WOLFF'S IDEAL PROBLEM IN THE MULTIPLIER ALGEBRA ON DIRICHLET SPACE

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ABSTRACT

In an effort to classify ideal membership for finitely-generated ideals in $H^\infty(\mathbb{D})$, Wolff [G] proved that: If

$$\{f_j\}_{j=1}^n \subset H^\infty(\mathbb{D}), H \in H^\infty(\mathbb{D}) \quad \text{and}$$
$$|H(z)| \leq \left(\sum_{j=1}^n |f_j(z)|^2\right)^{\frac{1}{2}} \quad \text{for all } \ z \in \mathbb{D}$$

then

$$H^3 \in \mathcal{I}(\{f_j\}_{j=1}^n), \text{ the ideal generated by } \{f_j\}_{j=1}^n \text{ in } H^\infty(\mathbb{D}).$$

In the first part of this dissertation, we establish an analogue of Wolff's result for the algebra of multipliers on Dirichlet space. In the second part, we extend this result to the multiplier algebra on weighted Dirichlet space with the weight $\alpha \in (0, 1)$.

Finally, we will prove a general ideal problem. We provide the best known sufficient condition for when H itself is in the ideal.

DEDICATION

This dissertation is dedicated to all those who supported me directly and indirectly throughout this journey.

LIST OF SYMBOLS

$\mathbb{C}$	The complex plane
$\mathbb{D}$	The open unit disk in the complex plane $\mathbb{C}$, $\mathbb{D} := \{z \in \mathbb{C} : \|z\| < 1\}$
RKHS	Reproducing Kernel Hilbert Space
k_w	Reproducing Kernel (r.k.)
$\partial\mathbb{D}$	Boundary of the unit disk, $\partial\mathbb{D} := \{z \in \mathbb{C} : \|z\| = 1\}$
$H^\infty(\mathbb{D})$	The space of all bounded analytic functions on $\mathbb{D}$
$B(H)$	The set of bounded linear operators from H to H
$A^\star$	The adjoint of the operator A
$\mathcal{I}\left(\{f_j\}_{j=1}^n\right)$	Ideal generated by $\{f_j\}_{j=1}^n$
$\mathcal{M}$	Maximal Ideal space of $H^\infty(\mathbb{D})$
$\mathcal{M}(\Omega)$	Multiplier Algebra of the RKHS Ω
M_ϕ	Multiplication operator defined by ϕ
$\bar{\partial}$	Partial derivative with respect to $\bar{z}$
$H^2(\mathbb{D})$	Hardy space: the space of all analytic functions on the unit disk whose coefficients are square-summable

$\mathcal{D}$ Dirichlet space: the space of all analytic functions on the unit disk whose coefficients are square - summable with the weight (n+1)

$$\|f\|^2_{\mathcal{D}} = \int_{-\pi}^{\pi} |f(e^{it})|^2 \, d\sigma(t) + \int_{\mathbb{D}} |f'(z)|^2 \, dA(z)$$

$\mathcal{D}_\alpha$ Weighted Dirichlet space: the space of all analytic functions on the unit disk whose coefficients are square - summable with the weight $(n+1)^\alpha$

$$\|f\|^2_{\mathcal{D}_\alpha} = \int_{-\pi}^{\pi} |f(e^{it})|^2 \, d\sigma(t) + \int_{\mathbb{D}} |f'(z)|^2 \left(1 - |z|^2\right)^{1-\alpha} \, dA(z)$$

$A^2(\mathbb{D})$ Bergman space: the space of all analytic functions on the unit disk whose coefficients are square - summable with the weight $\dfrac{1}{n+1}$

$\mathcal{F}(\mathbb{C})$ Fisher's space;

$$\mathcal{F}(\mathbb{C}) := \left\{ f : \mathbb{D} \to \mathbb{C} | \text{ f is entire and } \|f\|^2_{\mathcal{F}(\mathbb{C})} = \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} \frac{dm(z)}{\pi} < \infty \right\}$$

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CONTENTS

CHAPTER 1

PRELIMINARIES

1.1. Reproducing Kernel Hilbert Spaces (RKHS)

In this section, we give the basic definition of reproducing kernel Hilbert spaces and their reproducing kernels (r.k.). Moreover, we will discuss some standard properties of reproducing kernels. Some good references for this section are [**AM**] and [**S**].

1.1.1. Reproducing Kernel Hilbert Space (RKHS). Suppose $H(E)$ is a Hilbert space, whose elements are functions on a set E. Then if

(i) for all $e \in E,$ there exists $c_e < \infty$ so that $|h(e)| \leq c_e ||h||_H$, for all $h \in H$ and

(ii) if $h(e) = 0$ for all $e \in E,$ then $h = 0$ in H,

we say that $H(E)$ is a reproducing kernel Hilbert space on the set E.

1.1.2. Reproducing Kernel (r.k.). Suppose $H(E)$ is a reproducing kernel Hilbert space on a set E. Let e be a fixed element of E, then there exists $c_e < \infty$ s.t. $|h(e)| \leq c_e \|h\|_H$ for all $h \in H(E)$. Define

$$l(h) = h(e) \quad \text{for all} \quad h \in H(E).$$

It's clear that $l : H(E) \to \mathbb{C}$ is a linear functional. Also, by definition,

$$|l(h)| \leq c_e ||h||_H.$$

Hence, l is a bounded linear functional on $H(E)$. Thus, by the Riesz representation theorem there exists a unique $k_e \in H(E)$ so that

$$h(e) = l(h) = \langle h,\, k_e \rangle_{H(E)} \text{ for all } h \in H(E).$$

The function k_e is called the reproducing kernel for the RKHS $H(E)$.

1.1.3. Complete Nevanlinna-Pick Kernel. A r.k., k_w, is called a complete Nevanlinna-Pick kernel if it can be written as $\frac{1}{k_w(z)} = 1 - \sum\limits_{n=1}^{\infty} a_n (z\overline{w})^n$, $a_n > 0$.

1.1.3.1. *Example 1.* The Hardy space $H^2(\mathbb{D})$ with the inner product defined as

$$\langle f,\, g \rangle_{H^2} = \sum_{1}^{\infty} f_n \overline{g_n}, \text{ for all } f,\, g \in H^2(\mathbb{D})$$

is a RKHS with r.k.

$$k_w(z) = \frac{1}{1 - \overline{w}z}, \text{ for all z, w} \in \mathbb{D}.$$

Since $\frac{1}{k_w(z)} = 1 - \overline{w}z$, k_w is a complete Nevanlinna-Pick kernel.

1.1.3.2. *Example 2.* The Dirichlet space $\mathcal{D}$ with the inner product defined as

$$\langle f, g \rangle_{\mathcal{D}} = \sum_{1}^{\infty} (n+1)\, f_n \overline{g_n}, \text{ for all } f,\, g \in \mathcal{D}$$

is a RKHS with r.k.

$$k_w(z) = \frac{1}{z\overline{w}} \log\left(\frac{1}{1 - z\overline{w}}\right), \text{ for all z, w} \in \mathbb{D}.$$

It has been shown, in for example [**AM**], that

$$\frac{1}{k_w(z)} = 1 - \sum_{n=1}^{\infty} a_n (z\overline{w})^n,\ a_n > 0.$$

Thus, the Dirichlet space also has a complete Nevanlinna-Pick kernel.

1.1.3.3. *Example 3.* The weighted Dirichlet space $\mathcal{D}_\alpha$, $\alpha \in (0,1)$, defined as

$$\mathcal{D}_\alpha = \{\, f : \mathbb{D} \to \mathbb{C} \mid f \text{ is analytic on } \mathbb{D} \text{ and for } f(z) = \sum_{n=0}^{\infty} a_n\, z^n,$$

$$\|f\|_{\mathcal{D}_\alpha}^2 = \sum_{n=0}^{\infty} (n+1)^\alpha\, |a_n|^2 < \infty\},$$

with the inner product

$$\langle f, g\rangle_{\mathcal{D}_\alpha} = \sum_{n=0}^{\infty} (n+1)^\alpha\, f_n \overline{g_n}, \text{ for all } f,\, g \in \mathcal{D}_\alpha$$

is a RKHS with r.k.

$$k_w(z) = \sum_{n=0}^{\infty} \frac{1}{(n+1)^\alpha}\, (z\overline{w})^n \text{ for } z,\, w \in \mathbb{D}.$$

It can be seen in [**S**] and in [**KT**] that

$$\frac{1}{k_w(z)} = 1 - \sum_{n=1}^{\infty} c_n (z\overline{w})^n,\ c_n > 0 \text{ for all } n.$$

1.1.3.4. *Example 4.* Some examples of RKHS whose reproducing kernels are not complete Nevanlinna-Pick kernels are:

Bergman space, $A^2(\mathbb{D})$ with the r.k.

$$k_w(z) = \frac{1}{(1-\overline{w}z)^2}, \text{ for all } z, w \in \mathbb{D},$$

$H^2(\mathbb{D}^n)$ with the r.k

$$k_w(z) = \prod_{i=1}^{n} \frac{1}{(1-\overline{w_i}z_i)}, \text{ for all } z, w \in \mathbb{D}^n,$$

$H^2(\mathbb{B}^n)$ with the r.k.

$$k_w(z) = \frac{1}{(1-\overline{w}z)^n}, \text{ for all } z, w \in \mathbb{B}^n$$

l^2 with the r.k.

$$k_n(i) = \delta_{ni} \text{ for } n, i \in N, \text{ where } \delta_{ni} = \begin{cases} 1 & if\, n = i \\ 0 & if\, n \neq i \end{cases}$$

are some examples of RKHS whose reproducing kernels are not complete Nevanlinna-Pick kernels.

1.2. Multiplier Algebra

1.2.1. Definitions. Let $H(\Omega)$ be be a reproducing kernel Hilbert space on Ω. We define the multiplier algebra of $H(\Omega)$ as

$$\mathcal{M}(H(\Omega)) := \{\phi \in H(\Omega) : \phi f \in H(\Omega) \text{ for all } f \in H(\Omega)\}.$$

Similarly, we use M_ϕ to denote the multiplication operator and define as

$$M_\phi(f) = \phi f, \text{ for all } f \in H(\Omega).$$

By the closed graph theorem, it is clear that $M_\phi \in B(H(\Omega))$.

LEMMA 1.2.1. *Let $H(\Omega)$ be a RKHS. If M_ϕ is a multiplication operator and k_w is a r.k. for the RKHS $H(\Omega)$, then*

$$M_\phi^\star(k_w) = \overline{\phi(w)} k_w.$$

PROOF. Let $f \in H(\Omega)$, then

$$\langle f, M_\phi^\star k_w\rangle = \langle M_\phi f, k_w\rangle = \langle \phi f, k_w\rangle = \phi(w) f(w).$$

Therefore,

$$\langle f, M_\phi^\star k_w\rangle = \phi(w)\langle f, k_w\rangle = \langle f, \overline{\phi(w)}\, k_w\rangle.$$

This is true for all $f \in H(\Omega)$. Hence, $M_\phi^\star(k_w) = \overline{\phi(w)} k_w$. □

Using this lemma, we can see that that

$$|\phi(w)| \leq ||M_\phi||_{B(H)} \text{ for all z}\in\mathbb{D}.$$

Thus, if ϕ is a multiplier on $H(\Omega)$, then ϕ is bounded on Ω.

This implies that if $H(\mathbb{D})$ is a RKHS of analytic functions, then

$$\mathcal{M}(H(\mathbb{D})) \subset H^\infty(\mathbb{D}).$$

LEMMA 1.2.2. *Show that* $\mathcal{M}(H^2(\mathbb{D})) = H^\infty(\mathbb{D})$ *and* $||M_\phi||_{B(H^2(\mathbb{D}))} = ||\phi||_{\infty,\mathbb{D}}$.

PROOF. Let $\phi \in \mathcal{M}(H^2(\mathbb{D})$, then from above discussion we have that

$$||\phi||_{\infty,\mathbb{D}} \leq ||M_\phi||_{B(H^2(\mathbb{D}))} < \infty.$$

Thus, $\mathcal{M}(H^2(\mathbb{D})) \subseteq H^\infty(\mathbb{D})$.

Conversely, we will show that

$$H^\infty(\mathbb{D}) \subseteq \mathcal{M}(H^2(\mathbb{D})) \quad \text{and} \quad ||M_\phi||_{B(H^2(\mathbb{D}))} \leq ||\phi||_{\infty,\mathbb{D}}.$$

Let $\psi \in H^\infty(\mathbb{D})$ and $p \in H^2(\mathbb{D})$, then

$$\begin{aligned} ||M_\psi(p)||^2_{H^2(\mathbb{D})} &= \lim_{r\uparrow 1} \int_{-\pi}^{\pi} |(\psi p)(re^{it})|^2 \frac{d\sigma(t)}{2\pi} \\ &= \lim_{r\uparrow 1} \int_{-\pi}^{\pi} |\psi(re^{it})|^2 |p(re^{it})|^2 \frac{d\sigma(t)}{2\pi} \\ &\leq ||\psi||^2_{\infty,\mathbb{D}} \overline{\lim_{r\uparrow 1}} \int_{-\pi}^{\pi} |p(re^{it})|^2 \frac{d\sigma(t)}{2\pi} \\ &= ||\psi||^2_{\infty,\mathbb{D}} \overline{\lim_{r\uparrow 1}} \left(\sum_{n=0}^{\infty} |p_n|^2 r^{2n} \right) \\ &= ||\psi||^2_{\infty,\mathbb{D}} \, ||p||^2_{H^2(\mathbb{D})}. \end{aligned}$$

Hence, $\psi \in \mathcal{M}(H^2(\mathbb{D}))$ and $||M_\psi\|_{B(H^2(\mathbb{D}))} \leq \|\psi||_{\infty,\mathbb{D}}$. □

A similar argument shows that $\mathcal{M}(A^2(\mathbb{D})) = H^\infty(\mathbb{D})$.

1.2.2. Shift Operator. The shift operator M_z, multiplication by z, is defined as

$$M_z f(w) = zf(w) \text{ for all } f \in H(\Omega).$$

Concerning the boundness of M_z on $H(\Omega)$ the general answer is no. Let $\{\alpha_n\}_{n=0}^{\infty}$, $\alpha_n >$ for all n, $\lim_{n\to\infty} \alpha_n^{\frac{1}{n}} \geq 1$. Form

$$\mathcal{H}_{\{\alpha\}}(\mathbb{D}) = \left\{ f : \mathbb{D} \to \mathbb{C} \mid \text{ f is analytic on } \mathbb{D} \text{ and } \sum_0^\infty |f_n|^2 \alpha_n < \infty \right\}.$$

FACT 1.2.1. *M_z is bounded on $\mathcal{H}_{\{\alpha\}}(\mathbb{D}) \iff \sup \left\{ \frac{\alpha_{n+1}}{\alpha_n} \right\} < \infty$ and $||M_z|| = \sup_n \sqrt{\frac{\alpha_{n+1}}{\alpha_n}}$.*

For different values of α_n, we can observe that

$$\begin{aligned}
\alpha_n &= 1, \quad H(\mathbb{D}), \quad \sqrt{\frac{\alpha_{n+1}}{\alpha_n}} = 1 \\
\alpha_n &= \frac{1}{n+1}, \quad A^2(\mathbb{D}), \quad \sqrt{\frac{\alpha_{n+1}}{\alpha_n}} \nearrow 1 \\
\alpha_n &= (n+1), \quad \mathcal{D}, \quad \sqrt{\frac{\alpha_{n+1}}{\alpha_n}} \nearrow 1 \\
\alpha_n &= n!, \quad \mathcal{F}(\mathbb{C}), \quad \sqrt{\frac{\alpha_{n+1}}{\alpha_n}} \nearrow \infty.
\end{aligned}$$

Thus, M_z is not bounded on $\mathcal{F}(\mathbb{C})$. That means, $z \notin \mathcal{M}(\mathcal{F}(\mathbb{C}))$.

1.2.3. Multipliers of Dirichlet Space. At the beginning of this section, we defined the multiplier algebra of general reproducing kernel Hilbert spaces and discussed some basic properties. Since our main work is on Dirichlet space, we now concentrate more on its multiplier algebra.

An analytic function ϕ on the unit disk $\mathbb{D}$ is a multiplier of Dirichlet space if and only if the pointwise multiplication operator M_ϕ, defined by ϕ, is a bounded linear operator from $\mathcal{D}$ to $\mathcal{D}$. If $\phi \in \mathcal{M}(\mathcal{D})$, then $\phi \in \mathcal{D}$, since $1 \in \mathcal{D}$. Also from the fact above, we saw that $z \in \mathcal{M}(\mathcal{D})$. Thus, $\mathbb{C} \subsetneq \mathcal{M}(\mathcal{D}) \subseteq H^\infty(\mathbb{D})$. However, $\mathcal{M}(\mathcal{D}) \neq H^\infty(\mathbb{D})$. For example, if we take $f(z) = \sum\limits_{n=1}^{\infty} \frac{z^{n^3}}{n^2}$, then it's clear that $f \in H^\infty(\mathbb{D})$. But

$$\|f\|_{\mathcal{D}}^2 = \sum_{n=1}^{\infty} (n+1)\,|a_n|^2 = \frac{2}{1^4} + \frac{9}{2^4} + \frac{28}{3^4} + ... \geq \frac{1}{1^4} + \frac{8}{2^4} + \frac{27}{3^4} + ... = \sum_{n=1}^{\infty} \frac{1}{n},$$

which is a divergent series. So, $f \notin \mathcal{D}$ and thus not in $\mathcal{M}(\mathcal{D})$. Hence, $\mathcal{M}(\mathcal{D}) \subsetneq H^\infty(\mathbb{D}) \cap \mathcal{D}$. Therefore, $\mathbb{C} \subsetneq \mathcal{M}(\mathcal{D}) \subsetneq H^\infty(\mathbb{D})$.

1.2.4. Carleson Measure. A positive Borel measure μ on the unit disk $\mathbb{D}$ is a Carleson measure for $\mathcal{D}$ if there exists a constant $C > 0$ such that

$$\int_{\mathbb{D}} |g|^2 d\mu \leq C^2 \,||g||_{\mathcal{D}}^2 \text{ for all } g \in \mathcal{D}.$$

LEMMA 1.2.3. *Let* $\phi \in H^\infty(\mathbb{D})$, *then* $\phi \in \mathcal{M}(\mathcal{D})$ *if and only if* $|\phi'|^2 dA$ *is Carleson measure on* $\mathbb{D}$ *with the constant* $4||M_\phi||_{B(\mathcal{D})}^2$.

PROOF. To prove the lemma, we will show that

$$\int_{\mathbb{D}} |\phi'|^2 |g|^2 dA \leq 4||M_\phi||_{B(\mathcal{D})}^2 ||g||_{\mathcal{D}}^2 \text{ for all } g \in \mathcal{D}.$$

Let $g \in \mathcal{D}$, then

$$\begin{aligned}
\int_{\mathbb{D}} |\phi'|^2 |g|^2 dA &= \int_{\mathbb{D}} |\,(\phi g)' - \phi g'|^2 dA \\
&\leq 2 \int_{\mathbb{D}} |\,(\phi g)'\,|^2 dA + 2 \int_{\mathbb{D}} |\phi g'|^2 dA \\
&\leq 2 \int_{\mathbb{D}} |\phi g|^2 d\sigma + 2 \int_{\mathbb{D}} |\,(\phi g)'\,|^2 dA + 2 \int_{\mathbb{D}} |\phi g'|^2 dA. \\
&\leq 2\|\phi\|_{H^\infty(\mathbb{D})} \|g\|_{\mathcal{D}}^2 + 2\|\phi g\|_{\mathcal{D}}^2 \\
&\leq 4||M_\phi||_{B(\mathcal{D})}^2 ||g||_{\mathcal{D}}^2 \text{ for all } g \in \mathcal{D}.
\end{aligned}$$

Conversely, suppose that $|\phi'|^2 dA$ is Carleson measure on $\mathbb{D}$. Then we need to show that $\phi \in \mathcal{M}(\mathcal{D})$. For this we will show that $\phi h \in \mathcal{D}$ for all $h \in \mathcal{D}$. Now,

$$\begin{aligned} ||\phi h||_{\mathcal{D}}^2 &= \int_{-\pi}^{\pi} |\phi h|^2 d\sigma + \int_{\mathbb{D}} |(\phi h)'|^2 dA \\ &\leq ||\phi||_\infty \int_{-\pi}^{\pi} |h|^2 d\sigma + 2\int_{\mathbb{D}} |\phi h'|^2 dA + 2\int_{\mathbb{D}} |\phi' h|^2 dA \\ &\leq 2||\phi||_\infty ||h|_{\mathcal{D}}^2 + 2C||h||_{\mathcal{D}}^2 \end{aligned}$$

Therefore,

$$||\phi h||_{\mathcal{D}}^2 \leq (2||\phi||_\infty + 2C)\, ||h||_{\mathcal{D}}^2 < \infty$$

Thus, $\phi h \in \mathcal{D}$ for all $h \in \mathcal{D}$. □

1.3. Maximal Ideals and the Carleson's Corona Theorem in $H^\infty(\mathbb{D})$

In this section, we will discuss the space of maximal ideals of $H^\infty(\mathbb{D})$ and then state Carleson's corona theorem which proves that the open unit disk is dense in the maximal ideal space of $H^\infty(\mathbb{D})$. Moreover, we will talk about some extensions of Carleson's theorem. Also, we will point out the best estimates on the corona solutions. Useful references for this section are [**C**] and [**H**].

1.3.1. Maximal Ideal Space of $H^\infty(\mathbb{D})$. If we use pointwise addition and multiplication, together with the norm

$$||f||_\infty = \sup_{|z|<1} |f(z)|,$$

$H^\infty(\mathbb{D})$ is a commutative Banach algebra with identity 1. It's worthwhile to note that every ideal in $H^\infty(\mathbb{D})$ is contained in a maximal ideal of $H^\infty(\mathbb{D})$ and every maximal ideal in $H^\infty(\mathbb{D})$ is closed in the norm topology.

Let $\mathcal{M}$ denote the set of all multiplicative linear functionals (mlf) on $H^\infty(\mathbb{D})$;

$$\mathcal{M} := \{\varphi : \varphi \quad \text{is a mlf on } H^\infty(\mathbb{D}), \varphi \neq 0\}\,.$$

Using a Banach algebra argument, we can see that there is one to one correspondence between the maximal ideal space of $H^\infty(\mathbb{D})$ and $\mathcal{M}$. With each element f in $H^\infty(\mathbb{D})$, we can associate a complex valued function $\widehat{f}$ on $\mathcal{M}$ by

$$\widehat{f}(\phi) = \phi(f).$$

From the Banach-Alaoglu Theorem, it follows that $\mathcal{M}$ is a compact Hausdorff space when it is equipped with the weak-star topology.

Each $\widehat{f}$ is a continuous function on $\mathcal{M}$; indeed, by definition, a weak-star topology is the weakest topology on $\mathcal{M}$ which makes each $\widehat{f}$ continuous. Let $\widehat{H}^\infty(\mathbb{D})$ denote the set of all $\widehat{f}$, then we can construct a representation

$$f \to \widehat{f}$$

from $H^\infty(\mathbb{D})$ onto $\widehat{H}^\infty(\mathbb{D})$, an algebra of continuous complex-valued functions on $\mathcal{M}$. This is usually called the Gelfand representation. It's easy to see that

$$||\widehat{f}||_\infty \leq ||f||.$$

We now want to explore some of the structure of this space $\mathcal{M}$. What are the maximal ideals (complex homomorphisms) of $H^\infty(\mathbb{D})$? It seems clear that no concrete answer could ever be given, but Hoffman [**H**] has given some structures of $\mathcal{M}$ as follows: There are many complex homomorphisms of $H^\infty(\mathbb{D})$ but the only obvious complex homomorphisms of $H^\infty(\mathbb{D})$ are the point evaluations

$$\phi_\lambda(f) = f(\lambda),$$

where λ is a point in the open unit disk. Note that the point evaluations ϕ_λ show that the Gelfand representation $f \to \widehat{f}$ is one-one. Furthermore, we can see that the representation is isometric:

$$||f|| = ||\widehat{f}||_\infty = \sup_{\phi \in \mathcal{M}} |\widehat{f}(\phi)|.$$

We know that $||\widehat{f}||_\infty \leq ||f||$, and the ϕ_λ tells us that

$$||\hat{f}||_\infty \geq \sup_{|\lambda|<1} |\hat{f}(\phi_\lambda)| = ||f||.$$

Thus, $H^\infty(\mathbb{D})$ is isometrically isomorphic to $\widehat{H}^\infty(\mathbb{D})$, a uniformally closed subalgebra of the continuous complex-valued functions on the maximal ideal space $\mathcal{M}$.

Lemma 1.3.1. *The mapping π defined as $\pi(\phi) = \phi(z)$, $\phi \in \mathcal{M}$ is a continuous map of $\mathcal{M}$ onto the closed unit disk in the plane. Over the open unit disk $\mathbb{D}$, π is one to one, and π^{-1} maps $\mathbb{D}$ homeomorphically onto an open subset $\triangle$ of $\mathcal{M}$.*

It is convenient to picture π as a projection of $\mathcal{M}$ onto the closed disk. As we saw in the previous Lemma, π is one to one over $\mathbb{D}$, so the open unit disk is homeomorphically embedded in $\mathcal{M}$ by $\lambda \to \phi_\lambda$. The remainder of $\mathcal{M}$ is mapped by π onto the unit circle. If $|\alpha| = 1$, we shall call $\pi^{-1}(\alpha)$ the fiber of $\mathcal{M}$ over α and denote this fiber by $\mathcal{M}_\alpha$:

$$\mathcal{M}_\alpha := \pi^{-1}(\alpha) = \{\phi \in \mathcal{M} : \phi(z) = \alpha\}.$$

The fiber $\mathcal{M}_\alpha$ is a closed subset of $\mathcal{M}$ and consists of the homomorphisms of $H^\infty(\mathbb{D})$ which resemble evaluation at α.

Lemma 1.3.2. *If f is a function in $H^\infty(\mathbb{D})$ and α is a point of the unit circle, then the range of $\hat{f}$ on the fiber $\mathcal{M}_\alpha$ consists of all complex numbers ζ for which there is a sequence of points λ_n in $\mathbb{D}$ with $\lim_n \lambda_n = \alpha$ and $\lim_n f(\lambda_n) = \zeta$.*

Detailed proofs of these lemmas can be found in [**H**].

We can now clearly tell that the point evaluations ϕ_λ embed the open unit disk as an open subset $\triangle$ of $\mathcal{M}$. The remaining homomorphisms lie in the fibers $\mathcal{M}_\alpha$ and, as the last lemma shows, are in some sense limits of the points of $\triangle$. The natural question arises here: are these homomorphisms actually limits of the ϕ_λ in the topology of $\mathcal{M}$?

We define the term corona as the set $\mathcal{M} \setminus \overline{\mathbb{D}}$. In 1941, Kakutani asked whether the maximal ideal space $\mathcal{M}$ of $H^\infty(\mathbb{D})$ has a nontrivial corona.

Although it seems to be quite abstract, it is easily translated into a very concrete question about bounded analytic functions.

THEOREM 1.3.1. *A necessary and sufficient condition that the open unit disk* $\mathbb{D}$ *is dense in* $\mathcal{M}$ *is the following:*

If $f_1,\ f_2, ..., f_n$ *are bounded analytic functions in the open unit disk* $\mathbb{D}$ *such that*

$$|f_1(z)|^2 + ... + |f_n(z)|^2 \geqq \delta > 0, \ \ \text{for all} \ \ z \in \mathbb{D}. \tag{1.3.1}$$

Then there exist bounded analytic functions $g_1,\ g_2,\ ..., g_n$ *such that*

$$f_1(z)g_1(z) + ... + f_n(z)g_n(z) = 1, \ \ \text{for all} \ \ z \in \mathbb{D}.$$

In other words $1 \in \mathbb{I}(f_1, ..., f_n)$*, the ideal generated by* $f_1, ..., f_n$*.*

1.3.2. Corona Theorem. In 1962, Carleson [**C**] proved his famous "Corona Theorem" characterizing when a finitely generated ideal in $H^\infty(\mathbb{D})$ is actually all of $H^\infty(\mathbb{D})$. In other words, Carleson proved that the open unit disk is dense in the maximal ideal space of $H^\infty(\mathbb{D})$.

Carleson's Corona Theorem. *If* $f_1,\ f_2, ..., f_n$ *are bounded analytic functions in the open unit disk* $\mathbb{D}$ *such that*

$$|f_1(z)|^2 + ... + |f_n(z)|^2 \geqq \delta^2 > 0, \ \ \text{for all}\ z \in \mathbb{D}. \tag{1.3.2}$$

Then there exist bounded analytic functions $g_1,\ g_2,\ ...,g_n$ such that

$$f_1(z)g_1(z)+...+f_n(z)g_n(z)=1 \text{ for all } z\in\mathbb{D}.$$

Moreover, he provided estimates on the corona solutions.

Using Wolff's approach for solving the corona problem, Uchiyama improved these estimates to get the best known estimates

$$\sup_{z\in\mathbb{D}}\left\{\sum_{j=1}^{n}|g_j(z)|^2\right\}\leq\left(\frac{10}{\delta^2}\ln\frac{1}{\delta^2}\right)^2.$$

Independently, Rosenblum [R], Tolokonnikov [To], and Uchiyama gave an infinite version of Carleson's work on $H^\infty(\mathbb{D})$. Moreover, Fuhrmann proved that the matricial corona theorem holds for a matix of order $m\times n,\ m\,,\,n<\infty$, and later Vasyunin extended this result to the case $m\times\infty,\ m<\infty$. Trent [**Tr3**] gave the best estimates for the matricial corona theorem. As for the general corona theorem, it was shown by Treil [**T2**] that it fails for a matrix of order $\infty\times\infty$. Trent and Zhang proved that the result of Vasyunin can be extended to any algebra where the Corona Theorem holds.

1.4. Wolff's Ideal Problem in $H^\infty(\mathbb{D})$

The famous Carleson's Corona Theorem implies that if the functions $f_1,\ f_2,...,f_n\in H^\infty(\mathbb{D})$ satisfy

$$|f_1(z)|^2+...+|f_n(z)|^2\geqq 1 \text{ for all } z\in\mathbb{D},$$

then 1 belongs to the ideal generated by $f_1,\ f_2,\ ...,\ f_n$.

One can try to generalize this result, replacing 1 by an arbitrary function $h\in H^\infty(\mathbb{D})$. Namely, let $f_1,\ f_2,...,f_n\in H^\infty(\mathbb{D})$, and suppose $h\in H^\infty(\mathbb{D})$ satisfies

$$|f_1(z)|^2+...+|f_n(z)|^2\geqq|h(z)|^2 \text{ for all } z\in\mathbb{D}. \tag{1.4.1}$$

In light of the Corona Theorem it is natural to ask if (1.4.1) implies $h \in \mathcal{I}\left(\{f_j\}_{j=1}^n\right)$, the ideal generated by $\{f_j\}_{j=1}^n$. Note that the condition (1.4.1) is clearly a necessary condition.

An example by Rao [**G**] (Chapter VIII) shows that the answer is negative. However, it is plausible to ask if some power of h belongs to $h \in \mathcal{I}\left(\{f_j\}_{j=1}^n\right)$. It was realized in the early of $1880'$s that this condition is sufficient for $h^p \in \mathcal{I}(\{f_j\}_{j=1}^n)$ if $p > 2$ and it is not sufficient if $p < 2$.

In an effort to classify ideal membership for finitely-generated ideals in $H^\infty(\mathbb{D})$, Wolff [G] proved the following version:

Theorem [Wolff]. *If*

$$\{f_j\}_{j=1}^n \subset H^\infty(\mathbb{D}), H \in H^\infty(\mathbb{D}) \quad \textit{and}$$

$$|H(z)| \leq \left(\sum_{j=1}^n |f_j(z)|^2\right)^{\frac{1}{2}} \quad \textit{for all } z \in \mathbb{D}, \tag{1.4.2}$$

then

$$H^3 \in \mathcal{I}(\{f_j\}_{j=1}^n),$$

the ideal generated by $\{f_j\}_{j=1}^n$ *in* $H^\infty(\mathbb{D})$.

But the question for $p = 2$ remained open for almost 20 years, until Treil [**T1**] showed that for $p = 2$ the condition (1.4.2) is not sufficient either.

CHAPTER 2

WOLFF'S IDEAL PROBLEM IN $\mathcal{M}(\mathcal{D})$

In this chapter, we will extend the Wolff's Ideal Theorem in $H^\infty(\mathbb{D})$ to the multiplier algebra on Dirichlet space. For the algebra of multipliers on Dirichlet space, the analogue of the corona theorem was established in Tolokonnikov [**To**] and, for infinitely many generators, this was done in Trent [**Tr2**].

2.1. Wolff's Theorem for $\mathcal{M}(\mathcal{D})$

THEOREM 1. *Let $H, \{f_j\}_{j=1}^\infty \subset \mathcal{M}(\mathcal{D})$. Assume that*

$$\text{(a) } \|M_F^C\| \leq 1$$

$$\textit{and} \quad \text{(b) } |H(z)| \leq \sqrt{\sum_{j=1}^\infty |f_j(z)|^2} \quad \textit{for all } z \in \mathbb{D}.$$

Then there exist $\{g_j\}_{j=1}^\infty \subset \mathcal{M}(\mathcal{D})$ with

$$\|M_G^C\| < \infty$$

$$\textit{and} \quad F\,G^T = H^3.$$

We will use other equivalent norms for smooth functions in $\mathcal{D}$ as follows:

$$\|f\|_{\mathcal{D}}^2 = \int_{-\pi}^{\pi} |f|^2 d\sigma + \int_{\mathbb{D}} |f'(z)|^2 \, dA(z) \qquad \text{and}$$

$$\|f\|_{\mathcal{D}}^2 = \int_{-\pi}^{\pi} |f|^2 d\sigma + \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^2} \, d\sigma(t) d\sigma(\theta).$$

Also, we will consider $\overset{\infty}{\underset{1}{\oplus}} \mathcal{D}$ as an l^2-valued Dirichlet space. The norms in this case are exactly as above but we will replace the absolute value by l^2-norms. Moreover, we use $\mathcal{HD}$ to denote the harmonic Dirichlet space (restricted to the boundary of $\mathbb{D}$). The functions in $\mathcal{D}$ have only vanishing negative Fourier coefficients, whereas the functions in $\mathcal{HD}$ may have negative Fourier coefficients which do not vanish. Again, if f is smooth on ∂D, the boundary of the unit disk D, then

$$\|f\|^2_{\mathcal{HD}} = \int_{-\pi}^{\pi} |f|^2 \, d\sigma + \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^2} \, d\sigma(t) d\sigma(\theta).$$

2.1.1. Cauchy Transform. If $f \in C'(\mathbb{D})$, smooth on $\overline{\mathbb{D}}$, then the Cauchy transform of f is defined as

$$\widehat{f}(z) := -\frac{1}{\pi} \int_{\mathbb{D}} \frac{f(w)}{w - z} dA(w).$$

By the linear change of variables and by using Cauchy's formula we can show that $\overline{\partial} \widehat{f}(z) = f(z)$, for all $z \in \mathbb{D}$. See [A] for background on the Cauchy transform.

2.1.2. Beurling Transform. If ϕ is in $C^1(\overline{\mathbb{D}})$ and $\widehat{\phi}$ is the Cauchy transform of ϕ on $\mathbb{D}$, then we we define the Beurling transform by

$$\mathcal{B}(\phi) := \partial(\widehat{\phi}).$$

2.2. Row and Column Operators

Given $\{f_j\}_{j=1}^{\infty} \subset \mathcal{M}(\mathcal{D})$, we consider $F(z) = (f_1(z), f_2(z), \dots)$ for $z \in D$. We define the row operator $M_F^R : \overset{\infty}{\underset{1}{\oplus}} \mathcal{D} \to \mathcal{D}$ by

$$M_F^R \left(\{h_j\}_{j=1}^{\infty} \right) = \sum_{j=1}^{\infty} f_j h_j \text{ for } \{h_j\}_{j=1}^{\infty} \in \overset{\infty}{\underset{1}{\oplus}} \mathcal{D}.$$

Similarly, we define the column operator $M_F^C : \mathcal{D} \to \overset{\infty}{\underset{1}{\oplus}} \mathcal{D}$ by

$$M_F^C (h) = \{f_j h\}_{j=1}^{\infty} \text{ for } h \in \mathcal{D}.$$

Also, we define the analytic Toeplitz operators T_F^R and T_F^C on $\bigoplus_1^\infty H^2(\mathbb{D})$ and $H^2(\mathbb{D})$ in analogy to that of M_F^R and M_F^C. The norm of these Toeplitz operators are defined as

$$\left\|T_F^R\right\| = \left\|T_F^C\right\| = \sup_{z\in\mathbb{D}} \left(\sum_{j=1}^{\infty} |f_j(z)|^2\right)^{\frac{1}{2}}.$$

Also, it is worthwhile to note that the pointwise hypothesis $F(z)\,F(z)^\star \leq 1$, for $z \in \mathbb{D}$, implies that the analytic Toeplitz operators T_F^R and T_F^C are bounded. But, since $M(\mathcal{D}) \subsetneq H^\infty(\mathbb{D})$, the pointwise upper bound hypothesis will not be sufficient to conclude that M_F^R and M_F^C are bounded on Dirichlet space. However, $\left\|M_F^R\right\| \leq \sqrt{18}\left\|M_F^C\right\|$ from [**T2**].

2.3. Outline of the Proof of Theorem 1.

In this section, we will outline the method of our proof and give a detailed proof of some required lemmas. We will state some lemmas whose proofs can be found in the references.

Assume that $F \in \mathcal{M}_{l^2}(\mathcal{D})$ and $H \in \mathcal{M}(\mathcal{D})$ satisfy the hypotheses (a) and (b) of Theorem 1. Then we show that there exists a constant $K < \infty$, so that

$$M_{H^3}\, M_{H^3}^\star \leq K^2 M_F^R\, M_F^{\star R}. \tag{2.3.1}$$

Given (2.3.1), a commutant lifting theorem argument as it appears in, for example, Trent [Tr2] completes the proof by providing a $G \in \mathcal{M}_{l^2}(\mathcal{D})$ so that $\|M_G^C\| \leq K$ and $F\,G^T = H^3$.

Lemma 2.3.1. *There exists a constant $K < \infty$ so that, for any $h \in \mathcal{D}$, there exists $\underline{u}_h \in \bigoplus_1^\infty \mathcal{D}$ such that*

$$\text{(i)}\ \ M_F^R(\underline{u}_h) = H^3 h \qquad \textit{and}$$

$$\text{(ii)}\ \ \|\underline{u}_h\|_{\mathcal{D}} \leq K\,\|h\|_{\mathcal{D}} \tag{2.3.2}$$

if and only if

$$M_{H^3}\,M_{H^3}^{\star} \leq K^2 M_F^R\,M_F^{\star R}. \tag{2.3.3}$$

PROOF. Assume that there exists $K < \infty$ such that $M_H\,M_H^{\star} \leq K^2 M_F^R\,M_F^{\star R}$. Then by Douglas lemma, there exists a $C \in B\left(\mathcal{D}, \overset{\infty}{\underset{1}{\oplus}}\mathcal{D}\right)$ such that $M_F^R\,C = M_H$ and $||C|| \leq K$.

For $h \in \mathcal{D}$, let $\underline{u}_h = C(h)$. Then $\underline{u}_h \in \mathcal{D}$ and $M_F^R\,(\underline{u}_h) = M_H h = Hh$. Also, $\|\underline{u}_h\|_{\mathcal{D}} \leq K\,\|h\|_{\mathcal{D}}$. Hence, (2.3.2) follows.

Conversely, given (2.3.2), let $\underline{v}_h = P_{\left(KerM_F^R\right)^{\perp}}(\underline{u}_h)$. Since $M_F^R\,(\underline{v}_h) = Hh$ and $\|\underline{v}_h\|_{\mathcal{D}} \leq \|\underline{u}_h\|_{\mathcal{D}} \leq K\,\|h\|_{\mathcal{D}}$, (i) and (ii) hold for $\underline{v}_h$.

Define a densely defined operator, C, by $C\,(h) = \underline{v}_h$. So

$$||C(h)||_{\mathcal{D}} \leq ||\underline{v}_h||_{\mathcal{D}} \leq K\,||h||_{\mathcal{D}}.$$

Thus, C extends to an operator bounded by K. By (2.3.2) (i) we have $M_F^R C = M_H$. Therefore,

$$M_H M_H^{\star} = M_F^R C C^{\star} M_F^{\star R} \leq M_F^R K^2 M_F^{\star R} \leq K^2 M_F^R M_F^{\star R}$$

and thus (2.3.3) holds. □

Hence, our goal is to establish (2.3.2) from (a) and (b) rather than (2.3.1). For this we need a series of lemmas.

LEMMA 2.3.2. *Let* $\{c_j\}_{j=1}^{\infty} \in l^2$ *and* $C = (c_1, c_2, ...) \in B\,(l^2, \mathbb{C})$. *Then there exists* Q *such that the entries of* Q *are either* 0 *or* $\pm c_j$ *for some* j *and* $CC^{\star}I - C^{\star}C = QQ^{\star}$. *Also, range of* Q = *kernel of* C.

We will apply this lemma in our case with $C = F(z)$ for each $z \in \mathbb{D}$, when $F(z) \neq 0$. A proof of a more general version can be found in Trent [**Tr3**].

Given condition (b) of Theorem 1 for all $z \in \mathbb{D}$, $F \in \mathcal{M}_{l^2}(\mathcal{D})$ and $H \in \mathcal{M}(\mathcal{D})$, with H being not identically zero, we lose no generality assuming that $H(0) \neq 0$. If $H(0) = 0$, but $H(a) \neq 0$, let $\beta(z) = \frac{a-z}{1-\bar{a}\,z}$ for $z \in \mathbb{D}$. Then since (b) holds for all $z \in \mathbb{D}$, it holds for $\beta(z)$. So we may replace H and F by $H \circ \beta$ and $F \circ \beta$, respectively. If we prove our theorem for $H \circ \beta$ and $F \circ \beta$, then there exists $G \in \mathcal{M}_{l^2}(\mathcal{D})$ so that $(F \circ \beta)\, G = H \circ \beta$. Hence $F(G \circ \beta^{-1}) = H$ and $G \circ \beta^{-1} \in \mathcal{M}_{l^2}(\mathcal{D})$, so we were done. Thus, we may assume that $H(0) \neq 0$ in (b), so $\|F(0)\|_2 \neq 0$. This normalization will let us apply some relevant lemmas from [**Tr1**].

It suffices to establish (i) and (ii) for any dense set of functions in $\mathcal{D}$, so we will use polynomials. First, we will assume F and H are analytic on $\mathbb{D}_{1+\epsilon}(0)$. In this case, we write the most general solution of the pointwise problem on $\overline{\mathbb{D}}$ and find an analytic solution with uniform bounds. Then we remove the smoothness hypotheses on F and H.

For a polynomial, h, we take

$$\underline{u}_h(z) = F(z)^\star \left(F(z)F(z)^\star\right)^{-1} H^3\, h - Q(z)\underline{k}(z), \text{ where } \underline{k}(z) \in l^2 \text{ for } z \in \overline{\mathbb{D}}.$$

We have to find $\underline{k}(z)$ so that $\underline{u}_h \in \overset{\infty}{\underset{1}{\oplus}} \mathcal{D}$. Thus we want $\bar{\partial}_z\, \underline{u}_h = 0$ in $\mathcal{D}$. That is, we want $\underline{k} \in l^2$ satisfying $\langle \frac{F^\star Hh}{FF^\star} - Q\underline{k}, \overline{\underline{p}} \rangle_{H^2} = 0$ for all vector polynomial $\underline{p}$, such that $\underline{p}(0) = 0$. This gives us that $\underline{k} = \widehat{\frac{Q^\star F'^\star Hh}{(FF^\star)^2}}$, where $\widehat{k}$ is the Cauchy transform of k on $\mathbb{D}$. Therefore, we will try

$$\underline{u}_h = \frac{F^\star H^3 h}{FF^\star} - Q\, \widehat{W},$$

where

$$W = \frac{Q^\star F'^\star H^3 h}{\left(FF^\star\right)^2}$$

and $\widehat{W}$ is the Cauchy transform of W on $\mathbb{D}$. Note that for k smooth on $\overline{\mathbb{D}}$ and $z \in \mathbb{D}$,

$$\widehat{\underline{k}}(z) = -\frac{1}{\pi}\int_{\mathbb{D}} \frac{\underline{k}(w)}{w-z}\, dA(w) \text{ and } \bar{\partial}\,\widehat{\underline{k}}(z) = k(z) \text{ for } z \in \mathbb{D}.$$

Then it's clear that $M_F^R(\underline{u}_h) = H^3 h$ and $\underline{u}_h$ is analytic. Hence, we will be done in the smooth case if we are able to find $K < \infty$, independent of the polynomial h, and $\epsilon > 0$, such that

$$\|\underline{u}_h\|_{\mathcal{D}} \leq K \|h\|_{\mathcal{D}}.$$

LEMMA 2.3.3. *Let* $\underline{w}$ *be a harmonic function on* $\overline{\mathbb{D}}$*. Then*

$$\int_{\mathbb{D}} \|Q'\underline{w}\|_{l^2}^2\, dA \leq 8\, \|\underline{w}\|_{\mathcal{HD}}^2.$$

PROOF. Let $\underline{w}$ be a vector-valued harmonic function on $\overline{\mathbb{D}}$. Write $\underline{w} = \underline{x} + \underline{\bar{y}}$, where $\underline{x}$ and $\underline{\bar{y}}$ are respectively the analytic and co-analytic parts of $\underline{w}$. We have

$$\begin{aligned}\int_{\mathbb{D}} \|Q'\underline{w}\|_{l^2}^2\, dA &= \int_{\mathbb{D}} \|Q'\underline{x} + Q'\underline{\bar{y}}\|_{l^2}^2\, dA \\ &\leq 2\int_{\mathbb{D}} \|Q'\underline{x}\|_{l^2}^2\, dA + 2\int_{\mathbb{D}} \|Q'\underline{\bar{y}}\|_{l^2}^2\, dA.\end{aligned}$$

Now,

$$\begin{aligned}\int_{\mathbb{D}} \|Q'\underline{x}\|_{l^2}^2\, dA &= \int_{\mathbb{D}} \langle Q'^{\star}Q'\underline{x}\,,\, \underline{x}\rangle_{l^2} dA \\ &\leq \int_{\mathbb{D}} \langle F'F'^{\star}\underline{x},\, \underline{x}\rangle_{l^2} dA \\ &\leq \int_{\mathbb{D}} \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} |\bar{f}_j' x_k|^2\, dA \\ &\leq 2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \|M_{f_j} x_k\|_{\mathcal{D}}^2 + 2\sum_{k=1}^{\infty} \|x_k\|_{\mathcal{D}}^2\, dA \\ &\leq 2\sum_{k=1}^{\infty} \|M_F^C\|_{\mathcal{D}}^2\, \|x_k\|_{\mathcal{D}}^2 + 2\sum_{k=1}^{\infty} \|x_k\|_{\mathcal{D}}^2 \leq 4\, \|\underline{x}\|_{\mathcal{D}}^2.\end{aligned}$$

Similarly, we can show that $\int_{\mathbb{D}} \|Q'\underline{\bar{y}}\|_{l^2}^2 \, dA \leq 4 \, \|\underline{y}\|_{\mathcal{D}}^2$.

Thus,

$$\begin{aligned}\int_{\mathbb{D}} \|Q'\underline{w}\|_{l^2}^2 \, dA &\leq 8 \, \|\underline{x}\|_{\mathcal{D}}^2 + 8 \, \|\underline{y}\|_{\mathcal{D}}^2 \\ &= 8 \, \|\underline{x} + \underline{\bar{y}}\|_{\mathcal{HD}}^2 \\ &= 8 \, \|\underline{w}\|_{\mathcal{HD}}^2.\end{aligned}$$

□

LEMMA 2.3.4. *Let the operator T be defined on $L^2(\mathbb{D}, dA)$ by*

$$(Tf)(\lambda) = \int_{\mathbb{D}} \frac{f(z)}{(z-\lambda)\left(1 - z\,\bar{\lambda}\right)} \, dA(z),$$

for $\lambda \in \mathbb{D}$ and $f \in L^2(\mathbb{D}, dA)$.

Then the operator T is bounded on $L^2(\mathbb{D}, dA)$.

PROOF. To prove this lemma, using Schur's test, it's sufficient to show that there exists a measurable function $p(z) > 0$ a.e. on $\mathbb{D}$ and a constant $C > 0$ such that

$$\int_{\mathbb{D}} \frac{p(z)}{|z-\lambda||1 - z\bar{\lambda}|} \, dA(z) \leq C \, p(\lambda).$$

For this, taking $p(z) = (1 - |z|^2)^{-\frac{1}{2}}$, we will show that

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^{-\frac{1}{2}}}{|z-\lambda||1 - z\bar{\lambda}|} \, dA(z) \leq C \, \left(1 - |\lambda|^2\right)^{-\frac{1}{2}}.$$

Now,

$$\begin{aligned}\int_{\mathbb{D}} \frac{(1-|z|^2)^{-\frac{1}{2}}}{|z-\lambda||1 - z\bar{\lambda}|} \, dA(z) &= \int_{\mathbb{D}} \frac{(1-|z|^2)^{-\frac{1}{2}}}{|\varphi_\lambda(z)||1 - z\bar{\lambda}|^2} \, dA(z), \quad \text{where } \varphi_\lambda(z) = \frac{\lambda - z}{1 - z\bar{\lambda}} \\ &= \int_{\mathbb{D}} \frac{(1-|\varphi_\lambda(\zeta)|^2)^{-\frac{1}{2}}}{|\zeta||1 - \varphi_\lambda(\zeta)\bar{\lambda}|^2} \, |\varphi_\lambda'(\zeta)|^2 dA(\zeta)\end{aligned}$$

and

$$\int_{\mathbb{D}} \frac{(1-|\varphi_\lambda(\zeta)|^2)^{-\frac{1}{2}}}{|\zeta||1-\varphi_\lambda(\zeta)\bar{\lambda}|^2}\,|\varphi_\lambda'(\zeta)|^2 dA(\zeta) = \int_{\mathbb{D}} \frac{(1-|\lambda|^2)^{-\frac{1}{2}}\,(1-|\zeta|^2)^{-\frac{1}{2}}}{|\zeta||1-\bar{\lambda}\zeta|^{-1}\,|1-\varphi_\lambda(\zeta)\bar{\lambda}|^2}\;\frac{(1-|\lambda|^2)^2}{|1-\bar{\lambda}\zeta|^4} dA(\zeta)$$

$$= \left(1-|\lambda|^2\right)^{-\frac{1}{2}} \int_{\mathbb{D}} \frac{(1-|\zeta|^2)^{-\frac{1}{2}}}{|\zeta|\frac{(1-|\lambda|^2)^2}{|1-\bar{\lambda}\zeta|^2}}\;\frac{(1-|\lambda|^2)^2}{|1-\bar{\lambda}\zeta|^3} dA(\zeta)$$

$$= \left(1-|\lambda|^2\right)^{-\frac{1}{2}} \int_{\mathbb{D}} \frac{(1-|\zeta|^2)^{-\frac{1}{2}}}{|\zeta||1-\bar{\lambda}\zeta|}\,dA(\zeta).$$

Again,

$$\int_{\mathbb{D}} \frac{(1-|\zeta|^2)^{-\frac{1}{2}}}{|\zeta||1-\bar{\lambda}\zeta|}\,dA(\zeta) = \int_{\mathbb{D}_{\frac{1}{2}}} \frac{(1-|\zeta|^2)^{-\frac{1}{2}}}{|\zeta||1-\bar{\lambda}\zeta|}\,dA(\zeta)$$

$$+ \int_{\mathbb{D}-\left\{\mathbb{D}_{\frac{1}{2}}\right\}} \frac{(1-|\zeta|^2)^{-\frac{1}{2}}}{|\zeta||1-\bar{\lambda}\zeta|}\,dA(\zeta),$$

where $\mathbb{D}_{\frac{1}{2}}$ is a disk of radius $\frac{1}{2}$. For any $\zeta \in \mathbb{D}_{\frac{1}{2}}$ and $\lambda \in \mathbb{D}$,

$$\frac{1}{|1-\bar{\lambda}\zeta|} \le \frac{1}{1-\frac{1}{2}} = 2 \;\text{ and }\; \frac{1}{\sqrt{1-|\zeta|^2}} \le \frac{2}{\sqrt{3}}.$$

Therefore,

$$\int_{\mathbb{D}_{\frac{1}{2}}} \frac{(1-|\zeta|^2)^{-\frac{1}{2}}}{|\zeta||1-\bar{\lambda}\zeta|}\,dA(\zeta) \le \frac{4}{\sqrt{3}} \int_{\mathbb{D}_{\frac{1}{2}}} \frac{1}{|\zeta|}\,dA(\zeta) = \frac{4\pi}{\sqrt{3}}$$

and

$$\int_{\mathbb{D}-\left\{\mathbb{D}_{\frac{1}{2}}\right\}} \frac{(1-|\zeta|^2)^{-\frac{1}{2}}}{|\zeta||1-\bar{\lambda}\zeta|}\,dA(\zeta) \le 2\int_{\mathbb{D}-\left\{\mathbb{D}_{\frac{1}{2}}\right\}} \frac{(1-|\zeta|^2)^{-\frac{1}{2}}}{|1-\bar{\lambda}\zeta|}\,dA(\zeta)$$

$$\le 2\int_{\mathbb{D}} \frac{(1-|\zeta|^2)^{-\frac{1}{2}}}{|1-\bar{\lambda}\zeta|}\,dA(\zeta).$$

Using the Lemma 3.10 of [Z], we can see that

$$\int_{\mathbb{D}} \frac{(1-|\zeta|^2)^{-\frac{1}{2}}}{|1-\bar{\lambda}\zeta|}\,dA(\zeta) = \int_{\mathbb{D}} \frac{(1-|\zeta|^2)^{-\frac{1}{2}}}{|1-\bar{\lambda}\zeta|^{2-\frac{1}{2}-\frac{1}{2}}}\,dA(\zeta)$$

as a function of λ is uniformly bounded from above and bounded from below on $\mathbb{D}$.

Thus,

$$\sup_{\lambda\in\mathbb{D}}\int_{\mathbb{D}}\frac{(1-|\zeta|^2)^{-\frac{1}{2}}}{|1-\bar{\lambda}\zeta|}\,dA(\zeta)\leq C_0<\infty,$$

for some constant $C_0>0$.

Then, T is a bounded linear operator on $L^2(\mathbb{D},dA)$ with the norm

$$\|T\|\leq C,\text{where }\ C=\left(\frac{4\pi}{\sqrt{3}}+2C_0\right).$$

This finishes the proof of Lemma 2.3.4. □

LEMMA 2.3.5. *Let $\mathcal{B}$ denote the Beurling transform. Then*

$$||\mathcal{B}(f)||_A\leq 13\,||f||_A,\ \textit{for all}\ f\in L^2(\mathbb{D},\ \mathrm{dA}).$$

PROOF. To show that the Beurling transform, $\mathcal{B}$, is bounded on $L^2(\mathbb{D},\ dA)$, we we apply Zygmund's method of rotations [Z] and apply Schur's lemma an infinite number of times.

Let $f(z)=\sum\limits_{j=0}^{\infty}\sum\limits_{k=0}^{\infty}a_{jk}z^j\bar{z}^k$, where $a_{ij}=0$ except for a finite number of terms. For $z=r\,e^{i\theta}$, we relabel, so that

$$f(r\,e^{i\theta})=\sum_{l=-\infty}^{\infty}f_l(r)\,e^{il\theta},\ \text{where}\ \ f_l(r)=\sum_{k=0}^{\infty}a_{l+k\,k}\,r^{l+2k}.$$

Then

$$\|f\|_A^2=\sum_{l=-\infty}^{\infty}\|f_l(r)\|^2_{L^2[0,1]},$$

where the measure on $L^2[0,1]$ is "rdr". By definition, we have that

$$\widehat{f}(w)=-\frac{1}{\pi}\int_{\mathbb{D}}\frac{f(z)}{z-w}\,dA(z),\ \text{for}\ w\in\mathbb{D}.$$

Therefore,

$$
\begin{aligned}
\widehat{f}(w) &= -\frac{1}{\pi}\int_{\mathbb{D}} \frac{f(z)}{z-w}\,dA(z), \text{ for } w\in\mathbb{D} \\
&= -\frac{1}{\pi}\sum_{l=-\infty}^{\infty}\int_0^{2\pi}\int_0^1 \frac{f_l(r)e^{il\theta}}{re^{i\theta}-w}\,rdrd\theta \\
&\quad -\frac{1}{\pi}\sum_{l=-\infty}^{\infty}\int_0^{2\pi}\int_{|w|}^1 \frac{f_l(r)e^{il\theta}}{re^{i\theta}-w}\,rdrd\theta \\
&= -\frac{1}{\pi}\sum_{l=-\infty}^{\infty}\int_0^{2\pi}\int_0^{|w|} \frac{f_l(r)e^{il\theta}}{re^{i\theta}-w}\,rdrd\theta \\
&\quad -\frac{1}{\pi}\sum_{l=-\infty}^{\infty}\int_0^{2\pi}\int_0^{|w|} \frac{f_l(r)e^{il\theta}}{-w}\left(\sum_{n=0}^{\infty}\frac{r^n e^{in\theta}}{w^n}\right) rdrd\theta \\
&= \frac{1}{\pi}\sum_{l=-\infty}^{\infty}\sum_{n=0}^{\infty}\int_0^{2\pi}\int_0^{|w|} \frac{f_l(r)\; e^{i(l+n)\,\theta}}{w^{n+1}}\,r^{n+1}drd\theta \\
&\quad -\frac{1}{\pi}\sum_{l=-\infty}^{\infty}\sum_{n=0}^{\infty}\int_0^{2\pi}\int_{|w|}^1 \frac{f_l(r)\,e^{il\theta}w^n}{r^n\;e^{i(n+1)\theta}}drd\theta \\
&= \frac{1}{\pi}\sum_{l=-\infty}^{\infty}\sum_{n=0}^{\infty}\int_0^{2\pi}\int_0^{|w|} \frac{f_l(r)\; e^{i(l+n)\,\theta}}{w^{n+1}}\,r^{n+1}drd\theta \\
&\quad -\frac{1}{\pi}\sum_{l=-\infty}^{\infty}\sum_{n=0}^{\infty}\int_0^{2\pi}\int_{|w|}^1 \frac{f_l(r)\,e^{i\,(l-1-n)\,\theta}w^n}{r^n}drd\theta. \qquad (\star\star)
\end{aligned}
$$

If we take $l=0$ in $(\star\star)$, we get that

$$\widehat{f_0}(w) = \frac{2}{w}\int_0^{|w|} f_0(r)\,rdr.$$

So

$$\partial\widehat{f_0}(w) = -\frac{2}{w^2}\int_0^{|w|} f_0(r)\,rdr + \frac{2}{w}\,f_0\left(|w|\right)|w|\frac{\partial(|w|)}{\partial w}$$

Therefore,

$$\partial\widehat{f_0}(w) = \frac{-2}{w^2}\int_0^{|w|} f_0(r)\,r dr + \frac{\overline{w}}{w}\,f_0\left(|w|\right)$$
$$\left(\text{since } \overline{w} = \frac{\partial\left(|w|^2\right)}{\partial w} = 2|w|\frac{\partial|w|}{\partial w}\right)$$

Hence,

$$\mathcal{B}_0\left(f_0(se^{it})\right) = e^{-2it}\left[\frac{-2}{s^2}\int_0^s f_0(r)\,r dr + f_0\left(s\right)\right].$$

Similarly, if we take $l \geq 1$ in $(\star\star)$ we get that

$$\widehat{f_l}(w) = -2w^{l-1}\int_{|w|}^1 \frac{f_l(r)}{r^{l-1}}dr.$$

Therefore,

$$\begin{aligned}\partial\widehat{f_l}(w) &= -2(l-1)\,w^{l-2}\int_{|w|}^1 \frac{f_l(r)}{r^{l-1}}dr + -2\pi\,w^{l-1}\frac{f_l(|w|)}{|w|^{l-1}}\frac{\partial(|w|)}{\partial w}\\ &= -2(l-1)\,w^{l-2}\int_{|w|}^1 \frac{f_l(r)}{r^{l-1}}dr - \pi w^{l-1}\frac{f_l(|w|)}{|w|^{l-1}}\frac{\overline{w}}{|w|}\end{aligned}$$

So

$$\begin{aligned}\mathcal{B}\left(f_l(w)\right) &= -2(l-1)s^{l-2}e^{i(l-2)t}\int_s^1 \frac{f_l(r)}{r^{l-1}}dr - e^{i(l-1)t}s^{l-1}\frac{f_l(s)}{s^{l-1}}\frac{se^{-it}}{s}\\ &= e^{i(l-2)\,t}\left[-2(l-1)s^{l-2}\int_s^1 \frac{f_l(r)}{r^{l-1}}dr - f_l(s)\right].\end{aligned}$$

Again, if we take $l < 0$ (say $l = -k$, $k > 0$) in $(\star\star)$, then we get that

$$\widehat{f_l}(w) = 2\,w^{-(k+1)}\int_0^{|w|} f_l(r)\,r^{k+1}dr.$$

Therefore,

$$\partial\widehat{f_l}(w) = -2(k+1)w^{-(k+2)}\int_0^{|w|} f_l(r)r^{k+1}dr + 2w^{-(k+1)}f_l(|w|)\,|w|^{k+1}\,\frac{\overline{w}}{|w|}.$$

Hence,

$$\begin{aligned}\mathcal{B}_l\left(f_l(se^{it})\right) &= -2(k+1)s^{-k-2}e^{i(-k-2)t}\int_0^s f_l(r)r^{k+1}dr\ + e^{i(-k-2)it}f_l(s)\\ &= -2(k+1)s^{-k-2}e^{i(l-2)t}\int_0^s f_l(r)r^{k+1}dr\ + e^{i(l-2)it}f_l(s)\\ &= e^{i(l-2)t}\left[-2(k+1)s^{-k-2}\int_0^s f_l(r)r^{k+1}dr + f_l(s)\right].\end{aligned}$$

Therefore,

$$\mathcal{B}\left(f(w)\right) = e^{i(l-2)t}\sum_{l=-\infty}^{\infty}\mathcal{B}_l(f_l(s)),\ \text{where}$$

$$\mathcal{B}_l(f_l(s)) = \begin{cases} \dfrac{-2}{s^2}\displaystyle\int_0^s f_0(r)\,rdr + f_0\,(s)\ \text{ for } l=0\\ -\,2\,(l-1)s^{l-2}\displaystyle\int_s^1 \dfrac{f_l(r)}{r^{l-1}}dr - f_l(s)\ \text{ for } l\geq 1\\ -\,2(k+1)s^{-k-2}\displaystyle\int_0^s f_l(r)r^{k+1}dr + f_l(s)\ \text{ for } l<0.\end{cases}$$

Thus,

$$||\mathcal{B}f||_A^2 = \sum_{l=-\infty}^{\infty}||\mathcal{B}_l f_l||^2_{L^2[0,1],}$$

where the measure on $L^2\,[0,1]$ is "rdr".

Claim 1.

$$\sup_l ||\mathcal{B}_l||_{B(L^2[0,1],\,L^2[0,1])}\leq 13.$$

Once we prove Claim 1, we will have

$$||\mathcal{B}f(w)||_A^2 = \sum_{l=-\infty}^{\infty}||\mathcal{B}_l f_l(s)||^2_{L^2[0,1],}\leq (13)^2\sum_{l=-\infty}^{\infty}||f_l(s)||^2_{L^2[0,1]} = (13)^2\,||f\,||_A^2.$$

Without loss of generality, we may assume that $f_l(s)\geq 0$ for all l.

Case I. $l=0$

$$
\begin{aligned}
\int_0^1 |\mathcal{B}_0(f_0(s))|^2 sds &= \int_0^1 |\frac{-2}{s^2}\int_0^s f_0(r)\, rdr + f_0\,(s)\,|^2 sds \\
&\leq 4\int_0^1 |\frac{1}{s^2}\int_0^s f_0(r)\, rdr|^2 sds + 2\int_0^1 |f_0\,(s)\,|^2 sds \\
&= 4\int_0^1 \frac{1}{s^4}\left[\int_0^s f_0(u)\, udu \int_0^s f_0(v)\, vdv\right] sds \\
&\quad + 2\int_0^1 |f_0\,(s)\,|^2 sds \\
&= 4\int_0^1 \int_0^1 f_0(u) f_0(v) \left[\int_{\max\{u,v\}}^1 \frac{sds}{s^4}\right] uduvdv \\
&\quad + 2\int_0^1 |f_0\,(s)\,|^2 sds \\
&= 2\int_0^1 \int_0^1 f_0(u) f_0(v) \left[\int_{\max\{u^2,v^2\}}^1 \frac{ds}{s^2}\right] uduvdv \\
&\quad + 2\int_0^1 |f_0\,(s)\,|^2 sds. \\
&\leq 2\int_0^1 \int_0^1 f_0(u) f_0(v) \left[-\frac{1}{s}\right]_{\max\{u^2,v^2\}}^1 uduvdv \\
&\quad + 2\int_0^1 |f_0\,(s)\,|^2 sds \\
&= 2\int_0^1 \int_0^1 f_0(u) f_0(v) \left[\frac{(1-\max\{u^2,v^2\})}{\max\{u^2,v^2\}}\right] uduvdv \\
&\quad + 2\int_0^1 |f_0\,(s)\,|^2 sds \\
&= \int_0^1 \int_0^1 f_0(u) f_0(v) k\,(u,v)\, uduvdv + 2\int_0^1 |f_0\,(s)\,|^2 sds,
\end{aligned}
$$

where $k\,(u,v) = \frac{2\left(1-\max\left\{u^2,v^2\right\}\right)}{\max\{u^2,v^2\}}$.

By Schur's Test,

$$
\int_0^1 \int_0^1 f_0(u) f_0(v) k\,(u,v)\, uduvdv \leq \left(\frac{16}{3}\right)^2 \int_0^1 |f_0(u)|^2 udu
$$

if and only if there exists $p(u) \geq 0$ a.e. such that for a.e. v in $[0,1]$,

$$\int_0^1 k\,(u,v)\,p(u)udu \leq \frac{16}{3}p(v).$$

For this, we will show that

$$\int_0^v \left[\frac{2\,(1-v^2)}{v^2\,(1-v^2)}\right] p(u)\,udu \leq \frac{4}{3}p(v) \text{ and } \int_v^1 \left[\frac{2\,(1-u^2)}{u^2\,(1-v^2)}\right] p(u)udu \leq 4p(v).$$

If we take $p(u) = \frac{1}{\sqrt{u}}$, then

$$\begin{aligned}\int_0^v \left[\frac{2\,(1-v^2)}{v^2\,(1-v^2)}\right] p(u)\,udu &= \frac{2}{v^2}\int_0^v u^{\frac{1}{2}}du \\ &= \frac{4}{3v^2}v^{\frac{3}{2}} \\ &= \frac{4}{3}p(v).\end{aligned}$$

Similarly,

$$\begin{aligned}\int_v^1 \left[\frac{2\,(1-u^2)}{u^2\,(1-v^2)}\right] p(u)udu &\leq \int_v^1 \frac{2}{u^{\frac{3}{2}}}du, \quad \because \left(1-u^2\right) \leq \left(1-v^2\right) \\ &= \left[\frac{-4}{\sqrt{u}}\right]_v^1 \\ &= 4\left[\frac{1}{\sqrt{v}} - 1\right] \leq \frac{4}{\sqrt{v}}.\end{aligned}$$

Hence,

$$\int_0^1 k\,(u,v)\,p(u)udu \leq \frac{16}{3}p(v).$$

Thus,

$$\begin{aligned}\int_0^1 |\mathcal{B}_0(f_0(s))|^2 sds &\leq \left(\frac{16}{3}\right)^2 \int_0^1 |f_0(s)|^2 sds + 2\int_0^1 |f_0\,(s)\,|^2 sds \\ &\leq 31\int_0^1 |f_0\,(s)\,|^2 sds.\end{aligned}$$

Case II. $l = 1$

$$\int_0^1 |\mathcal{B}_1(f_1(s))|^2 sds = \int_0^1 |f_1(s)|^2 sds.$$

Case III. $l \geq 2$

$$\begin{aligned}\int_0^1 |\mathcal{B}_l(f_l(s))|^2 ds &= \int_0^1 | -2\,(l-1)s^{l-2} \int_s^1 \frac{f_l(r)}{r^{l-1}} dr - f_l(s)|^2 sds \\ &\leq 8(l-1)^2 \int_0^1 s^{2(l-2)} | \int_0^1 \chi_{(s,1)}(r)\, \frac{f_l(r)}{r^{l-1}} dr|^2 sds \\ &\quad + 2\int_0^1 |f_l(s)|^2 sds.\end{aligned}$$

Now

$$8(l-1)^2 \int_0^1 s^{2(l-2)} | \int_0^1 \chi_{(s,1)}(r)\, \frac{f_l(r)}{r^{l-1}} dr|^2 sds$$

$$\begin{aligned}&= 8\,(l-1) \int_0^1 \int_0^1 f_l(u) f_l(v) \left[\frac{1}{u^l}\, \frac{1}{v^l} \int_0^{\min\{u,v\}} s^{2(l-2)} sds \right] udu\, vdv \frac{f_l(v)}{v^{l-1}} dv \\ &= 8\,(l-1)^2 \int_0^1 s^{2(l-2)} \int_0^1 \chi_{(s,1)}(u)\, \frac{f_l(u)}{u^{l-1}} du \int_0^1 \chi_{(s,1)}(v)\, \frac{f_l(v)}{v^{l-1}} dv \\ &= 4\,(l-1)^2 \int_0^1 \int_0^1 f_l(u) f_l(v) \left[\frac{1}{u^l}\, \frac{1}{v^l} \int_0^{\min\left\{u^2,v^2\right\}} s^{(l-2)} ds \right] udu\, vdv \\ &\leq 4\,(l-1)^2 \int_0^1 \int_0^1 f_l(u) f_l(v) \left[\frac{1}{u^l}\, \frac{1}{v^l} \left[\frac{s^{l-1}}{l-1} \right]_0^{\min\left\{u^2,v^2\right\}} \right] udu\, vdv \\ &= \int_0^1 \int_0^1 f_l(u) f_l(v) \left[\frac{4\,(l-1)}{u^l}\, \frac{1}{v^l} \left(\min\left\{u^2, v^2\right\} \right)^{l-1} \right] udu\, vdv.\end{aligned}$$

We know that

$$\int_0^1 \int_0^1 f_l(u) f_l(v) \left[\frac{4\,(l-1)}{u^l}\, \frac{1}{v^l} \left(\min\left\{u^2, v^2\right\} \right)^{l-1} \right] udu\, vdv \leq (12)^2 \int_0^1 |f_l(u)|^2 udu$$

if and only if there exists $p(u) \geq 0$ a.e. such that for a.e $v \geq 0$

$$\int_0^1 \left[\frac{4\,(l-1)}{u^l}\,\frac{1}{v^l}\left(\min\left\{u^2,v^2\right\}\right)^{l-1}\right] p(u)udu \leq 12\,p(v).$$

Taking $p(u)=\frac{1}{\sqrt{u}}$, we will show that

$$\int_0^v \left[\frac{4\,(l-1)}{u^l}\,\frac{1}{v^l}u^{2(l-1)}\right] p(u)udu \leq 4\,p(v) \;\text{ and }\; \int_v^1 \left[\frac{4\,(l-1)}{u^l}\,\frac{1}{v^l}v^{2(l-1)}\right] p(u)udu \leq 8\,p(v)$$

Now

$$\begin{aligned}\int_0^v \left[\frac{4\,(l-1)}{u^l}\,\frac{1}{v^l}u^{2(l-1)}\right] p(u)udu &= \frac{4\,(l-1)}{v^l}\int_0^v u^{l-\frac{3}{2}}du \\ &= \frac{4(l-1)}{v^l}\,\frac{v^{l-\frac{1}{2}}}{l-\frac{1}{2}} \leq 4p(v).\end{aligned}$$

And

$$\begin{aligned}\int_v^1 \left[\frac{4\,(l-1)}{u^l}\,\frac{1}{v^l}v^{2(l-1)}\right] p(u)udu &= 4\,(l-1)\,v^{l-2}\int_v^1 \frac{1}{u^{l-\frac{1}{2}}}du \\ &= 4(l-1)v^{l-2}\left[\frac{v^{-l+\frac{3}{2}}}{-l+1.5}\right]_v^1 \\ &= 4(l-1)v^{l-2}\left[-\frac{v^{-l+\frac{3}{2}}}{l-1.5}\right]_v^1 \\ &\leq \frac{4(l-1)v^{l-2}}{l-1.5}\,\frac{1}{v^{l-\frac{3}{2}}} \leq 8p(v).\end{aligned}$$

Therefore,

$$\int_0^1 \left[\frac{2\,(l-1)}{u^l}\,\frac{1}{v^l}\left(\min\left\{u^2,v^2\right\}\right)^{l-1}\right] p(u)udu \leq (4+8)\,p(v) = 12p(v).$$

Hence,

$$\int_0^1 |\mathcal{B}_l(f_l(s))|^2 sds \leq (12)^2\int_0^1 |f_l(s)|^2 sds + 2\int_0^1 |f_l(s)|^2 sds = 146\int_0^1 |f_l(s)|^2 sds.$$

Case IV. $l < 0$ (say $l = -k$, $k > 0$)

$$\int_0^1 |\mathcal{B}_l(f_l(s))|^2 sds = \int_0^1 |-2(k+1)s^{-k-2}\int_0^s f_l(r)r^{k+1}dr + f_l(s)|^2 sds$$
$$\leq 8(k+1)^2 \int_0^1 s^{-2(k+2)} \left|\int_0^s f_l(r)r^{k+1}dr\right|^2 sds + 2\int_0^1 |f_l(s)|^2 sds.$$

We have

$$8(k+1)^2 \int_0^1 s^{-2(k+2)} \left|\int_0^s f_l(r)r^{k+1}dr\right|^2 sds$$

$$= 8(k+1)^2 \int_0^1 s^{-2(k+2)} \int_0^s f_l(u)u^{k+1}du \int_0^s f_l(v)v^{k+1}dvsds$$
$$= 8(k+1)^2 \int_0^1 s^{-2(k+2)} \left[\int_0^1 \chi_{(0,s)}(u)\, f_l(u)\, u^{k+1}du \int_0^1 \chi_{(0,s)}(v) f_l(v)\, v^{k+1}dv\right] sds$$
$$= 8(k+1)^2 \int_0^1 \int_0^1 f_l(u)f_l(v)\left[u^k v^k \int_{\max\{u,v\}}^1 s^{-2(k+2)}sds\right] udu\, vdv$$
$$= 4(k+1)^2 \int_0^1 \int_0^1 f_l(u)f_l(v)\left[u^k v^k \int_{\max\{u^2,v^2\}}^1 s^{-(k+2)}ds\right] udu\, vdv$$
$$= 4(k+1)^2 \int_0^1 \int_0^1 f_l(u)f_l(v)\left[u^k v^k \left[\frac{s^{-(k+1)}}{-(k+1)}\right]_{\{u^2,v^2\}}^1\right] udu\, vdv$$
$$\leq 4(k+1) \int_0^1 \int_0^1 f_l(u)f_l(v)\left[u^k v^k \left(\max\left\{u^2,v^2\right\}\right)^{-(k+1)}\right] udu\, vdv.$$

We know that

$$\int_0^1 \int_0^1 f_l(u)f_l(v)\left[4(k+1)u^k v^k \left(\max\left\{u^2,v^2\right\}\right)^{-(k+1)}\right] udu\, vdv \leq (12)^2 \int_0^1 |f_l(u)|^2 udu$$

if and only if there exists $p(u) \geq 0$ a.e. such that for a.e. v in $[0,1]$

$$\int_0^1 \left[4(k+1)u^k v^k \left(\max\left\{u^2,v^2\right\}\right)^{-(k+1)}\right] p(u)\, udu \leq 12p(v).$$

Taking $p(u) = 1$, we will show that

$$\int_0^v 4(k+1)u^k v^k v^{-2(k+1)}\, udu \leq 4 \text{ and } \int_v^1 \left[4(k+1)\, \tfrac{u^k v^k}{u^{2(k+1)}}\right] udu \leq 8.$$

Now

$$\begin{aligned}\int_0^v 4(k+1)u^k v^k v^{-2(k+1)}\, udu &= 4\,(k+1)\, v^{-k-2} \int_0^v u^{k+1} du \\ &= \frac{4\,(k+1)}{k+2} v^{-k-2} \left[u^{k+2}\right]_0^v \leq 4.\end{aligned}$$

and

$$\begin{aligned}\int_v^1 \left[4(k+1)\, \frac{u^k v^k}{u^{2(k+1)}}\right] udu &= 4\,(k+1)\, v^k \int_v^1 u^{-k-1} du \\ &= 4\,(k+1)\, v^k \left[\frac{u^{-k}}{-k}\right]_v^1 \\ &= \frac{4\,(k+1)\, v^k}{k} \left[\frac{1}{u^k}\right]_1^v \\ &\leq 8v^k \left[\frac{1}{v^k} - 1\right] \leq 8.\end{aligned}$$

Hence,

$$\int_0^1 |\mathcal{B}_l(f_l(s))|^2 sds \leq 146 \int_0^1 |f_l(s)|^2 sds.$$

Thus,

$$\sup_l ||\mathcal{B}_l||_{B(L^2[0,1],\, L^2[0,1])} \leq 13.$$

$\square$

Lemma 2.3.6. *If Q is a multiplier of $\mathcal{D}$, then*

$$\left(1 - |z|^2\right) |\, Q'(z)| \leq \|M_Q\|_{B(\mathcal{D})} \text{ for all } z \in \mathbb{D}.$$

PROOF. Define $\varphi : \mathbb{D} \to \mathbb{D}$ as $\varphi(z) = \frac{Q(z)}{||M_Q||}$ for all $z \in \mathbb{D}$.

Thus,

$$||\varphi||_\infty \leq \frac{||Q||_\infty}{||M_Q||} \leq 1.$$

Now, fixing w_0 and using the generalized Schwarz lemma; we get

$$\left|\frac{\varphi(z) - \varphi(w_0)}{1 - \varphi(z)\overline{\varphi(w_0)}}\right| \leq \left|\frac{z - w_0}{1 - z\,\overline{w_0}}\right| \text{ for all } z \in \mathbb{D}$$

$$\Longrightarrow \left|\frac{\varphi(z) - \varphi(w_0)}{z - w_0}\right| \leq \left|\frac{1 - \varphi(z)\overline{\varphi(w_0)}}{1 - z\,\overline{w_0}}\right|.$$

As $z \to w_0$,

$$|\varphi'(w_0)| \leq \left|\frac{|1 - |\varphi(w_0)|^2}{(1 - |w_0|^2)}\right|.$$

Thus,

$$\left(1 - |w_0|^2\right)|\varphi'(w_0)| \leq \left||1 - |\varphi(w_0)|^2\right| \leq 1, \tag{2.3.4}$$

and this is true for any fixed $w_0 \in \mathbb{D}$.

Hence, $(1 - |z|^2)\,|\varphi'(z)| \leq 1$ for all $z \in \mathbb{D}$.

This implies that $(1 - |z|^2)\,|Q'(z)| \leq ||M_Q||$ for all $z \in \mathbb{D}$. □

2.4. Proof of Theorem 1

We are now ready to prove Theorem 1.

PROOF. First, we will prove the theorem for smooth functions on $\overline{\mathbb{D}}$ and get a uniform bound. Then we will remove the smoothness hypothesis.

Assume that (a) and (b) of Theorem 1 hold for F and H and that F and H are analytic on $\mathbb{D}_{1+\epsilon}(0)$.

Then our main goal is to show that there exists a constant $K < \infty$, independent of ϵ, such that

$$\|\underline{u}_h\|_{\mathcal{D}}^2 \leq K \, \|h\|_{\mathcal{D}}^2,$$

where we have already taken $\underline{u}_h = \frac{F^\star H^3 h}{F F^\star} - Q\,\widehat{W}$.

We know that

$$\|\underline{u}_h\|_{\mathcal{D}}^2 = \int_{-\pi}^{\pi} \|\underline{u}_h(e^{it})\|^2 \, d\sigma(t) + \int_{\mathbb{D}} \| \left(\underline{u}_h(z)\right)' \|^2 \, dA(z).$$

Condition (b) implies that

$$\int_{-\pi}^{\pi} \left\| \frac{F^\star H^3 h}{F F^\star} - Q\,\widehat{W} \right\|^2 d\sigma(t) \leq \; C_1^2 \|h\|_{\sigma}^2,$$

where C_1 can be chosen to be 15 (See [**Tr1**]). Hence, we only need to show that

$$\int_{\mathbb{D}} \left\| \left(\frac{F^\star H^3 h}{F F^\star} - Q\,\widehat{W} \right)' \right\|^2 dA(z) \leq C_2^2 \, \|h\|_{\mathcal{D}}^2$$

for some $C_2 < \infty$. Now

$$\begin{aligned}
&\int_{\mathbb{D}} \left\| \left(\frac{F^\star H^3 h}{F F^\star} - Q\widehat{W} \right)' \right\|^2 dA(z) \\
&\quad \leq 2 \int_{\mathbb{D}} \left\| \left(\frac{F^\star H^3 h}{F F^\star} \right)' \right\|^2 dA(z) + 2 \int_{\mathbb{D}} \left\| \left(Q\widehat{W} \right)' \right\|^2 dA(z) \\
&\quad \leq 4 \times 9 \underbrace{\int_{\mathbb{D}} \left\| \frac{F^\star H^2 H' h}{F F^\star} \right\|^2 dA(z)}_{(a')} + 8 \underbrace{\int_{\mathbb{D}} \left\| \frac{F^\star H^3 h'}{F F^\star} \right\|^2 dA(z)}_{(b')} \\
&\qquad + 8 \underbrace{\int_{\mathbb{D}} \left\| \frac{F^\star H^3 h' F' F^\star}{(F F^\star)^2} \right\|^2 dA(z)}_{(c')} + 4 \underbrace{\int_{\mathbb{D}} \left\| Q' \,\widehat{W} \right\|^2 dA(z)}_{(d')} \\
&\qquad + 4 \underbrace{\int_{\mathbb{D}} \left\| Q \;\left(\widehat{W}\right)' \right\|^2 dA(z).}_{(e')}
\end{aligned}$$

Then

$$(a') = \int_{\mathbb{D}} \left\| \frac{F^{\star} 3 H^2 H' h}{F F^{\star}} \right\|^2 dA(z) = 9 \int_{\mathbb{D}} \left\| \frac{F^{\star}}{\sqrt{F F^{\star}}} \frac{H}{\sqrt{F F^{\star}}} H\, H'h \right\|^2 dA(z)$$
$$\leq 9 \int_{\mathbb{D}} \|H'h\|^2 \, dA(z)$$
$$\leq 18 \left(\|M_H\|^2 + \|H\|_{\infty}^2 \right) \|h\|_{\mathcal{D}}^2$$
$$\leq 36 \, \|M_H\|^2 \, \|h\|_{\mathcal{D}}^2.$$

$(b') = \int_{\mathbb{D}} \| \frac{F^{\star} H^3 h'}{F F^{\star}} \|^2 \, dA(z) \leq \int_{\mathbb{D}} \|h'\|^2 \, dA(z) \leq \|h\|_{\mathcal{D}}^2.$

$$(c') = \int_{\mathbb{D}} \left\| \frac{F^{\star} H^3 h F' F^{\star}}{(F F^{\star})^2} \right\|^2 dA(z) = \int_{\mathbb{D}} \left\| \frac{F^{\star} F' F^{\star}}{\sqrt{F F^{\star}}} \frac{H^2}{F F^{\star}} \frac{H}{\sqrt{F F^{\star}}} \, h \right\|^2 dA(z)$$
$$\leq \int_{\mathbb{D}} \left\| \frac{F^{\star} F' F^{\star}}{\sqrt{F F^{\star}}} \, h \right\|^2 dA(z)$$
$$\leq \int_{\mathbb{D}} \|F'^{\star} h\|^2 \, dA(z) \leq 4 \, \|h\|_{\mathcal{D}}^2.$$

By Lemma 2.3.5 on the Beurling transform,

$$(e') = \int_{\mathbb{D}} \|Q \left(\widehat{W}\right)' \|^2 \, dA(z)$$
$$\leq \int_{\mathbb{D}} \| \left(\widehat{W}\right)' \|^2 \, dA(z)$$
$$\leq (13)^2 \int_{\mathbb{D}} \|W\|^2 \, dA(z)$$
$$\leq (13)^2 \int_{\mathbb{D}} \|F'^{\star} h\|^2 \, dA(z)$$
$$\leq (13)^2 .4. \, \|h\|_{\mathcal{D}}^2 = 676 \|h\|_{\mathcal{D}}^2.$$

So we only need to estimate (d'). For this we have

$$\int_{\mathbb{D}} \|Q' \; \widehat{W}\|^2 \, dA(z) \leq 2 \underbrace{\int_{\mathbb{D}} \|Q' \widehat{W} - Q' \widetilde{\widehat{W}}\|^2 \, dA(z)}_{(f')} + 2 \int_{\mathbb{D}} \|Q' \widetilde{\widehat{W}}\|^2 \, dA(z),$$

where $\widetilde{\widehat{W}}(z) = \int_{-\pi}^{\pi} \frac{1-|z|^2}{|1-e^{-it}z|} \widehat{W}(e^{it})\, d\sigma(t)$ is the harmonic extension of $\widehat{W}$ from $\partial\mathbb{D}$ to $\mathbb{D}$.

Lemma 2.3.3 tells us that

$$\int_{\mathbb{D}} \|Q^{'}\widetilde{\widehat{W}}\|^2\, dA(z) \leq 8\, \|\widetilde{\widehat{W}}\|^2_{\mathcal{HD}}.$$

Also, a lemma of [**Tr2**] implies that

$$\|\widetilde{\widehat{W}}\|^2_{\mathcal{HD}} \leq \|W\|^2_A + \|\widehat{W}\|^2_\sigma.$$

But, as we showed above,

$$\|W\|^2_A = \int_{\mathbb{D}} \left\| \frac{Q^\star F^{'\star} H^3 h}{(FF^\star)^2} \right\|^2 dA(z) \leq \int_{\mathbb{D}} \|F^{'\star} h\|^2\, dA(z) \leq 4\, \|h\|^2_{\mathcal{D}}.$$

and

$$\|\widehat{W}\|^2_\sigma = \int_{-\pi}^{\pi} \left\| \left(\frac{Q^\star \widehat{F^{'\star} H^3} h}{(FF^\star)^2} \right) \right\|^2 d\sigma(t) \leq 15\, \|h\|^2_\sigma \text{ (see [\textbf{Tr1}]).}$$

Thus,

$$\int_{\mathbb{D}} \|Q^{'}\widetilde{\widehat{W}}\|^2\, dA(z) \leq 8\, \left[4\, \|h\|^2_{\mathcal{D}} + 15\, \|h\|^2_\sigma \right].$$

Now we are just left with estimating (f').

$$\begin{aligned}
(f') &= \int_{\mathbb{D}} \|Q^{'}\widehat{W} - Q^{'}\widetilde{\widehat{W}}\|^2\, dA(z) \\
&= \int_{\mathbb{D}} \left\| Q^{'} \left[-\frac{1}{\pi} \int_D \frac{W(u)}{u-z} dA(u) - \int_{-\pi}^{\pi} \frac{1-|z|^2}{|1-e^{-it}z|} \widehat{W}(e^{it}) d\sigma(t) \right] \right\|^2 dA(z) \\
&= \int_{\mathbb{D}} \left\| Q^{'} [-\frac{1}{\pi} \int_{\mathbb{D}} \frac{W(u)}{u-z} dA(u) - \int_{-\pi}^{\pi} \frac{1-|z|^2}{|1-e^{-it}z|} \left\{ -\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{W(u)}{(u-e^{it})} dA(u) \right\} d\sigma(t)] \right\|^2 dA(z) \\
&= \frac{1}{\pi^2} \int_{\mathbb{D}} \left\| Q^{'} \int_{\mathbb{D}} W(u) [\frac{1}{u-z} + \int_{-\pi}^{\pi} \frac{1-|z|^2}{|1-e^{-it}z|}\ e^{-it} \frac{1}{1-ue^{-it}} d\sigma(t)] dA(z) \right\|^2 dA(z)
\end{aligned}$$

Therefore,

$$
\begin{aligned}
(f') &= \frac{1}{\pi^2}\int_{\mathbb{D}}\left\|Q' \int_{\mathbb{D}} W(u)\left[\frac{1}{u-z}+\int_{-\pi}^{\pi}\frac{1-|z|^2}{|1-e^{-it}z|}\,e^{-it}\frac{1}{1-ue^{-it}}\,d\sigma(t)\right]dA(u)\right\|^2 dA(z)\\
&= \frac{1}{\pi^2}\int_{\mathbb{D}}\left\|Q' \int_{\mathbb{D}} W(u)\left[\frac{1}{u-z}+\frac{\bar z}{1-u\bar z}\right]dA(u)\right\|^2 dA(z)\\
&= \frac{1}{\pi^2}\int_{\mathbb{D}}\left\|Q' \int_{\mathbb{D}} W(u)\left[\frac{1-|z|^2}{(u-z)(1-u\bar z)}\right]dA(u)\right\|^2 dA(z)\\
&= \frac{1}{\pi^2}\int_{\mathbb{D}}\left\|Q'(z)\,(1-|z|^2)\,T(W)(z)\right\|^2 dA(z)\\
&\le \frac{\|M_Q\|^2}{\pi^2}\|T(W)\|_A^2 \;\text{ by Lemma 2.3.6}\\
&\le C^2\,\frac{\|M_Q\|^2}{\pi^2}\,\|W\|^2 \;\text{ by Lemma 2.3.4}\\
&\le \frac{(86C)^2}{\pi^2}\|h\|_{\mathcal{D}}^2\left(\text{since }\;\|M_Q\|_{\left(\overset{\infty}{\underset{1}{\oplus}}\mathcal{HD}\right)}\le\sqrt{86}\right).
\end{aligned}
$$

Combining all these pieces, we see that in the smooth case

$$
\|\underline{u}_h\|_{\mathcal{D}}^2 \le K^2\,\|h\|_{\mathcal{D}}^2\,,
$$

for some constant $K<\infty$, which is independent of h and $\epsilon>0$. By the proof of Theorem 1 in the smooth case, we have

$$
M_{H_r^3}M_{H_r^3}^{\star} \le K^2 M_{F_r}^R (M_{F_r}^R)^{\star} \;\text{ for } 0\le r<1,
$$

where $F_r(z)=F(rz)$.

Using a commutant lifting argument, there exists $G_r\in\mathcal{M}(\mathcal{D},\overset{\infty}{\underset{1}{\oplus}}\mathcal{D})$ so that $M_{F_r}^R M_{G_r}^C = M_{H_r^3}$ and $\|M_{G_r}^C\|\le K$. Then $M_{F_r}^R\to M_F^R$ and $M_{H_r^3}\to M_{H^3}$ as $r\uparrow 1$ in the $\star$-strong topology.

By compactness, we may choose a net with $G^{\star}_{r_\alpha} \to G^{\star}$ as $r_\alpha \to 1^-$. Since the multiplier algebra (as operators) is WOT closed, $G \in \mathcal{M}(\mathcal{D}, \bigoplus_1^\infty \mathcal{D})$. Also, since $F^{\star}_{r_\alpha} \xrightarrow{s} F^{\star}$, we get $M^{\star}_{H_r^3} = M^{\star C}_{G_r} M^{\star R}_{F_r} \overset{WOT}{\to} M^{\star C}_{G} M^{\star R}_{F}$ and so $M^R_F M^C_G = M_{H^3}$ with entries of G in $\mathcal{M}(\mathcal{D})$ and $\|M^C_G\| \le K$.

This ends our proof. $\square$

CHAPTER 3

WOLFF'S IDEAL PROBLEM IN $\mathcal{M}(\mathcal{D}_\alpha)$

In this chapter, we will extend our previous result into the multiplier algebra on weighted Dirichlet spaces. The analogue of the corona theorem for the algebra of multipliers on weighted Dirichlet spaces was established in Kidane-Trent [KT] . It seems plausible that Wolff type ideal results could be extended to the algebra of multipliers on weighted Dirichlet spaces. The techniques used in this chapter are similar to those of Chapter 2 for the classical Dirichlet space. Moreover, we will prove the boundedness of a certain singular integral operator (Lemma 3.3.2) and the boundedness of the Beurling transform (Lemma 3.3.3) on some L^2 spaces with weights. Also, we will be using several lemmas from [**KT**] to extend the norms from $\mathcal{D}$ to $\mathcal{D}_\alpha$.

3.1. Weighted Dirichlet Spaces and Corresponding Norms

In Chapter 2, we discussed weighted Dirichlet spaces and its reproducing kernel. In this section, we will discuss the multiplier algebra on weighted Dirichlet spaces and the corresponding norms. The weighted Dirichlet spaces $\mathcal{D}_\alpha$, $\alpha \in (0,1)$, are defined as

$$\mathcal{D}_\alpha = \{\, f : \mathbb{D} \to \mathbb{C} \mid f \text{ is analytic on } \mathbb{D} \text{ and}$$
$$\text{for } f(z) = \sum_{n=0}^{\infty} a_n\, z^n, \|f\|^2_{\mathcal{D}_\alpha} = \sum_{n=0}^{\infty} (n+1)^\alpha\, |a_n|^2 < \infty \}.$$

We will use other equivalent norms for smooth functions in $\mathcal{D}_\alpha$ as follows,

$$\|f\|^2_{\mathcal{D}_\alpha} = \int_{-\pi}^{\pi} |f|^2 d\sigma + \int_{\mathbb{D}} |f\,'(z)|^2 \left(1 - |z|^2\right)^{1-\alpha} dA(z) \qquad \text{and}$$

$$\|f\|_{\mathcal{D}_\alpha}^2 = \int_{-\pi}^{\pi} |f|^2 d\sigma(t) + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^{1+\alpha}}\, d\sigma(t)\, d\sigma(\theta).$$

For ease of notation, we will denote $(1-|z|^2)^{1-\alpha}\, dA(z)$ by $dA_\alpha(z)$. As before, we use $\mathcal{M}(\mathcal{D}_\alpha)$ to denote the multiplier algebras of weighted Dirichlet spaces, defined as:

$$\mathcal{M}(\mathcal{D}_\alpha) = \{\phi \in \mathcal{D}_\alpha : \phi f \in \mathcal{D}_\alpha \text{ for all } f \in \mathcal{D}_\alpha\}.$$

REMARK 1. *How big is* $\mathcal{M}(\mathcal{D}_\alpha)$ *?*

We know that $\mathcal{M}(\mathcal{D}_\alpha) \subseteq H^\infty(\mathbb{D})$ and the inclusion is strict, $\mathcal{M}(\mathcal{D}_\alpha) \subsetneq H^\infty(\mathbb{D})$. For example, if we take $f(z) = \sum\limits_{n=1}^{\infty} \frac{z^{n^{4m+1}}}{n^{2m\alpha}}$, $m = \left[\frac{1}{\alpha}\right] + 1$, $z \in \mathbb{D}$, then $f \in H^\infty(\mathbb{D})$ but is not in $\mathcal{D}_\alpha$ and so neither in $\mathcal{M}(\mathcal{D}_\alpha)$), see [**KT**] for detail.

Hence,

$$\mathcal{M}(\mathcal{D}_\alpha) \subsetneq H^\infty(\mathbb{D}) \cap \mathcal{D}_\alpha.$$

LEMMA 3.1.1. $\left\|M_F^R\right\|_{B(\mathcal{D}_\alpha)} \leq \sqrt{10}\left\|M_F^C\right\|_{B(\mathcal{D}_\alpha)}$.

Proof of this lemma can be found in [**KT**].

3.2. Wolff's Theorem for $\mathcal{M}(\mathcal{D}_\alpha)$

THEOREM 2. *Let* H, $\{f_j\}_{j=1}^\infty \subset \mathcal{M}(\mathcal{D}_\alpha)$. *Assume that*

$$\text{(A) } \|M_F^C\| \leq 1$$

$$\textit{and} \quad \text{(B) } |H(z)| \leq \sqrt{\sum_{j=1}^{\infty} |f_j(z)|^2} \ \textit{ for all } z \in \mathbb{D}.$$

Then there exists $K(\alpha) < \infty$ *and* $\{g_j\}_{j=1}^\infty \subset \mathcal{M}(\mathcal{D}_\alpha)$ *with*

$$\|M_G^C\| \leq K(\alpha)$$

$$\textit{and} \quad F\, G^T = H^3.$$

3.3. Outline of the Proof of Theorem 2

Assume that $F \in \mathcal{M}_{l^2}(\mathcal{D}_\alpha)$ and $H \in \mathcal{M}(\mathcal{D}_\alpha)$ satisfy hypotheses (A) and (B) of Theorem 2. Then we will show that there exists a constant $K(\alpha) < \infty$, so that

$$M_{H^3}\, M^{\star}_{H^3} \leq K(\alpha)^2 M^R_F\, M^{\star R}_F. \tag{3.3.1}$$

Using a commutant lifting theorem for multipliers on weighted Dirichlet spaces from [**KT**] (see [**BTV**] for details), there exists $G \in \mathcal{M}(\mathcal{D}_\alpha, \overset{\infty}{\underset{1}{\oplus}} \mathcal{D}_\alpha)$ so that $M^R_F M^C_G = M_{H^3}$ and $\|M^C_G\| \leq K(\alpha)$. The reader should note that such contractions can always be found for the multiplier algebra on reproducing kernel Hilbert spaces with complete Nevanlinna-Pick kernels.

But, from lemma 2.3.1, we know that (3.3.1) is equivalent to the following: there exists a constant $K(\alpha) < \infty$ so that, for any $h \in \mathcal{D}_\alpha$, there exists $\underline{u}_h \in \overset{\infty}{\underset{1}{\oplus}} \mathcal{D}_\alpha$ such that

$$\text{(iii)}\ \ M^R_F(\underline{u}_h) = H^3 h \quad \text{and} \tag{3.3.2}$$

$$\text{(iv)}\ \ \|\underline{u}_h\|_{\mathcal{D}_\alpha} \leq K(\alpha)\, \|h\|_{\mathcal{D}_\alpha}. \tag{3.3.3}$$

Hence, our goal is to show that (3.3.2) and (3.3.3) follow from (A) and (B).

As in Theorem 1, we will assume that F and H are analytic on $\mathbb{D}_{1+\epsilon}(0)$ and for a polynomial, h, we take

$$\underline{u}_h(z) = F(z)^\star\, (F(z)F(z)^\star)^{-1}\, H^3\, h - Q(z)\underline{k}(z), \text{where } \underline{k}(z) \in l^2 \text{ for } z \in \mathbb{D}.$$

We have to find $\underline{k}(z)$ so that $\underline{u}_h \in \overset{\infty}{\underset{1}{\oplus}} \mathcal{D}_\alpha$. Thus we want $\bar{\partial}_z\, \underline{u}_h = 0$ in $\mathbb{D}$.

For this, we will try

$$\underline{u}_h = \frac{F^\star H^3 h}{FF^\star} - Q\, \widehat{W},$$

where $W = \left(\frac{Q^\star F'^\star H^3 h}{(FF^\star)^2}\right)$ and $\widehat{W}$ is the Cauchy transform of W on $\mathbb{D}$.

Then it is clear that $M_F^R(\underline{u}_h) = H^3 h$ and $\underline{u}_h$ is analytic. Hence, we will be done in the smooth case if we are able to find $K(\alpha) < \infty$, only depending on α and thus independent of the polynomial, h, such that

$$\|\underline{u}_h\|_{\mathcal{D}_\alpha} \leq K(\alpha) \|h\|_{\mathcal{D}_\alpha}. \tag{3.3.4}$$

LEMMA 3.3.1. *Let $\underline{w}$ be a harmonic function on $\overline{\mathbb{D}}$, then*

$$\int_{\mathbb{D}} \|Q'\underline{w}\|_{l^2}^2 \, dA_\alpha \leq 8 \|\underline{w}\|_{\mathcal{HD}_\alpha}^2.$$

The proof follows exactly as in Lemma 2.3.3, just replacing the area measure by the weighted measure dA_α.

The next lemma shows that the linear operator T, defined as in Lemma 2.3.4, is bounded on $L^2(\mathbb{D}, dA_\alpha)$. This can be obtained by replacing the measure dA by dA_α and applying Schur's Test in Lemma 2.3.4. In fact, we get a stronger bound with this weight. However, since we are using the weight $\alpha \in (0,1)$, we are interested in getting the exact bound in terms of α. Therefore, we present an alternative proof for this lemma.

LEMMA 3.3.2. *Let the operator T be defined on $L^2(\mathbb{D}, dA_\alpha)$ by*

$$(Tf)(z) = \int_{\mathbb{D}} \tfrac{f(u)}{(u-z)(1-u\bar{z})} dA_\alpha,$$

for $z \in \mathbb{D}$ and $f \in L^2(\mathbb{D}, dA_\alpha)$. Then

$$||Tf||_{A_\alpha}^2 \leq 4\pi^2 C_\alpha^2 ||f||_{A_\alpha}^2,$$

where $C_\alpha = \frac{8}{\alpha^2}$.

PROOF. To show that the singular integral operator, T, is bounded on $L^2(\mathbb{D}, dA_\alpha)$, as in Lemma 2.3.5, we again apply Zygmund's method of rotations [Z] and apply

Schur's lemma an infinite number of times.

Let $f(z) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{jk} z^j \bar{z}^k$, where $a_{ij} = 0$ except for a finite number of terms. For $z = r\,e^{i\theta}$, we relabel to get

$$f(r\,e^{i\theta}) = \sum_{l=-\infty}^{\infty} f_l(r)\,e^{il\theta}, \text{ where } f_l(r) = \sum_{k=0}^{\infty} a_{l+k\,k}\,r^{l+2k}.$$

Then

$$\|f\|^2_{A_\alpha} = \sum_{l=-\infty}^{\infty} \|f_l(r)\|^2_{L^2_\alpha[0,1]},$$

where the measure on $L^2_\alpha[0,1]$ is "$(1-r^2)^{1-\alpha}\,rdr$." Now

$$\begin{aligned}
(Tf)(\lambda) &= \int_{\mathbb{D}} \frac{f(z)}{(z-\lambda)\left(1-z\bar{\lambda}\right)}\,dA(z) \\
&= \sum_{n=0}^{\infty} \overline{\lambda}^n \int_{\mathbb{D}} \left[\frac{1}{z-\lambda}\right] z^n f(z)\,dA(z) \\
&= \sum_{n=0}^{\infty} \overline{\lambda}^n \left[\int_{|z|<|\lambda|} \frac{1}{z-\lambda} + \int_{|\lambda|<|z|} \frac{1}{z-\lambda}\right] z^n f(z)\,dA(z) \\
&= \sum_{n=0}^{\infty} \overline{\lambda}^n \left[\frac{1}{-\lambda}\int_{|z|<|\lambda|} \sum_{p=0}^{\infty}\frac{z^p}{\lambda^p} + \int_{|\lambda|<|z|} \frac{1}{z}\sum_{p=0}^{\infty}\frac{\lambda^p}{z^p}\right] z^n f(z)\,dA(z) \\
&= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}(-1)\,\overline{\lambda}^n\,\frac{1}{\lambda}\int_{|z|<|\lambda|} \frac{z^{n+p}}{\lambda^p}\left(\sum_{l=-\infty}^{\infty} f_l(r)e^{il\theta}\right)\,dA(z) \\
&\quad + \sum_{n=0}^{\infty}\overline{\lambda}^n\sum_{p=0}^{\infty}\int_{|\lambda|<|z|} \frac{\lambda^p}{z^{p+1}} z^n \left(\sum_{l=-\infty}^{\infty} f_l(r)e^{il\theta}\right)\,dA(z) \\
&= \sum_{l=-\infty}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}(-1)\,s^n e^{-int}\frac{e^{-it}}{s}\int_{-\pi}^{\pi}\int_0^s \frac{r^{n+p}e^{i(n+p+l)\theta}}{s^p e^{ipt}}\,f_l(r)rdrd\theta \\
&+ \sum_{l=-\infty}^{\infty}\sum_{n=0}^{\infty} s^n e^{-int}\sum_{p=0}^{\infty}\int_{-\pi}^{\pi}\int_s^1 \frac{s^p e^{ipt}}{r^{p+1}e^{i(p+1)\theta}} r^n e^{in\theta}e^{il\theta} f_l(r)\,rdr\,d\theta. \qquad (\star)
\end{aligned}$$

Taking $l = 0$ and $\lambda = se^{it}$ in $(\star)$, we get that

$$(Tf_0)\,(se^{it}) = \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}(-1)s^n e^{-int}\,\frac{e^{-it}}{s}\int_{-\pi}^{\pi}\int_0^s \frac{r^{n+p}e^{i(n+p)\theta}}{s^p e^{ipt}}\,f_0(r)\,rdr\,d\theta$$
$$+\sum_{n=0}^{\infty}s^n e^{-int}\sum_{p=0}^{\infty}\int_{-\pi}^{\pi}\int_s^1 \frac{s^p e^{ipt}}{r^{p+1}e^{i(p+1)\theta}}r^n e^{in\theta}f_l\,(r)rdr\,d\theta.$$

We know that everything vanishes in the first term except in the case $p + n = 0 \implies p = 0$ and $n = 0$. Similarly, in the second term everything vanishes except the term $n = p + 1$. So $p = n - 1 \geq 0 \implies n \geq 1$.

Thus

$$(Tf_0)\,(se^{it}) = (-2\pi)\int_0^s \frac{f_0(r)\,e^{-it}}{s}\,rdr$$
$$+\,2\pi\sum_{n=1}^{\infty}s^n e^{-int}\int_s^1 \frac{s^{n-1}e^{i(n-1)t}}{r^{n-1+1}}r^n f_0(r)\,rdrd\theta$$

Simplifying the above equality we get,

$$(Tf_0)\,(se^{it}) = -2\pi\int_0^1 \chi_{(0,s)}(r)\,\frac{f_0(r)\,e^{-it}}{s}\,rdr$$
$$+\,2\pi s\,e^{-it}\sum_{n=0}^{\infty}s^{2n}\int_0^1 \chi_{(s,1)}(r)\,f_0(r)\,rdr.$$

So

$$(Tf_0)\,(se^{it}) = 2\,\pi e^{-it}\,(T_0f_0)\,(s),$$

where we define T_0 on $L^2([0,1], rdr)$ by

$$(T_0f_0)\,(s) = -\int_0^1 \chi_{(0,s)}(r)\left(\frac{r}{s}\right)f_0(r)\,dr + \frac{s}{1-s^2}\int_0^1 \chi_{(s,1)}(r)\,f_0(r)\,rdr.$$

A similar calculation shows that when $l \geq 1$, then

$$\left(Tf_l(r)\,e^{il\theta}\right)(se^{it}) = 2\pi e^{i(l-1)t}\,(T_lf_l)\,(s),$$

where

$$\left(T_l f_l\right)(s) = \frac{1}{1-s^2} \int_0^1 \chi_{(s,1)}(r) \left(\frac{s}{r}\right)^{l-1} f_0(r)\, r dr.$$

Similarly, when $l < 0$,

$$\left(T f_l(r) e^{il\theta}\right)\left(s e^{it}\right) = 2\pi e^{i(l-1)t} \left(T_l f_l\right)(s),$$

where

$$\begin{aligned}\left(T_l f_l\right)(s) = &-\left(\sum_{n=0}^{-l} s^{2n}\right) \int_0^1 \chi_{(0,s)}(r) \left(\frac{r}{s}\right)^{1-l} f_l(r) dr \\ &+ \frac{1}{1-s^2} \int_0^1 \chi_{(s,1)}(r)\ (rs)^{1-l} f_l(r) dr.\end{aligned}$$

Hence,

$$\left(T f\right)\left(s e^{it}\right) = 2\pi \sum_{l=-\infty}^{\infty} e^{i(l-1)t} \left(T_l f_l\right)(s),$$

$$\text{for } \left(T_l f_l\right)(s) = \begin{cases} -(\sum\limits_{n=0}^{-l} s^{2n}) \int_0^1 \chi_{(0,s)}(r) \left(\frac{r}{s}\right)^{1-l} f_l(r)\, dr & \\ \quad + \frac{1}{1-s^2} \int_0^1 \chi_{(s,1)}(r)\ (rs)^{1-l} f_l(r)\, dr & \text{for } l \leq 0 \\ \frac{1}{1-s^2} \int_0^1 \chi_{(s,1)}(r) \left(\frac{s}{r}\right)^{l-1} f_0(r)\, r dr & \text{for } l > 0. \end{cases}$$

By our construction,

$$\|T f\|_{A_\alpha}^2 = 4\,\pi^2 \sum_{l=-\infty}^{\infty} ||T_l f_l||_{L_\alpha^2[0,1]}^2,$$

where the measure on $L^2[0,1]$ is "$\left(1-r^2\right)^{1-\alpha} r dr$". Thus, to prove our lemma it suffices to show that

$$\sup_l \|T_l\|_{B(L_\alpha^2[0,1])} \leq C_\alpha < \infty.$$

To illustrate the techniques, we show a detailed estimate for $\|T_0\|_{B(L_\alpha^2[0,1])}$. The other cases follow similarly. For this,

$$\int_0^1 \left|T_0 f_0(se^{it})\right|^2 (1-s^2)^{1-\alpha} sds$$
$$= 2\int_0^1\int_0^1 f_0(u) f_0(v)\left(\int_{\max\{u,v\}}^1 \frac{(1-s^2)^{1-\alpha}ds}{s}\right) udu\, vdv$$
$$+ 2\int_0^1\int_0^1 f_0(x)\, f_0(y)\left[\int_0^{\min\{x,\,y\}} \frac{s^2\left(1-s^2\right)^{1-\alpha}}{(1-s^2)^2} sds\right] xdx\, ydy.$$

Claim 2.

$$\int_0^1\int_0^1 f_0(u) f_0(v)\left(\int_{\max\{u,v\}}^1 \frac{(1-s^2)^{1-\alpha}ds}{s}\right) udu\, vdv$$
$$\leq \frac{25}{16}\int_0^1 |f_0(u)|^2 \left(1-u^2\right)^{1-\alpha} u\, du.$$

We have

$$\int_0^1\int_0^1 f_0(u) f_0(v)\left(\int_{\max\{u,v\}}^1 \frac{(1-s^2)^{1-\alpha}ds}{s}\right) udu\, vdv$$
$$\leq \int_0^1\int_0^1 f_0(u)\, f_0(v)\left[\frac{(1-\max(u^2,v^2))^{1-\alpha}}{\left(1-u^2\right)^{1-\alpha}\left(1-v^2\right)^{1-\alpha}} \ln\left(\frac{1}{\max\{u,v\}}\right)\right]$$
$$\left(1-u^2\right)^{1-\alpha}\left(1-v^2\right)^{1-\alpha} udu\, vdv.$$

Applying Schur's Test with $p(u)=1$, we get that

$$\int_0^v \left[\frac{\left(1-v^2\right)^{1-\alpha}}{\left(1-u^2\right)^{1-\alpha}\left(1-v^2\right)^{1-\alpha}} \ln\left(\frac{1}{v}\right)\right]\left(1-u^2\right)^{1-\alpha} udu$$
$$= \frac{1}{2}\, ln\left(\frac{1}{v^2}\right)\frac{v^2}{2} \leq \frac{1}{4}.$$

Similarly,

$$\int_v^1 \left[\frac{\left(1-u^2\right)^{1-\alpha}}{\left(1-u^2\right)^{1-\alpha}\left(1-v^2\right)^{1-\alpha}} \ln\left(\frac{1}{u}\right)\right]\left(1-u^2\right)^{1-\alpha} udu \leq 1.$$

Therefore,

$$\int_0^1 \left[\frac{(1-\max(u^2,v^2))^{1-\alpha}}{(1-u^2)^{1-\alpha}(1-v^2)^{1-\alpha}} \ln\left(\frac{1}{\max\{u,v\}}\right)\right] p(u)\left(1-u^2\right)^{1-\alpha} udu$$
$$\leq \frac{5}{4}\, p(v).$$

CLAIM 3.

$$\int_0^1\int_0^1 f_0(x)\, f_0(y)\left[\int_0^{\min\{x,\,y\}} \frac{s^2\left(1-s^2\right)^{1-\alpha}}{(1-s^2)^2} sds\right] xdx\, ydy$$
$$\leq \frac{4}{\alpha^2}\int_0^1 |f_0(x)|^2 \left(1-x^2\right)^{1-\alpha} xdx.$$

We have

$$\int_0^1\int_0^1 f_0(x)\, f_0(y)\left[\int_0^{\min\{x,\,y\}} \frac{s^2\left(1-s^2\right)^{1-\alpha}}{(1-s^2)^2} sds\right] xdx\, ydy$$
$$= \int_0^1\int_0^1 f_0(x)\, f_0(y)\left[\frac{1}{2}\int_0^{\min\left\{x^2,y^2\right\}} \frac{s}{(1-s)^{1+\alpha}} ds\right] xdx\, ydy$$
$$\leq \int_0^1\int_0^1 f_0(x)\, f_0(y)\left[\frac{1}{2\alpha}\frac{\min\{x^2,y^2\}}{(1-\min\{x^2,y^2\})^{\alpha}}\right] xdx\, ydy.$$

For this term, we take $p(x) = \frac{1}{(1-x^2)^{\beta}}$, where $\beta = 1-\frac{\alpha}{2}$. Then, calculating, we get that

$$\int_0^y \frac{1}{2\alpha}\frac{x^2}{(1-x^2)^{\alpha+\beta}}\frac{1}{(1-y^2)^{1-\alpha}} xdx \leq \frac{1}{4\alpha\,(\beta+\alpha-1)}\frac{1}{(1-y^2)^{\beta}}.$$

Similarly,

$$\int_y^1 \frac{1}{2\alpha}\frac{y^2}{(1-y^2)^{\alpha}}\frac{1}{(1-y^2)^{1-\alpha}}\frac{1}{(1-x^2)^{\beta}} xdx \leq \frac{1}{4\alpha\,(\beta-1)}\frac{1}{(1-y^2)^{\beta}}.$$

Therefore,

$$\int_0^1 \left[\frac{1}{2\alpha}\frac{\min\{x^2,y^2\}}{(1-\min\{x^2,y^2\})^{\alpha}(1-x^2)^{1-\alpha}(1-y^2)^{1-\alpha}}\right]p(x)\left(1-x^2\right)^{1-\alpha}xdx$$
$$=\left(\frac{1}{4\alpha(\beta+\alpha-1)}+\frac{1}{4\alpha(1-\beta)}\right)p(y)$$
$$=\frac{1}{(4\beta+\alpha-1)(1-\beta)}p(y)=\frac{1}{\alpha^2}p(y).$$

Hence,

$$\int_0^1 |T_0 f_0(s)|^2 (1-s^2)^{1-\alpha} sds \le C_{\alpha_0}^2 \int_0^1 |f_0(s)|^2 \left(1-s^2\right)^{1-\alpha} sds,$$

where $C_{\alpha_0}=\left[\frac{5}{2}+\frac{2}{\alpha^2}\right]\le\frac{5}{\alpha^2}$.

Applying Schur's test for $l>1$ with $p(x)=\frac{1}{(1-x^2)^{\beta}}$, $\beta=1-\frac{\alpha}{2}$, we get the estimate $C_l\le\frac{5}{\alpha^2}$, independent of l. Similarly, for $l<0$ with $p(x)=1$ and $p(x)=\frac{1}{(1-x^2)^{\beta}}$, for each of the two terms, respectively, we get the estimate $C_l\le 6+\frac{2}{\alpha^2}$, independent of l. Thus we conclude that

$$\sup_l \|T_l\|_{B(L^2_\alpha[0,1])}\le\frac{8}{\alpha^2}.$$

This finishes the proof of the lemma. □

LEMMA 3.3.3. *Let $\mathcal{B}$ denote the Beurling transform. Then*

$$||\mathcal{B}(f)||_{A_\alpha}\le\frac{23}{\alpha}||f||_{A_\alpha},\ f\in L^2(\mathbb{D},\ \mathrm{dA}_\alpha).$$

PROOF. Proof of this lemma resembles the proof of lemma 2.3.5. We will replace the weight "rdr" by " $(1-r^2)^{1-\alpha}rdr$". From lemma 2.3.5, we can directly write that

$$\mathcal{B}(f)(se^{it})=\sum_{l=-\infty}^{\infty}e^{i(l-2)t}\mathcal{B}_l f_l(s),$$

$$\text{for } \mathcal{B}_l f_l(s) = \begin{cases} \dfrac{-2}{s^2}\displaystyle\int_0^s f_0(r)\, r dr + f_0\,(s) \quad \text{for} \quad l = 0 \\ \\ -2\,(l-1)s^{l-2}\displaystyle\int_s^1 \frac{f_l(r)}{r^{l-1}} dr - f_l(s) \quad \text{for} \quad l \geq 1 \\ \\ -2(1-l)s^{l-2}\int_0^s f_l(r) r^{1-l} dr + f_l(s) \quad \text{for} \quad l < 0. \end{cases}$$

Thus,

$$||\mathcal{B}f||_{A_\alpha}^2 = \sum_{l=-\infty}^{\infty} ||\mathcal{B}_l f_l||_{L^2_\alpha[0,1]}^2,$$

where the measure on $L^2_\alpha\,[0,1]$ is "$(1-r^2)^{1-\alpha}\, rdr$".

Claim 4.

$$\sup_l ||\mathcal{B}_l||_{B(L^2_\alpha[0,1])} \leq \frac{23}{\alpha} < \infty.$$

Without loss of generality we may assume that $f_l(s) \geq 0$ for all l. For $l < 2$, applying Schur's test with $p(u) = 1$ or $p(u) = \frac{1}{\sqrt{u}}$, we get that $||\mathcal{B}_l||_{B(L^2[0,1])} \leq 7$. Taking the case $l = 0$, we get that

$$\begin{aligned} \int_0^1 |\mathcal{B}_0(f_0(s))|^2 (1-s^2)^{1-\alpha} sds &= \int_0^1 |\frac{-2}{s^2}\int_0^s f_0(r)\, rdr + f_0\,(s)\,|^2 (1-s^2)^{1-\alpha} sds \\ &\leq 4\int_0^1 |\frac{1}{s^2}\int_0^s f_0(r)\, rdr|^2 (1-s^2)^{1-\alpha} sds \\ &+ 2\int_0^1 |f_0\,(s)\,|^2 (1-s^2)^{1-\alpha} sds \\ &= 4\int_0^1 \int_0^1 f_0(u) f_0(v) \left[\int_{\max\{u,v\}}^1 \frac{(1-s^2)^{1-\alpha} sds}{s^4}\right] udu vdv \\ &+ 2\int_0^1 |f_0\,(s)\,|^2 (1-s^2)^{1-\alpha} sds \end{aligned}$$

Now, $4\int_0^1\int_0^1 f_0(u)f_0(v)\left[\int_{\max\{u,v\}}^1 \frac{(1-s^2)^{1-\alpha}sds}{s^4}\right]udu vdv$

$$= 4\int_0^1 \frac{1}{s^4}\left[\int_0^s f_0(u)\,udu\int_0^s f_0(v)\,vdv\right](1-s^2)^{1-\alpha}sds$$
$$= 4\int_0^1\int_0^1 f_0(u)f_0(v)\left[\int_{\max\{u,v\}}^1 \frac{(1-s^2)^{1-\alpha}sds}{s^4}\right]uduvdv$$
$$= 2\int_0^1\int_0^1 f_0(u)f_0(v)\left[\int_{\max\{u^2,v^2\}}^1 \frac{(1-s)^{1-\alpha}ds}{s^2}\right]uduvdv$$
$$\leq 2\int_0^1\int_0^1 f_0(u)f_0(v)\left[-\frac{(1-s)^{1-\alpha}ds}{s}\right]_{\max\{u^2,v^2\}}^1 uduvdv$$
$$= 2\int_0^1\int_0^1 f_0(u)f_0(v)\left[\frac{(1-\max\{u^2,v^2\})^{\,1-\alpha}}{\max\{u^2,v^2\}}\right]uduvdv$$
$$= \int_0^1\int_0^1 f_0(u)f_0(v)k(u,v)\left(1-u^2\right)^{1-\alpha}udu\left(1-v^2\right)^{1-\alpha}vdv,$$

where

$$k(u,v) = \frac{2\,(1-\max\{u^2,v^2\})^{\,1-\alpha}}{\max\{u^2,v^2\}\,(1-u^2)^{1-\alpha}\,(1-v^2)^{1-\alpha}}.$$

By Schur's Test

$$\int_0^1\int_0^1 f_0(u)f_0(v)k(u,v)\left(1-u^2\right)^{1-\alpha}udu\left(1-v^2\right)^{1-\alpha}vdv$$

$$\leq \left(\frac{16}{3}\right)^2\int_0^1 |f_0(u)|^2\left(1-u^2\right)^{1-\alpha}udu$$

if and only if there exists $p(u)\geq 0$ a.e. such that for a.e. $v\geq 0$

$$\int_0^1 k(u,v)\,p(u)\left(1-u^2\right)^{1-\alpha}udu \leq \frac{16}{3}p(v).$$

For this, we will show that

$$\int_0^v \left[\frac{2\,(1-v^2)^{\,1-\alpha}}{v^2\,(1-v^2)^{1-\alpha}}\right]p(u)\,udu \leq \frac{4}{3}p(v) \text{ and } \int_v^1 \left[\frac{2\,(1-u^2)^{\,1-\alpha}}{u^2\,(1-v^2)^{1-\alpha}}\right]p(u)udu \leq 4p(v).$$

If we take $p(u) = \frac{1}{\sqrt{u}}$, then

$$\int_0^v \left[\frac{2\left(1-v^2\right)^{1-\alpha}}{v^2\left(1-v^2\right)^{1-\alpha}}\right] p(u)\, udu = \frac{2}{v^2}\int_0^v u^{\frac{1}{2}} du$$
$$= \frac{4}{3v^2} v^{\frac{3}{2}}$$
$$= \frac{4}{3} p(v).$$

Similarly,

$$\int_v^1 \left[\frac{2\left(1-u^2\right)^{1-\alpha}}{u^2\left(1-v^2\right)^{1-\alpha}}\right] p(u) udu \leq \int_v^1 \frac{2}{u^{\frac{3}{2}}} du, \quad (\text{since } \left(1-u^2\right)^{1-\alpha} \leq \left(1-v^2\right)^{1-\alpha})$$
$$= \left[\frac{-4}{\sqrt{u}}\right]_v^1$$
$$= 4\left[\frac{1}{\sqrt{v}} - 1\right] \leq \frac{4}{\sqrt{v}}.$$

Hence,

$$\int_0^1 k\left(u, v\right) p(u) \left(1-u^2\right)^{1-\alpha} udu \leq \frac{16}{3} p(v).$$

Thus,

$$\int_0^1 |\mathcal{B}_0(f_0(s))|^2 (1-s^2)^{1-\alpha} sds \leq \left(\frac{16}{3}\right)^2 \int_0^1 |f_0(s)|^2 \left(1-s^2\right)^{1-\alpha} sds$$
$$+ 2\int_0^1 |f_0\left(s\right)|^2 (1-s^2)^{1-\alpha} sds$$
$$\leq 31 \int_0^1 |f_0\left(s\right)|^2 (1-s^2)^{1-\alpha} sds.$$

The main cases occur for $l \geq 2$. So let $l \geq 2$ be fixed. Then

$$||\mathcal{B}_l f_l||_{L^2_\alpha[0,1]} \leq 2\left(\int_0^1 |-(l-1)s^{l-2}\int_s^1 \frac{f_l(r)}{r^{l-1}} dr|^2 (1-s^2)^{1-\alpha} sds\right)^{\frac{1}{2}}$$
$$+ ||f_l||_{L^2_\alpha[0,1]}.$$

Now,

$$
\begin{aligned}
&(l-1)^2 \int_0^1 s^{2(l-2)} \left| \int_0^1 \chi_{(s,1)}(r)\, \frac{f_l(r)}{r^{l-1}} dr \right|^2 (1-s^2)^{1-\alpha} s ds \\
&= \int_0^1 \int_0^1 f_l(u) f_l(v) \mathcal{K}(u,v) (1-u^2)^{1-\alpha} (1-v^2)^{1-\alpha} udu\, vdv,
\end{aligned}
$$

where $\mathcal{K}(u,v) = (l-1)^2 \frac{1}{u^l} \frac{1}{v^l} \frac{\int_0^{\min\{u,v\}} s^{2(l-2)}(1-s^2)^{1-\alpha} sds}{(1-u^2)^{1-\alpha}(1-v^2)^{1-\alpha}}$.

We know that

$$
\int_0^1 f_l(u) f_l(v) \mathcal{K}(u,v) \left(1-u^2\right)^{1-\alpha} \left(1-v^2\right)^{1-\alpha} udu\, vdv \leq \left(\frac{4}{\alpha}\right)^2 \int_0^1 |f_l(u)|^2 \left(1-u^2\right)^{1-\alpha} udu \tag{3.3.5}
$$

if and only if there exists $p(u) \geq 0$ a.e. s.t. for a.e. $v \geq 0$

$$
\int_0^1 \mathcal{K}(u,v) p(u) \left(1-u^2\right)^{1-\alpha} udu \leq \frac{4}{\alpha} p(v).
$$

Taking $p(u) = \frac{1}{(1-u^2)^{1-\alpha}}$, it is sufficient to show that

$$
\int_0^1 \left[(l-1)^2 \frac{1}{u^l} \frac{\int_0^{\min\{u,v\}} s^{2l-3}(1-s^2)^{1-\alpha} ds}{(1-u^2)^{1-\alpha}} \right] udu \leq \frac{4}{\alpha} v^l.
$$

Since $(1+s)^{1-\alpha} \leq 2$ and $\frac{1}{2} \leq \frac{1}{(1+u)^{1-\alpha}} \leq 1$, we will be done if we are able to show

$$
\int_0^1 \left[(l-1)^2 \frac{1}{u^l} \frac{\int_0^{\min\{u,v\}} s^{2l-3}(1-s)^{1-\alpha} ds}{(1-u)^{1-\alpha}} \right] udu \leq \frac{4}{\alpha} v^l.
$$

So we are trying to prove that

$$
\begin{aligned}
&\int_0^v \left[(l-1)^2 \frac{1}{u^l} \frac{\int_0^u s^{2l-3}(1-s)^{1-\alpha} ds}{(1-u)^{1-\alpha}} \right] udu \leq \frac{2}{\alpha} v^l \quad \text{and} \\
&\int_v^1 \left[(l-1)^2 \frac{1}{u^l} \frac{\int_0^v s^{2l-3}(1-s)^{1-\alpha} ds}{(1-u)^{1-\alpha}} \right] udu \leq \frac{2}{\alpha} v^l.
\end{aligned}
$$

The first inequality holds as follows

$$
\begin{aligned}
&\int_0^v \left[(l-1)^2 \frac{1}{u^l} \frac{\int_0^u s^{2l-3}(1-s)^{1-\alpha} ds}{(1-u)^{1-\alpha}}\right] u du \\
&= \int_0^v \left[(l-1)^2 s^{2l-3}(1-s)^{1-\alpha} \int_s^v \frac{du}{u^{l-1}(1-u)^{1-\alpha}}\right] ds.
\end{aligned}
$$

Letting $t=(1-u)^\alpha$ and changing variables, we get that

$$
\begin{aligned}
&\int_0^v \left[(l-1)^2 s^{2l-3}(1-s)^{1-\alpha} \int_s^v \frac{du}{u^{l-1}(1-u)^{1-\alpha}}\right] ds \\
&= \int_0^v \left[\frac{1}{\alpha}(l-1)^2 s^{2l-3}(1-s)^{1-\alpha} \int_{(1-v)^\alpha}^{(1-s)^\alpha} \frac{dt}{\left(1-t^{\frac{1}{\alpha}}\right)^{(l-2)+1}}\right] ds \\
&= \int_0^v \left[\frac{1}{\alpha}(l-1)^2 s^{2l-3}(1-s)^{1-\alpha} \sum_{p=0}^{\infty} \binom{l-2+p}{p} \int_{(1-v)^\alpha}^{(1-s)^\alpha} t^{\frac{p}{\alpha}} dt\right] ds.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\int_0^v \left[(l-1)^2 \frac{1}{u^l} \frac{\int_0^u s^{2l-3}(1-s)^{1-\alpha} ds}{(1-u)^{1-\alpha}}\right] u du \\
&\leq \int_0^v \left[\frac{1}{\alpha}(l-1)^2 s^{2l-3}(1-s)^{1-\alpha} \sum_{p=0}^{\infty} \frac{(l-2+p)!}{(l-2)!\,p!}\left[\frac{((1-s)^\alpha)^{\frac{p}{\alpha}+1}}{\frac{p}{\alpha}+1}\right]\right] ds \\
&\leq \frac{2}{\alpha}\int_0^v \left[(l-1)\, s^{2l-3}(1-s)^{1-\alpha} \sum_{q=1}^{\infty} \frac{(l-3+q)!}{(l-3)!\,q!} \frac{(1-s)^q}{(1-s)^{1-\alpha}}\right] ds \\
&= \frac{2}{\alpha}\int_0^v \left[(l-1)\, s^{2l-3}\left(\frac{1}{(1-(1-s))^{l-3+1}} - 1\right)\right] ds \\
&\leq \frac{2}{\alpha}\int_0^v \left[(l-1)\, s^{2l-3}\left(\frac{1}{s^{l-2}}\right)\right] ds \leq \frac{2}{\alpha} v^l.
\end{aligned}
$$

Changing the variable with $t=(1-u)^\alpha$, the second inequality holds as follows:

$$
\begin{aligned}
&\int_v^1 \left[(l-1)^2 \frac{1}{u^l} \frac{\int_0^v s^{2l-3}(1-s)^{1-\alpha} ds}{(1-u)^{1-\alpha}} \right] u du \\
&= \int_0^v \left[(l-1)^2 s^{2l-3}(1-s)^{1-\alpha} \int_v^1 \frac{du}{u^{l-1}(1-u)^{1-\alpha}} \right] ds. \\
&= \int_0^v \left[\frac{1}{\alpha} (l-1)^2 s^{2l-3}(1-s)^{1-\alpha} \int_0^{(1-v)^\alpha} \frac{dt}{\left(1-t^{\frac{1}{\alpha}}\right)^{l-1}} \right] ds \\
&= \int_0^v \left[\frac{1}{\alpha} (l-1)^2 s^{2l-3}(1-s)^{1-\alpha} \sum_{p=0}^{\infty} \begin{pmatrix} l-2+p \\ p \end{pmatrix} \int_0^{(1-v)^\alpha} t^{\frac{p}{\alpha}} dt \right] ds \\
&= \int_0^v \left[\frac{1}{\alpha} (l-1)^2 s^{2l-3}(1-s)^{1-\alpha} \sum_{p=0}^{\infty} \frac{(l-3+p+1)!}{(l-2)(l-3)!\,p!} \left[\frac{(1-v)^{p+\alpha}}{p+1} \right] \right] ds \\
&\leq \frac{2}{\alpha} \int_0^v \left[(l-1) s^{2l-3}(1-s)^{1-\alpha} \sum_{q=1}^{\infty} \frac{(l-3+q)!}{(l-3)!\,q!} \frac{(1-v)^q}{(1-v)^{1-\alpha}} \right] ds \\
&= \frac{2}{\alpha} \int_0^v \left[(l-1) s^{2l-3}(1-s)^{1-\alpha} \left(\frac{1}{(1-(1-v))^{l-3+1}} - 1 \right) \frac{1}{(1-v)^{1-\alpha}} \right] ds \\
&\leq \frac{2}{\alpha} \int_0^v \left[(l-1) s^{2l-3}(1-s)^{1-\alpha} \left(\frac{1-v^{l-2}}{v^{l-2}} \right) \frac{(1-v)^\alpha}{(1-v)} \right] ds \\
&\leq \frac{2}{\alpha} \int_0^v \left[(l-1) s^{2l-3}(1-s)^{1-\alpha}(1-s)^\alpha \left(\frac{1-v^{l-2}}{1-v} \right) \frac{1}{v^{l-2}} \right] ds \\
&= \frac{2(l-1)}{\alpha} \int_0^v \left[\left(s^{2l-3} - s^{2l-2} \right) \left(\frac{1-v^{l-2}}{1-v} \right) \frac{1}{v^{l-2}} \right] ds \\
&= \frac{2(l-1)}{\alpha} \left[\left(\frac{v^{2l-2}}{2l-2} - \frac{v^{2l-1}}{2l-1} \right) \left(\frac{1-v^{l-2}}{1-v} \right) \frac{1}{v^{l-2}} \right] \\
&= \frac{2(l-1)v^l}{\alpha} \left[\left(\frac{(1-v)}{2l-2} + v \left(\frac{1}{2l-2} - \frac{1}{2l-1} \right) \right) \left(\frac{1-v^{l-2}}{1-v} \right) \right] \\
&\leq \frac{1}{\alpha}\, v^l + \frac{2(l-1)v^{l+1}}{\alpha} \left[\left(\frac{1}{2(l-1)(2l-1)} \right) \left(\frac{1-v^{l-2}}{1-v} \right) \right] \\
&\leq \frac{1}{\alpha} v^l + \frac{v^{l+1}}{\alpha} \frac{(l-2)}{(2l-1)} \leq \frac{2}{\alpha} v^l.
\end{aligned}
$$

Hence,

$$\int_0^1 \mathcal{K}(u,v)p(u)\left(1-u^2\right)^{1-\alpha} udu \leq \frac{4}{\alpha}p(v).$$

Case IV. $l<0$. Let's say $l=-k,\ k>0$.

$$\int_0^1 |\mathcal{B}_l(f_l(s))|^2 \left(1-s^2\right)^{1-\alpha} sds$$
$$= \int_0^1 |-2(k+1)s^{-k-2}\int_0^s f_l(r)r^{k+1}dr + f_l(s)|^2 \left(1-s^2\right)^{1-\alpha} sds$$
$$\leq 8(k+1)^2 \int_0^1 s^{-2(k+2)} \left|\int_0^s f_l(r)r^{k+1}dr\right|^2 \left(1-s^2\right)^{1-\alpha} sds$$
$$+ 2\int_0^1 |f_l(s)|^2 \left(1-s^2\right)^{1-\alpha} sds$$

Now,

$$8(k+1)^2 \int_0^1 s^{-2(k+2)} \left|\int_0^s f_l(r)r^{k+1}dr\right|^2 \left(1-s^2\right)^{1-\alpha} sds$$
$$= 8(k+1)^2 \int_0^1 s^{-2(k+2)} \int_0^s f_l(u)u^{k+1}du \int_0^s f_l(v)v^{k+1}dv \left(1-s^2\right)^{1-\alpha} sds$$
$$= 8(k+1)^2 \int_0^1 s^{-2(k+2)} [\int_0^1 \chi_{(0,s)}(u)\, f_l(u)\, u^{k+1}du$$
$$\int_0^1 \chi_{(0,s)}(v) f_l(v)\, v^{k+1}dv] \left(1-s^2\right)^{1-\alpha} sds$$
$$= 8(k+1)^2 \int_0^1\int_0^1 f_l(u)f_l(v)\left[u^k v^k \int_{\max\{u,v\}}^1 s^{-2(k+2)}\left(1-s^2\right)^{1-\alpha} sds\right] uduvdv$$
$$= 4(k+1)^2 \int_0^1\int_0^1 f_l(u)f_l(v)\left[u^k v^k \int_{\max\{u^2,v^2\}}^1 s^{-(k+2)}\left(1-s\right)^{1-\alpha} ds\right] udu\, vdv$$
$$\leq 4(k+1)^2 \int_0^1\int_0^1 f_l(u)f_l(v)\left[u^k v^k \left[\frac{s^{-(k+1)}\left(1-s\right)^{1-\alpha}}{-(k+1)}\right]_{\{u^2,v^2\}}^1\right] udu\, vdv$$
$$\leq \int_0^1\int_0^1 f_l(u)f_l(v)\mathcal{K}(u,v)\, udu\, vdv,$$

where $\mathcal{K}(u,v) = \frac{4(k+1)u^k v^k \left(\max\left\{u^2,v^2\right\}\right)^{-(k+1)}\left(1-\max\left\{u^2,v^2\right\}\right)^{1-\alpha}}{\left(1-u^2\right)^{1-\alpha}\left(1-v^2\right)^{1-\alpha}}$.

We know that

$$\int_0^1 \int_0^1 f_l(u) f_l(v) \mathcal{K}(u,v) \left(1-u^2\right)^{1-\alpha} \left(1-v^2\right)^{1-\alpha} udu\, vdv \leq (12)^2 \int_0^1 |f_l(u)|^2 \left(1-u^2\right)^{1-\alpha} udu \tag{3.3.6}$$

if and only if there exists $p(u) \geq 0$ a.e. s.t. for a.e. $v \geq 0$

$$\int_0^1 \mathcal{K}(u,v) p(u) \left(1-u^2\right)^{1-\alpha} p(u) udu \leq 12 p(v).$$

Taking $p(u) = 1$, this is obviously true as follows,

$$\begin{aligned}
&\int_0^v \left[\frac{4(k+1)u^k v^k \left(v^2\right)^{-(k+1)} \left(1-v^2\right)^{1-\alpha}}{\left(1-u^2\right)^{1-\alpha}\left(1-v^2\right)^{1-\alpha}}\right] \left(1-u^2\right)^{1-\alpha} udu \\
&= 4\left(k+1\right) v^{-k-2} \int_0^v u^{k+1} du \\
&= \frac{4\left(k+1\right)}{k+2} v^{-k-2} \left[u^{k+2}\right]_0^v \\
&\leq 4.
\end{aligned}$$

And

$$\begin{aligned}
&\int_v^1 \left[4(k+1) \frac{u^k v^k}{u^{2(k+1)}} \frac{\left(1-u^2\right)^{1-\alpha}}{\left(1-v^2\right)^{1-\alpha}}\right] udu \\
&\leq 4\left(k+1\right) v^k \int_v^1 u^{-k-1} du \\
&= \frac{4\left(k+1\right) v^k}{k} \left[\frac{1}{u^k}\right]_1^v \\
&\leq 4.2 v^k \left[\frac{1}{v^k} - 1\right] \leq 8.
\end{aligned}$$

Therefore,

$$\int_0^1 \mathcal{K}(u,v)p(u)\left(1-u^2\right)^{1-\alpha} p(u)udu \leq (4+8)\, p(v) = 12p(v). \tag{3.3.7}$$

Hence,

$$\begin{aligned}
&\int_0^1 |\mathcal{B}_l(f_l(s))|^2 \left(1-s^2\right)^{1-\alpha} sds \\
&\leq 144 \int_0^1 |f_l(s)|^2 \left(1-s^2\right)^{1-\alpha} sds + 2\int_0^1 |f_l(s)|^2 \left(1-s^2\right)^{1-\alpha} sds \\
&= 146 \int_0^1 |f_l(s)|^2 \left(1-s^2\right)^{1-\alpha} sds.
\end{aligned}$$

Thus, we conclude that

$$\sup_l ||\mathcal{B}_l||_{B(L^2_\alpha[0,1])} \leq 15 + \frac{8}{\alpha} \leq \frac{23}{\alpha}.$$

□

LEMMA 3.3.4. *If Q is a multiplier of $\mathcal{D}_\alpha$, then*

$$\left(1-|z|^2\right)|\,Q'(z)| \leq \|M_Q\|_{B(\mathcal{D}_\alpha)} \text{ for all } z \in \mathbb{D}.$$

PROOF. Define $\varphi : \mathbb{D} \to \mathbb{D}$ as $\varphi(z) = \frac{Q(z)}{\|M_Q\|_{B(\mathcal{D}_\alpha)}}$ for all $z \in \mathbb{D}$. Now use the Schwartz lemma and the fact that $\|\varphi\|_{\infty,\mathbb{D}} \leq \|M_\varphi\|_{B(\mathcal{D}_\alpha)}$ to complete the proof. □

LEMMA 3.3.5. *If $H \in \mathcal{M}(\mathcal{D}_\alpha)$, then $|H'|^2 dA_\alpha$ is a $\mathcal{D}_\alpha$-Carleson measure with the constant* $4||M_H||^2_{B(\mathcal{D}_\alpha)}$.

PROOF. To prove the lemma, we need to show that

$$\int_{\mathbb{D}} |H'|^2 |g|^2 dA_\alpha \leq 4||M_H||^2_{B(\mathcal{D}_\alpha)}||g||^2_{\mathcal{D}_\alpha} \text{ for all } g \in \mathcal{D}_\alpha.$$

Let $g \in \mathcal{D}_\alpha$, then

$$\begin{aligned}\int_{\mathbb{D}} |H'|^2 |g|^2 dA_\alpha &= \int_{\mathbb{D}} |\,(Hg)' - Hg'|^2 dA_\alpha \\ &\leq 2\int_{\mathbb{D}} |\,(Hg)'\,|^2 dA_\alpha + 2\int_{\mathbb{D}} |Hg'|^2 dA_\alpha \\ &\leq 2\int_{\mathbb{D}} |Hg|^2 d\sigma + 2\int_{\mathbb{D}} |\,(Hg)'\,|^2 dA_\alpha + 2\int_{\mathbb{D}} |Hg'|^2 dA_\alpha \\ &\leq 2||M_H g||^2_{\mathcal{D}_\alpha} + 2||M_H||^2_{B(\mathcal{D}_\alpha)}||g||^2_{\mathcal{D}_\alpha} \\ &\leq 4||M_H||^2_{B(\mathcal{D}_\alpha)}||g||^2_{\mathcal{D}_\alpha}.\end{aligned}$$

This proves the lemma. □

3.4. Proof of Theorem 2

Proof. Assume that (A) and (B) of Theorem 2 hold for F and H, and also that F and H are analytic on $\mathbb{D}_{1+\epsilon}(0)$. Our main goal is to show that there exists a constant, $K(\alpha) < \infty$, independent of ϵ, so that for any polynomial, h, there exists $\underline{u}_h \in \bigoplus_1^\infty \mathcal{D}_\alpha$ such that

$$\|\underline{u}_h\|^2_{\mathcal{D}_\alpha} \leq K(\alpha)\,\|h\|^2_{\mathcal{D}_\alpha},$$

where we take $\underline{u}_h = \frac{F^\star H^3 h}{FF^\star} - Q\widehat{W},\ W = \frac{Q^\star F'^\star H^3 h}{(FF^\star)^2}$. Then $\underline{u}_h$ is analytic and

$$M_F^R(\underline{u}_h) = H^3 h.$$

We know that

$$\|\underline{u}_h\|^2_{\mathcal{D}_\alpha} = \int_{-\pi}^{\pi} \|\underline{u}_h(e^{it})\|^2\, d\sigma(t) + \int_D \|\,(\underline{u}_h(z))'\,\|^2\, dA_\alpha(z).$$

Condition (B) implies that

$$\int_{-\pi}^{\pi} \left\| \frac{F^\star H^3 h}{FF^\star} - Q\,\widehat{W} \right\|^2 d\sigma(t) \leq 15\|h\|^2_\sigma \text{ (see [Tr1]).}$$

Hence, we only need to show that

$$\int_{\mathbb{D}} \left\| \left(\frac{F^\star H^3 h}{F F^\star} - Q\,\widehat{W} \right)' \right\|^2 dA_\alpha(z) \le K(\alpha)^2 \|h\|^2_{\mathcal{D}_\alpha},$$

for some $K(\alpha) < \infty$.

Now,

$$\begin{aligned}
&\int_{\mathbb{D}} \left\| \left(\frac{F^\star H^3 h}{F F^\star} - Q\,\widehat{W} \right)' \right\|^2 dA_\alpha(z) \\
&\quad \le 5 \underbrace{\int_{\mathbb{D}} \left\| \frac{F^\star 3 H^2 H' h}{F F^\star} \right\|^2 dA_\alpha(z)}_{(a'')} + 5 \underbrace{\int_{\mathbb{D}} \left\| \frac{F^\star H^3 h'}{F F^\star} \right\|^2 dA_\alpha(z)}_{(b'')} \\
&\qquad + 5 \underbrace{\int_{\mathbb{D}} \left\| \frac{F^\star H^3 h' F' F^\star}{(F F^\star)^2} \right\|^2 dA_\alpha(z)}_{(c'')} + 5 \underbrace{\int_{\mathbb{D}} \left\| Q'\,\widehat{W} \right\|^2 dA_\alpha(z)}_{(d'')} \\
&\qquad + 5 \underbrace{\int_{\mathbb{D}} \left\| Q\,\left(\widehat{W}\right)' \right\|^2 dA_\alpha(z)}_{(e'')}.
\end{aligned}$$

Then

$$\begin{aligned}
(a'') = \int_{\mathbb{D}} \left\| \frac{F^\star 3 H^2 H' h}{F F^\star} \right\|^2 dA_\alpha(z) &= 9 \int_{\mathbb{D}} \left\| \frac{F^\star}{\sqrt{F F^\star}} \frac{H}{\sqrt{F F^\star}} H\, H' h \right\|^2 dA_\alpha(z) \\
&\le 9 \int_{\mathbb{D}} \|H' h\|^2\, dA_\alpha(z) \\
&\le 36\, \|M_H\|^2_{B(\mathcal{D}_\alpha)}\, \|h\|^2_{\mathcal{D}_\alpha} \quad \text{by Lemma 6.}
\end{aligned}$$

$$(b'') = \int_{\mathbb{D}} \left\| \frac{F^\star H^3 h'}{F F^\star} \right\|^2 dA_\alpha(z) \le \int_{\mathbb{D}} \|h'\|^2\, dA_\alpha(z) \le \|h\|^2_{\mathcal{D}_\alpha}.$$

$$(c'') = \int_{\mathbb{D}} \left\| \frac{F^\star H^3 h F' F^\star}{(FF^\star)^2} \right\|^2 dA_\alpha(z) = \int_{\mathbb{D}} \left\| \frac{F^\star F' F^\star}{\sqrt{FF^\star}} \frac{H^2}{FF^\star} \frac{H}{\sqrt{FF^\star}} h \right\|^2 dA_\alpha(z)$$

$$\leq \int_{\mathbb{D}} \left\| \frac{F^\star F' F^\star}{\sqrt{FF^\star}} h \right\|^2 dA_\alpha(z)$$

$$\leq \int_{\mathbb{D}} \|F'^\star h\|^2 \, dA_\alpha(z) \leq 4 \, \|h\|^2_{\mathcal{D}_\alpha}.$$

We use condition (B) and Lemma 3.3.3 to estimate (e'').

$$\begin{aligned} (e'') &= \int_{\mathbb{D}} \|Q \, \widehat{W}'\|^2 \, dA_\alpha(z) \\ &\leq \int_{\mathbb{D}} \| \left(\widehat{W}\right)' \|^2 \, dA_\alpha(z) && \text{(since } \|Q(z)\|_{B(l^2)} \leq 1) \\ &\leq \left(\frac{23}{\alpha}\right)^2 \int_{\mathbb{D}} \left\| \frac{Q^\star F'^\star H^3 h}{(FF^\star)^2} \right\|^2 dA_\alpha(z) && \text{(by Lemma 3.3.3)} \\ &\leq 4 \left(\frac{23}{\alpha}\right)^2 \|h\|^2_{\mathcal{D}_\alpha}. \end{aligned}$$

So we only need estimates for (d''). For this, we have

$$\int_{\mathbb{D}} \|Q' \, \widehat{W}\|^2 \, dA_\alpha(z) \leq 2 \underbrace{\int_{\mathbb{D}} \|Q' \widehat{W} - Q' \widetilde{\widehat{W}}\|^2 \, dA_\alpha(z)}_{(f'')} + 2 \int_{\mathbb{D}} \|Q' \widetilde{\widehat{W}}\|^2 \, dA_\alpha(z),$$

where $\widetilde{\widehat{W}}(z) = \int_{-\pi}^{\pi} \frac{1-|z|^2}{|1-e^{-it}z|} \widehat{W}(e^{it}) \, d\sigma(t)$ is the harmonic extension of $\widehat{w}$ from $\partial\mathbb{D}$ to $\mathbb{D}$.

Lemma 3.3.1 tells us that

$$\int_{\mathbb{D}} \|Q' \widetilde{\widehat{W}}\|^2 \, dA_\alpha(z) \leq 8 \, \|\widetilde{\widehat{W}}\|^2_{\mathcal{HD}_\alpha}.$$

Also, Lemmas 10 and 11 of [KT] imply that there exists a constant, $C_1 < \infty$, independent of w and α, satisfying

$$\|\widetilde{\widehat{W}}\|^2_{\mathcal{HD}_\alpha} \le C_1 \|W\|^2_{A_\alpha}.$$

But, as we showed above

$$\|W\|^2_{A_\alpha} = \int_{\mathbb{D}} \left\| \frac{Q^\star F' H^3 h}{(FF^\star)^2} \right\|^2 dA_\alpha(z) \le \int_{\mathbb{D}} \|F'^\star h\|^2 \, dA_\alpha(z) \le 4\,\|h\|^2_{\mathcal{D}_\alpha}.$$

Thus,

$$\int_{\mathbb{D}} \|Q' \, \widetilde{\widehat{W}}\|^2 \, dA_\alpha(z) \le C_2 \, \|h\|^2_{\mathcal{D}_\alpha},$$

where $C_2 < \infty$ is independent of W and α.

Now we are just left with estimating (f''). We have

$$\begin{aligned}
(f'') &= \int_{\mathbb{D}} \left\| Q' \, \widehat{W} - Q' \, \widetilde{\widehat{W}} \right\|^2 dA_\alpha(z) \\
&= \int_{\mathbb{D}} \left\| Q' \left[-\frac{1}{\pi} \int_{\mathbb{D}} \frac{W(u)}{u-z} dA(u) - \int_{-\pi}^{\pi} \frac{1-|z|^2}{|1-e^{-it}z|} \widehat{W}(e^{it}) d\sigma(t) \right] \right\|^2 dA_\alpha(z) \\
&= \frac{1}{\pi^2} \int_{\mathbb{D}} \left\| Q' \int_{\mathbb{D}} W(u) \left[\frac{1}{u-z} + \int_{-\pi}^{\pi} \frac{1-|z|^2}{|1-e^{-it}z|} \, e^{-it} \frac{1}{1-ue^{-it}} \, d\sigma(t) \right] dA(u) \right\|^2 dA_\alpha(z) \\
&= \frac{1}{\pi^2} \int_{\mathbb{D}} \left\| Q' \int_{\mathbb{D}} W(u) \left[\frac{1}{u-z} + \frac{\bar{z}}{1-u\bar{z}} \right] dA(u) \right\|^2 dA_\alpha(z) \\
&= \frac{1}{\pi^2} \int_{\mathbb{D}} \left\| Q' \int_{\mathbb{D}} W(u) \left[\frac{1-|z|^2}{(u-z)(1-u\bar{z})} \right] dA(u) \right\|^2 dA_\alpha(z) \\
&= \frac{1}{\pi^2} \int_{\mathbb{D}} \left\| Q'(z) \, (1-|z|^2) \, T(W)(z) \right\|^2 dA_\alpha(z) \\
&\le \frac{\|M_Q\|^2}{\pi^2} \, \|T(W)\|^2_{A_\alpha} \qquad \text{by Lemma 3.3.4} \\
&\le \frac{256}{\alpha^4} \, \|M_Q\|^2 \, \|W\|^2_{A_\alpha} \quad \text{by Lemma 3.3.2} \\
&\le \frac{1024}{\alpha^4} \|M_Q\|^2 \, \|h\|^2_{\mathcal{D}_\alpha}.
\end{aligned}$$

By Lemma 9 of [KT], we have that $\|M_Q\|_{B\left(\overset{\infty}{\underset{1}{\oplus}} \mathcal{HD}_\alpha\right)} \le \sqrt{86}$.

Combining all these pieces, we see that in the smooth case

$$\|\underline{u}_h\|_{\mathcal{D}_\alpha}^2 \leq K(\alpha)^2 \|h\|_{\mathcal{D}_\alpha}^2 ,$$

where $K(\alpha) = K_1 \|M_H\|_{B(\mathcal{D}_\alpha)} + \frac{K_2}{\alpha^2}$, where $K_1 < \infty$ and $K_2 < \infty$ are constants independent of h, ϵ, and α.

Now applying the compactness argument as in Theorem 1, we can conclude the theorem.

This ends our proof. □

CHAPTER 4

GENERALIZED IDEAL PROBLEM

In this chapter, by providing a sufficient condition, we will obtain a general result for the "Ideal Problem". Independently, Cegrel, Pau, Trent, and Treil generalized the ideal problem in $H^\infty(\mathbb{D})$. The strongest positive known result in this direction is due to Treil [**T3**]. Our result is a consequence of Trent's [**Tr1**] ideal problem in $H^\infty(\mathbb{D})$.

4.1. Background of Ideal Problem

As we stated in Chapter 1, let $f_1,\ f_2, ..., f_n \in H^\infty(\mathbb{D})$, and suppose $H \in H^\infty(\mathbb{D})$. Clearly, the condition

$$|H(z)| \leq \sqrt{F(z)F(z)^\star} \text{ for all } z \in \mathbb{D}$$

is necessary for the solvability of the Bezout equation $F(z)G(z)^T = H(z)$, for all $z \in \mathbb{D}$. However, this condition, as we explained above, is obviously not sufficient.

One can ask if a stronger condition

$$|H(z)| \leq C\left(F(z)F(z)^\star\right)^{\frac{p}{2}} \text{ for all } z \in \mathbb{D}, \tag{4.1.1}$$

for $p > 2$ is sufficient for the existence of $G = \{g_j\}_{j=1}^n, g_j \in \mathcal{M}(\mathbb{D})$ such that $F(z)G(z)^T = H(z)$. It was understood in the early 1980s that the condition is sufficient for $p > 2$, but it is not sufficient if $p < 2$. The question for $p = 2$ remained open for almost 20 years, until Treil [**T1**] showed that the condition (4.1.1) is not sufficient for $p = 2$ as well.

We can consider a more general question, namely to ask for which strictly increasing function α the condition

$$|H(z)| \leq F(z)F(z)^{\star}\alpha\left(F(z)F(z)^{\star}\right) \text{ for all } z \in \mathbb{D}$$

implies the solvability of the equation $F(z)G(z)^T = H(z)$.

It was shown by Trent [**Tr1**] that the answer is affirmative for any increasing functions $\alpha : [0,1] \to [0,1]$ such that

$$\int_0^1 \frac{\alpha(t)}{t} dt < \infty \text{ and } \int_0^1 \frac{1}{t} \int_0^t \alpha(u) du\, dt < \infty.$$

For example, any function of the type

$$\alpha(t) = \frac{1}{\left(\ln\left(\frac{1}{t}\right)\right)^{\frac{3}{2}} \left(\ln\ln\left(\frac{1}{t}\right)\right) \dots \left(\underbrace{\ln\ln\dots\ln\left(\frac{1}{t}\right)}_{m\, times}\right)\left(\underbrace{\ln\ln\dots\ln\left(\frac{1}{t}\right)}_{m+1\, times}\right)^{1+\epsilon}},\ \epsilon > 0$$

works.

Then it was shown by Treil [**T3**] that the answer is affirmative for any increasing functions $\alpha : [0,1] \to [0,1]$ such that

$$\int_0^1 \frac{\alpha(t)}{t} dt < \infty.$$

That means, the exponent "$\frac{3}{2}$" in Trent's result is not a critical one for $H^\infty(\mathbb{D})$. One can take any function of the type

$$\alpha(t) = \frac{1}{\left(\ln\left(\frac{1}{t}\right)\right) \left(\ln\ln\left(\frac{1}{t}\right)\right) \dots \left(\underbrace{\ln\ln\dots\ln\left(\frac{1}{t}\right)}_{m\, times}\right)\left(\underbrace{\ln\ln\dots\ln\left(\frac{1}{t}\right)}_{m+1\, times}\right)^{1+\epsilon}},\ \epsilon > 0.$$

In this chapter, we will extend Trent's result to the multiplier algebra on Dirichlet space. However, it seems quite unlikely that a size condition (C) alone should characterize the solvability of the Bezout equation $F(z)G(z)^T = H(z)$ in the multiplier algebra of Dirichlet space. Because of the involvement of the derivative term in the Dirichlet norm

$$\|f\|_{\mathcal{D}}^2 = \int_{-\pi}^{\pi} \|f(e^{it})\|^2 \, d\sigma(t) + \int_D \|f'\|^2 \, dA(z),$$

we require some additional condition in terms of H'. Thus a condition as in (D) is quite natural.

4.2. Generalized Ideal Theorems on $\mathcal{M}(\mathcal{D})$

THEOREM 3. *Let* $\{f_j\}_{j=1}^{\infty} \subset \mathcal{M}(\mathcal{D})$ *and* $H \in \mathcal{M}(\mathcal{D})$. *Assume that*

(C) $|H(z)| \leq (F(z)\, F(z)^{\star})^{\frac{3}{2}}$ *and*

(D) $|H'(z)| \leq C|\partial(F(z)\, F(z)^{\star})^{\frac{3}{2}}|$ *for all* $z \in \mathbb{D}$ *and for some constant* $C < \infty$,

then there exists $G(z) = (g_1(z), g_2(z), ...)$, $g_j \in \mathcal{M}(\mathcal{D})$ *so that*

$$H(z) = F(z)G(z)^T \text{ for all } z \in \mathbb{D} \quad \text{and}$$

$$||M_G|| \leq C_0 \text{ for some constant } C_0 < \infty.$$

PROOF. Applying the argument as in Theorem 1, and with given conditions (C) and (D), it's enough to show that there exists $C < \infty$ so that for any fixed $h \in \mathcal{D}$, there exists $\underline{u}_h \in \bigoplus_1^{\infty} \mathcal{D}$ such that

$$F\underline{u}_h = Hh, \text{ and } ||u_h||_{\mathcal{D}}^2 \leq C||h||_{\mathcal{D}}^2.$$

Here, we will take

$$\underline{u}_h = \frac{F^\star H h}{F F^\star} - Q\,\widehat{W}, \;\; W = \frac{Q^\star F'^\star H h}{(FF^\star)^2}.$$

One should understand that the functions F and H are taken to be smooths up to the boundary and the obtained constant C would be a uniform constant so that we can allow the compactness argument, as before, to complete the proof.

Using the definition of Dirichlet norm,

$$\|\underline{u}_h\|^2_{\mathcal{D}} = \int_{-\pi}^{\pi} \|\underline{u}_h(e^{it})\|^2 \, d\sigma(t) + \int_{\mathbb{D}} \|\left(\underline{u}_h(z)\right)'\|^2 \, dA(z).$$

Condition (C) implies that

$$\int_{-\pi}^{\pi} \left\| \frac{F^\star H h}{F F^\star} - Q\,\widehat{W} \right\|^2 d\sigma(t) \le (15)^2 \|h\|^2_\sigma, \;\; \text{(see [Tr1])}.$$

Hence, we only need to show that

$$\int_{\mathbb{D}} \left\| \left(\frac{F^\star H h}{F F^\star} - Q\,\widehat{W} \right)' \right\|^2 dA(z) \le C_1^2 \|h\|^2_{\mathcal{D}}$$

for some $C_1 < \infty$.

Now

$$\begin{aligned}
\int_{\mathbb{D}} \le \left\| \left(\frac{F^\star H h}{F F^\star} - Q\widehat{W} \right)' \right\|^2 dA(z) &\le 2 \int_{\mathbb{D}} \left\| \left(\frac{F^\star H h}{F F^\star} \right)' \right\|^2 dA(z) + 2 \int_{\mathbb{D}} \left\| \left(Q\widehat{W} \right)' \right\|^2 dA(z) \\
&\le 5 \underbrace{\int_{\mathbb{D}} \left\| \frac{F^\star H' h}{F F^\star} \right\|^2 dA(z)}_{(a''')} + 5 \underbrace{\int_{\mathbb{D}} \left\| \frac{F^\star H h'}{F F^\star} \right\|^2 dA(z)}_{(b''')} \\
&+ 5 \underbrace{\int_{\mathbb{D}} \left\| \frac{F^\star H h F' F^\star}{(FF^\star)^2} \right\|^2 dA(z)}_{(c''')} + 5 \underbrace{\int_{\mathbb{D}} \left\| Q'\,\widehat{W} \right\|^2 dA(z)}_{(d''')} \\
&+ 5 \underbrace{\int_{\mathbb{D}} \|Q \left(\widehat{W}\right)'\|^2 \, dA(z)}_{(e''')}.
\end{aligned}$$

Then,

$$(a''') = \int_{\mathbb{D}} \left\| \frac{F^\star H' h}{FF^\star} \right\|^2 dA(z) \leq \int_{\mathbb{D}} \left\| \frac{F^\star}{\sqrt{FF^\star}} \frac{H'}{\sqrt{FF^\star}} h \right\|^2 dA(z)$$
$$\leq \|h\|_{\mathcal{D}}^2 \ \text{(using (D))}.$$

$$(b''') = \int_{\mathbb{D}} \left\| \frac{F^\star H h'}{FF^\star} \right\|^2 dA(z) \leq \int_{\mathbb{D}} \|h'\|^2 \, dA(z)$$
$$\leq \|h\|_{\mathcal{D}}^2.$$

$$\begin{aligned}(c''') = \int_{\mathbb{D}} \left\| \frac{F^\star H h F' F^\star}{(FF^\star)^2} \right\|^2 dA(z) &= \int_{\mathbb{D}} \left\| \frac{F^\star}{\sqrt{FF^\star}} \frac{F' F^\star}{\sqrt{FF^\star}} \frac{H}{FF^\star} \, h \right\|^2 dA(z) \\ &\leq \int_{\mathbb{D}} \| \frac{F' F^\star}{\sqrt{FF^\star}} \, h \|^2 \, dA(z) \\ &= \int_{\mathbb{D}} \left\| \frac{F F'^\star F' F^\star}{FF^\star} \, h \right\| dA(z) \\ &\leq \int_{\mathbb{D}} ||F'^\star h||^2 dA(z) \\ &\leq 4 \, \|h\|_{\mathcal{D}}^2.\end{aligned}$$

Using Lemma 2.3.5 on the Beurling transform, we get that

$$\begin{aligned}(e''') = \int_{\mathbb{D}} \left\| Q \, \widehat{W}' \right\|^2 dA(z) &\leq (13)^2 \int_{\mathbb{D}} \left\| Q \frac{Q^\star F'^\star H h}{(FF^\star)^2} \right\|^2 dA(z) \\ &= (13)^2 \int_{\mathbb{D}} \left\| \frac{Q}{\sqrt{FF^\star}} \frac{Q^\star}{\sqrt{FF^\star}} \frac{H}{FF^\star} F'^\star h \right\|^2 dA(z) \\ &\leq (13)^2 \int_{\mathbb{D}} \left\| F'^\star h \right\|^2 dA(z) \\ &\leq (26)^2 \|h\|_{\mathcal{D}}^2.\end{aligned}$$

Now we are left with estimating (d'''). But we can easily see that the (d''') is dominated by $4(d')$ of Theorem 1 as follows:

$$
\begin{aligned}
\|W\|_A^2 &= \int_{\mathbb{D}} \left\| \frac{Q^\star F^{'\star} H h}{(FF^\star)^2} \right\|^2 dA(z) \\
&= \int_{\mathbb{D}} \left\| \frac{Q^\star}{\sqrt{FF^\star}}\ F^{'\star} h\ \frac{H}{(FF^\star)^{3/2}} \right\|^2 dA(z) \\
&\le \int_{\mathbb{D}} \left\| F^{'\star} h \right\|^2 dA(z) \\
&\le 4\|h\|_{\mathcal{D}}^2 .
\end{aligned}
$$

Hence,

$$
(d''') = \int_{\mathbb{D}} \|Q'\ \widehat{W}\|^2 \, dA(z) \le 2 \underbrace{\int_{\mathbb{D}} \|Q'\widehat{W} - Q'\widetilde{\widehat{W}}\|^2 \, dA(z)}_{(f''')} + 2 \int_{\mathbb{D}} \|Q'\widetilde{\widehat{W}}\|^2 \, dA(z).
$$

As in Theorem 1,

$$
\int_{\mathbb{D}} \|Q'\widetilde{\widehat{W}}\|^2 \, dA(z) \le 8 \left(\|W\|_A^2 + \|\widehat{W}\|_\sigma^2 \right).
$$

Also, we have just shown above that

$$
\|W\|_A^2 \le 4 \, \|h\|_{\mathcal{D}}^2
$$

and

$$
\|\widehat{W}\|_\sigma^2 = \int_{-\pi}^{\pi} \left\| \left(\widehat{W} \right) \right\|^2 d\sigma(t) \le 15 \, \|h\|_\sigma^2 .
$$

Thus,

$$
\int_{\mathbb{D}} \|Q'\widetilde{\widehat{W}}\|^2 \, dA(z) \le 8 \ \left[4 \, \|h\|_{\mathcal{D}}^2 + 15 \, \|h\|_\sigma^2 \right].
$$

Again, as in (f'), we have that

$$(f''') \leq C^2 \frac{\|M_Q\|^2}{\pi^2} \|w\|^2 \leq \left(\frac{172C}{\pi^2}\right)^2 \|h\|_{\mathcal{D}}^2.$$

Collecting all these pieces, we see that

$$\|\underline{u}_h\|_{\mathcal{D}}^2 \leq K \|h\|_{\mathcal{D}}^2 ,$$

where K is a uniform constant, independent of h.

This ends our proof. □

THEOREM 4. *Let $\{f_j\}_{j=1}^{\infty} \subset \mathcal{M}(\mathcal{D})$ and $H \in \mathcal{M}(\mathcal{D})$. Assume that*

$$(C)\ \ |H(z)| \leq (F(z)\, F(z)^{\star})\, \alpha(F(z)F(z)^{\star}) \ \ \textit{and}$$

$$(D)\ \ |H'(z)| \leq |\partial\, (F(z)\, F(z)^{\star})\,|\ \alpha(F(z)F(z)^{\star})\, \textit{for all } z \in \mathbb{D},$$

where α is an increasing, onto C^1-smooth function such that $\alpha(0) = 0$,

$$\alpha(t) \leq \sqrt{t},\ \int_0^1 \frac{\alpha(t)}{t} dt < \infty \ \ \textit{and} \ \ \int_0^1 \frac{1}{t} \int_0^t \alpha(u) du\, dt < \infty.$$

Then there exists $G(z) = (g_1(z),\, g_2(z), ...)\, ,\, g_j \in \mathcal{M}(\mathcal{D})$ so that

$$H(z) = F(z)G(z)^T \ \ \textit{for all } z \in \mathbb{D} \quad \textit{and}$$

$$||M_G|| \leq C_0 \ \textit{for some constant } C_0 < \infty.$$

PROOF. Proof of this Theorem follows from the proof of Theorem 3. □

REMARK 2. *Both Theorems 3 and 4 hold true on weighted Dirichlet spaces.*

CHAPTER 5

FUTURE RESEARCH

As we solved the ideal problem in Dirichlet space and weighted Dirichlet spaces, this led us to think about similar problems in other RKHS with complete Nevanlinna-Pick kernels. In this chapter, we will mention some open problems and our progress towards the results.

5.1. Corona Theorem and Wolff's Ideal Problem in Q_p Spaces.

DEFINITION 5.1.1. *For $p \geq 0$, Q_p spaces are the space of analytic functions with norm*

$$\|f\|_{Q_p}^2 = \int_{-\pi}^{\pi} |f|^2 d\sigma + \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 \left(1 - |\varphi_a(z)|^2\right)^p dA(z) < \infty,$$

where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ is a möbius map and $p \in (0, \infty)$.

The usual "Corona Theorem" in the Banach algebra $H^\infty \cap Q_p$ was proven by Nicolau and Xiao in [NX]; and later Pau [P] proved the corona theorem on the multiplier algebra of Q_p spaces. Both proofs were valid only for a finite number of generators. Two important questions arise:

PROBLEM 5.1.1. *Can we extend the corona theorem for an infinite number of generators on $\mathcal{M}(\mathbb{D})$?*

PROBLEM 5.1.2. *Extension of Wolff's theorem on $\mathcal{M}(Q_p)$.*

5.2. Ideal Problem in Drury-Arveson Spaces.

DEFINITION 5.2.1. *Let* $\mathbb{B}_n$ *be the unit ball in* $\mathbb{C}^n$. *Let* dz *be Lebesgue measure on* $\mathbb{C}^n$ *and let* $d\lambda_n(z) = \left(1-|z|^2\right)^{-n-1} dz$ *be the invariant measure on the ball. For an integer* $m \geq 0$ *and for* $0 \leq \sigma < \infty$, $1 < p < \infty$, $m + \sigma < \frac{n}{p}$, *we define the analytic Besov-Sobolev spaces* $B_p^\sigma(\mathbb{B}_n)$ *to consist of those holomorphic functions* f *on the ball such that*

$$\left\{ \sum_{k=0}^{m-1} |f^{(k)}(0)|^p + \int_{\mathbb{B}_n} \left| \left(1-|z|^2\right)^{m+\sigma} f^{(m)}(z) \right|^p d\lambda_n(z) \right\}^{1/p} < \infty.$$

Here $f^{(m)}$ *is the* m^{th} *order complex derivative of* f. *The spaces* $B_p^\sigma(\mathbb{B}_n)$ *are independent of* m *and are Banach spaces with the norm given above.*

For $p = 2$, *these are Hilbert spaces with the obvious inner product. This scale of spaces includes the Dirichlet spaces* $B_2(\mathbb{B}_n) = B_2^0(\mathbb{B}_n)$, *weighted Dirichlet-type spaces with* $0 < \sigma < \frac{1}{2}$, *the Drury-Arveson Hardy spaces* $H_n^2 = B_2^{\frac{1}{2}}(\mathbb{B}_n)$ *(also known as the symmetric Fock spaces over* $\mathbb{C}^n$*), the Hardy spaces* $H^2(\mathbb{B}_n) = B_2^{\frac{n}{2}} b(\mathbb{B}_n)$, *and the weighted Bergman spaces with* $\sigma < \frac{n}{2}$.

PROBLEM 5.2.1. *Let* $f, h \in \mathcal{M}(H_n^2)$ *and* $|h(z)| \leq |f(z)|$ *for all* $z \in \mathbb{B}^n$. *Can we show*

$$h^3 \in \mathcal{I}(\{f\})?$$

For this, we aim to show that

$$\frac{h^3}{f} \in \mathcal{M}(H_n^2)?$$

PROBLEM 5.2.2. *Extension of this result to a finite number of multipliers.*

5.3. Ideal Problem in Dirichlet-Type Spaces.

DEFINITION 5.3.1. *Let μ be a positive finite Borel measure on the unit circle $\mathbb{T}$, and let P_μ be its harmonic extension to $\mathbb{D}$, i.e.*

$$P_\mu = \int_{\mathbb{T}} \frac{1-|z|^2}{|\xi - z|^2} d\mu(\xi), \;\; z \in \mathbb{D}.$$

The corresponding $\mathcal{D}(\mu)$ space is defined to be the set of analytic functions f for which

$$\mathcal{D}_\mu(f) := \int_{\mathbb{D}} |f'(z)|^2 P_\mu(z)\, dA(z) < \infty.$$

In particular, if μ is taken to be the normalized Lebesgue measure on $\mathbb{T}$, then $P_\mu(z) = 1$, $z \in \mathbb{D}$ and therefore $\mathcal{D}(\mu) = \mathcal{D}$, the classical Dirichlet spaces.

Shimorin [**S**] showed that the $\mathcal{D}(\mu)$ spaces have complete Nevanlinna-Pick kernels.

Recently, Luo [L] has proved the analogue of "Carleson's Corona Theorem" in $\mathcal{M}(\mathcal{D}(\mu))$.

PROBLEM 5.3.1. *Can we extend our result to $\mathcal{M}(\mathcal{D}(\mu))$?*

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