$ on $\llbracket \overleftarrow{T} \rrbracket$, for $p_1, p_2 \in \mathcal{B}\Gamma_n$, we impose the relation:

$$m_{2,\bullet}(\xi \otimes p_1 p_2) = m_{2,\bullet}(m_{2,\bullet}(\xi \otimes p_1) \otimes p_2). \tag{11.1}$$

For this definition to be well defined, we need to prove the following proposition:

Proposition 57. *If two products of the generators p_1 and p_2 define equal elements in $\mathcal{B}\Gamma_n$ then $m_{2,\bullet}(\xi \otimes p_1) = m_{2,\bullet}(\xi \otimes p_2)$.*

Proof. It suffices to prove $m_{2,\bullet}(\xi \otimes p) = 0$ if p is a relation defining $\mathcal{B}\Gamma_n$. First of all, we recall the two following facts:

1. $m_2(\xi \otimes p) = 0$ as in [18, Proposition 20].

2. $m_{2,\bullet}(\xi \otimes e) = m_2(\xi \otimes e)$ for every generator of $\mathcal{B}\Gamma_n$ unless $e = \overleftarrow{e_C}$.

Combining these two facts, we have that if p does not involve $\overleftarrow{e_C}$, then $m_{2,\bullet}(\xi \otimes p) = m_2(\xi \otimes p) = 0$. If p involves $\overleftarrow{e_C}$, let p_1 be an element of $\mathcal{B}\Gamma_n$ obtained from p by substituting $\overrightarrow{e_C}$ for each term $\overleftarrow{e_C}$ in p. In this situation, we have the following cases:

Case I. If p_1 is a relation defining $\mathcal{B}\Gamma_n$, then we have two possibilities:

• If $p = \overleftarrow{e_C}\overleftarrow{e_D} + \overleftarrow{e_D}\overleftarrow{e_C}$: Since $m_{2,\bullet}(\xi \otimes \overleftarrow{e_C}) = m_2(\xi \otimes \overleftarrow{e_C}) + \overleftarrow{w_C} m_2(\xi \otimes \overrightarrow{e_C})$ and $m_{2,\bullet}(\xi \otimes \overleftarrow{e_D}) = m_2(\xi \otimes \overleftarrow{e_D}) + \overleftarrow{w_D} m_2(\xi \otimes \overrightarrow{e_D})$, we have:

$m_{2,\bullet}(\xi \otimes p)$

$$= m_{2,\bullet}\big(\xi \otimes (\overleftarrow{e_C}\overleftarrow{e_D} + \overleftarrow{e_D}\overleftarrow{e_C})\big)$$

$$= m_{2,\bullet}(m_{2,\bullet}(\xi \otimes \overleftarrow{e_C}) \otimes \overleftarrow{e_D}) + m_{2,\bullet}(m_{2,\bullet}(\xi \otimes \overleftarrow{e_D}) \otimes \overleftarrow{e_C})$$

$$= m_{2,\bullet}\big(\big[m_2(\xi \otimes \overleftarrow{e_C}) + \overleftarrow{w_C} m_2(\xi \otimes \overrightarrow{e_C})\big] \otimes \overleftarrow{e_D}\big) + m_{2,\bullet}\big(\big[m_2(\xi \otimes \overleftarrow{e_D}) + \overleftarrow{w_D} m_2(\xi \otimes \overrightarrow{e_D})\big] \otimes \overleftarrow{e_C}\big)$$

$$= \big[m_2(\xi \otimes \overleftarrow{e_C}\overleftarrow{e_D}) + m_2(\xi \otimes \overleftarrow{e_D}\overleftarrow{e_C})\big] + \overleftarrow{w_D}\big[m_2(\xi \otimes \overleftarrow{e_C}\overrightarrow{e_D}) + m_2(\xi \otimes \overrightarrow{e_D}\overleftarrow{e_C})\big] +$$

$$\overleftarrow{w_C}\left[m_2(\xi\otimes\overleftarrow{e_D}\overrightarrow{e_C})+m_2(\xi\otimes\overrightarrow{e_C}\overleftarrow{e_D})\right]+\overleftarrow{w_C}\overleftarrow{w_D}\left[m_2(\xi\otimes\overrightarrow{e_C}\overrightarrow{e_D})+m_2(\xi\otimes\overrightarrow{e_D}\overrightarrow{e_C})\right]$$

$= 0$ since $m_2(\xi\otimes q) = 0$ for every q which is a relation defining $\mathcal{B}\Gamma_n$ and each of the terms in brackets is a relation defining $\mathcal{B}\Gamma_n$.

• If $\overleftarrow{e_C}$ is the only left decoration element involved in the relation p. We have: $m_{2,\bullet}(\xi\otimes p) = m_2(\xi\otimes p)+\overleftarrow{w_C}m_2(\xi\otimes p_1) = 0$ because both p and p_1 are relations defining $\mathcal{B}\Gamma_n$.

Case II. If p_1 is not a relation defining $\mathcal{B}\Gamma_n$, we see that the only relations which involve left decoration elements and satisfy that p_1 is not a relation defining $\mathcal{B}\Gamma_n$ come from either merging two + cleaved circles or dividing a + cleaved circle by surgering along a bridge γ. Since the proof of the case of division is similar, we just present the proof of the case of merging. In this case, $p = \overleftarrow{e_{C_1}}m_{\gamma_1}+\overleftarrow{e_{C_2}}m_{\gamma_2}+m_\gamma\overleftarrow{e_C}$. We need to prove:

$$m_{2,\bullet}(\xi\otimes\overleftarrow{e_{C_1}}m_{\gamma_1})+m_{2,\bullet}(\xi\otimes\overleftarrow{e_{C_2}}m_{\gamma_2})+m_{2,\bullet}(\xi\otimes m_\gamma\overleftarrow{e_C}) = 0. \tag{11.2}$$

Rewriting the left side:

$\Leftrightarrow m_{2,\bullet}(m_{2,\bullet}(\xi\otimes\overleftarrow{e_{C_1}})\otimes m_{\gamma_1})+m_{2,\bullet}(m_{2,\bullet}(\xi\otimes\overleftarrow{e_{C_2}})\otimes m_{\gamma_2})+m_{2,\bullet}(m_{2,\bullet}(\xi\otimes m_\gamma)\otimes\overleftarrow{e_C}) = 0.$

$\Leftrightarrow \left[m_2(m_2(\xi\otimes\overleftarrow{e_{C_1}})\otimes m_{\gamma_1})+m_2(m_2(\xi\otimes\overleftarrow{e_{C_2}})\otimes m_{\gamma_2})+m_2(m_2(\xi\otimes m_\gamma)\otimes\overleftarrow{e_C})\right]+$

$\left[m_2(\overleftarrow{w_{C_1}}m_2(\xi\otimes\overrightarrow{e_{C_1}})\otimes m_{\gamma_1})+m_2(\overleftarrow{w_{C_2}}m_2(\xi\otimes\overrightarrow{e_{C_2}})\otimes m_{\gamma_2})+\overleftarrow{w_C}m_2(m_2(\xi\otimes m_\gamma)\otimes\overrightarrow{e_C})\right] = 0.$

The first bracket equals 0 since $p = \overleftarrow{e_{C_1}}m_{\gamma_1}+\overleftarrow{e_{C_2}}m_{\gamma_2}+m_\gamma\overleftarrow{e_C}$ is a relation defining $\mathcal{B}\Gamma_n$. Therefore, it suffices to prove:

$$m_2(\overleftarrow{w_{C_1}}m_2(\xi\otimes\overrightarrow{e_{C_1}})\otimes m_{\gamma_1})+m_2(\overleftarrow{w_{C_2}}m_2(\xi\otimes\overrightarrow{e_{C_2}})\otimes m_{\gamma_2})+\overleftarrow{w_C}m_2(m_2(\xi\otimes m_\gamma)\otimes\overrightarrow{e_C}) = 0.$$

Since $\overleftarrow{w_C} = \overleftarrow{w_{C_1}}+\overleftarrow{w_{C_2}}$, we can rewrite the left side:

$\Leftrightarrow \overleftarrow{w_{C_1}}\big[m_2(\xi \otimes \overrightarrow{e_{C_1}} m_{\gamma_1}) + m_2(\xi \otimes m_\gamma \overrightarrow{e_C})\big] + \overrightarrow{w_{C_2}}\big[m_2(\xi \otimes \overrightarrow{e_{C_2}} m_{\gamma_2}) + m_2(\xi \otimes m_\gamma \overrightarrow{e_C})\big] = 0.$

The sums of the first and second brackets equal 0 since $\overrightarrow{e_{C_1}} m_{\gamma_1} + m_\gamma \overrightarrow{e_C}$ and $\overrightarrow{e_{C_2}} m_{\gamma_2} + m_\gamma \overrightarrow{e_C}$ are the relations defining $\mathcal{B}\Gamma_n$. Thus, Equation (11.2) is true and as a consequence, $m_{2,\bullet}$ is well-defined. $\diamond$

Next, we will prove that $(\langle\!\langle \overleftarrow{T}]\!], m_{1,\bullet}, m_{2,\bullet})$ is an A_∞ module over the differential graded algebra $\mathcal{B}\Gamma_n$ with $m_{n,\bullet} = 0$ for $n \geq 3$.

Proposition 58. *Let $\xi = (r, s)$ be a generator of $\langle\!\langle \overleftarrow{T}]\!]$ and $p_1, p_2 \in \mathcal{B}\Gamma_n$. The maps $m_{1,\bullet}$ and $m_{2,\bullet}$ satisfy the following relations:*

1. $m_{1,\bullet}(m_{1,\bullet}(\xi)) = 0.$

2. $m_{2,\bullet}(m_{1,\bullet}(\xi) \otimes p_1) + m_{2,\bullet}(\xi \otimes d_{\Gamma_n}(p_1)) + m_{1,\bullet}(m_{2,\bullet}(\xi \otimes p_1)) = 0.$

3. $m_{2,\bullet}(\xi \otimes p_1 p_2) = m_{2,\bullet}(m_{2,\bullet}(\xi \otimes p_1) \otimes p_2).$

Proof. The first identity comes from the fact that $m_{1,\bullet}$ is the differential on the complex $\langle\!\langle \overleftarrow{T}]\!]$ (see [20, Proposition 3.6]). The third identity comes from the fact that the construction of $m_{2,\bullet}$ is well defined, as shown above. Therefore, we only need to verify the second identity. By using an inductive argument described at the beginning of [18, Proposition 21], it suffices to prove the second identity when p_1 is a generator of $\mathcal{B}\Gamma_n$. We have the following two cases:

Case I. If $p_1 \neq \overleftarrow{e_C}$, using the fact that $d_{\Gamma_n}(p_1) = 0$, $m_{1,\bullet} = m_1 + \partial_{\mathcal{V}}$, and the relations between $m_{2,\bullet}$ and m_2, we rewrite the second identity as

$$\big[m_2(m_1(\xi)\otimes p_1) + m_2(\xi \otimes d_{\Gamma_n}(p_1)) + m_1(m_2(\xi \otimes p_1))\big] + \big[m_2(\partial_{\mathcal{V}}(\xi) \otimes p_1) + \partial_{\mathcal{V}}(m_2(\xi \otimes p_1))\big] = 0.$$

Due to the fact that $(\langle\!\langle \overleftarrow{T}]\!], m_1, m_2)$ is an A_∞ module over the differential graded algebra $\mathcal{B}\Gamma_n$, the first bracket is 0. Therefore, we only need to prove that:

$$m_2(\partial_{\mathcal{V}}(\xi) \otimes p_1) + \partial_{\mathcal{V}}(m_2(\xi \otimes p_1)) = 0. \tag{11.3}$$

There are three possibilities for p_1:

1. If p_1 is an idempotent $I_{(L,\sigma)}$, then both terms will equal $\partial_{\mathcal{V}}(\xi)$ if $\partial(r,s) = I_{(L,\sigma)}$ and they are both 0 if $\partial(r,s) \neq I_{(L,\sigma)}$. As the result, Equation (11.3) is true in this case.

2. If $p_1 = \overrightarrow{e_C}$, then $m_2(\partial_{\mathcal{V}}(\xi) \otimes p_1) = \partial_{\mathcal{V}}(m_2(\xi \otimes p_1)) = \sum_D \overleftarrow{w_D}(r, s_{C,D})$ where the sum is over all + free circles D of ξ.

3. If p_1 is a bridge element corresponding to surgering along a bridge γ, then

$$m_2(\partial_{\mathcal{V}}(\xi) \otimes p_1) = \partial_{\mathcal{V}}(m_2(\xi \otimes p_1)) = \sum_{\alpha,D}(r_{\alpha,D}, s_{\alpha,D})$$

   where 1) if p_1 is a left bridge element, the sum is over all active resolution bridges α which map to γ and all + free circles D of ξ, or 2) if p_1 is a right bridge element, $\alpha = \gamma$ and the sum is over all + free circles D of ξ. Therefore, Equation (11.3) is true in this case.

Case II. If $p_1 = \overleftarrow{e_C}$, we need to prove the following:
$m_{2,\bullet}(m_{1,\bullet}(\xi) \otimes \overleftarrow{e_C}) + m_{2,\bullet}(\xi \otimes d_{\Gamma_n}(\overleftarrow{e_C})) + m_{1,\bullet}(m_{2,\bullet}(\xi \otimes \overleftarrow{e_C})) = 0.$

$$\Leftrightarrow m_{2,\bullet}(m_1(\xi) \otimes \overleftarrow{e_C}) + m_{2,\bullet}(\partial_{\mathcal{V}}(\xi) \otimes \overleftarrow{e_C}) + m_2(\xi \otimes d_{\Gamma_n}(\overleftarrow{e_C})) + m_{1,\bullet}(m_2(\xi \otimes \overleftarrow{e_C}))+$$
$$\overleftarrow{w_C} m_{1,\bullet}(m_2(\xi \otimes \overrightarrow{e_C})) = 0.$$

$$\Leftrightarrow \ \left[m_2(m_1(\xi) \otimes \overleftarrow{e_C}) + \sum_\gamma \overleftarrow{w_{C_\gamma}} m_2(\xi_\gamma \otimes \overrightarrow{e_C})\right] + \left[m_2(\partial_{\mathcal{V}}(\xi) \otimes \overleftarrow{e_C}) + \overleftarrow{w_C} m_2(\partial_{\mathcal{V}}(\xi) \otimes \overrightarrow{e_C})\right]+$$
$$m_2(\xi \otimes d_{\Gamma_n}(\overleftarrow{e_C})) + \left[m_1(m_2(\xi \otimes \overleftarrow{e_C})) + \partial_{\mathcal{V}}(m_2(\xi \otimes \overleftarrow{e_C}))\right] + \left[\overleftarrow{w_C} m_1(m_2(\xi \otimes \overrightarrow{e_C}))+\right.$$

$$\overleftarrow{w_C}\partial_{\mathcal{V}}(m_2(\xi\otimes\overrightarrow{e_C}))\big] = 0.$$

where ξ_γ is in the image of $m_1(\xi)$, C_γ corresponds to C in $\partial(\xi_\gamma)$, and γ is taken over all active resolution bridges which contribute to m_1. Rewriting the left hand side:

$$\Leftrightarrow \big[m_2(m_1(\xi)\otimes\overleftarrow{e_C}) + m_2(\xi\otimes d_{\Gamma_n}(\overleftarrow{e_C})) + m_1(m_2(\xi\otimes\overleftarrow{e_C}))\big]+$$

$$\overleftarrow{w_C}\big[m_2(\partial_{\mathcal{V}}(\xi)\otimes\overrightarrow{e_C}) + v(m_2(\xi\otimes\overrightarrow{e_C}))\big]+$$

$$\big[\sum_\gamma \overleftarrow{w_{C_\gamma}}m_2(\xi_\gamma\otimes\overrightarrow{e_C}) + m_2(\partial_{\mathcal{V}}(\xi)\otimes\overleftarrow{e_C}) + \partial_{\mathcal{V}}(m_2(\xi\otimes\overleftarrow{e_C})) + \overleftarrow{w_C}m_1(m_2(\xi\otimes\overrightarrow{e_C}))\big] = 0.$$

The first bracket is 0 since $(\langle\!\langle \overleftarrow{T}\,]\!], m_1, m_2)$ is an A_∞ module over the differential graded algebra $\mathcal{B}\Gamma_n$. The second bracket is also 0 by Equation (11.3). Therefore, we only need to make sure the third sum is also 0:

$$\sum_\gamma \overleftarrow{w_{C_\gamma}}m_2(\xi_\gamma\otimes\overrightarrow{e_C}) + m_2(\partial_{\mathcal{V}}(\xi)\otimes\overleftarrow{e_C}) + \partial_{\mathcal{V}}(m_2(\xi\otimes\overleftarrow{e_C})) + \overleftarrow{w_C}m_1(m_2(\xi\otimes\overrightarrow{e_C})) = 0. \quad (11.4)$$

Since we have the following identity:

$$m_2(m_1(\xi)\otimes\overrightarrow{e_C}) = m_2(m_1(\xi)\otimes d_{\Gamma_n}(\overrightarrow{e_C})) + m_1(m_2(\xi\otimes\overrightarrow{e_C})) = m_1(m_2(\xi\otimes\overrightarrow{e_C})),$$

we can rewrite Equation (11.4) as following:

$$\sum_\gamma m_2((\overleftarrow{w_{C_\gamma}} + \overleftarrow{w_C})\xi_\gamma\otimes\overrightarrow{e_C}) + m_2(\partial_{\mathcal{V}}(\xi)\otimes\overleftarrow{e_C}) + \partial_{\mathcal{V}}(m_2(\xi\otimes\overleftarrow{e_C})) = 0. \quad (11.5)$$

The second term contains pairs of (D,γ) from changing the decoration on a + free circle D first, then resolving an active resolution bridge γ which changes the decoration on C from + to $-$. On the other hand, the third term will be the sum of pairs (γ_1, D_1) coming from surgering an active resolution bridge γ_1 to change the decoration on C first, then changing a + free circle D_1 of ξ_{γ_1} to $-$. Taking the sum of the second and the third terms, the pairs (D,γ) in the second term will be canceled out by the reverse pair (γ, D) if (γ, D) belongs to the third term and vice versa. However, there are two exceptional cases when reversing a pair of the second (third) term does not belong to the third (second) term: 1) (D, γ) where D is a + free circle of ξ and γ has one foot on D and another on C or 2) (γ, D_γ) where γ is an active resolution bridge whose feet belongs C and D_γ is new + free circle which is created by surgering γ. On the other hand, the first term of Equation 11.5 contains the generators whose coefficients $\overleftarrow{w_{C_\gamma}} + \overleftarrow{w_C} \neq 0$ if and only if the active bridge γ has at least one foot on C. By comparing the "weight" coefficients, those generators will be canceled out by generators in the above two exceptional cases. As a result, Equation 11.5 is true. $\diamond$

CHAPTER 12

A SPANNING TREE MODEL FOR THE TYPE A STRUCTURE

In this chapter, we will define the type A structure $(\llbracket \overleftarrow{CT} \rrbracket, m_{1,T}, m_{2,T})$ described in Chapter 1 and prove that it is A_∞ homotopy equivalent to $(\llbracket \overleftarrow{T} \rrbracket, m_{1,\bullet}, m_{2,\bullet})$.

Let $\mathrm{ST}_n(\overleftarrow{T})$ be the collection of states of $\overleftarrow{T}$, consisting of those states that do not have any free circles in their resolutions. Recall that $\llbracket \overleftarrow{CT} \rrbracket$ is a vector space over $\mathbb{F}_{\overleftarrow{T}}$ generated by $\mathrm{ST}_n(\overleftarrow{T})$.

Next, we will describe two maps:

$$m_{1,T} : \llbracket \overleftarrow{CT} \rrbracket \to \llbracket \overleftarrow{CT} \rrbracket[-1].$$

$$m_{2,T} : \llbracket \overleftarrow{CT} \rrbracket \otimes_{\mathcal{I}_n} \mathcal{B}\Gamma_n \longrightarrow \llbracket \overleftarrow{CT} \rrbracket.$$

Let $\xi = (r, s)$ be a generator of $\llbracket \overleftarrow{CT} \rrbracket$ and e be a generator of $\mathcal{B}\Gamma_n$.

For $m_{1,T}$: we denote $m_{1,T}(\xi) := \sum_{(r',s') \in \mathrm{ST}_n(\overleftarrow{T})} \big\langle (r, s), (r', s') \big\rangle (r', s')$ where $\big\langle (r, s), (r', s') \big\rangle$ is defined the same as in Chapter 7.

For $m_{2,T}$: we first define the action of a generator of $\mathcal{B}\Gamma_n$ on $\llbracket \overleftarrow{CT} \rrbracket$.

- If $e \neq \overleftarrow{e_C}$, then $m_{2,T}(\xi \otimes e) := m_{2,\bullet}(\xi \otimes e) = m_2(\xi \otimes e)$ where e is either an idempotent, or a bridge element, or a right decoration element.
- If $e = \overleftarrow{e_C}$, then $m_{2,T}(\xi \otimes \overleftarrow{e_C}) := \overleftarrow{w_C}(r, s_C)$.

To define the action of a general element of $\mathcal{B}\Gamma_n$ on $\llbracket \overleftarrow{CT} \rrbracket$, for $p_1, p_2 \in \mathcal{B}\Gamma_n$, we impose the relation:

$$m_{2,T}(\xi \otimes p_1 p_2) = m_{2,T}(m_{2,T}(\xi \otimes p_1) \otimes p_2). \tag{12.1}$$

Instead of proving that the definition of $m_{2,T}$ is well-defined and $(\langle\!\langle \overleftarrow{CT} \rrbracket, m_{1,T}, m_{2,T})$ is a type A structure directly, we will show that after a simplification process applied to the type A structure $(\langle\!\langle \overleftarrow{T} \rrbracket, m_{1,\bullet}, m_{2,\bullet})$, the resulting type A structure is actually the same as $(\langle\!\langle \overleftarrow{CT} \rrbracket, m_{1,T}, m_{2,T})$. In order to describe this simplification process, let us recall a standard result in the study of A_∞ module which can be found in [18, Section 5]:

Proposition 59. *Let $(M, \{m_i\})$ be a strictly unital, right A_∞ module over an A_∞ algebra $(A, \{\mu_i\})$, and let $(\overline{M}, \overline{m}_1)$ be a chain complex. Suppose there exist chain maps $\iota : (\overline{M}, \overline{m}_1) \longrightarrow (M, m_1)$ and $\pi : (M, m_1) \longrightarrow (\overline{M}, \overline{m}_1)$, and a map $H : M \longrightarrow M[1]$ satisfying*

$$\pi \circ \iota = \mathbb{I}_{\overline{M}} \tag{12.2}$$

$$\iota \circ \pi - \mathbb{I}_M = m_1 \circ H + H \circ m_1 \tag{12.3}$$

$$H \circ \iota = 0 \tag{12.4}$$

$$\pi \circ H = 0 \tag{12.5}$$

$$H^2 = 0. \tag{12.6}$$

Then there are maps $\overline{m}_i : \overline{M} \otimes A^{\otimes (i-1)} \to \overline{M}$ for $i \geq 2$ such that $\{\overline{m}_i\}_{i=1}^\infty$ defines a strictly unital right A_∞ module structure on $\overline{M}$. This structure is homotopy equivalent to $(M, \{m_i\})$ through strictly unital homomorphisms which extend π and ι.

We note that $\{\overline{m}_i\}_{i=1}^\infty$ and the homomorphisms can be computed explicitly in the proof of this proposition. For our purpose, we only need to recall the formulas for $\{\overline{m}_i\}_{i=1}^\infty$. For $n \geq 2$, $\Sigma_n : M \otimes A^{\otimes(n-1)} \longrightarrow M[n-2]$ is defined by:

$$\Sigma_n = \sum_{i_j \geq 2, i_1 + \ldots + i_k = n-1+k} m_{i_1}(H \otimes \mathbb{I}^{\otimes(i_1-1)})(m_{i_2} \otimes \mathbb{I}^{\otimes(i_1-1)}) \ldots\ldots (H \otimes \mathbb{I}^{\otimes(n-i_k)})(m_{i_k} \otimes \mathbb{I}^{\otimes(n-i_k)}).$$

Then, $\overline{m}_n : \overline{M} \otimes A^{\otimes(n-1)} \longrightarrow \overline{M}[n-2]$ is defined by:

$$\overline{m}_n := \pi \circ \Sigma_n \circ (\iota \otimes \mathbb{I}^{\otimes(n-1)}).$$

We next will recall the general strategy to define H, ι, and π. Suppose that we have a chain complex (M, m_1) where the generators of M_k are $\{x_1, ..., x_n\}$ and those of M_{k+1} are $\{y_1, ..., y_l\}$. We also suppose that $m_1(x_i) = \sum a_{i,j} y_j$ where $a_{1,1} = u$ is an unit. Canceling the pair (x_1, y_1), we obtain a new chain complex $(\overline{M}, \overline{m}_1)$ on where the other chain groups and boundary maps are taken to be the same, but $\overline{M}_k$ is spanned by $x'_2, ..., x'_n$ and $\overline{M}_{k+1}$ is spanned by $y'_2, ..., y'_l$. $\pi : M \to \overline{M}$ is a natural projection. The new boundary map $\overline{m}_1 : \overline{M}_k \to \overline{M}_{k+1}$ is given by $\overline{m}_1(x'_i) = (\pi \circ m_1)(x_i - a_{i,1} u^{-1} x_1)$. Additionally, we let $\iota(x'_i) = x_i - a_{i,1} u^{-1} x_1$, $H(y_1) = u^{-1} x_1$, and $H(z) = 0$ otherwise. With these formulas in hand, we can find the explicit formula for $\overline{m}_n$.

Proposition 60. *$(\llangle \overleftarrow{CT} \rrbracket, m_{1,T}, m_{2,T})$ is an A_∞ module over the differential graded algebra $\mathcal{B}\Gamma_n$ with $m_{n,T} = 0$ for $n \geq 3$. Furthermore, $(\llangle \overleftarrow{T} \rrbracket, m_{1,\bullet}, m_{2,\bullet})$, defined in Chapter 11, is A_∞ homotopy equivalent to $(\llangle \overleftarrow{CT} \rrbracket, m_{1,T}, m_{2,T})$.*

Proof. Starting with $(\llangle \overleftarrow{T} \rrbracket, m_{1,\bullet}, m_{2,\bullet})$, we will use Proposition 59 to cancel all of the mutual pairs (see Proposition 32 for the definition of a mutual pair) to get an A_∞ module $(\llangle \overleftarrow{CT} \rrbracket, \{\overline{m}_n\})$ homotopy equivalent to $(\llangle \overleftarrow{T} \rrbracket, m_{1,\bullet}, m_{2,\bullet})$. After canceling all of the mutual pairs, the resulting differential $\overline{m}_1$ is actually $m_{1,T}$ (the proof can be found in [20, Proposition 5.1]). Therefore, we only need to prove $\overline{m}_2 = m_{2,T}$ and $\overline{m}_n = 0$ for $n \geq 3$.

Suppose there are l mutual pairs $\{(\xi^i, \xi^i_{C_i})\}_{i=\overline{1,l}}$ in $\textsc{State}(\overleftarrow{T})$. We will cancel them in turn by using Proposition 59 and obtain the collection of $(l+1)$ strictly unital right A_∞ modules $\{(M^j, \{m^j_i\})\}_{j=\overline{0,l}}$ where $(M^0, \{m^0_i\}) = (\llangle \overleftarrow{T} \rrbracket, m_{1,\bullet}, m_{2,\bullet})$ and $(M^l, \{m^l_i\}) = (\llangle \overleftarrow{CT} \rrbracket, \{\overline{m}_n\})$. We analyze how new actions can be created in the simplification process by using the formulas to compute H, π, ι, Σ_n, and the new actions as in Proposition 59. With a slight abuse of notations, we use the same notations π, ι, and Σ_n at each step of the whole simplification process. Let $\overline{\xi} \in \textsc{State}(\overleftarrow{T})$ and let $e_1, ..., e_{n-1}$ be non-idempotent elements of $\mathcal{B}\Gamma_n$. We will observe that after canceling the mutual pair $(\xi^i, \xi^i_{C_i})$, there are exactly three ways the new actions can show up:

• **Case I.** If $\langle m_1^{i-1}(\overline{\xi}), \xi^i_{C_i}\rangle = \alpha \neq 0$ and $m_n^{i-1}(\xi^i \otimes e_1 \otimes ... \otimes e_{n-1}) \neq 0$, then

$$m_n^i(\overline{\xi} \otimes e_1 \otimes ... \otimes e_{n-1}) = \pi \circ \Sigma_n \circ (\iota(\overline{\xi}) \otimes e_1 \otimes ... \otimes e_{n-1}).$$

We note that there is only one term (the one not involving H) in Σ_n which is nonzero in $\Sigma_n \circ (\iota(\overline{\xi}) \otimes e_1 \otimes ... \otimes e_{n-1})$. That is because of two following reasons: 1a) $H(Q) = 0$ if Q is a linear combination supported on states not equal to $\xi^i_{C_i}$, and 1b) $\langle m_k^{i-1}(\overline{\xi} \otimes e_1 \otimes ... \otimes e_{k-1}), \xi^i_{C_i}\rangle = \langle m_k^{i-1}(\xi^i \otimes e_1 \otimes ... \otimes e_{k-1}), \xi^i_{C_i}\rangle = 0$ for any $k \in \{2, ..., n\}$ (it is true because $\partial(\overline{\xi}) = \partial(\xi^i) = \partial(\xi^i_{C_i})$ and $e_1, ..., e_{n-1}$ are non-idempotent elements). Therefore,

$$m_n^i(\overline{\xi} \otimes e_1 \otimes ... \otimes e_{n-1}) = \pi \circ m_n^i(\overline{\xi} \otimes e_1 \otimes ... \otimes e_{n-1}) + (\overleftarrow{w_{C_i}})^{-1} \cdot \alpha \cdot m_n^{i-1}(\xi^i \otimes e_1 \otimes ... \otimes e_{n-1}).$$

In this case, the last term of the right hand side is a new term in the action.

• **Case II.** If $m_n^{i-1}(\overline{\xi} \otimes e_1 \otimes ... \otimes e_{n-1}) = \alpha \cdot \xi^i_{C_i} + Q_1$ and $m_1^{i-1}(\xi^i) = \overleftarrow{w_{C_i}} \cdot \xi^i_{C_i} + Q$, by using a similar argument as in above case, we obtain:

$$\begin{aligned} m_n^i(\overline{\xi} \otimes e_1 \otimes ... \otimes e_{n-1}) &= \pi \circ \Sigma_n \circ (\iota(\overline{\xi}) \otimes e_1 \otimes ... \otimes e_{n-1}) \\ &= \pi(m_n^{i-1}(\overline{\xi} \otimes e_1 \otimes ... \otimes e_{n-1})) \\ &= \pi(\alpha \cdot \xi^i_{C_i} + Q_1) \\ &= \alpha \cdot (\overleftarrow{w_{C_i}})^{-1} \cdot Q + Q_1. \end{aligned}$$

In this case, the first term of the right hand side is a new term in the action.

• **Case III.** If $m_n^{i-1}(\overline{\xi} \otimes e_1 \otimes ... \otimes e_{n-1}) = \alpha \cdot \xi^i_{C_i} + Q$ and $m_{n-p+1}^{i-1}(\xi^i \otimes e_p \otimes ... \otimes e_{n-1}) \neq 0$, then

$$m_n^i(\overline{\xi} \otimes e_1 \otimes ... \otimes e_{n-1}) = \pi \circ \Sigma_n \circ (\iota(\overline{\xi}) \otimes e_1 \otimes ... \otimes e_{n-1})$$

$$= \pi \circ \big[m_n^{i-1}(\overline{\xi} \otimes e_1 \otimes ... \otimes e_{n-1}) + m_{n-p+1}^{i-1}(H \circ (m_p^{i-1}(\overline{\xi} \otimes e_1 \otimes ... \otimes e_{p-1})) \otimes e_p \otimes ... \otimes e_{n-1}) \big]$$

$$= \pi \circ m_n^{i-1}(\overline{\xi} \otimes e_1 \otimes ... \otimes e_{n-1}) + \pi \circ (H(\alpha \cdot \xi_{C_i}^i + Q) \otimes e_p \otimes ... \otimes e_{n-1})$$

$$= \pi \circ m_n^{i-1}(\overline{\xi} \otimes e_1 \otimes ... \otimes e_{n-1}) + \alpha \cdot (\overleftarrow{w_{C_i}})^{-1} \pi \circ m_{n-p+1}^{i-1}(\xi^i \otimes e_p \otimes ... \otimes e_{n-1}).$$

In this case, the second term of the right hand side is a new term in the action.

Using the above observation about how new actions can be created, we analyze the action $\overline{m}_n$ for $n \geq 2$ as follows. Let $\xi = (r, s)$ and $\xi' = (r', s')$ be two states in $\mathrm{ST}_n(\overleftarrow{T})$, and let $e_1, ..., e_{n-1}$ be non-idempotent generators in $\mathcal{B}\Gamma_n$ such that $\langle \overline{m}_n(\xi \otimes e_1 \otimes ... \otimes e_{n-1}), \xi' \rangle \neq 0$. Therefore, there exists a following sequence of transitions of states in $\mathrm{State}(\overleftarrow{T})$:

$$(r, s) = (r_0, s_0^+) \to (r_1, s_1^-) \to (r_1, s_1^+) \to .... \to (r_k, s_k^+) \to (r_{k+1}, s_{k+1}^-) = (r', s')$$

where each transition $(r_i, s_i^-) \to (r_i, s_i^+)$ comes from a mutual pair and let C_i be the free circle where $s_i^-(C_i) = -$ and $s_i^+(C_i) = +$. For each transition $(r_i, s_i^+) \to (r_{i+1}, s_{i+1}^-)$, it comes from either of two following cases: 1) $\langle m_{1,\bullet}(r_i, s_i^+), (r_{i+1}, s_{i+1}^-) \rangle \neq 0$, or 2) there exists $i_0 \in \{1, ..., n-1\}$ such that $\langle m_{2,\bullet}((r_i, s_i^+) \otimes e_{i_0}), (r_{i+1}, s_{i+1}^-) \rangle \neq 0$.

We now can apply a similar argument as in the proof of Proposition 32. We record the number of $+$ and $-$ free circles for each state (r, s) of $\overleftarrow{T}$ by $J(r, s) = (J_+(r, s), J_-(r, s))$. We see that $J(r_i, s_i^+) - J(r_i, s_i^-) = (1, -1)$ for each $i \in \{1, ..., k\}$. Additionally, we evaluate $J_i = (J_{i_+}, J_{i_-}) = J(r_{i+1}, s_{i+1}^-) - J(r_i, s_i^+)$ as the following cases:

1. If $\langle m_{1,\bullet}(r_i, s_i^+), (r_{i+1}, s_{i+1}^-)\rangle \neq 0$, J_i belongs to $(-1,0), (0,1), (-1,1)$.

2. If $\langle m_{2,\bullet}((r_i, s_i^+) \otimes e_{i_0}), (r_{i+1}, s_{i+1}^-)\rangle \neq 0$, $J_{i_+} \geq 0$ and $J_{i_-} \leq 0$. This is because the actions of bridge and right decoration elements do not change the number of $\pm$ free circles. Also, the action of a left decoration element will either 1) change the decoration on a cleaved circle from $+$ to $-$, or 2) create a new $+$ free circle by splitting a $+$ cleaved circle, or 3) merge a $-$ free circle to a $+$ cleaved circle.

We also note that $J_0 = (0,1)$ and $J_k = (-1,0)$. Since (r,s) and (r',s') belong to $\mathrm{ST}_n(\overleftarrow{T})$, we have $J(r,s) = J(r',s') = (0,0)$. Furthermore, we have:

$$J(r',s') - J(r,s) = \sum_{i=1}^{k} \left[J(r_i, s_i^+) - J(r_i, s_i^-)\right] + \sum_{i=0}^{k} J_i = (0,0).$$

Therefore, $\sum_{i=0}^{k} J_i = (-k,k)$. By looking through all of possible cases of J_i, we see that k has to be either 0 or 1. If $k=1$, we need to have $\partial(\xi) = \partial(r_1, s_1^-) = \partial(r_1, s_1^+) = \partial(\xi')$. However, it is a contradiction since $\langle \overline{m}_n(\xi \otimes e_1 \otimes ... \otimes e_{n-1}), \xi'\rangle \neq 0$ and $n \geq 2$. If $k=0$, n has to be 2. In this case, ξ' is obtained from ξ by either changing the decoration on a $+$ circle of ξ or surgering along a bridge of r. In this case, $\langle \overline{m}_2(\xi \otimes e_1), \xi'\rangle = \langle m_{2,T}(\xi \otimes e_1), \xi'\rangle$. Additionally, since n has to be 2, it implies that $\overline{m}_n = 0$ for $n \geq 3$. Due to this fact, we obtain $\overline{m}_2(\xi \otimes p_1 p_2) = \overline{m}_2(\overline{m}_2(\xi \otimes p_1) \otimes p_2)$ for $p_1, p_2 \in \mathcal{B}\Gamma_n$. As a result, $\overline{m}_2(\xi \otimes e) = m_{2,T}(\xi \otimes e)$ for any $e \in \mathcal{B}\Gamma_n$. It implies that the definition of $m_{2,T}$ is well-defined and that $(\langle\!\langle \overleftarrow{T} \rrbracket, m_{1,\bullet}, m_{2,\bullet})$ is A_∞ homotopy equivalent to $(\langle\!\langle \overleftarrow{CT} \rrbracket, m_{1,T}, m_{2,T})$. $\diamond$

CHAPTER 13

INVARIANCE OF THE TYPE A STRUCTURE UNDER THE WEIGHT MOVES AND REIDEMEISTER MOVES

13.1 Invariance under the weight moves

In Chapter 8, we proved that the type D structure in the twisted tangle homology described in Chapter 5 is invariant under the weight moves by using the trick in [20] or [23]. Similarly, in this section, we will prove that under the weight moves, the type A structure in the twisted tangle homology is an invariant. Additionally, based on the construction of the type A and the type D structures, there exists a type DA bimodule version in twisted tangle homology for a tangle subordinate to an annulus (This work is joint with L. Roberts and is in preparation to submit). Then thanks to the gluing process in [12] which pairs the type A and type DA structures, it suffices to prove the invariance for the local case (see Figure 13.1), and the invariance for the global case follows as a consequence.

Before giving a proof of the invariance under the weight moves for these cases, we will state precisely the necessary results of the bimodule paper ([22]) and describe precisely how the invariance of the type A structure for the global case follows these results and the invariance for the local case. Let T be a tangle diagram subordinate to an annulus Σ (see the middle picture in Figure 13.2). In [22], by combining the constructions of the type A and type D structures, we have the following theorem:

Theorem 61. *There exists a type DA structure $(\llbracket T \rrbracket, \delta_1, \delta_2)$ associated to T which satisfies the structure equations from [12, Definition 2.2.43]. Also, the homotopy class of $(\llbracket T \rrbracket, \delta_1, \delta_2)$ as a type DA structure is an invariant of the tangle defined by T.*

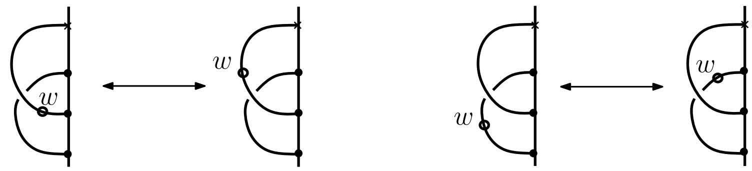


Figure 13.1: This figure illustrates the tangles before and after the weights are moved along the crossings in the local case.

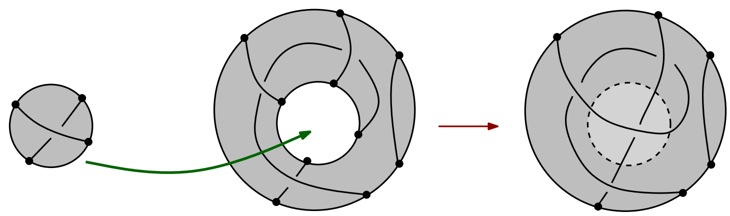


Figure 13.2: This figure illustrates how to obtain a new tangle by filling a tangle embedded in a disk in the middle of a tangle subordinate to an annulus.

Additionally, let T_1 be a tangle embedded in a disk $\mathbb{D}$ (see the left picture in Figure 13.2) so that it is well-defined to glue along the boundary of $\mathbb{D}$ and the inner boundary component of Σ to obtain a new tangle $T_0 = T_1 \# T$. We note that in Chapter 11, we associate the type A structure to a tangle embedded in $\overleftarrow{\mathbb{H}}$. By compactification, we can think of this tangle being embedded in a disk and we still associate the same type A structure to it. Using the gluing process in [12, Definition 2.3.9], we can pair the type A structure $(\langle\!\langle \overleftarrow{T_1}]\!], m_{1,\bullet}, m_{2,\bullet})$ and the type DA structure $([\![T]\!], \delta_1, \delta_2)$ to obtain a type A structure. Not surprisingly, this type A structure is actually $(\langle\!\langle \overleftarrow{T_0}]\!], m_{1,\bullet}, m_{2,\bullet})$ (a proof of this result will be described in [22] as well). Therefore, it suffices to prove the invariance for the local case because the invariance for the global case will be followed by the result that paring homotopy equivalences of the type A structure and the type DA structure results in homotopy equivalences of the type A structure (see [12, Lemma 2.3.13]).

We will only give a proof for transition on the left in Figure 13.1, since the proof of the right case is similar. Let $\overleftarrow{T}$ be the weighted tangle before the weight w moved along the crossing c and let $\overleftarrow{T'}$ be the weighted tangle after the movement of w. Let $\{m_{1,\bullet}, m_{2,\bullet}\}$ and

$\{m'_{1,\bullet}, m'_{2,\bullet}\}$ be the maps defining the type A structures for $\langle\!\langle \overleftarrow{T} \rrbracket$ and $\langle\!\langle \overleftarrow{T'} \rrbracket$ respectively. Note that in this case, both $m_{1,\bullet}$ and $m'_{1,\bullet}$ are trivial maps. We will construct an A_∞ morphism $\Psi = \{\psi_1, \psi_2\}$ from $\langle\!\langle \overleftarrow{T} \rrbracket$ to $\langle\!\langle \overleftarrow{T'} \rrbracket$ as follows: • $\psi_1 : \langle\!\langle \overleftarrow{T} \rrbracket \longrightarrow \langle\!\langle \overleftarrow{T'} \rrbracket$ is the identity map since the generators of $\langle\!\langle \overleftarrow{T} \rrbracket$ can be canonically identified with the generators of $\langle\!\langle \overleftarrow{T'} \rrbracket$.

• $\psi_2 : \langle\!\langle \overleftarrow{T} \rrbracket \otimes_{\mathcal{I}_2} \mathcal{B}\Gamma_2 \longrightarrow \langle\!\langle \overleftarrow{T'} \rrbracket[1]$. To define ψ_2, it suffices to specify its values on $\xi \otimes e$ where $\xi = (r, s)$ is a generator of $\langle\!\langle \overleftarrow{T} \rrbracket$ and e is a generator of $\mathcal{B}\Gamma_2$. If ξ and e satisfy that $\partial(r, s) = s(e)$, $r(c) = 1$, and e is a left bridge element; we define $\psi_2(\xi \otimes e) = w \cdot (r_\gamma, s_\gamma)$ where r_γ is obtained by surgering along the inactive bridge resolution γ at crossing c and s_γ is computed by using the Khovanov Frobenius algebra (note that r_γ is a resolution of $\overleftarrow{T'}$ under the identification of $\overleftarrow{T}$ and $\overleftarrow{T'}$) and $(r_\gamma, s_\gamma) = t(e)$. Otherwise, we define $\psi_2(\xi \otimes e) = 0$.

We also illustrate the definition of ψ_2 as the thick red arrows of Figure 13.3. Then we define ψ_2 when e is any element in $\mathcal{B}\Gamma_2$ by imposing the following relation for each $\xi \in \langle\!\langle \overleftarrow{T} \rrbracket$ and $e_1, e_2 \in \mathcal{B}\Gamma_2$:

$$\psi_2(\xi \otimes e_1 e_2) = m'_{2,\bullet}(\psi_2(\xi \otimes e_1) \otimes e_2) + \psi_2(m_{2,\bullet}(\xi \otimes e_1) \otimes e_2). \tag{13.1}$$

For ψ_2 to be well-defined, we need to verify that with the relation we just imposed, $\psi_2(\xi \otimes p) = 0$ for each relation p defining $\mathcal{B}\Gamma_2$. Since for each e generating $\mathcal{B}\Gamma_2$, $\psi_2(\xi \otimes e) = 0$ unless e is a left bridge element, $\psi_2(\xi \otimes p)$ is trivially 0 if p does not involve left bridge element(s). Therefore, there are two cases to verify: 1) if p is Relation (3) in the **Group I** and 2) if p is a relation in the **Group III**. With the aid of Figure 13.3, the proof is straightforward. We will do one example to illustrate the method. Since $p = e_{\gamma_1} e_{\overrightarrow{\eta_1}} + e_{\gamma_2} e_{\overrightarrow{\eta_2}} = 0$, we need to verify $\psi_2(\xi \otimes e_{\gamma_1} e_{\overrightarrow{\eta_1}}) = \psi_2(\xi \otimes e_{\gamma_2} e_{\overrightarrow{\eta_2}})$ where ξ is the right top corner state in Figure 13.3. Indeed, since $\xi(c) = r(c) = 1$, using Relation (13.1) and the fact that $m_{2,\bullet}(\xi \otimes e_{\gamma_1}) = 0$ (because its resolution bridge is inactive), we have:

$$\psi_2(\xi \otimes e_{\gamma_1} e_{\overrightarrow{\eta_1}}) = m'_{2,\bullet}(\psi_2(\xi \otimes e_{\gamma_1}) \otimes e_{\overrightarrow{\eta_1}}).$$

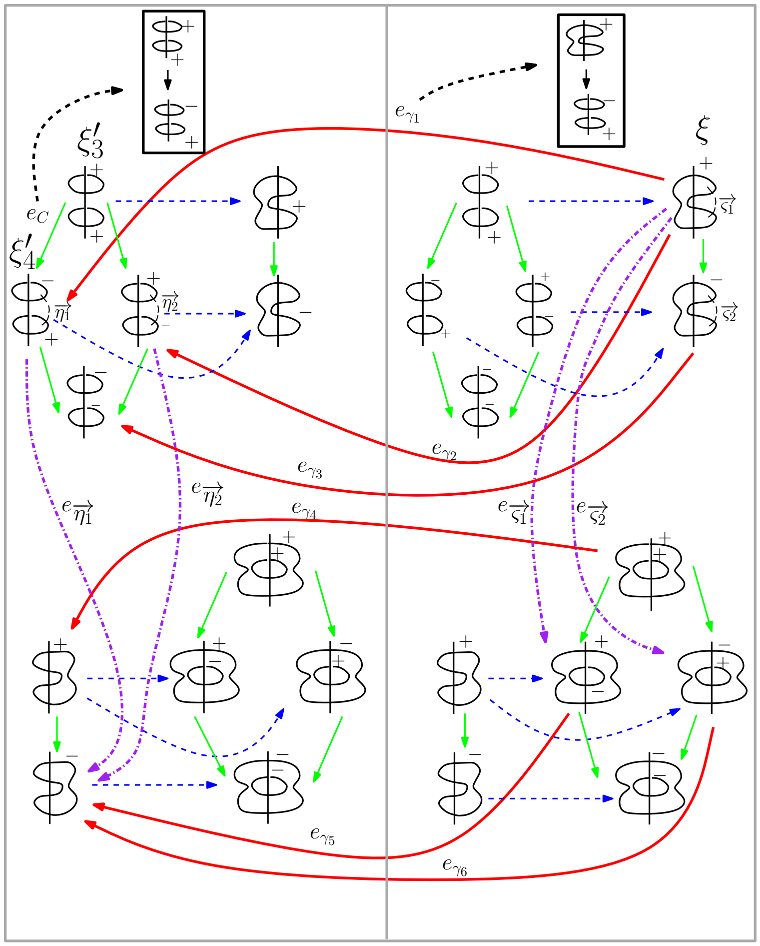


Figure 13.3: In this figure, the right and left columns contain the generators of the complexes associated to the two tangles respectively. The thick red arrows define the map ψ_2. The symbol above each arrow specifies the element in $\mathcal{B}\Gamma_2$ acting on the complex. For example, $\psi_2(\xi \otimes e_{\gamma_1}) = w \cdot \xi_4'$ where e_{γ_1} is a bridge element and illustrated as in the box of the right column. Additionally, the dashed dotted purple, the blue and the green arrows stand for the actions of the right, left bridge elements, and the right (or left) decoration elements of the complexes on themselves respectively. For example, $m'_{2,\bullet}(\xi_3' \otimes e_C) = \xi_4'$ where e_C is a right (or left) decoration element and illustrated as in the box of the left column.

Thus, in Figure 13.3, we go along the thick red arrow to get the result of e_{γ_1} acting on ξ and follow the dashed dotted purple curve due to the action of $e_{\overrightarrow{\eta_1}}$. Similarly, $\psi_2(\xi \otimes e_{\gamma_2} e_{\overrightarrow{\eta_2}})$ is calculated by first going along the red arrow under the action of e_{γ_2} and then following the dashed dotted purple arrow $e_{\overrightarrow{\eta_2}}$. Since the result will be the same if we follow either of those two paths, we finish the proof that ψ_2 is well-defined in this case. The proof of other cases can be handled by the same method and we leave the verification to the readers.

Proposition 62. $\Psi = \{\psi_1, \psi_2\}$ *is an* A_∞ *morphism from* $\langle\!\langle \overleftarrow{T} \rrbracket$ *to* $\langle\!\langle \overleftarrow{T'} \rrbracket$.

Proof. We need to verify the three following conditions:

1. ψ_1 is the chain map from $(\langle\!\langle \overleftarrow{T} \rrbracket, m_{1,\bullet})$ to $(\langle\!\langle \overleftarrow{T'} \rrbracket, m'_{1,\bullet})$.

2. $m'_{2,\bullet}(\psi_1(\xi) \otimes e) + m'_{1,\bullet}(\psi_2(\xi \otimes e)) + \psi_1(m_{2,\bullet}(\xi \otimes e)) + \psi_2(m_{1,\bullet}(\xi) \otimes e) + \psi_2(\xi \otimes d_{\Gamma_2}(e)) = 0.$

3. $\psi_2(\xi \otimes e_1 e_2) = \psi_2(m_{2,\bullet}(\xi \otimes e_1) \otimes e_2) + m'_{2,\bullet}(\psi_2(\xi \otimes e_1) \otimes e_2)$.

The first condition is trivially true since both $m_{1,\bullet}$ and $m'_{1,\bullet}$ are zero maps. The third one comes from Relation (13.1) that we impose on ψ_2. For the second condition to be verified, it suffices to prove that:

$$\psi_2(\xi \otimes d_{\Gamma_2}(e)) = m'_{2,\bullet}(\psi_1(\xi) \otimes e) + \psi_1(m_{2,\bullet}(\xi \otimes e)). \tag{13.2}$$

First of all, we will prove that Equation (13.2) is true when e is a (length 0 or 1) generator of $\mathcal{B}\Gamma_2$. If e is not a left decoration element, the left hand side is 0 by the definition of d_{Γ_2}. Similarly, the right hand side is also 0 because the action of e on $\langle\!\langle \overleftarrow{T'} \rrbracket$ does not change after we move the weight and thus, $m'_{2,\bullet}(\psi_1(\xi) \otimes e) = \psi_1(m_{2,\bullet}(\xi \otimes e))$. If e is a left decoration element $\overleftarrow{e_C}$ where C is + cleaved circle of $\partial(\xi)$, there are two possibilities:

1. If C is the only cleaved circle of $\partial(\xi)$ then:

$$m'_{2,\bullet}(\psi_1(\xi) \otimes e) + \psi_1(m_{2,\bullet}(\xi \otimes e)) = 0$$

because both terms in the identity are $\overleftarrow{w_C}(r, s_C)$ (the moved weight w is still on C in this case)

2. If C is not the only one, then:

$$m'_{2,\bullet}(\psi_1(\xi) \otimes e) + \psi_1(m_{2,\bullet}(\xi \otimes e)) = w \cdot (r, s_C).$$

On the other hand, by the definition of d_{Γ_2} and Relation (13.1), $\psi_2(\xi \otimes d_{\Gamma_2}(\overleftarrow{e_C}))$ is calculated as the sum of paths starting at ξ, then following either the dashed blue arrow (the action of a left bridge element) or the thick red arrow (the action of ψ_2), and then following either the thick red arrow or the dashed blue arrow (see Figure 13.3). Note that if C is the only cleaved circle of ξ (as in the previous case) then there are two such paths and their sum will be canceled out. Otherwise, we have only one such path and the end point of this path is (r, s_C). The reason why we have only one path in this case is that according to the definition:

$$\psi_2(\xi \otimes d_{\Gamma_2}(\overleftarrow{e_C})) = \psi_2(\xi \otimes \overleftarrow{e_\gamma}\overleftarrow{e_{\gamma^\dagger}}) = \psi_2(m_{2,\bullet}(\xi \otimes \overleftarrow{e_\gamma}) \otimes \overleftarrow{e_{\gamma^\dagger}}) + m'_{2,\bullet}(\psi_2(\xi \otimes \overleftarrow{e_\gamma}) \otimes \overleftarrow{e_{\gamma^\dagger}})$$

and depending on whether $\xi(c) = 0$ or $\xi(c) = 1$, the second term or the first term in the latest sum will disappear. Furthermore, the weight w comes from the chain map ψ_2.

Therefore, we finish the proof when e is of length 0 or 1.

Next we prove that Identity (13.2) is true when e is a general element in $\mathcal{B}\Gamma_2$. Let e_1 and e_2 be two elements of $\mathcal{B}\Gamma_2$. For ease of notation, we let m_2, m'_2 and d stand for $m_{2,\bullet}$, $m'_{2,\bullet}$ and d_{Γ_2} respectively. Then

$\psi_2(\xi \otimes d(e_1 e_2))$

$$= \psi_2(\xi \otimes (de_1)e_2) + \psi_2(\xi \otimes e_1(de_2))$$

$$= \psi_2(m_2(\xi \otimes de_1) \otimes e_2) + m_2'(\psi_2(\xi \otimes de_1) \otimes e_2) + \psi_2(m_2(\xi \otimes e_1) \otimes de_2) + m_2'(\psi_2(\xi \otimes e_1) \otimes de_2)$$

$$= \big[\psi_2(m_2(\xi \otimes de_1) \otimes e_2) + m_2'(\psi_2(\xi \otimes e_1) \otimes de_2)\big] + m_2'(\psi_2(\xi \otimes de_1) \otimes e_2) + \psi_2(m_2(\xi \otimes e_1) \otimes de_2)$$

$$= \big[\psi_2(m_2(\xi \otimes de_1) \otimes e_2) + m_2'(\psi_2(\xi \otimes e_1) \otimes de_2)\big] + \big[m_2'(m_2'(\psi_1(\xi) \otimes e_1) \otimes e_2) + m_2'((\psi_1(\xi) \otimes e_1) \otimes e_2)\big] + \big[\psi_1(m_2(m_2(\xi \otimes e_1) \otimes e_2)) + m_2'((\psi_1(\xi) \otimes e_1) \otimes e_2)\big]$$

$$= m_2'(m_2'(\psi_1(\xi) \otimes e_1) \otimes e_2) + \psi_1(m_2(m_2(\xi \otimes e_1) \otimes e_2)).$$

The fourth equality is true because by induction Equation (13.2) is true for e_1 and e_2. The last equality is obtained from the fourth equality because $\psi_2(m_2(\xi \otimes de_1) \otimes e_2) = m_2'(\psi_2(\xi \otimes e_1) \otimes de_2) = 0$. Indeed, de_i $(i = 1, 2)$ is either 0 or a sum of product(s) which contains a factor of the form $\overleftarrow{e_\gamma}\overleftarrow{e_{\gamma^\dagger}}$. As a result, the actions of de_i on our complexes $\langle\!\langle \overleftarrow{T} \rrbracket$ and $\langle\!\langle \overleftarrow{T'} \rrbracket$ are trivial since $\overleftarrow{T}$ and $\overleftarrow{T'}$ have only one crossing. Therefore,

$\psi_2(\xi \otimes d(e_1 e_2))$

$$= m_2'(m_2'(\psi_1(\xi) \otimes e_1) \otimes e_2) + \psi_1(m_2(m_2(\xi \otimes e_1) \otimes e_2))$$

$$= m_2'(\psi_1(\xi) \otimes e_1 e_2) + \psi_1(m_2(\xi \otimes e_1 e_2)).$$

As a result, we have proved that $\Psi = \{(\psi_1, \psi_2)\}$ is an A_∞ morphism from $\langle\!\langle \overleftarrow{T} \rrbracket$ to $\langle\!\langle \overleftarrow{T'} \rrbracket$. $\diamond$

If we define Φ identically as Ψ but reversing the roles of $\langle\!\langle \overleftarrow{T'} \rrbracket$ and $\langle\!\langle \overleftarrow{T} \rrbracket$, we immediately have the following:

1. $(\Psi \circ \Phi)_1 = I_{\langle\!\langle \overleftarrow{T} \rrbracket}$ since ψ_1 and ϕ_1 are identity maps.

2. $(\Psi \circ \Phi)_2 = \psi_1 \circ \phi_2 + \psi_2 \circ (\phi_1 \otimes \mathbb{I}) = 0$ since both terms are ϕ_2 under the canonical

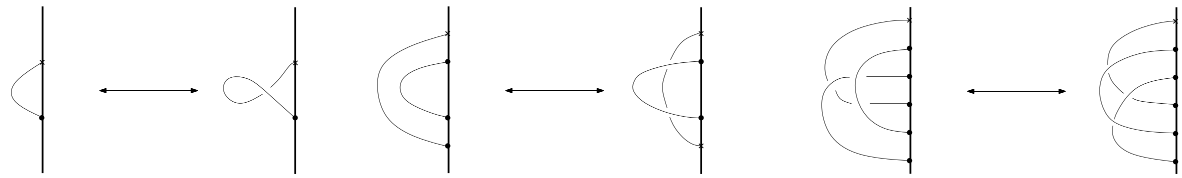


Figure 13.4: The three local Reidemeister moves.

identification of generators of $\langle\!\langle \overleftarrow{T} \rrbracket$ and $\langle\!\langle \overleftarrow{T'} \rrbracket$.

3. $(\Psi \circ \Phi)_3 = \psi_2(\phi_2 \otimes \mathbb{I}) = 0$ since both ψ_2 and ϕ_2 are supported on states ξ where $\xi(c) = 1$ and their images contain states whose crossing c is resolved by 0-resolution.

Therefore, $\Psi \circ \Phi = 1_{\langle\!\langle \overleftarrow{T} \rrbracket}$. Similarly, we have $\Phi \circ \Psi = 1_{\langle\!\langle \overleftarrow{T'} \rrbracket}$. As a result, $(\langle\!\langle \overleftarrow{T} \rrbracket, m_{1,\bullet}, m_{2,\bullet})$ is isomorphic to $(\langle\!\langle \overleftarrow{T'} \rrbracket, m'_{1,\bullet}, m'_{2,\bullet})$ as type A structures.

13.2 Invariance under Reidemeister Moves

Due to the bimodule structure introduced at the beginning of this chapter, we only need to prove the invariance under the Reidemeister moves in Figure 13.4. The strategy is to use the isomorphism in Section 13.1 to move the weights to any arcs whose one of end points belongs to P_n. We then can modify Roberts's proofs of the invariance in the untwisted case (see [18, Section 6]). Let $\overleftarrow{T}$ and $\overleftarrow{T'}$ be the tangles before and after the Reidemeister moves. We will briefly sketch his arguments for the untwisted case and how we can apply those arguments for our case:

1. Since the weights are close to the y-axis, we have $m_{1,\bullet} = m_1$.

2. Decomposing the complex of tangle $\langle\!\langle \overleftarrow{T'} \rrbracket$ as a direct sum where each summand corresponds to a resolution of the crossing(s) of the tangle $\overleftarrow{T'}$, there always exists a summand V consisting of states which have a free circle in the generating resolutions. We then decompose $V = V_+ \oplus V_-$, based on the decoration on the free circle.

3. For the differential m_1, there are two types of isomorphisms from merging a + free circle to a decorated circle or dividing out a decorated circle to get a new − free circle. We can use this isomorphism to cancel out the summand V_+ and its image $d_{APS}(V_+)$ in the case of the Reidemeister move I, or V_+ and $d_{APS}(V_+)$ first and then cancel out V_- and its preimage under the dividing isomorphism in the case of the Reidemeister moves II and III. After the cancellation, we get exactly the same chain module as before the Reidemeister move with the possibility that the higher order actions might appear.

4. Roberts proves that the higher order actions actually do not show up because of two properties. The first property is that the image of m_1 on V_+ is another summand of complex, which is canceled out by the cancellation process. The second property is that the images of higher order actions always lie on V_+ and this will be canceled out at the end.

5. For our case, the same technique can be used to prove there is no higher order actions. Since we have $m_{1,\bullet} = m_1$, we definitely have the first property. (**Note**. If we do not move weights close to the y-axis, the image of $m_{1,\bullet}$ on V_+ also intersects V_-). Additionally, since $m_{2,\bullet}$ is different from m_2 only on the action of left decoration elements, the image of $m_{2,\bullet}(V_+)$ lies in V_+ (again, it is due to the fact that the weights are near the boundary of the tangle) and therefore the image of the higher order actions lie in V_+. At the end, they all disappear when we cancel out V_+. The same argument can be applied for V_- if needed (as in Reidemeister move II or III).

6. After canceling out those terms, we obtain almost the same type A structure associated to the tangles before the Reidemeister moves. The only difference is that we are working over different ground fields. This issue can be handled exactly like the case of the type D structure (see Chapter 10) by using the stable equivalence relation, described in Section 9.2.

Therefore, pursuant to the bimodule version in the twisted tangle homology, we have the following theorem:

Theorem 63. *Let* $\overleftarrow{\mathcal{T}}$ *be a left tangle with a diagram* $\overleftarrow{T}$*. The stable homotopy class of* $(\llangle \overleftarrow{T} \rrbracket, m_{1,\bullet}, m_{2,\bullet})$*, defined as in Section 11, is an invariant of the left tangle* $\overleftarrow{\mathcal{T}}$*.*

CHAPTER 14

RELATION TO THE TOTALLY TWISTED KHOVANOV HOMOLOGY BY GLUING LEFT AND RIGHT TANGLES

As described in Chapter 1, let T be a link diagram for a link $\mathcal{T}$ which is divided by the y-axis into two parts: a left tangle $\overleftarrow{T}$ and a right one $\overrightarrow{T}$. Using the pairing technique in Section 9.4, we will prove that the chain complex $(\langle\!\langle \overleftarrow{T}]\!] \boxtimes_{\bullet} [\![\overrightarrow{T} \rangle\!\rangle, \partial^{\boxtimes}_{\bullet})$ obtained by gluing the type A structure $(\langle\!\langle \overleftarrow{T}]\!], m_{1,\bullet}, m_{2,\bullet})$ and the type D structure $([\![\overrightarrow{T} \rangle\!\rangle, \overrightarrow{\delta_{T,\bullet}})$ is chain isomorphic to the totally twisted Khovanov complex of T.

Let $\mathbb{F}_T$ be the field of fractions of $\mathbb{P}_T = \mathbb{Z}_2[x_f | f \in \text{ARC}(T)]$ where $\text{ARC}(T)$ is the set of segments whose endpoints are either the crossings of T or the intersection points of T and the y-axis. Let $[\![T]\!]$ be the graded Khovanov complex over $\mathbb{F}_T$, equipped with a totally twisted differential $\widetilde{\partial} = \partial_{KH} + \partial_{\mathcal{V},T}$ where ∂_{KH} is the regular Khovanov map and $\partial_{\mathcal{V},T}$ is a Koszul map defined similar to $\partial_{\mathcal{V}}$ (see [19] for more details). Recall that $([\![T]\!], \widetilde{\partial})$ is a link invariant.

There are natural injections $\phi_l : \mathbb{F}_{\overleftarrow{T}} \hookrightarrow \mathbb{F}_T$ and $\phi_r : \mathbb{F}_{\overrightarrow{T}} \hookrightarrow \mathbb{F}_T$, which come from the fact that $\text{ARC}(T)$ is the disjoint union of $\text{ARC}(\overleftarrow{T})$ and $\text{ARC}(\overrightarrow{T})$. To describe the glued complex, we first need to describe the type A structure $(\mathcal{F}_{\phi_l}(\langle\!\langle \overleftarrow{T}]\!]), \mathcal{F}_{\phi_l}(m_{1,\bullet}), \mathcal{F}_{\phi_l}(m_{2,\bullet}))$ and the type D structure $(\mathcal{G}_{\phi_r}([\![\overrightarrow{T} \rangle\!\rangle), \mathcal{G}_{\phi_r}(\overrightarrow{\delta_{T,\bullet}}))$. By using the formulas in Definitions 40 and 48, over $\mathbb{F}_T$, the generators of $\mathcal{F}_{\phi_l}(\langle\!\langle \overleftarrow{T}]\!])$ and $\mathcal{G}_{\phi_r}([\![\overrightarrow{T} \rangle\!\rangle)$ are identified with the generators of $\langle\!\langle \overleftarrow{T}]\!]$ (as a vector space over $\mathbb{F}_{\overleftarrow{T}}$) and $[\![\overrightarrow{T} \rangle\!\rangle$ (as a vector space over $\mathbb{F}_{\overrightarrow{T}}$), respectively. Additionally, under this identification, the maps $\mathcal{F}_{\phi_l}(m_{1,\bullet})$, $\mathcal{F}_{\phi_l}(m_{2,\bullet})$ and $\mathcal{G}_{\phi_r}(\overrightarrow{\delta_{T,\bullet}})$ are identical to $m_{1,\bullet}$, $m_{2,\bullet}$ and $\overrightarrow{\delta_{T,\bullet}}$ respectively.

Therefore, we can use $(\langle\!\langle \overleftarrow{T} \rrbracket, m_{1,\bullet}, m_{2,\bullet})$ to stand for $(\mathcal{F}_{\phi_l}(\langle\!\langle \overleftarrow{T} \rrbracket), \mathcal{F}_{\phi_l}(m_{1,\bullet}), \mathcal{F}_{\phi_l}(m_{2,\bullet}))$ and $(\llbracket \overrightarrow{T} \rangle\!\rangle, \overrightarrow{\delta_{T,\bullet}})$ to stand for $(\mathcal{G}_{\phi_r}(\llbracket \overrightarrow{T} \rangle\!\rangle), \mathcal{G}_{\phi_r}(\overrightarrow{\delta_{T,\bullet}}))$. Keep in mind that from now to the end of this chapter, $\langle\!\langle \overleftarrow{T} \rrbracket$ and $\llbracket \overrightarrow{T} \rangle\!\rangle$ are vector spaces over $\mathbb{F}_T$ while $m_{1,\bullet}$, $m_{2,\bullet}$ and $\overrightarrow{\delta_{T,\bullet}}$ are defined in Chapters 5 and 11. We also note that in the type A structure (respectively the type D structure), the weight of a cleaved circle in a state is calculated as the sum of weights on the left-side (respectively right-side) arcs which belong to this circle. The module structure of the glued complex then can be described as:

$$\langle\!\langle \overleftarrow{T} \rrbracket \boxtimes_\bullet \llbracket \overrightarrow{T} \rangle\!\rangle = \langle\!\langle \overleftarrow{T} \rrbracket \otimes_{\mathcal{I}_n} \llbracket \overrightarrow{T} \rangle\!\rangle.$$

Additionally, the differential of this complex is given by the following formula (see Section 9.4):

$$\partial_\bullet^{\boxtimes}(x \otimes y) = m_{1,\bullet}(x) \otimes y + (m_{2,\bullet} \otimes \mathbb{I})(x \otimes \overrightarrow{\delta_{T,\bullet}}(y)).$$

We now prove the gluing theorem:

Theorem 64. *$(\langle\!\langle \overleftarrow{T} \rrbracket \boxtimes_\bullet \llbracket \overrightarrow{T} \rangle\!\rangle, \partial_\bullet^{\boxtimes})$ is isomorphic to $(\llbracket T \rrbracket, \widetilde{\partial})$.*

Proof. Due to the module structures of $\langle\!\langle \overleftarrow{T} \rrbracket$, $\llbracket \overrightarrow{T} \rangle\!\rangle$ as vector spaces over $\mathbb{F}_T$ and the action of $\mathcal{I}_n$ on them, $\langle\!\langle \overleftarrow{T} \rrbracket \boxtimes_\bullet \llbracket \overrightarrow{T} \rangle\!\rangle$ is a vector space over $\mathbb{F}_T$ whose generators are identified with the generators of $\llbracket T \rrbracket$ (see [18, Section 7] for more details). Furthermore, this identification was proved to preserve the bigrading, and thus, it is ζ-grading preserving. Therefore, it suffices to prove that, under this identification, $\partial_\bullet^{\boxtimes} = \widetilde{\partial}$. Since $m_{1,\bullet} = d_{APS} + \partial_{\mathcal{V}}$ and $\overrightarrow{\delta_{T,\bullet}} = \overrightarrow{\delta_T} + \overrightarrow{\delta_{\mathcal{V}}}$, the differential $\partial_\bullet^{\boxtimes}$ can be decomposed as follows:

$$\partial_\bullet^{\boxtimes}(x \otimes y) = \left[d_{APS}(x) \otimes y + (m_2 \otimes \mathbb{I})(x \otimes \overrightarrow{\delta_T}(y))\right] + \left[\partial_{\mathcal{V}}(x) \otimes y + ((m_{2,\bullet} - m_2) \otimes \mathbb{I})(x \otimes \overrightarrow{\delta_T}(y))\right.$$

$$+(m_{2,\bullet} \otimes \mathbb{I})(x \otimes \overrightarrow{\delta_{\mathcal{V}}}(y))\big].$$

From [18, Proposition 36], we know that:

$$d_{APS} \otimes \mathbb{I} + (m_2 \otimes \mathbb{I})(\mathbb{I} \otimes \overrightarrow{\delta_T}) = \partial_{KH}.$$

Therefore, we will complete the proof of the theorem if we can show:

$$\partial_{\mathcal{V}}(x) \otimes y + ((m_{2,\bullet} - m_2) \otimes \mathbb{I})(x \otimes \overrightarrow{\delta_T}(y)) + (m_{2,\bullet} \otimes \mathbb{I})(x \otimes \overrightarrow{\delta_{\mathcal{V}}}(y)) = \partial_{\mathcal{V},T}(x \otimes y) \quad (14.1)$$

for x and y are generators of $\langle\!\langle \overleftarrow{T}]\!]$ and $[\![\overrightarrow{T} \rangle\!\rangle$ such that $I_{\partial(x)} = I_{\partial(y)}$.

We note that $x \otimes y$ specifies a resolution (r, s) of $[\![T]\!]$, consisting of a collection of circles which can be divided into four groups:

1. Circles decorated by $-$.

2. Left free circles decorated by $+$.

3. Right free circles decorated by $+$.

4. Cleaved circles decorated by $+$.

Therefore, the right hand side of Equation (14.1) can be written as follows:

$$\partial_{\mathcal{V},T}(x \otimes y) = \sum_{C \in (2)\dot{\bigcup}(3)\dot{\bigcup}(4)} w_C(r, s_C)$$

where $w_C = \sum_{f \in \textsc{arc}(C)} x_f$ and $\dot{\bigcup}$ stands for the disjoint union.
On the other hand,

$$\partial_{\mathcal{V}}(x) \otimes y = \sum_{C \in (2)} w_C(r, s_C).$$

Additionally, according to the construction of $m_{2,\bullet}$, the action $m_{2,\bullet} - m_2$ is supported only on left decoration elements. As a result:

$$((m_{2,\bullet} - m_2) \otimes \mathbb{I})(x \otimes \overrightarrow{\delta_T}(y)) = \sum_{C \in (4)} ((m_{2,\bullet} - m_2) \otimes \mathbb{I})(x \otimes \overleftarrow{e_C} \otimes y_C)$$

$$= \sum_{C \in (4)} \overleftarrow{w_C} x_C \otimes y_C$$

$$= \sum_{C \in (4)} \overleftarrow{w_C}(r, s_C).$$

From the definition of $\overrightarrow{\delta_{\mathcal{V}}}$, we can calculate the last term of the left hand side of Equation (14.1) as:

$$(m_{2,\bullet} \otimes \mathbb{I})(x \otimes \overrightarrow{\delta_{\mathcal{V}}}(y)) = \sum_{C \in (4)} \overrightarrow{w_C}(m_{2,\bullet} \otimes \mathbb{I})(x \otimes \overrightarrow{e_C} \otimes y_C) + \sum_{C \in (3)} w_C(m_{2,\bullet} \otimes \mathbb{I})(x \otimes I_{\partial(y_C)} \otimes y_C)$$

$$= \sum_{C \in (4)} \overrightarrow{w_C}(r, s_C) + \sum_{C \in (3)} w_C(r, s_C).$$

Rewriting the left hand side of Equation (14.1), we have:

$$LHS = \sum_{C \in (2)} w_C(r, s_C) + \sum_{C \in (3)} w_C(r, s_C) + \sum_{C \in (4)} (\overleftarrow{w_C} + \overrightarrow{w_C})(r, s_C).$$

Since $w_C = \overleftarrow{w_C} + \overrightarrow{w_C}$ for each cleaved circle, we can conclude that Equation (14.1) is true and thus, $\partial^{\boxtimes}_{\bullet} = \widetilde{\partial}$. Thus, $(\langle\!\langle \overleftarrow{T} \rrbracket \boxtimes_{\bullet} \llbracket \overrightarrow{T} \rangle\!\rangle, \partial^{\boxtimes}_{\bullet})$ is chain isomorphic to $(\llbracket T \rrbracket, \widetilde{\partial})$. $\diamond$

CHAPTER 15

EXAMPLES OF THE TYPE D AND THE TYPE A STRUCTURES

In this chapter, we will give examples calculating the type D and type A structures for several knots and links.

Example 1 : Hopf link

Consider the Hopf link T which is transverse to the y-axis at 4 points. It divides the link T into two parts: the left tangle $\overleftarrow{T}$ and the right one $\overrightarrow{T}$. Label each arc of T as in Figure 15.1.

We first describe the type D structure on $[\![\overrightarrow{T} \rangle\!\rangle$. As shown in Figure 15.2, $[\![\overrightarrow{T} \rangle\!\rangle$ can be thought as a vector space over $\mathbb{F}_T$ (by the argument as in the third and fourth paragraphs of Chapter 14), generated by six elements: $\overrightarrow{\xi_1}$,..., $\overrightarrow{\xi_6}$ corresponding to the bottom row of the figure from left to right. We should notice that for each generator, the decoration on the circle passing through the marked point is always $-$ due to our setting (see the construction of $[\![\overrightarrow{T} \rangle\!\rangle$ in Chapter 4). Their ζ-gradings are $\frac{-1}{4}$, $\frac{1}{4}$, 0, $\frac{1}{4}$, $\frac{3}{4}$ and $\frac{1}{2}$ respectively. For example: since the bigrading of $\overrightarrow{\xi_4}$ is $(-1, -5/2)$ due to $n_+(\overrightarrow{T}) = 0$ and $n_-(\overrightarrow{T}) = 2$, the ζ-grading of $\overrightarrow{\xi_4}$ is $-1 + 5/4 = 1/4$. We also denote $\{\overleftarrow{\eta_i}\}_{i=1,..,4}$ the left bridges as in the figure, and $\{\overrightarrow{\gamma_j}\}_{j=1,2}$ and β the right active resolution bridges as in the below figure. Let C and D be the $+$ circles of $\overrightarrow{\xi_1}$ and $\overrightarrow{\xi_4}$ respectively.

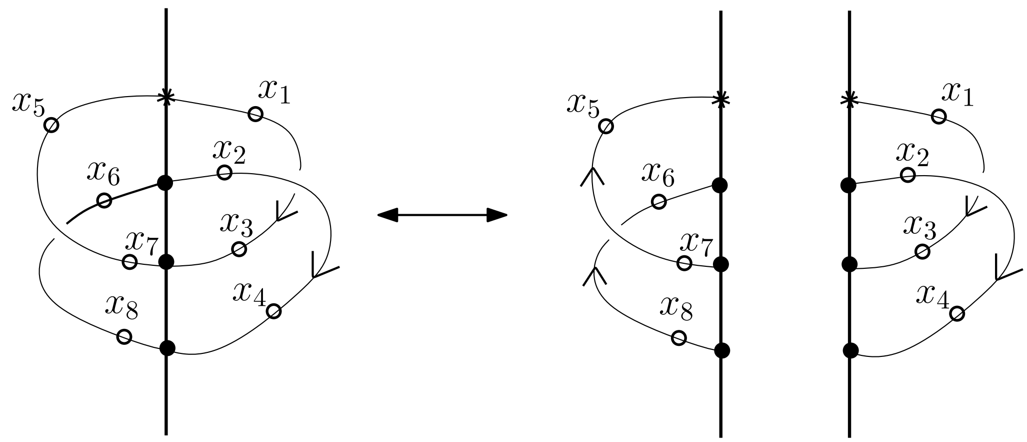


Figure 15.1: Hofp link

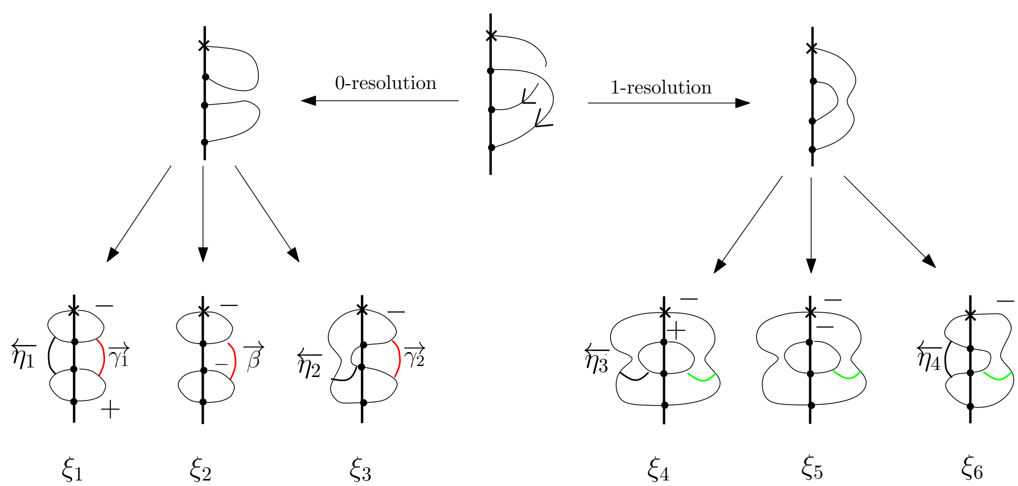


Figure 15.2: Generators of the complex obtained from the tangle

We now can describe $\overrightarrow{\delta_{T,\bullet}}$:

$$\begin{aligned}
\overrightarrow{\delta_{T,\bullet}}(\overrightarrow{\xi_1}) &= e_{\overleftarrow{\eta_1}} \otimes \overrightarrow{\xi_3} + e_{\overrightarrow{\gamma_1}} \otimes \overrightarrow{\xi_6} + [\overleftarrow{e_C} + (x_3 + x_4)\overrightarrow{e_C}] \otimes \overrightarrow{\xi_2} \\
\overrightarrow{\delta_{T,\bullet}}(\overrightarrow{\xi_3}) &= e_{\overrightarrow{\gamma_2}} \otimes \overrightarrow{\xi_5} + e_{\overleftarrow{\eta_2}} \otimes \overrightarrow{\xi_2} \\
\overrightarrow{\delta_{T,\bullet}}(\overrightarrow{\xi_4}) &= e_{\overleftarrow{\eta_3}} \otimes \overrightarrow{\xi_6} + [\overleftarrow{e_D} + (x_2 + x_3)\overrightarrow{e_D}] \otimes \overrightarrow{\xi_5} \\
\overrightarrow{\delta_{T,\bullet}}(\overrightarrow{\xi_6}) &= e_{\overleftarrow{\eta_4}} \otimes \overrightarrow{\xi_5}.
\end{aligned} \tag{15.1}$$

We next describe the type A structure on $\langle\!\langle \overleftarrow{T}]\!]$. Similarly to $[\![\overrightarrow{T} \rangle\!\rangle$, $\langle\!\langle \overleftarrow{T}]\!]$ is a six dimensional vector space over $\mathbb{F}_T$, generated by $\{\overleftarrow{\xi_i}\}_{i=\overline{1,6}}$ where $\overleftarrow{\xi_i}$ has the same boundary as $\overrightarrow{\xi_i}$. We label the bridges of each generator $\overleftarrow{\xi_i}$ exactly as we labeled the ones for $\overrightarrow{\xi_i}$ in the type D structure. For example: $\overleftarrow{\xi_1}$ has a left active resolution bridge $\overleftarrow{\eta_1}$ and right bridge $\overrightarrow{\gamma_1}$. We also let $\overrightarrow{\gamma_3}$ and $\overrightarrow{\gamma_4}$ be the right bridges of $\overleftarrow{\xi_4}$ and $\overleftarrow{\xi_6}$ respectively. Additionally, the ζ-grading of $\overleftarrow{\xi_1}$,...,$\overleftarrow{\xi_6}$ are $\frac{-1}{4}$, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$ and 0, respectively. Since none of generators has a free circle, the type A structure on $\langle\!\langle \overleftarrow{T}]\!]$ can be described as follows:

$$\begin{aligned}
m_{1,\bullet} &= 0 \\
m_{2,\bullet}(\overleftarrow{\xi_1} \otimes e_{\overleftarrow{\eta_1}}) &= \overleftarrow{\xi_3} \\
m_{2,\bullet}(\overleftarrow{\xi_1} \otimes e_{\overrightarrow{\gamma_1}}) &= \overleftarrow{\xi_6} \\
m_{2,\bullet}(\overleftarrow{\xi_1} \otimes \overrightarrow{e_C}) &= \overleftarrow{\xi_2} \\
m_{2,\bullet}(\overleftarrow{\xi_1} \otimes \overleftarrow{e_C}) &= (x_7 + x_8)\overleftarrow{\xi_2} \\
m_{2,\bullet}(\overleftarrow{\xi_3} \otimes e_{\overrightarrow{\gamma_2}}) &= \overleftarrow{\xi_5} \\
m_{2,\bullet}(\overleftarrow{\xi_4} \otimes e_{\overrightarrow{\gamma_3}}) &= \overleftarrow{\xi_3} \\
m_{2,\bullet}(\overleftarrow{\xi_4} \otimes \overrightarrow{e_D}) &= \overleftarrow{\xi_5} \\
m_{2,\bullet}(\overleftarrow{\xi_4} \otimes \overleftarrow{e_D}) &= (x_6 + x_7)\overleftarrow{\xi_5} \\
m_{2,\bullet}(\overleftarrow{\xi_6} \otimes e_{\overrightarrow{\gamma_4}}) &= \overleftarrow{\xi_2} \\
m_{2,\bullet}(\overleftarrow{\xi_6} \otimes e_{\overleftarrow{\eta_4}}) &= \overleftarrow{\xi_5}.
\end{aligned} \tag{15.2}$$

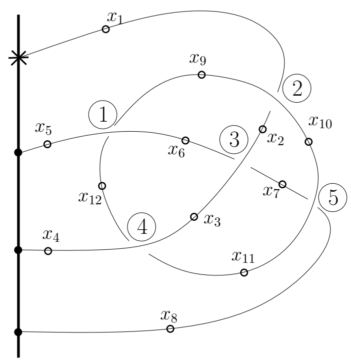


Figure 15.3: An example about a decorated tangle

Then $\langle\!\langle \overleftarrow{T} \rrbracket \boxtimes \llbracket \overrightarrow{T} \rangle\!\rangle$ is a graded vector space over $\mathbb{F}_T$, generated by $\{\xi_i\}_{i=\overline{1,6}}$ where $\xi_i = \overleftarrow{\xi_i} \otimes_{\mathcal{I}_2} \overrightarrow{\xi_i}$. Since $\zeta(\xi_i) = \zeta(\overleftarrow{\xi_i}) + \zeta(\overrightarrow{\xi_i})$, the ζ-grading of ξ_1,...,ξ_6 are $\frac{-1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{3}{2}$ and $\frac{1}{2}$ respectively. Since $m_{1,\bullet} = 0$, the differential of $\langle\!\langle \overleftarrow{T} \rrbracket \boxtimes \llbracket \overrightarrow{T} \rangle\!\rangle$ is:

$$\partial^{\boxtimes}(x \otimes y) = (m_{2,\bullet} \otimes \mathbb{I})(x \otimes \overrightarrow{\delta_{T,\bullet}}(y)).$$

Therefore, using Formulas (15.1) and (15.2), we have:

$$\begin{aligned}
\partial^{\boxtimes}(\xi_1) &= \xi_3 + \xi_6 + (x_3 + x_4 + x_7 + x_8)\xi_2 \\
\partial^{\boxtimes}(\xi_3) &= \xi_5 \\
\partial^{\boxtimes}(\xi_4) &= (x_2 + x_3 + x_6 + x_7)\xi_5 \\
\partial^{\boxtimes}(\xi_6) &= \xi_5.
\end{aligned} \tag{15.3}$$

The above description of ζ-complex $(\langle\!\langle \overleftarrow{T} \rrbracket \boxtimes \llbracket \overrightarrow{T} \rangle\!\rangle, \partial^{\boxtimes})$ is exactly the same as the totally twisted Khovanov homology of Hopf link whose homology is $\mathbb{F}_T \oplus \mathbb{F}_T$, occurring in the ζ-grading $\frac{1}{2}$.

Example 2 : Consider the tangle diagram $\overrightarrow{T}$ as in Figure 15.3.

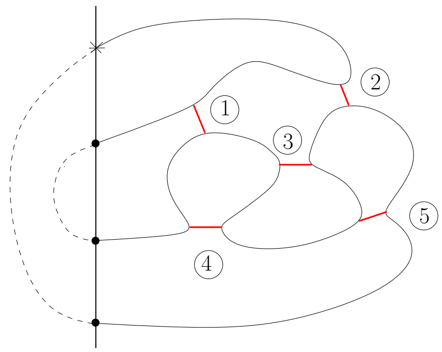


Figure 15.4: A complete resolution of the tangle diagram

We label the five crossings of $\overrightarrow{T}$ as in the above figure. In this example, we will investigate the image of $\overrightarrow{\delta_{n,T}}$ on some generators of $[\![\overrightarrow{CT}]\!\rangle$. Recall: $([\![\overrightarrow{CT}]\!\rangle, \overrightarrow{\delta_{n,T}})$ is the deformation retraction of $([\![\overrightarrow{T}]\!\rangle, \overrightarrow{\delta_{T,\bullet}})$, defined in Section 7. For this tangle $\overrightarrow{T}$, there are two left planar matchings p_1 (containing an arc which connects the marked point and the bottommost point on the y-axis) and p_2. A generator of $[\![\overrightarrow{CT}]\!\rangle$ is obtained by resolving the crossings of $\overrightarrow{T}$ by either a 0-resolution or a 1-resolution in such a way that it has no free circles, then capping off by either p_1 or p_2 and finally, decorating the unmarked cleaved circle (it might not exist) by $\pm$. Therefore, we can encode a generator of $[\![\overrightarrow{CT}]\!\rangle$ in the form of either $\xi_{*****,i}$ or $\xi_{*****,i,\pm}$ where each $*$ receives the value in $\{0,1\}$, depending on the resolution of the corresponding crossing. i receives the value of either 1 or 2, depending on the way we choose either left planar matching p_1 or p_2. Additionally, if the generator has an unmarked cleaved circle, we use the second form to specify the decoration on the unmarked circle. Otherwise, we use the first form to represent this generator. For example, Figure 15.4 represents $\xi_{00000,1}$. We have:

$$
\begin{aligned}
\overrightarrow{\delta_{n,T}}(\xi_{00000,1}) = [x_3 + x_6 {+} x_{12}]^{-1} I_{\partial(\xi_1)} \otimes \xi_{10010,1} + [x_2 + x_7 + x_{10}]^{-1} I_{\partial(\xi_1)} \otimes \xi_{01001,1} \\
+ ([x_3 + x_7 + x_{11}]^{-1} + [x_2 + x_7 + x_{10}]^{-1}) I_{\partial(\xi_1)} \otimes \xi_{00101,1} \\
+ ([x_3 + x_6 + x_{12}]^{-1} + [x_3 + x_7 + x_{11}]^{-1}) I_{\partial(\xi_1)} \otimes \xi_{00110,1} \\
+ e_{\overrightarrow{\gamma_1}} \otimes \xi_{10000,1,-} + e_{\overrightarrow{\gamma_2}} \otimes \xi_{01000,1,-} \\
+ e_{\overleftarrow{\eta_1}} \otimes \xi_{00000,2,-}.
\end{aligned} \tag{15.4}
$$

where $I_{\partial(\xi_1)} := I_{\partial(\xi_{00000,1})}$, $\{e_{\overrightarrow{\gamma_i}}\}_{i=1,2}$ are the right bridge elements of $\mathcal{B}\Gamma_2$, corresponding to the change of the cleaved links after surgering $\xi_{00000,1}$ along the active resolution bridge at crossing i. $\overleftarrow{\eta_1}$ is the unique left bridge of the left planar matching p_1.

As we can see, in the right hand side of Equation (15.4), the first (or second) term comes from a generator, obtained from $\xi_{00000,1}$ by surgering along two active resolution bridges. Depending on which crossing we resolve first, there are two ways to make this surgery. However, there is only one way which creates a free circle, while the other way will change the boundary. On the other hand, both ways to surger $\xi_{00000,1}$ to obtain a generator of the third (or fourth) term contain free circles, and, as the result, we count the coefficients for both paths.

Similarly, we compute $\overrightarrow{\delta_{n,T}}(\xi_{00000,2,+})$:

$$\begin{aligned}\overrightarrow{\delta_{n,T}}(\xi_{00000,2,+}) &= [x_3+x_6+x_{12}]^{-1} I_{\partial(\xi_2)} \otimes \xi_{10010,2,+} + [x_2+x_7+x_{10}]^{-1} I_{\partial(\xi_2)} \otimes \xi_{01001,2,+} \\ &+ ([x_3+x_7+x_{11}]^{-1} + [x_2+x_7+x_{10}]^{-1}) I_{\partial(\xi_2)} \otimes \xi_{00101,2,+} \\ &+ ([x_3+x_6+x_{12}]^{-1} + [x_3+x_7+x_{11}]^{-1}) I_{\partial(\xi_2)} \otimes \xi_{00110,2,+} \\ &+ e_{\overrightarrow{\gamma_1}} \otimes \xi_{10000,2} + e_{\overrightarrow{\gamma_2}} \otimes \xi_{01000,2} \\ &+ e_{\overleftarrow{\eta_2}} \otimes \xi_{00000,2} + [\overleftarrow{e_C} + (x_2+x_3+x_4+x_6+x_7+x_8+x_{10}+x_{11}+x_{12})\overrightarrow{e_C}] \otimes \xi_{00000,2,-}.\end{aligned} \tag{15.5}$$

where $I_{\partial(\xi_2)} = I_{\partial(\xi_{00000,2,+})}$. $\overleftarrow{\eta_2}$ is a unique left bridge of p_2 and C stands for the unmarked cleaved circle of $\xi_{00000,2,+}$. We note that the main difference between Equation (15.4) and Equation (15.5) is that the image of $\overrightarrow{\delta_{n,T}}$ on $\xi_{00000,2,+}$ contains a term coming from a decoration element.

REFERENCES

[1] M. Asaeda, J. Przytycki, A. Sikora, *Categorification of the Kauffman bracket skein module of I-bundles over surfaces. Algebr. Geom. Topol.* 4 (2004), 11771210 (electronic).

[2] M. Asaeda, J. Przytycki, A. Sikora, *Categorification of the skein module of tangles.* Primes and knots, 18, Contemp. Math., 416, Amer. Math. Soc., Providence, RI, 2006.

[3] D. Bar-Natan, *On Khovanov's categorification of the Jones polynomial. Algebr. Geom. Topol.* 2:337–370 (2002).

[4] D. Bar-Natan, *Khovanov's homology for tangles and cobordisms. Geom. Topol.* 9:1443-1499 (2005).

[5] A. Champanerkar & I. Kofman, *Spanning trees and Khovanov homology. Proceedings of the American Mathematical Society* 137.6 (2009): 2157-2167.

[6] M. Khovanov, *A categorification of the Jones polynomial. Duke Math. J.* 101(3):359–426 (2000).

[7] M. Khovanov, *A categorification of the Jones polynomial. Duke Math. J.* 101(3):359–426 (2000).

[8] M. Khovanov, *A functor-valued invariant of tangles. Algebr. Geom. Topol.* 2:665-741 (2002).

[9] P. B. Kronheimer & T. S. Mrowka, *Khovanov homology is an unknot-detector. Publications mathématiques de l'IHÉS* 113.1 (2011): 97-208.

[10] A. D. Lauda & H. Pfeiffer, *Open-closed TQFTS extend Khovanov homology from links to tangles. J. Knot Theory Ramifications* 18(1)87150 (2009).

[11] R. Lipshitz, P. S. Ozsvath, & D. P. Thurston. *Bordered Heegaard Floer homology: Invariance and pairing.* arXiv:0810.0687.

[12] R. Lipshitz, P. S. Ozsvath, & D. P. Thurston. *Bimodules in bordered Heegaard Floer homology.* arXiv:1003.0598 (2010).

[13] R. Lipshitz, P. S. Ozsvath, & D. P. Thurston, *Heegaard Floer homology as morphism spaces.* arXiv:1005.1248.

[14] I. Petkova & V. Vertesi, *Combinatorial tangle Floer homology.* arXiv:1410.2161v1.

[15] E. S. Lee, *An endomorphism of the Khovanov invariant. Adv. Math.* 197(2):554-586 (2005).

[16] O. Viro, *Khovanov homology, its definition and ramifications. Fund. Math.* 184:317–342 (2004).

[17] L. P. Robets, *A type D structure in Khovanov Homology.* arXiv:1304.0463v3.

[18] L. P. Roberts, *A type A structure in Khovanov Homology.* arXiv:1304.0465v3.

[19] L. P. Roberts, *Totally Twisted Khovanov Homology.* arXiv:math.GT/1109.0508.

[20] N. D. Duong & L. P. Roberts, *Twisted skein homology.* arXiv:1209.2967v1.

[21] N. D. Duong, *Twisting bordered Khovanov homology.* arXiv:1403.0959v2

[22] N. D. Duong & L. P. Roberts, *Planar algebra structure in bordered Khovanov homology.* In preparation.

[23] Jaeger, Thomas C, *A remark on Roberts' totally twisted Khovanov Homology. Journal of Knot Theory and Its Ramifications* 22.06 (2013).

[24] M. B. Thistlethwaite, *A spanning tree expansion of the Jones polynomial. Topology* 26.3 (1987): 297-309.

[25] S. Wehrli, *A spanning tree model for Khovanov homology. Journal of Knot Theory and Its Ramifications* 17 (2008), no. 12, 1561-1574.