

COFINITE GRAPHS AND THEIR PROFINITE COMPLETIONS

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ABSTRACT

We generalize the work of B. Hartley, to the category of topological graphs.

The completion of a topological group in Hartley's work can be thought of as a particular example of the completion of a general uniform space. We were not able to find a reference to any obvious way of topologizing a graph and constructing their completions, defined accordingly. While topological groups share the beautiful property that any continuous homomorphism between two topological groups is uniformly continuous, a similar statement is not true in the case of topological graphs. So we had to consider topologies that are induced by uniformities, in order to topologize our graph structures, and talk about their completions.

In this dissertation generalize the idea of **cofinite groups**, due to B. Hartley, [2]. First we define **cofinite spaces** in general. Then, as a special case, we study **cofinite graphs and their uniform completions**. We are able to show that these completions are also cofinite graphs and being compact Hausdorff and totally disconnected they are rather regarded as profinite graphs.

The idea of constructing a cofinite graph starts with defining a uniform topological graph Γ , in an appropriate fashion. We endow abstract graphs with uniformities corresponding to separating filter bases of equivalence relations with finitely many equivalence classes over Γ . By taking finitely many equivalence classes, we want to ensure the production of profinite structures over our **topological graphs** on taking the projective limit of the corresponding quotient graphs. It is established that for any cofinite graph there exists a unique cofinite completion.

Generalizing Hartley's idea of cofinite groups and obtaining the structure for cofinite graphs we start establishing a parallel theory of cofinite graphs which in

many ways can also be thought of as generalizations of the well-known works on profinite graphs by Pavel Zalesskii and Luis Ribes.

Suitably defining the concept of cofinite connectedness of a cofinite graph we find that many of the properties of connectedness of topological spaces have analogs for cofinite connectedness. As an immediate consequence we obtain the following generalized characterization of the connected Cayley graphs of cofinite groups:

G be a cofinite group and let $\Gamma = \Gamma(G, X)$ be the Cayley graph. Then Γ can be given a suitable cofinite uniform topological structure so that X generates G (topologically) iff Γ is cofinitely connected.

Our immediate next concern is developing group actions on cofinite graphs. Defining the action of an abstract group over a cofinite graph in the most natural way we are able to characterize a unique way of uniformizing an abstract group with a cofinite structure, obtained from the cofinite structure of the graph in the underlying action, so that the afore said action becomes uniformly continuous. We show that the aforesaid actions can actually be extended to the structures' of corresponding cofinite completions, preserving the underlying character of the original group action. It should be noted here that this enables us to produce cofinite groups out of an ordinary group action over a cofinite graph.

DEDICATION

I dedicate this dissertation to my parents Susim Kumar Das and Kajal Das, who have sacrificed all of their joys only to see me joyous.

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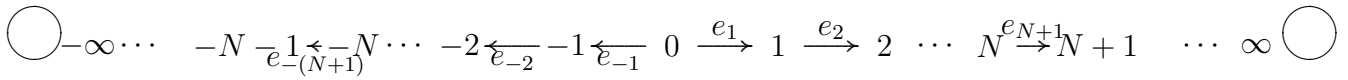
$$q_N(x) = \begin{cases} x & |x| \leq N+1 \\ N+1 & x \geq N+2 \\ -(N+1) & x \leq -(N+2) \end{cases}, \quad q_N(e_x) = \begin{cases} e_x & |x| \leq N+1 \\ e & x \geq N+2 \\ e' & x \leq -(N+2) \end{cases}$$

Consider the uniformity Φ_1 over Γ which is induced by the fundamental system of entourages $R_N = (q_N \times q_N)^{-1}[D(\Gamma_N)]$. Clearly, with respect to Φ_1 , q_N is uniformly continuous for all $N \in \mathbb{N} \cup \{0\}$. Let τ_{Φ_1} be the topology induced by Φ_1 . If $x \in V(\Gamma)$, then $R_{|x|}[x] = \{x\}$. Similarly, if $e_x \in E(\Gamma)$, then $R_{|x|}[e_x] = \{e_x\}$. Hence τ_{Φ_1} represents the discrete topology over Γ .

Now let us define $\varphi_{ij}: \Gamma_j \rightarrow \Gamma_i$ for all $i \leq j \in \mathbb{N} \cup \{0\}$ via $\varphi_{ij}(x) =$

$$\begin{cases} x & |x| \leq i \\ i & x \geq i+1 \\ -i & x \leq -(i+1) \end{cases}, \quad \varphi_{ij}(e_x) = \begin{cases} e_x & |x| \leq i \\ e & x \geq i+1 \\ e' & x \leq -(i+1) \end{cases}, \quad \varphi_{ij}(e) = e, \varphi_{ij}(e') = e'.$$

Clearly, each φ_{ij} is a uniformly continuous map of graphs and for $i = j$, $\varphi_{ii} = \text{id}_{\Gamma_i}$. Also if $i \leq j \leq k$, then $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$. Hence $(\Gamma_i, \varphi_{ij})_{i \leq j \in \mathbb{N} \cup \{0\}}$ forms an inverse system of finite discrete graphs. Then by Theorem 6.5, we can deduce that $\hat{\Gamma} = \varprojlim_{i \in \mathbb{N} \cup \{0\}} \Gamma_i$ is a profinite completion of Γ . Now let us consider the following graph Δ :



with $V(\Delta) = \mathbb{Z} \cup \{-\infty, \infty\}$, $E(\Gamma) = \{e_x \mid x \in V(\Gamma) \setminus \{0, \infty, -\infty\}\} \cup \{e, e'\}$ with $s(e_x) = x - 1$, if $x > 0$, $s(e_x) = x + 1$, if $x < 0$, $t(e_x) = x$, $s(e) = \infty = t(e)$, $s(e') = -\infty = t(e')$. Now let $G_1 = \{x \mid x > 0\}$, $G_2 = \{e_x \mid x > 0\}$, $G_3 = \{x \mid x < 0\}$, $G_4 = \{e_x \mid x < 0\}$, $p_1 = \infty$, $p_2 = e$, $p_3 = -\infty$, $p_4 = e'$. Then $\tau = \{O \subseteq \Delta \mid O \cap (\Delta \setminus [\cup_{i=1}^4 \{p_i\}]) \text{ is open in } (\Delta \setminus [\cup_{i=1}^4 \{p_i\}]) \text{ with respect to its discrete topology and } p_i \in O \text{ implies } [(\Delta \setminus [\cup_{i=1}^4 \{p_i\}]) \setminus O] \cap G_i \text{ is finite}\}$ forms the topology over Δ . Hence Δ is compact, Hausdorff, totally disconnected and thus a compactification (4 point compactification) of $\Delta \setminus \{\infty, -\infty, e, e'\}$ and thus a compactification of the graph Γ .

Let us define maps $\theta_N: \Delta \rightarrow \Gamma_N$ via

$$\theta_N(x) = \begin{cases} x & |x| \leq N+1 \\ i & x \geq N+2 \\ -i & x \leq -(N+2) \end{cases}, \quad \theta_N(e_x) = \begin{cases} e_x & |x| \leq N+1 \\ e & x \geq N+2 \\ e' & x \leq -(N+2) \end{cases},$$

$$\theta_N(\infty) = N+1, \theta_N(-\infty) = -N-1, \theta_N(e) = e, \theta_N(e') = e'.$$

Clearly each θ_N is a uniformly continuous map of graphs.

$$\begin{array}{ccc} & \Delta & \\ & \swarrow \theta_j & \searrow \theta_i \\ \Gamma_j & \xrightarrow{\varphi_{ij}} & \Gamma_i \end{array}$$

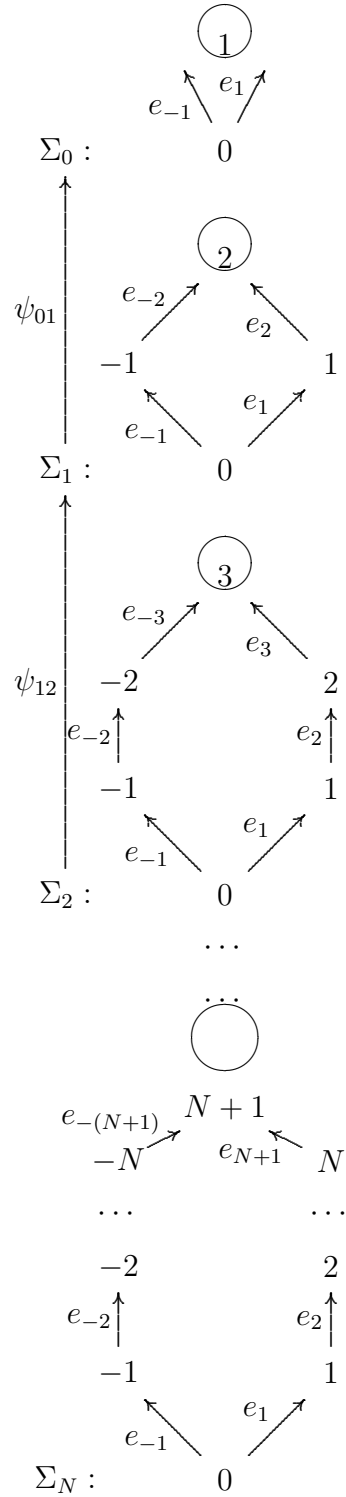
Thus $(\Delta, \theta_N)_{N \in \mathbb{N} \cup \{0\}}$ is compatible with the inverse system $(\Gamma_i, \varphi_{ij})_{i \leq j \in \mathbb{N} \cup \{0\}}$ and thus there exists a uniformly continuous map of graphs $\theta: \Delta \rightarrow \widehat{\Gamma}$ such that for the canonical projection maps $\varphi_N: \widehat{\Gamma} \rightarrow \Gamma_N$ the following diagram commutes for all $i \leq j \in \mathbb{N} \cup \{0\}$:

$$\begin{array}{ccc} & \Delta & \\ & \swarrow \theta & \searrow \theta_N \\ \widehat{\Gamma} & \xrightarrow{\varphi_N} & \Gamma_N \end{array}$$

Since for all $N \in \mathbb{N} \cup \{0\}$, θ_N is surjective, we obtain $\Gamma_N = \theta_N(\Delta) = \varphi_N(\theta(\Delta))$. Thus $\overline{\theta(\Delta)} = \widehat{\Gamma}$. But since Δ is compact and $\widehat{\Gamma}$ is Hausdorff, $\theta(\Delta)$ is a closed subset of $\widehat{\Gamma}$ and thus $\theta(\Delta) = \overline{\theta(\Delta)} = \widehat{\Gamma}$. Hence θ is onto. Also let δ_1, δ_2 in Δ be such that $\theta(\delta_1) = \theta(\delta_2)$ and thus $\varphi_N(\theta(\delta_1)) = \varphi_N(\theta(\delta_2))$. Then for all $N \in \mathbb{N} \cup \{0\}$, $\theta_N(\delta_1) = \theta_N(\delta_2)$ and thus $\delta_1 = \delta_2$. Hence θ is one one and thus θ is a continuous bijection from a compact space Δ to a Hausdorff space $\widehat{\Gamma}$ and thus a homeomorphism. Hence Δ is the cofinite completion of Γ .

Next, assume finite discrete graphs Σ_N , where $V(\Sigma_N) = \{-N, \dots, 0, \dots, N, N+1\}$, $E(\Sigma_N) = \{e_x \mid x \in V(\Sigma_N) \setminus \{0\}\} \cup \{e\}$ and $s(e_x) = x-1$, if $x > 0$, $s(e_x) = x+1$,

if $x < 0$, $t(e_x) = x$, if $-N \leq x \leq N + 1$, $t(e_{-(N+1)}) = N + 1$, $s(e) = N + 1 = t(e)$.



Let us now define map of graphs $q'_N: \Gamma \rightarrow \Sigma_N$ via

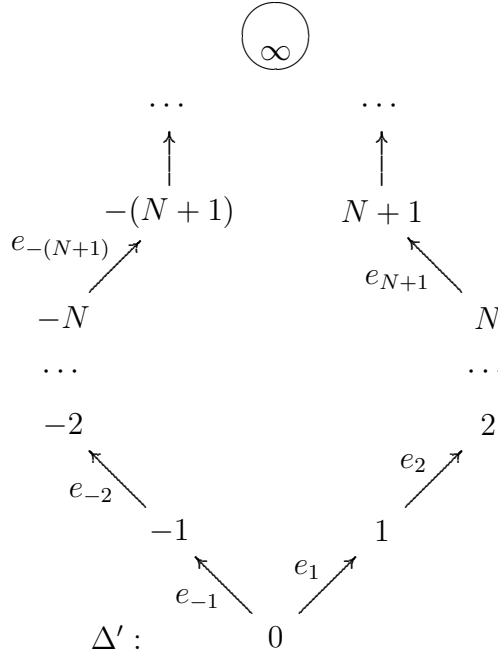
$$q'_N(x) = \begin{cases} x & |x| \leq N \\ N+1 & |x| \geq N+1 \end{cases}, \quad q'_N(e_x) = \begin{cases} e_x & |x| \leq N+1 \\ e & |x| \geq N+2 \end{cases}$$

and consider the uniformity Φ_2 over Γ , consisting of the entourages $S_N = (q'_N \times q'_N)^{-1}[D(\Sigma_N)]$. Clearly, with respect to Φ_2 , q'_N is uniformly continuous for all $N \in \mathbb{N} \cup \{0\}$. Let τ_{Φ_2} be the topology induced by Φ_2 . Now if $x \in V(\Gamma)$, then $R_{|x|}[x] = \{x\}$, similarly, if $e_x \in E(\Gamma)$, $R_{|x|}[e_x] = \{e_x\}$. Hence τ_{Φ_2} represents the discrete topology over Γ too.

Now let us define $\psi_{ij}: \Sigma_j \rightarrow \Sigma_i$ for all $i \leq j \in \mathbb{N} \cup \{0\}$ as follows:

$$\psi_{ij}(x) = \begin{cases} x & |x| \leq i \\ i & |x| \geq i+1 \end{cases}, \quad \psi_{ij}(e_x) = \begin{cases} e_x & |x| \leq i+1 \\ e & |x| \geq i+2 \end{cases}, \quad \psi_{ij}(e) = e.$$

Clearly, each ψ_{ij} is a uniformly continuous map of graphs and for $i = j$, $\psi_{ii} = \text{id}_{\Sigma_i}$, $i \leq j \leq k$, $\psi_{jk} \circ \psi_{ij} = \psi_{ik}$. Hence $(\Sigma_i, \psi_{ij})_{i \leq j \in \mathbb{N} \cup \{0\}}$ forms an inverse system of finite discrete graphs. Then by Theorem 6.5, we deduce that $\widehat{\Sigma} = \varprojlim_{i \in \mathbb{N} \cup \{0\}} \Sigma_i$ is a profinite completion of Γ . Now let us consider the following graph Δ' :



with $V(\Delta') = \mathbb{Z} \cup \{\infty\}$, $E(\Delta') = \{e_x \mid x \in V(\Delta') \setminus \{0, \infty\}\} \cup \{e\}$ with $s(e_x) = x - 1$, if $x > 0$, $s(e_x) = x + 1$, if $x < 0$, $t(e_x) = x$, $s(e) = \infty = t(e)$. Now let $G_1 = \{x \mid x \in \mathbb{Z}\}$, $G_2 = \{e_x \mid x \in \mathbb{Z} \setminus \{0\}\}$, $p_1 = \infty$, $p_2 = e$. Then $\tau' = \{O \subseteq \Delta' \mid O \cap (\Delta' \setminus [\cup_{i=1}^2 \{p_i\}]) \text{ is open in } (\Delta' \setminus [\cup_{i=1}^2 \{p_i\}]) \text{ with respect to its discrete topology and } p_i \in O \text{ implies } [(\Delta' \setminus [\cup_{i=1}^2 \{p_i\}]) \setminus O] \cap G_i \text{ is finite}\}$ forms the topology over Δ' . Hence Δ' is compact, Hausdorff, totally disconnected and thus a compactification (2 point compactification) of $\Delta' \setminus \{\infty, e\}$ and thus a compactification of the graph Γ .

Let us define maps $\zeta_N: \Delta' \rightarrow \Sigma_N$ via

$$\zeta_N(x) = \begin{cases} x & |x| \leq N \\ e & |x| \geq N + 1 \end{cases}, \quad \zeta_N(e_x) = \begin{cases} e_x & |x| \leq N + 1 \\ e & |x| \geq N + 2 \end{cases}, \quad \zeta_N(\infty) = N + 1,$$

$\zeta_N(e) = e$.

Clearly each ζ_N is a uniformly continuous map of graphs.

$$\begin{array}{ccc} & \Delta' & \\ & \swarrow \zeta_j & \searrow \zeta_i \\ \Sigma_j & \xrightarrow{\psi_{ij}} & \Sigma_i \end{array}$$

Thus $(\Delta', \zeta_N)_{N \in \mathbb{N} \cup \{0\}}$ is compatible with the inverse system $(\Sigma_i, \psi_{ij})_{i \leq j \in \mathbb{N} \cup \{0\}}$ and thus there exists a uniformly continuous map of graphs $\zeta: \Delta' \rightarrow \widehat{\Sigma}$ such that for the canonical projection maps $\psi_N: \widehat{\Sigma} \rightarrow \Sigma_N$ the following diagram commutes for all $i \leq j \in \mathbb{N} \cup \{0\}$:

$$\begin{array}{ccc} & \Delta' & \\ & \swarrow \zeta & \searrow \zeta_N \\ \widehat{\Sigma} & \xrightarrow{\psi_N} & \Sigma_N \end{array}$$

Since for all $N \in \mathbb{N} \cup \{0\}$, ζ_N is surjective, we get $\Sigma_N = \zeta_N(\Delta') = \psi_N(\zeta(\Delta'))$, Thus $\overline{\zeta(\Delta')} = \widehat{\Sigma}$. But since Δ' is compact and $\widehat{\Sigma}$ is Hausdorff, $\zeta(\Delta')$ is a closed subset of $\widehat{\Sigma}$ and thus $\zeta(\Delta') = \overline{\zeta(\Delta')} = \widehat{\Sigma}$. Hence ζ is onto. Also let $\delta'_1, \delta'_2 \in \Delta'$ be such

that $\zeta(\delta'_1) = \zeta(\delta'_2)$ and thus $\psi_N(\zeta(\delta'_1)) = \psi_N(\zeta(\delta'_2))$. Then for all $N \in \mathbb{N} \cup \{0\}$, $\zeta_N(\delta'_1) = \zeta_N(\delta'_2)$ and thus $\delta'_1 = \delta'_2$. Hence ζ is one one and thus ζ is a continuous bijection from a compact space Δ' to a Hausdorff space $\widehat{\Sigma}$ and thus a homeomorphism. Hence Δ' is the cofinite completion of Γ .

But Δ is not isomorphic to Δ' as they are the two point and 4 point compactifications for Γ respectively. So the example indicates that different uniformities that induces the same topology on a graph can lead us to two non isomorphic completions, which is different from the case of residually finite groups. Hence in our work we had to work with the uniform topology induced on our graphs.

Also while topological groups share the beautiful property that any continuous homomorphism between two topological groups is uniformly continuous, a similar statement is not true in the case of topological graphs.

In our work we want to reproduce the work done by B. Hartley, [2], but in the category of uniform topological graphs. In the dissertation we try to generalize the idea of cofinite groups, due to B. Hartley, first as generalized cofinite spaces. Then, as a special case, we study the cofinite graphs and their uniform completions. We are able to show that these completions are also cofinite graphs and being compact Hausdorff and totally disconnected they may be regarded as profinite graphs. We introduce the notion of cofinite connectedness in the category of cofinite graphs and studied about the cofinite connectedness of Cayley graphs of cofinite groups suitably endowed with a unique choice of cofinite uniformity. We also study the action of cofinite groups on cofinite graphs.

The idea of constructing a cofinite graph starts by defining cofinite equivalence relations. We define a cofinite equivalence relation as an equivalence relation over a topological space with finitely many equivalence classes such that it induces the discrete quotient space. Accordingly, a space is called a cofinite space if the cofinite equivalence relations form a fundamental system of cofinite entourages. It is also

establish that subspaces, product spaces, inverse limits of cofinite spaces and the finite uniform sum of cofinite spaces are also cofinite spaces.

We introduce the Uniform quotient space as follows:

Suppose we are given a cofinite space X , and K an equivalence relation over X . Let $q: X \rightarrow X/K$ be the corresponding quotient map. Then, $K_q = K$. Let I be a fundamental system of cofinite entourages over X . Let $I' = \{R \in I \mid K \subseteq R\}$. By the correspondence theorem, the collection $J = \{(q \times q)[R] \mid R \in I'\}$ is a fundamental system of entourages for a uniformity on the set of equivalence classes over X/K where each such $(q \times q)[R]$ has finitely many equivalence classes. The uniformity induced by J is called the *quotient uniformity of X modulo K* .

In general, the topology induced by the quotient uniformity of X modulo K is not as fine as the quotient topology on X/K . For this reason, we write $X//K$ for the set X/K endowed with the quotient uniformity of X modulo K and the topology it induces, reserving the notation X/K for the quotient space (with the quotient topology).

If K is an equivalence relation on a cofinite space X , then $X//K$ is called the *uniform quotient space of X modulo K* .

In chapter 3 we discuss topological graphs. A *topological graph* is a topological space Γ that is partitioned into two closed subsets $V(\Gamma)$ and $E(\Gamma)$ together with two continuous functions $s, t: E(\Gamma) \rightarrow V(\Gamma)$ and a continuous function $\bar{\cdot}: E(\Gamma) \rightarrow E(\Gamma)$ satisfying the following properties: for every $e \in E(\Gamma)$,

- (1) $\bar{e} \neq e$ and $\bar{\bar{e}} = e$;
- (2) $t(\bar{e}) = s(e)$ and $s(\bar{e}) = t(e)$.

The elements of $V(\Gamma)$ are called *vertices*. An element $e \in E(\Gamma)$ is called a (*directed*) *edge* with *source* $s(e)$ and *target* $t(e)$; the edge \bar{e} is called the *reverse* or *inverse* of e . A *map of graphs* $f: \Gamma \rightarrow \Delta$ is a function that maps vertices to vertices, edges to

edges, and preserves sources, targets, and inverses of edges. Analogously, we will call a map of graphs a *graph isomorphism* if and only if it is a bijection. An *orientation* of a topological graph Γ is a closed subset $E^+(\Gamma)$ consisting of exactly one edge in each pair $\{e, \bar{e}\}$. In this situation, setting $E^-(\Gamma) = \{e \in E(\Gamma) \mid \bar{e} \in E^+(\Gamma)\}$ we see that $E(\Gamma)$ is a disjoint union of the two closed (hence also open) subsets $E^+(\Gamma)$, $E^-(\Gamma)$. An equivalence relation R on a graph Γ is *compatible* if the following properties hold:

- (1) $R = R_V \cup R_E$ where R_V , R_E are equivalence relations on $V(\Gamma)$, $E(\Gamma)$, precisely the restrictions of R ;
- (2) if $(e_1, e_2) \in R$, then $(s(e_1), s(e_2)) \in R$, $(t(e_1), t(e_2)) \in R$, and $(\bar{e}_1, \bar{e}_2) \in R$;
- (3) for all $e \in E(\Gamma)$, $(e, \bar{e}) \notin R$;

If K is a compatible equivalence relation on Γ , then there is a unique way to make Γ/K into a graph such that the canonical map $\Gamma \rightarrow \Gamma/K$ is a map of graphs. It is defined by setting $s(K[e]) = K[s(e)]$, $t(K[e]) = K[t(e)]$, and $\overline{K[e]} = K[\bar{e}]$. Conversely, if Δ is a graph and $f: \Gamma \rightarrow \Delta$ is a surjective map of graphs, then $K = f^{-1}f = \{(a, b) \in \Gamma \times \Gamma \mid f(a) = f(b)\}$ is a compatible equivalence relation on Γ and f induces an isomorphism of graphs $\Gamma/K \cong \Delta$. If R_1 and R_2 are compatible equivalences on Γ , then so is $R_1 \cap R_2$.

We then establish that if R is any cofinite equivalence relation on a topological graph Γ , then there exists a compatible cofinite equivalence relation S on Γ such that $S \subseteq R$. In the course of the proof we noticed that if Γ is a topological graph with a specified closed orientation $E^+(\Gamma)$, then for any cofinite equivalence relation R on Γ , there exists a compatible orientation preserving cofinite equivalence relation S on Γ such that $S \subseteq R$.

If Γ is a compact Hausdorff totally disconnected topological graph, then its compatible cofinite equivalence relations form a fundamental system of entourages for the unique uniform structure that induces the topology of Γ .

We recall that a compact Hausdorff totally disconnected topological graph Γ is called a *profinite graph*. As for any compact Hausdorff space, we will view a profinite graph as a uniform space endowed with the unique uniformity that induces its topology. Thus, Corollary 4.7 states that the collection of all compatible cofinite equivalence relations on a profinite graph Γ form a fundamental system of entourages.

In order to introduce cofinite graphs we developed the concept of uniform topological graphs. By a *uniform topological graph* we mean a topological graph Γ endowed with a uniform structure that induces its topology such that Γ is the uniform sum of its uniform subspaces $V(\Gamma)$, $E(\Gamma)$ and the maps $s, t: E(\Gamma) \rightarrow V(\Gamma)$ and $\bar{\cdot}: E(\Gamma) \rightarrow E(\Gamma)$ are uniformly continuous. If $f: \Gamma \rightarrow \Delta$ is a uniformly continuous map of uniform topological graphs then for any compatible cofinite equivalence relation R over Δ , $(f \times f)^{-1}(R)$ is a compatible cofinite equivalence relation over Γ . Then we defined cofinite graphs as follows: A *cofinite graph* is an abstract graph Γ endowed with a Hausdorff uniformity such that the compatible cofinite entourages of Γ form a fundamental system of entourages (i.e. every entourage of Γ contains a compatible cofinite entourage). In particular, a profinite graph is also cofinite graph. It follows that if Γ is a cofinite graph and Z is a cofinite space, then the map $f: \Gamma \rightarrow Z$ is uniformly continuous if and only if both the restrictions $f|_{V(\Gamma)}$ and $f|_{E(\Gamma)}$ are uniformly continuous.

We prove the following theorem which is needed for our definition of the cofinite completion for a graph. Let Γ be a cofinite graph contained as a dense subgraph in a compact Hausdorff topological graph $\bar{\Gamma}$. Then given any compact Hausdorff topological graph Δ and any uniformly continuous map of graphs $\varphi: \Gamma \rightarrow \Delta$,

- (1) $\overline{V(\Gamma)} = V(\bar{\Gamma})$ and $\overline{E(\Gamma)} = E(\bar{\Gamma})$;
- (2) there exists a unique continuous map of graphs $\bar{\varphi}: \bar{\Gamma} \rightarrow \Delta$ extending φ .

As in the previous theorem, $\overline{\bar{\varphi}(\bar{\Gamma})} = \overline{\varphi(\Gamma)}$.

So we define a cofinite completion of a cofinite graph as follows: Let Γ be a cofinite graph. Then any compact Hausdorff topological graph $\bar{\Gamma}$ that contains Γ as a dense subgraph is called a *completion* of Γ . It turns out that the completion of a cofinite graph Γ is unique up to an isomorphism extending the identity map on Γ . Existence is seen by the theorem 6.5, which establishes the following. Let Γ be a cofinite graph and let I be a fundamental system of compatible cofinite entourages of Γ , directed by the opposite of inclusion. Then the inverse limit $\hat{\Gamma} = \varprojlim \Gamma/R$ ($R \in I$) is a compact Hausdorff topological graph and the natural map $\Gamma \rightarrow \hat{\Gamma}$ embeds Γ as a dense subgraph of $\hat{\Gamma}$.

Similar to Hartley's work, we were able to show that if $\bar{\Gamma}$ is the completion of a cofinite graph Γ and R is a compatible cofinite entourage of Γ , then \bar{R} is a compatible cofinite entourage of $\bar{\Gamma}$ and $\bar{R} \cap (\Gamma \times \Gamma) = R$.

We conclude chapter 5 with the following result (Theorem 6.7): Let Γ be a cofinite graph and let I be the filter base of all compatible cofinite entourages of Γ . Then the completion $\bar{\Gamma}$ is also a cofinite graph and $\{\bar{R} \mid R \in I\}$ is the filter base of all compatible cofinite entourages of $\bar{\Gamma}$.

In Generalizing Hartley's idea of cofinite groups to obtain the notion of cofinite graphs in this way, we have actually started to establish a parallel theory of cofinite graphs. In many ways, our theory of cofinite graphs generalizes well known results on Profinite graphs by Pavel Zalesskii and Luis Ribes.

Cofinite connectedness is discussed in the chapter 6. A cofinite graph Γ is said to be *cofinitely connected* if for each compatible cofinite equivalence relation R on Γ , the quotient graph Γ/R is path connected.

Similar to the standard connectedness arguments for finite graphs or general topological spaces we were able to establish that the following statements are equivalent for any cofinite graph Γ :

- (1) Γ is cofinitely connected;
- (2) Γ is not the union of two disjoint nonempty subgraphs.

As an immediate consequence we obtained the following generalized characterization of connected Cayley graphs of cofinite groups:

Let G be a cofinite group and let $\Gamma = \Gamma(G, X)$ be the Cayley graph. Then Γ can be given a suitable cofinite topological graph structure so that X generates G (topologically) iff Γ is cofinitely connected.

Our final chapter is concerned with cofinite group actions on cofinite graphs.

Let G be a group and Γ be a cofinite graph. We say that the group G acts over Γ if and only if

- (1) For all x in Γ , for all g in G , $g.x$ is in Γ ;
- (2) For all x in Γ , for all g_1, g_2 in G , $g_1.(g_2.x) = (g_1g_2).x$;
- (3) For all x in Γ , $1.x = x$;
- (4) For all $v \in V(\Gamma)$, $e \in E(\Gamma)$, $g \in G$, we have, $g.v \in V(\Gamma)$, $g.e \in E(\Gamma)$;
- (5) For all $e \in E(\Gamma)$, for all $g \in G$, $g.s(e) = s(g.e)$, $g.t(e) = t(ge)$, $g.(\bar{e}) = \overline{g.e}$;
- (6) There exists a G -invariant orientation $E^+(\Gamma)$ of Γ .

Note that the aforesaid group action restricted to a singleton group element $g \in G$ can be treated as a well defined map of graphs, $\Gamma \rightarrow \Gamma$ taking $x \mapsto g.x$.

A group G is said to act uniformly equicontinuously over a cofinite graph Γ if and only if for each entourage W over Γ there exists an entourage V over Γ such that for all g in G , $(g \times g)[V] \subseteq W$. If G acts uniformly equicontinuously over a cofinite graph Γ , then there exists a fundamental system of entourages consisting of G -invariant compatible cofinite entourages over Γ , i.e. for any entourage U over Γ there exists a compatible cofinite entourage R over Γ such that for all $g \in G$, $(g \times g)[R] \subseteq R \subseteq U$. We say a group G acts faithfully on a cofinite space Γ , if for all g in $G \setminus \{1\}$ there exists x in Γ such that gx is not equal to x in Γ . Let G acts on a cofinite graph Γ

uniformly equicontinuously. Then G acts on Γ/R and G/N_R acts on Γ/R as well, where R is a G -invariant compatible cofinite entourage over Γ and N_R is the kernel of the action $G \times \Gamma/R \rightarrow \Gamma/R$. If $\{R \mid R \in I\}$ is a fundamental system of G -invariant compatible cofinite entourages over Γ , then $\{N_R \mid R \in I\}$ forms a fundamental system of cofinite congruences for some uniformity over G . We say that a group G acts on a cofinite graph Γ residually freely, if there exists a fundamental system of G -invariant compatible cofinite entourages R over Γ such that the induced group action of G/N_R over Γ/R is a free action. $N_R[1]$ is a finite indexed normal subgroup of G and $G/N_R[1]$ is isomorphic with G/N_R . More generally, if N is a congruence on G , then $N[1]$ is a normal subgroup of G and $G/N[1] \cong G/N$. The induced uniform topology over G as earlier is Hausdorff if and only if G acts faithfully over Γ . Suppose that G is a group acting uniformly equicontinuously on a cofinite graph Γ and give G the induced uniformity. Then the action $G \times \Gamma \rightarrow \Gamma$ is uniformly continuous. If G acts on Γ uniformly equicontinuously and faithfully then \widehat{G} acts on $\widehat{\Gamma}$ uniformly equicontinuously.

CHAPTER 2

Cofinite Spaces

We define a cofinite equivalence relation as an equivalence relation over a topological space with finitely many equivalence classes such that it induces the discrete quotient space. Accordingly, a space is called a cofinite space if the cofinite equivalence relations form a fundamental system of entourages. In this chapter it is also established that subspaces, product spaces, inverse limits of cofinite spaces and the finite uniform sum of cofinite spaces are also cofinite spaces. We also introduce uniform quotient maps and uniform quotient spaces of cofinite spaces.

2.1. Cofinite Spaces

In order to define cofinite spaces we first define cofinite equivalence relations. Let us first see the following result about an equivalence relation on a general topological space.

2.1.1. Cofinite equivalence relations on topological spaces. Note that if R is an equivalence relation on a set X , then

$$R = R^3 = \bigcup\{R[a] \times R[b] \mid (a, b) \in R\}.$$

PROOF. $R^3 = R^2R = RR = R^2 = R$, as R is an equivalence relation.

By A.3, we get $\bigcup\{R[a] \times R[b] \mid (a, b) \in R\} = RRR^{-1} = RRR = R^3$. □

For topological spaces, this leads to the following observation.

LEMMA 2.1. *Let R be an equivalence relation on a topological space X . Then R is an open subset of the product space $X \times X$ if and only if $R[a]$ is an open subset of X , for each a in X .*

PROOF. Suppose first that R is open in $X \times X$. Now let $a \in X$. If $b \in R[a]$ then $(a, b) \in R$. However R is open in $X \times X$, so there exists open sets U and V in X , such that $(a, b) \in U \times V \subseteq R$. Now, for all $v \in V$, $(a, v) \in R$ so that $v \in R[a]$. Hence $b \in V \subseteq R[a]$. Thus $R[a]$ is an open subset of X for all $a \in X$. On the other hand let us now assume that $R[a]$ is open in X for all $a \in X$. Now $(r, s) \in R$ implies that $s \in R[r]$ and $r \in R[r]$. Also $(x, y) \in R[r] \times R[r]$ implies that $(r, x), (r, y) \in R$ and thus $(x, r), (r, y) \in R$ and so $(x, y) \in R$. Hence $(r, s) \in R[r] \times R[r] \subseteq R$. Since $R[r]$ is open in X , we see that R is open in $X \times X$. \square

Notice that in the situation of 2.1, the quotient space X/R has the discrete topology. Hence, we will refer to such an equivalence relation R as being *co-discrete*. It should be noted that the term 'open' for an equivalence relation on a topological space X typically means something else, namely that the quotient map $X \rightarrow X/R$ is an open mapping.

DEFINITION 2.2 (Cofinite equivalence relation). Let X be a topological space. A *cofinite equivalence relation* on X is an equivalence relation R such that the quotient space X/R is a finite discrete space.

In other words, an equivalence relation R on X is cofinite if and only if R is co-discrete and there are only finitely many equivalence classes of X modulo R .

LEMMA 2.3. *Cofinite equivalence relations on topological spaces satisfy the following elementary properties:*

- (1) *The intersection $R_1 \cap R_2$ of two cofinite equivalence relations R_1, R_2 on a space X is also cofinite.*

- (2) Let S be an equivalence relation on a space X . If S contains a cofinite equivalence relation, then S itself is cofinite.
- (3) If R_1, R_2 are commuting equivalence relations on a space X , and if one of R_1, R_2 is cofinite, then the product $R_1 R_2$ is also a cofinite equivalence relation.
- (4) If $f: X \rightarrow Y$ is a continuous map of topological spaces and R is a cofinite equivalence relation on Y , then $(f \times f)^{-1}[R]$ is a cofinite equivalence relation on X .
- (5) If A is a subspace of a topological space X and R is a cofinite equivalence relation on X , then the restriction $R \cap (A \times A)$ is a cofinite equivalence relation on A .
- (6) If X is compact, then every co-discrete equivalence relation on X is cofinite.

PROOF. We will prove the above claims in the order in which they appear.

- (1) Clearly, $X/(R_1 \cup R_2)$ is discrete. Let us define a map $\theta: X/(R_1 \cap R_2) \rightarrow X/R_1 \times X/R_2$ via $\theta((R_1 \cap R_2)[x]) = (R_1[x], R_2[x])$, for all $x \in X$. Now $(R_1 \cap R_2)[x_1] = (R_1 \cap R_2)[x_2] \Leftrightarrow (x_1, x_2) \in R_1 \cap R_2 \Leftrightarrow (x_1, x_2) \in R_1$ and $(x_1, x_2) \in R_2 \Leftrightarrow R_1[x_1] = R_1[x_2]$ and $R_2[x_1] = R_2[x_2] \Leftrightarrow (R_1[x_1], R_2[x_1]) = (R_1[x_2], R_2[x_2])$. Hence θ is well defined and an injection. Thus

$$|X/(R_1 \cap R_2)| \leq |X/R_1 \times X/R_2| < \infty.$$

- (2) Let R be a cofinite equivalence relation on X such that $R \subseteq S$. Let $x \in X$ and $y \in S[x]$. If $v \in R[y]$ we have $(y, v) \in R \subseteq S$. Hence $(x, v) \in S$ implying that $v \in S[x]$. Hence $y \in R[y] \subseteq S[x]$. Thus

$$S[x] = \bigcup_{y \in S[x]} R[y]$$

and hence each $S[x]$ is open. Also there is a natural surjection $X/R \rightarrow X/S$ and X/R is finite. So X/S is also finite. Thus S is a cofinite equivalence relation.

- (3) Since R_1, R_2 are commuting equivalence relations, R_2R_1 is an equivalence relation on X . Now let R_1 be a cofinite equivalence relation. Note that $(x, y) \in R_1$ implies that $(x, y) \in R_2R_1$, as $(y, y) \in R_2, \forall y \in Y$. Hence $R_1 \subseteq R_2R_1$. Thus, by (2), the claim follows.
- (4) By Theorem A.8 we know that $(f \times f)^{-1}[R]$ is an equivalence relation over X with finitely many equivalence classes. Since R is a cofinite equivalence relation on Y , it is an open subset of $Y \times Y$. Thus $(f \times f)^{-1}[R]$ is an open subset of X , as $f \times f$ is continuous. And thus our claim follows.
- (5) Let us first see that $R \cap (A \times A)$ is an equivalence relation. For all $a \in A$, $(a, a) \in D(X) \subseteq R$ implies that for all $(a, a) \in D(A)$, $(a, a) \in R \cap (A \times A)$. Next $(a_1, a_2) \in (R \cap (A \times A))^{-1} \Leftrightarrow (a_2, a_1) \in R \cap (A \times A) \Leftrightarrow (a_1, a_2) \in R^{-1} \cap (A \times A) \Leftrightarrow (a_1, a_2) \in R \cap (A \times A)$. So $R \cap (A \times A) = (R \cap (A \times A))^{-1}$. Furthermore $(a_1, a_2) \in (R \cap (A \times A))^2$ implies that there exists $a \in A$ such that $(a_1, a) \in R \cap (A \times A)$ and $(a, a_2) \in R \cap (A \times A)$. Hence $(a_1, a_2) \in R^2 \cap (A \times A) = R \cap (A \times A)$. Thus $(R \cap (A \times A))^2 \subseteq R \cap (A \times A)$. Hence $R \cap (A \times A)$ is an equivalence relation.

Now let us show that $|A/[R \cap (A \times A)]| < \infty$. For let us define i from $A/[R \cap (A \times A)]$ to X/R via $i([R \cap (A \times A)][a]) = R[a]$. For all $a_1, a_2 \in A$, $[R \cap (A \times A)][a_1] = [R \cap (A \times A)][a_2] \Leftrightarrow (a_1, a_2) \in R \cap (A \times A) \Leftrightarrow (a_1, a_2) \in R \Leftrightarrow R[a_1] = R[a_2]$. Hence i is an well defined injection and thus $|A/[R \cap (A \times A)]| \leq |X/R| < \infty$.

Now for some $a \in A, a' \in [R \cap (A \times A)][a] \Leftrightarrow (a, a') \in R \cap (A \times A) \Leftrightarrow (a, a') \in R \Leftrightarrow a' \in R[a] \cap A$. Hence $[R \cap (A \times A)][a] = R[a] \cap A, \forall a \in A$. Hence $[R \cap (A \times A)][a]$ is open in A for all a in A . Thus our claim follows.

- (6) Let R be a co-discrete equivalence relation over a compact space X . Now since X/R is a continuous surjective image of X , X/R is compact as well.

But as R is co-discrete X/R is a discrete topological space and hence finite, so our claim follows.

□

2.1.2. Cofinite Spaces. We now turn our attention to uniform spaces. Unless otherwise stated, the topology on a uniform space will always be the one induced by its uniformity. (See Appendix ?? for basic facts about uniform spaces.)

Let X be a uniform space. By a *cofinite entourage* on X we will mean an entourage R which is also a cofinite equivalence relation on X . As consequences of 2.3, we see that cofinite entourages satisfy the following elementary properties:

LEMMA 2.4. *Let X and Y be uniform spaces.*

- (1) *The intersection $R_1 \cap R_2$ of two cofinite entourages R_1, R_2 of X is also a cofinite entourage of X .*
- (2) *Let S be an equivalence relation on X . If S contains a cofinite entourage, then S itself is a cofinite entourage.*
- (3) *If R_1, R_2 are commuting equivalence relations on a space X , and if one of R_1, R_2 is a cofinite entourage, then the product R_1R_2 is also a cofinite entourage.*
- (4) *If $f: X \rightarrow Y$ is a uniformly continuous map and R is a cofinite entourage of Y , then $(f \times f)^{-1}[R]$ is a cofinite entourage of X .*
- (5) *If X is compact and Hausdorff, then every co-discrete equivalence relation on X is a cofinite entourage.*

DEFINITION 2.5 (Cofinite space). A *cofinite (uniform) space* is a uniform space X whose cofinite entourages form a fundamental system of entourages (i.e., every entourage of X contains a cofinite entourage).

LEMMA 2.6. *For a cofinite space X with a fundamental system of cofinite entourages, say, I , the set $\beta = \{R[x] \mid x \in X, R \in I\}$ forms the basis of the corresponding uniform topology and each $R[x]$ is clopen.*

PROOF. Since, for each $R \in I$, X/R is a discrete topological space, we deduce our claim that $R[x]$ is clopen in X for all x in X . Now let U be any open subset of X . Then for all $u \in U$ there exists some $R_U \in I$ such that $u \in R_U[u] \subseteq U$. Hence the claim follows. \square

EXAMPLES 2.7. 1. Let G be a cofinite group, i.e., a Hausdorff topological group in which the set of all open normal subgroups of finite index forms a neighborhood base of the identity $1 \in G$; see (Appendix ??). Then for each open normal subgroup N of G , the subset $R_N = \{(a, b) \in G \times G \mid ab^{-1} \in N\}$ is a cofinite equivalence relation on G . Furthermore, the set $I = \{R_N \mid N \text{ is an open normal subgroup of } G\}$ is a fundamental system of entourages for a uniformity on G that induces its topology. In this way, we view G as a cofinite space.

2. Let X be a compact Hausdorff totally disconnected space. Then, endowed with the unique uniform structure compatible with its topology, X is a cofinite space. (See 3.3).

3. Let X be any set and let I be a separating filter base of equivalence relations on X , each of which has only finitely many equivalence classes. By this we mean that I is a set of equivalence relations that have only finitely many equivalence classes satisfying the two conditions:

- (i) If $R_1, R_2 \in I$, then there exists $R_3 \in I$ such that $R_3 \subseteq R_1 \cap R_2$.
- (ii) The intersection of all members of I is the diagonal $D(X)$.

Then I is a fundamental system of entourages for a uniform structure making X into a Hausdorff cofinite space.

Cofinite spaces have the following elementary properties:

This following lemma is an analogue to similar works done in [2], but in the category of general cofinite spaces.

LEMMA 2.8. *Let X be a cofinite space and let I be a fundamental system of cofinite entourages of X . Then the following properties hold:*

- (1) *If $W \subseteq X \times X$, then $\overline{W} = \bigcap_{R \in I} (R \times R)[W] = \bigcap_{R \in I} RWR$ and each RWR is a clopen neighborhood of W in $X \times X$.*
- (2) *If $A \subseteq X$, then $\overline{A} = \bigcup_{R \in I} R[A]$ and each $R[A]$ is a clopen neighborhood of A in X .*

PROOF.

- (1) We have $(R \times R)[W] = RWR^{-1} = RWR$, for all $R \in I$ and hence $\bigcap_{R \in I} (R \times R)[W] = \bigcap_{R \in I} RWR$.

Now let $(x, y) \in \overline{W}$. Then for all $R \in I$, $(R[x] \times R[y]) \cap W \neq \emptyset$. Let $(t, s) \in (R[x] \times R[y]) \cap W$. Then $(x, t) \in R$, $(y, s) \in R$ and so $(s, y) \in R$. Together with $(t, s) \in W$, this implies that $(x, y) \in RWR$, for all $R \in I$, whence $(x, y) \in \bigcap_{R \in I} RWR$.

Conversely, let $(x, y) \in \bigcap_{R \in I} RWR$. Now let $U \times V$ be a basic open set in $X \times X$, such that $(x, y) \in U \times V$. Then there exists $R_1, R_2 \in I$ such that $(x, y) \in R_1[x] \times R_2[y] \subseteq U \times V$. Then $(x, y) \in (R_1 \cap R_2)[x] \times (R_1 \cap R_2)[y] \subseteq U \times V$. Also $(x, y) \in (R_1 \cap R_2)W(R_1 \cap R_2)$, so there exists $z \in X$, such that $(x, z) \in (R_1 \cap R_2)$, $(z, y) \in (R_1 \cap R_2)W$. Hence there exists $w \in X$, such that $(z, w) \in W$ and $(w, y) \in (R_1 \cap R_2)$, i.e. $(y, w) \in (R_1 \cap R_2)$. So $z \in (R_1 \cap R_2)[x]$ and $w \in (R_1 \cap R_2)[y]$ and $(z, w) \in (R_1 \cap R_2)[x] \times (R_1 \cap R_2)[y] \subseteq U \times V$. Hence $(z, w) \in W \cap (U \times V)$. Therefore $W \cap (U \times V) \neq \emptyset$ and it follows that $(x, y) \in \overline{W}$.

(2) Let $x \in \bar{A}$. Now for all $R \in I$, $R[x] \cap A \neq \emptyset$. Let $a_x \in R[x] \cap A$, which implies that $(x, a_x) \in R$ and so $(a_x, x) \in R$. Thus $x \in R[a_x] \subseteq \bigcup_{a \in A} R[a] = R[A]$, for all $R \in I$ and $x \in \bigcap_{R \in I} R[A]$.

Conversely, let $x \in \bigcap_{R \in I} R[A]$. Now let U be any open set in X such that $x \in U$. So there is some $R_U \in I$ such that $x \in R_U[x] \subseteq U$. But, as $x \in R_U[A]$, implies that there exists $a \in A$ such that $x \in R_U[a]$. Then $(a, x) \in R_U$, so, $(x, a) \in R_U$ and $a \in R_U[x] \cap A$. Hence $a \in U \cap A$ so $x \in \bar{A}$.

Now $R[A] = \bigcup_{a \in A} R[a]$. Since each $R[a]$ is open in X , $R[A]$ is open in X , for all $R \in I$.

Let $t \in X \setminus R[A]$. Now if possible, let $x \in R[t] \cap R[A]$. Then there exists $a_1 \in A$ such that $x \in R[a_1]$. Then $(x, a_1) \in R$ and since $(t, x) \in R$ it follows that $(t, a_1) \in R$ and thus $(a_1, t) \in R$. Hence $t \in R[a_1] \subseteq \bigcup_{a_1 \in A} R[a_1] = R[A]$, a contradiction. Therefore $t \in R[t] \subseteq X \setminus R[A]$. Thus $R[A]$ is closed in X , for all $R \in I$.

□

LEMMA 2.9. *Let X be a cofinite space and let I be a fundamental system of cofinite entourages of X . Then the following statements are equivalent:*

- (1) X is Hausdorff;
- (2) X is totally disconnected;
- (3) $\bigcap_{R \in I} R = D(X)$;

PROOF.

(1) \Rightarrow (2) : Let $A \subseteq X$ be connected and if possible let $x \neq y$ in X be such that $x, y \in A$. Since X is Hausdorff, there exists $R \in I$ such that $y \notin R[x]$. Now, as $R[x]$ is a clopen subset of X , we deduce $R[x] \cap A$ and $(X \setminus R[x]) \cap A$ are both nonempty open subsets of A . Then $(R[x] \cap A) \cup ((X \setminus R[x]) \cap A) = (R[x] \cup (X \setminus R[x])) \cap A = A$. That contradicts the connectedness of A . Hence

any connected subset of X has at most one element. Thus X is totally disconnected.

(2) \Rightarrow (3) : Since X is totally disconnected so is $X \times X$. Now for all $a \in X$, $\{(a, a)\} = \overline{\{(a, a)\}} = \bigcap_{R \in I} (R \times R)[\{(a, a)\}]$. Let for some $b \neq a$ in X such that $(a, b) \in \bigcap_{R \in I} R$. Then $((a, b), (a, b)) \in R \times R$ for all $R \in I$. Hence $(b, b) \in \bigcap_{R \in I} (R \times R)[\{(a, a)\}]$, a contradiction. Thus $\bigcap_{R \in I} R = D(X)$.

(3) \Rightarrow (1) Let us first assume $a \neq b$ in X . Since $\bigcap_{R \in I} R = D(X)$ there exists some R such that $(a, b) \notin R$. Hence $a \in R[a], b \in R[b]$ and as $R[a] \cap R[b] = \emptyset$ we deduce our claim.

□

Next we consider a general process for constructing cofinite spaces, using what is called by N. Bourbaki, [1], "initial uniformities".

DEFINITION 2.10. *Let X be a set, let $(X_i)_{i \in I}$ be a family of sets, and let $F = (f_i: X \rightarrow X_i)_{i \in I}$ be a family of functions for X . We call F a separating family of maps if for all $x \neq y$ in X , then exists $i \in I$, such that $f_i(x) \neq f_i(y)$ in X_i .*

PROPOSITION 2.11. *Let X be a set, let $(X_i)_{i \in I}$ be a family of cofinite spaces, and let $(f_i: X \rightarrow X_i)_{i \in I}$ be a family of functions for X . Let \mathbf{S} be the set of all equivalence relations on X of the form $(f_i \times f_i)^{-1}[R_i]$, where $i \in I$ and R_i runs through a fundamental system of cofinite entourages of X_i . Finally, let \mathbf{B} be the set of all finite intersections of members of \mathbf{S} . Then \mathbf{B} is a fundamental system of entourages of a uniformity on X which is the coarsest uniformity on X for which all the mappings f_i are uniformly continuous. Endowed with this uniform structure, X becomes a cofinite space. Moreover if each X_i is Hausdorff and $(f_i: X \rightarrow X_i)_{i \in I}$ is a separating family of functions for X , then X is Hausdorff as well.*

PROOF. Let us first see that \mathbf{B} forms the fundamental system for a uniformity over X . By definition of \mathbf{B} , for all $(f_i \times f_i)^{-1}[R_i] \in \mathbf{S}$, $(f_i \times f_i)^{-1}[R_i] \in \mathbf{B}$. Then,

- (1) $D(X) \subseteq (f_i \times f_i)^{-1}[R_i]$, for all $(f_i \times f_i)^{-1}[R_i] \in \mathbf{S}$ as each $(f_i \times f_i)^{-1}[R_i]$ is an equivalence relation over X and hence is reflexive.
- (2) If $(f_i \times f_i)^{-1}[R_i], (f_i \times f_i)^{-1}[S_j] \in \mathbf{S}$, then $(f_i \times f_i)^{-1}[R_i] \cap (f_i \times f_i)^{-1}[S_j] \in \mathbf{B}$, as \mathbf{B} is the set of all finite intersections of members in \mathbf{S} .
- (3) For all $(f_i \times f_i)^{-1}[R_i] \in \mathbf{S}$, $((\mathbf{f}_i \times \mathbf{f}_i)^{-1}[\mathbf{R}_i])^2 \subseteq (\mathbf{f}_i \times \mathbf{f}_i)^{-1}[\mathbf{R}_i]$, as $(f_i \times f_i)^{-1}[R_i]$ is an equivalence relation over X and hence is transitive.
- (4) For all $(f_i \times f_i)^{-1}[R_i] \in \mathbf{S}$, $((f_i \times f_i)^{-1}[R_i])^{-1} = (f_i \times f_i)^{-1}[R_i]$ as $(f_i \times f_i)^{-1}[R_i]$ is an equivalence relation over X and hence is symmetric.

Now let $\Phi_{\mathbf{B}}$ be the corresponding uniformity over X . Hence each $f_i, i \in I$ is uniformly continuous w.r.t. $\Phi_{\mathbf{B}}$.

Let Φ be any other uniformity over X such that each $f_i, i \in I$ is uniformly continuous w.r.t. Φ . Hence $(f_i \times f_i)^{-1}[R_i] \in \Phi$. Thus $\Phi_{\mathbf{B}} \subseteq \Phi$. Hence the claim that $\Phi_{\mathbf{B}}$ is the coarsest uniformity on X for which all the mappings f_i are uniformly continuous.

Now, as f_i is uniformly continuous for all $i \in I$, by 2.4, $(f_i \times f_i)^{-1}[R_i]$ is cofinite entourage over X , for all $(f_i \times f_i)^{-1}[R_i] \in \mathbf{S}$. Hence $(X, \Phi_{\mathbf{B}})$ is a cofinite space.

Finally, let $(x, y) \notin D(X)$ and assume that each X_i is Hausdorff. Assuming $(f_i: X \rightarrow X_i)_{i \in I}$ is a separating family of functions for X , there exists $i \in I$, such that $f_i(x) \neq f_i(y)$ in X_i . Hence, there exists some R_i , cofinite entourage over X_i , such that $(f_i(x), f_i(y)) \notin R_i$. Hence $(x, y) \notin (f_i \times f_i)^{-1}[R_i]$. Hence $x \in (f_i \times f_i)^{-1}[R_i][x], y \in (f_i \times f_i)^{-1}[R_i][y]$ and $(f_i \times f_i)^{-1}[R_i][x] \cap (f_i \times f_i)^{-1}[R_i][y] = \emptyset$. Thus $(X, \Phi_{\mathbf{B}})$ is a Hausdorff cofinite space. \square

Here are two corollaries of this construction. Let X be as in Proposition 2.11.

COROLLARY 2.12. *If $h: Y \rightarrow X$ is a mapping from a uniform space Y , then h is uniformly continuous if and only if each mapping $f_i \circ h: Y \rightarrow X_i$ is uniformly continuous.*

PROOF. Let us first take $h: Y \rightarrow X$ to be uniformly continuous. Now since f_i , for all $i \in I$ is uniformly continuous so is $f_i \circ h: Y \rightarrow X_i$, for all $i \in I$. Conversely, let's take $f_i \circ h: Y \rightarrow X_i$, for all $i \in I$ to be uniformly continuous. Now let $(f_i \times f_i)^{-1}[R_i] \in \mathbf{S}$ be any cofinite entourage over X , for some cofinite entourage R_i over X_i . Then $(f_i \times f_i)^{-1}[R_i] \in \mathbf{S}$ and so $(h \times h)^{-1}[(f_i \times f_i)^{-1}[R_i]] = (h \circ f_i \times h \circ f_i)^{-1}[R_i]$ is a cofinite entourage over Y . Thus h is uniformly continuous. \square

COROLLARY 2.13. *The topology on X induced by the above uniformity is the coarsest topology for which the f_i are continuous.*

PROOF. Let $\tau_{\Phi_{\mathbf{B}}}$ be the topology on X induced by the uniformity $\Phi_{\mathbf{B}}$. Then f_i , for all $i \in I$ is continuous. Now let τ be any other topology over X such that f_i , for all $i \in I$ is continuous. For any cofinite entourage R_i of X_i , $R_i[x_i]$ is an open subset of X_i , for all $x_i \in X_i$. Hence, $f_i^{-1}(R_i[x_i]) \in \tau$, for all $x_i \in X_i$, for each cofinite entourage R_i over X_i , $\forall i \in I$. Now let $x \in f_i^{-1}(R_i[x_i])$, so, $f_i(x) \in R_i[x_i]$. Also, $U = (f_i \times f_i)^{-1}[R_i][x]$ is a basic open set in $(X, \tau_{\Phi_{\mathbf{B}}})$. Now $y \in U \Leftrightarrow (x, y) \in (f_i \times f_i)^{-1}[R_i] \Leftrightarrow (x_i, f_i(y)) \in R_i \Leftrightarrow f_i(y) \in R_i[x_i] \Leftrightarrow y \in f_i^{-1}(R_i[x_i])$. Thus $U = f_i^{-1}(R_i[x_i]) \in \tau$. Hence $\tau_{\Phi_{\mathbf{B}}} \subseteq \tau$. \square

2.1.3. Uniform subspaces of cofinite spaces. Recall that a *uniform subspace* of a uniform space X is a subset A , endowed with the coarsest uniformity for which the inclusion mapping $A \rightarrow X$ is uniformly continuous. This uniformity is called the *uniformity induced on A* by that of X .

In the case of a uniform subspace, Proposition 2.11, can be stated as follows.

PROPOSITION 2.14. *Let A be a uniform subspace of a cofinite space X . Then the family of all sets of the form $R \cap (A \times A)$, where R runs through a fundamental system of cofinite entourages of X , is a fundamental system of entourages of A . In particular, A is a cofinite space.*

PROOF. Let us consider the inclusion map $i: A \rightarrow X$. Then by Proposition 2.11, $(i \times i)^{-1}[R]$, where R runs through the fundamental system of cofinite entourages over X , forms the fundamental system of cofinite entourages for the coarsest uniformity over A , such that i is uniformly continuous. Thus the the corresponding inherited uniformity coincides with the subspace uniformity over A . Hence A is a cofinite space. Now $R \cap (A \times A) = (i \times i)^{-1}[R]$, as, $(a_1, a_2) \in R \cap (A \times A) \Leftrightarrow (a_1, a_2) \in (i \times i)^{-1}[R]$. Our claim follows. \square

By Corollary 2.13, we see that the topology induced on a uniform subspace A of a cofinite space X by its uniformity is the same as the subspace topology on A . Recall that the subspace topology on A is the coarsest topology on A such that i is continuous. Furthermore, we next observe that restrictions of uniformly continuous maps to uniform subspaces are uniformly continuous.

PROPOSITION 2.15. *Let $f: X \rightarrow Y$ be a uniformly continuous map of cofinite spaces and let A, B be uniform subspaces of X, Y such that $f(A) \subseteq B$. Then the restriction $f|_A: A \rightarrow B$ is also uniformly continuous.*

PROOF. Since $f: X \rightarrow Y$ is uniformly continuous, for each cofinite entourage S over Y , $(f \times f)^{-1}[S]$ is a cofinite entourage over X . Now $(f \times f)^{-1}[S \cap (B \times B)] = (f \times f)^{-1}[S] \cap (f \times f)^{-1}[B \times B]$. Since $A \times A \subseteq (f \times f)^{-1}[f(A) \times f(A)] \subseteq (f \times f)^{-1}[B \times B]$, we obtain, $(f \times f)^{-1}[S] \cap (A \times A) \subseteq (f \times f)^{-1}[S] \cap (f \times f)^{-1}[(B \times B)]$. However $(f \times f)^{-1}[S] \cap (A \times A)$ is a cofinite entourage over A , so our claim that $f|_A: A \rightarrow B$ is uniformly continuous follows. \square

2.1.4. Products of cofinite spaces. Recall that if $(X_i)_{i \in I}$ is a family of uniform spaces, then the coarsest uniformity on the Cartesian product

$$X = \prod_{i \in I} X_i$$

for which the projections $\pi_i: X \rightarrow X_i$ are uniformly continuous is called the *product uniformity*. The set X together with its product uniformity is called the *product uniform space* of this family.

In the case of a Cartesian product of cofinite spaces, Proposition 2.11 yields the follow result.

PROPOSITION 2.16. *If X is the product uniform space of a family $(X_i)_{i \in I}$ of cofinite spaces, then X is a cofinite space.*

PROOF. Let $X = \prod_{i \in I} X_i$ and let $\pi_i: X \rightarrow X_i$, for all $i \in I$ be the projections. Let \mathbf{S} be the set of all equivalence relations on X of the form $(\pi_i \times \pi_i)^{-1}[R_i]$, where $i \in I$ and R_i runs through a fundamental system of cofinite entourages of X_i . Finally, let \mathbf{B} be the set of all finite intersections of members of \mathbf{S} . Then \mathbf{B} is a fundamental system of entourages of a uniformity on X which is the coarsest uniformity on X for which all the projections π_i are uniformly continuous, and so coincides with the product uniformity. Endowed with this uniform structure, X becomes a cofinite space. \square

By Corollary 2.13, the topology induced on a product uniform space $X = \prod_{i \in I} X_i$ of a family of cofinite spaces is the same as the product topology on X . Recall that the product topology on X is the coarsest topology on X such that each projection π_i , for all $i \in I$ is continuous. In this situation, Corollary 2.12 says: if f is a function from a uniform space Y into the product uniform space X , then f is uniformly continuous if and only if the coordinate functions $f_i = \pi_i \circ f$ are uniformly continuous.

Note that, alternatively, one can show that $\{R \mid R = \prod_{i \in I} R_i\}$ forms a fundamental system of cofinite entourages for the product uniformity over X , whenever R_i is equal to $X_i \times X_i$ for all but finitely many i 's in I and for the rest of the finitely many $i \in I$, R_i is a cofinite entourage over X_i .

Let $\{R_{i_n}\}_{n=1}^N$ be those finitely many cofinite entourages over X_{i_n} , as mentioned above. Then we claim that $R = \bigcap_{i=1}^N (\pi_{i_n} \times \pi_{i_n})^{-1}[R_{i_n}]$. For, $(x, y) \in R \Leftrightarrow$ for all i , $(\pi_i \times \pi_i)((x, y)) \in R_i \Leftrightarrow (x, y) \in (\pi_{i_n} \times \pi_{i_n})^{-1}[R_{i_n}]$, for all $n = 1, 2, \dots, N \Leftrightarrow (x, y) \in \bigcap_{i=1}^N (\pi_{i_n} \times \pi_{i_n})^{-1}[R_{i_n}]$. Thus our claim follows and so whenever convenient, we will use the aforesaid fundamental system of cofinite entourages for the product uniformity over X .

2.1.5. Inverse limits of cofinite spaces. Let (X_i, ϕ_{ij}) be an inverse system of sets indexed by a directed set I . We say that (X_i, ϕ_{ij}) is an *inverse system of uniform spaces* if (i) each X_i is a uniform space, and (ii) for all $i \leq j$, $\phi_{ij}: X_j \rightarrow X_i$ is uniformly continuous. The set $X = \varprojlim X_i$ endowed with the coarsest uniformity for which the canonical maps $\phi_i: X \rightarrow X_i$ are uniformly continuous is called the *inverse limit of the inverse system of uniform spaces*.

Equivalently, the inverse limit of an inverse system of uniform spaces (X_i, ϕ_{ij}) is the uniform subspace of the product uniform space $\prod_{i \in I} X_i$ consisting of all points x such that

$$\pi_i(x) = \phi_{ij}(\pi_j(x))$$

whenever $i \leq j$, and π_i is the regular projection map. Also the induced topology on the uniform space $X = \varprojlim X_i$ is the same as the inverse limit of the topologies on the X_i ; see [1, Chapter II, §2, no. 7].

In the case of an inverse system of cofinite spaces, Proposition 2.11 can be stated as follows:

PROPOSITION 2.17. *Let (X_i, ϕ_{ij}) be an inverse system of cofinite spaces and let $X = \varprojlim X_i$ be the inverse limit. For each $i \in I$, let $\phi_i: X \rightarrow X_i$ be the canonical map. Then the collection of all sets $(\phi_i \times \phi_i)^{-1}[R_i]$, where i runs through I and R_i runs through a fundamental system of cofinite entourages of X_i , is a fundamental system of cofinite entourages of X . In particular, X is a cofinite space.*

As in any category, inverse limits of cofinite spaces are characterized by a universal property: Let (X_i, ϕ_{ij}) be an inverse system of cofinite spaces, let Y be a cofinite space, and let $(g_i: Y \rightarrow X_i)_{i \in I}$ be a compatible family of uniformly continuous maps. Here *compatible* means that $\phi_{ij}g_j = g_i$ whenever $i \leq j$. Denote the inverse limit by $X = \varprojlim X_i$ and denote the canonical maps by $\phi_i: X \rightarrow X_i$. Then there is a unique uniformly continuous map $g: Y \rightarrow X$ such that $\phi_i g = g_i$ for all $i \in I$. The map g exists and is unique by the general theory of inverse limits of sets, and it is uniformly continuous by Corollary 2.12.

2.1.6. Sums of cofinite spaces. To begin with, let $(X_i)_{i \in I}$ be an arbitrary family of uniform spaces. The *uniform sum* of this family is the disjoint union $X = \coprod_{i \in I} X_i$ endowed with the uniformity having a fundamental system of entourages consisting of all sets of the form $\bigcup_{i \in I} V_i$, where each V_i is an entourage of X_i . Note that each X_i , when identified with its image in X under the canonical inclusion map, is a uniform subspace of X . For let $i_{X_i}: X_i \rightarrow X$ be the corresponding inclusion map. Now let $U = \bigcup_{i \in I} U_i$ be an entourage over X . Let $(x_i, y_i) \in U_i$. Then $(x_i, y_i) \in U$. So $U_i \subseteq (i_{X_i} \times i_{X_i})^{-1}[U]$. Hence i_{X_i} is uniformly continuous. Also, $(x_i, y_i) \in U \cap ((i_{X_i} \times i_{X_i})[X_i \times X_i]) \Leftrightarrow (x_i, y_i) \in U_i$. Hence $U \cap ((i_{X_i} \times i_{X_i})[X_i \times X_i]) = U_i$.

Conversely, the next lemma gives a criterion for when a partition of a uniform space constitutes a uniform sum decomposition.

LEMMA 2.18. *Let X be a uniform space and let $(X_i)_{i \in I}$ be a family of uniform subspaces that forms a partition of X . Suppose that whenever U_i is an entourage of*

X_i for each $i \in I$, then $\bigcup_{i \in I} U_i$ is an entourage of X . Then $X = \coprod_{i \in I} X_i$, the uniform sum.

PROOF. Let U be an entourage over X . Then $U_i = U \cap (X_i \times X_i)$ is an entourage over $X_i, \forall i \in I$. Now $\bigcup_{i \in I} U_i$ is an entourage over X which is contained in U . Thus all sets of the form $\bigcup_{i \in I} V_i$, where each V_i is an entourage of X_i , forms a fundamental system of entourages for the uniformity over X . Hence $X = \coprod_{i \in I} X_i$. \square

As a direct consequence of the above lemma one can claim that

COROLLARY 2.19. *For a compact, Hausdorff topological space X if $(X_i)_{i \in I}$ is a family of open subspaces that forms a partition of X then $X = \coprod_{i \in I} X_i$.*

PROOF. Let U_i be an entourage over X_i , for all $i \in I$. So without loss of generality U_i is open in $X_i \times X_i$ and thus in X and contains $D(X_i)$. Then $U = \bigcup_{i \in I} U_i$ is an open set in $X \times X$ that contains the diagonal $D(X)$. Hence U is an entourage over X . \square

It should be noted that the underlying topological space of a uniform sum X of uniform spaces $(X_i)_{i \in I}$ is the same as the topological sum of the underlying topological spaces of the X_i .

Uniform sums satisfy the following pasting lemma for uniformly continuous maps.

LEMMA 2.20. *Let X be the uniform sum of a family $(X_i)_{i \in I}$ of uniform spaces. If f is a function from X to a uniform space Y , then f is uniformly continuous if and only if each restriction $f|_{X_i}: X_i \rightarrow Y$ is uniformly continuous.*

PROOF. We already have noted that the inclusion maps $i_{X_i}: X_i \rightarrow X$ are uniformly continuous for all $i \in I$.

Now let $f: X \rightarrow Y$ be uniformly continuous. Then $f|_{X_i}: X_i \rightarrow Y$ can be realized as $f \circ i_{X_i}: X_i \rightarrow Y$ and hence is uniformly continuous for all $i \in I$. Conversely, let

each restriction $f|_{X_i}: X_i \rightarrow Y$ be uniformly continuous. Then for any entourage U over Y , the set $U_i = (f|_{X_i} \times f|_{X_i})^{-1}(U)$ is an entourage over X_i for all $i \in I$. Thus $R = \bigcup_{i \in I} U_i$ is an entourage over X . Let $(x, y) \in R$ so that there exists $i \in I$ such that $(x, y) \in U_i$. Now $(f|_{X_i} \times f|_{X_i})(x, y) \in U$ so that $(f \times f)(x, y) \in U$ which implies that $(x, y) \in (f \times f)^{-1}[U]$. Hence $R \subseteq (f \times f)^{-1}[U]$. Thus f is uniformly continuous. \square

In general, the uniform sum of a family of cofinite spaces may not be a cofinite space. However, this is true for *finite* uniform sums:

PROPOSITION 2.21. *The uniform sum X of a finite family $(X_i)_{i=1}^n$ of cofinite spaces is a cofinite space.*

PROOF. Note that if R_i is a cofinite entourage of X_i for $1 \leq i \leq n$, then $R_1 \cup \dots \cup R_n$ is a cofinite entourage of X . (For X/R to be finite we need the family to be finite.) Moreover, the collection of all entourages of X of this form is a fundamental system of entourages of X . Hence X is a cofinite space. \square

2.1.7. Quotients of cofinite spaces. In general, there is no obvious way to form quotients of uniform spaces. However, there is a nice way to do this in the special case of cofinite spaces. First let us recall the correspondence theorem from set theory.

2.22 (Correspondence Theorem). *Let $q: X \rightarrow Y$ be a surjective function and let $K = q^{-1}q = \{(x_1, x_2) \in X \times X \mid q(x_1) = q(x_2)\}$. Then there is a one-to-one correspondence between the set of all equivalence relations R on X such that $K \subseteq R$ and the set of all equivalence relations on Y given by*

$$R \mapsto (q \times q)[R] = qRq^{-1}.$$

In the case of the Correspondence Theorem, if R is an equivalence relation on X such that $K \subseteq R$ and $S = (q \times q)[R]$, then there is a bijection $X/R \rightarrow Y/S$ given by $R[x] \mapsto S[q(x)]$.

Let $R[x_1] = R[x_2]$. The equality implies that $(x_1, x_2) \in R$ so that $(q(x_1), q(x_2)) \in S$ and thus $S[q(x_1)] = S[q(x_2)]$. Conversely, let $S[q(x_1)] = S[q(x_2)]$, which implies that $(q(x_1), q(x_2)) \in S$. Then there exists $(a_1, a_2) \in R$ such that $q(a_1) = q(x_1)$ and $q(a_2) = q(x_2)$. Thus $(x_1, a_1) \in K \subseteq R, (a_1, a_2) \in R, (a_2, x_2) \in K \subseteq R$ and hence $(x_1, x_2) \in R$ which yields that $R[x_1] = R[x_2]$. So the aforesaid map is a well defined injection and for all $S[y] \in Y/S$ there is $x \in X$ such that $q(x) = y$, since q is surjective. Hence the function above will map $R[x]$ to $S[q(x)] = S[y]$ proving the surjectiveness of the map. So, in particular, $|X/R| = |Y/S|$. In the case when X and Y are cofinite spaces, if each cofinite entourage of X containing K corresponds to a cofinite entourage of Y , then we will call q a uniform quotient map. More precisely, we have,

DEFINITION 2.23 (Uniform quotient map). Let X and Y be cofinite spaces. A map $q: X \rightarrow Y$ is called a *uniform quotient map* if q is surjective and if for each equivalence relation R on Y , R is a cofinite entourage if and only if $(q \times q)^{-1}[R]$ is a cofinite entourage.

Uniform quotient maps of cofinite spaces satisfy a fundamental property analogous to that of quotient maps of topological spaces.

PROPOSITION 2.24. *Let $q: X \rightarrow Y$ be a uniform quotient map of cofinite spaces. Then*

- (1) *q is uniformly continuous;*
- (2) *a function f from Y to a uniform space Z is uniformly continuous if and only if $f \circ q$ is uniformly continuous.*

PROOF.

- (1) Let R be a cofinite entourage over Y . Then by definition $(q \times q)^{-1}[R]$ is a cofinite entourage over X . Hence q is uniformly continuous.
- (2) Let us first assume that f is uniformly continuous. Then $f \circ q$ is also uniformly continuous. Conversely, let $f \circ q$ be uniformly continuous. Now let S be an entourage over Z . Then $(f \circ q \times f \circ q)^{-1}[S] = (q \times q)^{-1}[(f \times f)^{-1}[S]]$ is an entourage in X . Hence $(f \times f)^{-1}[S]$ is an entourage over Y . Thus f is uniformly continuous.

□

COROLLARY 2.25. *If $q: X \rightarrow Y$ is a uniform quotient map of cofinite spaces, then a function f from Y to a cofinite space Z is a uniform quotient map if and only if $f \circ q$ is a uniform quotient map.*

PROOF. Let us first assume that f is a uniform quotient map. As both f and q are surjective, so is $f \circ q$. Now S is a cofinite entourage over $Z \Leftrightarrow (f \times f)^{-1}[S]$ is a cofinite entourage over $Y \Leftrightarrow (q \times q)^{-1}[(f \times f)^{-1}[S]]$ is a cofinite entourage over X . Hence S is a cofinite entourage over $Z \Leftrightarrow (f \circ q \times f \circ q)^{-1}[S]$ is a cofinite entourage over X and thus $f \circ q$ is a uniform quotient map.

Conversely, assume that $f \circ q$ is a uniform quotient map. Then $f \circ q$ is surjective. Now let $z \in Z$. Then there is $x \in X$ such that $f(q(x)) = z$ and as $q(x) \in Y$, f is surjective. Furthermore, S is a cofinite entourage over $Z \Leftrightarrow (f \circ q \times f \circ q)^{-1}[S]$ is a cofinite entourage over $X \Leftrightarrow (q \times q)^{-1}[(f \times f)^{-1}[S]]$ is a cofinite entourage over $X \Leftrightarrow (f \times f)^{-1}[S]$ is a cofinite entourage over Y . Hence f is a uniform quotient map.

□

Now we turn to constructing uniform quotients of a cofinite space. Let X be a cofinite space and let I denote its filter base of cofinite entourages. Let an equivalence

relation K on X be given and set $I' = \{R \in I \mid K \subseteq R\}$. Denote the canonical map from X to the set of equivalence classes X/K by $q: X \rightarrow X/K$.

By the correspondence theorem, the collection $J = \{(q \times q)[R] \mid R \in I'\}$ is a filter base of equivalence relations on Y , each having finitely many equivalence classes. We see that J is a fundamental system of cofinite entourages for a uniformity on the set of equivalence classes X/K . We call this uniformity the *quotient uniformity of X modulo K* .

In general, the topology induced by the quotient uniformity of X modulo K is not as fine as the quotient topology on X/K . For this reason, we write $X//K$ for the set X/K endowed with the quotient uniformity of X modulo K and the topology it induces, reserving the notation X/K for the quotient space (with the quotient topology).

DEFINITION 2.26 (Uniform quotient space). If K is an equivalence relation on a cofinite space X , then $X//K$ is called the *uniform quotient space of X modulo K* .

LEMMA 2.27. *Let X be a cofinite space and let K be an equivalence relation on X . Then the canonical map $q: X \rightarrow X//K$ is a uniform quotient map.*

PROOF. It is obvious that q is surjective. Now let S be a cofinite entourage over $X//K$ so $(q \times q)[R] \subseteq S$ for some cofinite entourage R over X , that contains K . This implies that $R \subseteq (q \times q)^{-1}[(q \times q)[R]] \subseteq (q \times q)^{-1}[S]$ implying that $(q \times q)^{-1}[S]$ is a cofinite entourage over X . Now let $(q \times q)^{-1}[T]$ is a cofinite entourage over X , for some equivalence relation T over $X//K$. Note that $(x, y) \in K$ implies that $q(x) = q(y)$ so $(q(x), q(y)) \in T$, (as T is reflexive) yielding that $(x, y) \in (q \times q)^{-1}[T]$. Thus $K \subseteq (q \times q)^{-1}[T]$. Then $(q \times q)[(q \times q)^{-1}[T]] = T$ is a cofinite entourage over $X//K$. Hence q is a uniform quotient map. \square

PROPOSITION 2.28. *Let $f: X \rightarrow Y$ be a uniform quotient map of cofinite spaces and let $K = f^{-1}f$. Then there is an isomorphism of uniform spaces $X//K \rightarrow Y$ given by $K[x] \mapsto f(x)$.*

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 \downarrow q & \nearrow & \\
 X//K & &
 \end{array}$$

PROOF. Let us define $\theta: X//K \rightarrow Y$ via $\theta(K[x]) = f(x)$. Now $K[x] = K[y] \Leftrightarrow (x, y) \in K \Leftrightarrow f(x) = f(y)$. Hence θ is well defined and an injection. Now let $y \in Y$. Since f is a surjection, there exists $x \in X$ such that $f(x) = y$. Then $\theta(K[x]) = f(x) = y$. Thus θ is surjection as well. Now let R be a cofinite entourage over Y . Then $(f \times f)^{-1}[R]$ is an cofinite entourage over X containing K . Thus $(q \times q)[(f \times f)^{-1}[R]]$ is a cofinite entourage over $X//K$. Let $(K[x], K[y]) \in (q \times q)[(f \times f)^{-1}[R]]$. Then there exists $(p, r) \in (f \times f)^{-1}[R]$ such that $K[x] = K[p]$ and $K[y] = K[r]$ which implies that $(\theta(K[x]), \theta(K[y])) = (\theta(K[p]), \theta(K[r])) = (f(p), f(r)) \in R$. this shows that $\theta \times \theta[(q \times q)[(f \times f)^{-1}[R]]] \subseteq R$ and thus $(q \times q)[(f \times f)^{-1}[R]] \subseteq (\theta \times \theta)^{-1}[R]$. Hence θ is uniformly continuous.

Now let S be a cofinite entourage over $X//K$. Then there exists T a cofinite entourage over X , containing K such that $(q \times q)[T] \subseteq S$. But then $(f \times f)[T]$ is a cofinite entourage over Y . Also $(f \times f)[T] = (\theta \times \theta)[(q \times q)[T]] \subseteq (\theta \times \theta)[S] = (\theta^{-1} \times \theta^{-1})[S]$. Hence θ^{-1} is uniformly continuous as well. Thus our claim follows. \square

It should be noted that, although a uniform quotient space $X//K$ has a fundamental system of entourages consisting of cofinite entourages, it may not be Hausdorff, even if X is a cofinite Hausdorff space. We give the following answer to the question as to when $X//K$ is a Hausdorff cofinite space.

PROPOSITION 2.29. *Let X be a cofinite space and let I be the filter base of cofinite entourages of X . If K is any equivalence relation on X , then the following conditions are equivalent:*

- (1) $X//K$ is a Hausdorff cofinite space;
- (2) $\bigcap\{R \mid R \in I \text{ and } K \subseteq R\} = K$.

PROOF. (1) \Rightarrow (2):

Let $X//K$ be Hausdorff. Since $K \subseteq R$ for all R in I , we obtain $K \subseteq \bigcap\{R \mid R \in I \text{ and } K \subseteq R\}$. Now let $(x, y) \in \bigcap\{R \mid R \in I \text{ and } K \subseteq R\}$. This implies $(q(x), q(y)) \in (q \times q)[R]$, for all $R \in I$ whenever R contains K . But $X//K$ is Hausdorff so we conclude that $q(x) = q(y)$. Thus $K[x] = K[y]$. Hence $(x, y) \in K$. So,

$$\bigcap\{R \mid R \in I, K \subseteq R\} = K$$

(2) \Rightarrow (1):

Let us now take $\bigcap\{R \mid R \in I \text{ and } K \subseteq R\} = K$. Now if $K[x] \neq K[y]$ in $X//K$, we have $(x, y) \notin K$. Hence there exists some $R \in I$ containing K such that $(x, y) \notin R$. But then $(q(x), q(y)) = (K[x], K[y]) \notin (q \times q)[R]$. Otherwise there will be $(t, s) \in R$ such that $q(t) = q(x), q(y) = q(s)$. Then $(x, t) \in K \subseteq R, (t, s) \in R, (s, y) \in K \subseteq R$, which implies $(x, y) \in R$, a contradiction. Hence $X//K$ is a Hausdorff cofinite space. \square

Note that we do not even require X to be Hausdorff in the above cases.

In some important special cases, the uniform quotient space of a cofinite space X modulo an equivalence relation K is equal to its quotient space X/K (as topological spaces). To give a necessary and sufficient condition for this to hold, we first make some general observations about quotients of topological spaces.

Let K be an equivalence relation on a topological space X and denote the canonical quotient map by $q: X \rightarrow X/K$. We say that a subset $B \subseteq X$ is K -saturated

if $K[B] = B$. It is easy to check that the intersection of any family of K -saturated subsets is again K -saturated.

Let $\{B_\lambda \mid \lambda \in \Lambda\}$ be a family of K -saturated subsets of X . Then for all λ in Λ , $K[B_\lambda] = B_\lambda$. Let $B = \bigcap_{\lambda \in \Lambda} B_\lambda$ and $x \in K[B]$. Then there exists some b in B such that $x \in K[b]$. Hence $x \in K[b] \subseteq K[B_\lambda] = B_\lambda$, for all $\lambda \in \Lambda$. Thus $x \in \bigcap_{\lambda \in \Lambda} B_\lambda = B$. So $K[B] \subseteq B \subseteq K[B]$. Thus $K[B] = B$.

Hence, for any subset A of X , there is a unique smallest K -saturated closed subset \overline{A}^s of X with $A \subseteq \overline{A}^s$; simply let \overline{A}^s be the intersection of the family of all closed K -saturated subsets of X that contain A .

LEMMA 2.30. *For any subset A of X , we have $q(\overline{A}^s) = \overline{q(A)}$.*

PROOF. Let us first see that $q^{-1}(q(\overline{A}^s)) = K[\overline{A}^s] = \overline{A}^s$.

Now $x \in q^{-1}(q(\overline{A}^s)) \Leftrightarrow q(x) \in q(\overline{A}^s) \Leftrightarrow$ there exists $t \in \overline{A}^s$ such that $q(t) = q(x) \Leftrightarrow (t, x) \in K \Leftrightarrow x \in K[t] \subseteq K[\overline{A}^s]$. So $q^{-1}(q(\overline{A}^s)) = \overline{A}^s$ and hence $q(\overline{A}^s)$ is closed in X/K . Then $A \subseteq \overline{A}^s$ implies that $q(A) \subseteq q(\overline{A}^s)$ and thus $\overline{q(A)} \subseteq \overline{q(\overline{A}^s)} = q(\overline{A}^s)$.

Since $\overline{q(A)}$ is closed in X/K , $q^{-1}(\overline{q(A)})$ is closed in X . Clearly, $A \subseteq \overline{A} \subseteq q^{-1}(\overline{q(A)}) \subseteq q^{-1}(q(\overline{A}^s))$. Now let $x \in K[q^{-1}(\overline{q(A)})]$. This implies that there exists $t \in q^{-1}(\overline{q(A)})$ such that $x \in K[t]$. Then $(t, x) \in K$, where $q(t) \in \overline{q(A)}$, so $q(x) = q(t) \in \overline{q(A)}$ and thus $x \in q^{-1}(\overline{q(A)})$. Hence $K[q^{-1}(\overline{q(A)})] \subseteq q^{-1}(\overline{q(A)}) \subseteq K[q^{-1}(\overline{q(A)})]$. So $K[q^{-1}(\overline{q(A)})] = q^{-1}(\overline{q(A)})$. Thus $q^{-1}(\overline{q(A)})$ is a K -saturated closed subset of X containing A and hence $\overline{A}^s \subseteq q^{-1}(\overline{q(A)})$. So $q(\overline{A}^s) \subseteq q(q^{-1}(\overline{q(A)})) = \overline{q(A)}$. Our claim that $q(\overline{A}^s) = \overline{q(A)}$ follows. \square

THEOREM 2.31. *Let X be a cofinite space and let K be an equivalence relation on X .*

- (1) *The identity map $\text{id}: X/K \rightarrow X//K$ is a continuous bijection.*
- (2) *The identity map $\text{id}: X/K \rightarrow X//K$ is a homeomorphism (i.e., the topology induced by the quotient uniformity of X modulo K and the quotient topology*

are the same) if and only if K satisfies the property: for each subset $A \subseteq X$, the K -saturated closure $\overline{A}^s = \bigcap R[A]$, as R runs through all cofinite entourages of X such that $K \subseteq R$.

PROOF. We will prove the results in the order they appear.

(1) Its obvious that $\text{id}: X/K \rightarrow X//K$ is a bijection. Now let O be open in $X//K$. Let $x \in q^{-1}(\text{id}^{-1}(O))$ so that $K[x] \in \text{id}^{-1}(O) = O$. Hence there is a cofinite entourage R over X such that $K[x] \in (q \times q)[R][K[x]] \subseteq O$. Now let $t \in R[x]$. Hence $(x, t) \in R$ which implies that $(K[x], K[t]) \in (q \times q)[R]$. Therefore $K[t] \in (q \times q)[R][K[x]] \subseteq O$, so $t \in q^{-1}(O)$. Hence $x \in R[x] \subseteq q^{-1}(O)$. Hence $q^{-1}(O)$ is open in X . Thus O is open in X/K , proving the continuity of id .

(2) Let us first assume that id is a homeomorphism between X/K and $X//K$. Let $I' = \{R \mid R \text{ is a cofinite entourage over } X \text{ and } K \subseteq R\}$. Now for any subset Q of $X//K$, $\overline{Q} = \bigcap_{R \in I'} (q \times q)[R][Q]$. As id is a homeomorphism it is also a closed map. Now writing $q(A) = A_K$ and $\bigcap_{R \in I'} (q \times q)[R][A_K] = A_{K_q}$, it follows from Lemma 2.30 it follows that

$$\overline{A}^s = q^{-1}(\overline{A_K}^{X/K}) = q^{-1}(\overline{A_K}^{X//K}) = q^{-1}(A_{K_q}).$$

If $x \in \overline{A}^s$ it follows that $q(x) \in A_{K_q}$ which implies that for all $R \in I'$ there exists $a_R \in A$ such that $q(x) \in (q \times q)[R][q(a_R)]$. Then $(q(a_R), q(x)) \in (q \times q)[R]$. Hence there exists $(t_{1_R}, t_{2_R}) \in R$, for all $R \in I'$ such that $q(t_{1_R}) = q(a_R)$ and $q(t_{2_R}) = q(x)$. So $(a_R, t_{1_R}) \in K \subseteq R$, $(t_{1_R}, t_{2_R}) \in R$, $(t_{2_R}, x) \in K \subseteq R$, for all $R \in I'$. Thus $(a_R, x) \in R$, for all $R \in I'$ implying that $x \in R[a_R]$, for all $R \in I'$ and thus $x \in \bigcap_{R \in I'} R[A]$.

On the other hand, let us take $y \in \bigcap_{R \in I'} R[A]$. This implies that for all R in I' there exists $b_R \in A$ such that $y \in R[b_R]$. So $(b_R, y) \in R$, for all $R \in I'$.

This implies that $(q(b_R), q(y)) \in (q \times q)[R]$, for all $R \in I'$, so $q(y) \in A_{K_q}$ and therefore $y \in q^{-1}(A_{K_q}) = \bar{A}^s$. Thus $\bar{A}^s = \bigcap_{R \in I'} R[A]$.

It is worth noting that $A_{K_q} = \bigcap_{R \in I'} (qRq^{-1})[A_K] = \bigcap (qRK)[A] = \bigcap q(R[A])$ as $K \subseteq R$ for all $R \in I'$, Hence $A_{K_q} = (qq^{-1})(\bigcap q(R[A])) = q(\bigcap (KR)[A]) = q(\bigcap_{R \in I'} R[A])$.

Conversely, let us assume that $\bar{A}^s = \bigcap_{R \in I'} R[A]$.

We will first see that for any subset A of X/K , $q^{-1}(A)$ is K -saturated. For, $x \in K[q^{-1}(A)]$ implies that there exists $a \in q^{-1}(A)$, such that $(a, x) \in K$ and then $q(x) = q(a) \in A$. Hence $x \in q^{-1}(A)$. So $K[q^{-1}(A)] \subseteq q^{-1}(A) \subseteq K[q^{-1}(A)]$. Hence $K[q^{-1}(A)] = q^{-1}(A)$.

Now let B be closed in X/K . Then $C = q^{-1}(B)$ is closed in X . Hence C is a closed K -saturated subset of X . So $C = \bar{C}^s = \bigcap_{R \in I'} R[C]$.

We now claim that $B = q(C) = \bigcap_{R \in I'} (q \times q)[R][q(C)]$

To see this let $s \in q(C)$. This implies that there exists $t \in C$ such that $s = q(t)$ and for some $b_R \in R[C]$, for all $R \in I'$, $(b_R, t) \in R$. Then $(q(b_R), q(t)) \in (q \times q)[R]$, for all $R \in I'$ so that, for all $R \in I'$, $s \in (q \times q)[R][q(b_R)] \subseteq (q \times q)[R][q(C)]$ and thus $s \in \bigcap_{R \in I'} (q \times q)[R][q(C)]$.

For the other way, let $z \in \bigcap_{R \in I'} (q \times q)[R][q(C)]$. This implies there exists $c_R \in C$ such that $(q(c_R), z) \in (q \times q)[R]$, for all $R \in I'$ and so for all $R \in I'$, there exists (m, n) in R such that $q(m) = q(c_R), q(n) = z$. Then, for all $R \in I'$, $(c_R, m) \in K \subseteq R$ and so $(c_R, n) \in R$, for all $R \in I'$. Consequently, for all R in I' , n in $R[c_R] \subseteq R[C]$, so $n \in \bigcap_{R \in I'} R[C] = C$. Thus $z = q(n) \in q(C)$. Hence,

$$B = \bigcap_{R \in I'} (q \times q)[R][q(C)] = \bar{B}^{X//K}$$

So id is a closed map and thus is a homeomorphism.

□

COROLLARY 2.32. *If K is an equivalence relation on a cofinite space X such that X/K is compact and $\bigcap\{R \mid R \in I \text{ and } K \subseteq R\} = K$, then $\text{id}: X/K \rightarrow X//K$ is a homeomorphism.*

PROOF. By Proposition 2.29, $X//K$ is Hausdorff and so id is a continuous bijection from a compact space to a Hausdorff space and thus is a homeomorphism. \square

COROLLARY 2.33. *If X is a cofinite space and R is a cofinite entourage of X , then $\text{id}: X/R \rightarrow X//R$ is a homeomorphism.*

PROOF. First let us take $I = \{S \mid S \text{ is a cofinite entourage over } X \text{ and } R \subseteq S\}$. Since R is a cofinite entourage, X/R is finite discrete and thus compact. Also

$$\bigcap\{S \in I; R \subseteq S\} = R$$

and thus by Proposition 2.29, $X//R$ is Hausdorff, so by the previous corollary, $\text{id}: X/R \rightarrow X//R$ is a homeomorphism. \square

CHAPTER 3

Inverse limits of compact Hausdorff spaces

Most of the results in this chapter are already well known; see for example [6]. We are including the proofs as some of these techniques are used for proving new results in the next three chapters of this dissertation. Also in one of the occasions we did not find the proof perfectly penned in the aforesaid reference. In Note 3.3 we introduce a new property and prove that it is equivalent to the other well known properties cited there.

3.1. Inverse limits of compact Hausdorff spaces

We begin with some observations about general inverse systems of topological spaces. Let (X_i, ϕ_{ij}) be an inverse system of topological spaces indexed by a directed set I .

3.1. Let X denote the inverse limit of (X_i, ϕ_{ij}) and let $\phi_i: X \rightarrow X_i$ be the canonical map for each $i \in I$.

- (1) The family of sets $\phi_i^{-1}(U_i)$, where $i \in I$ and U_i is open in X_i , is a basis for the topology of X .
- (2) Let A be a subset of X and write $A_i = \phi_i(A)$ for each $i \in I$. Then

$$\bar{A} = \bigcap_{i \in I} \phi_i^{-1}(\bar{A}_i) = \varprojlim \bar{A}_i.$$

- (3) If A is a subset of X satisfying $\phi_i(A) = X_i$ for all $i \in I$, then A is dense in X .

- (4) If $f: Y \rightarrow X$ is a function from a space Y , then f is continuous if and only if each composition $\phi_i f$ is continuous.

PROOF.

- (1) Let $C = \prod_{i \in I} X_i$. Let $\pi_i: C \rightarrow X_i$ be the canonical projection map and hence $\pi_i|_X: X \rightarrow X_i \equiv \phi_i$. A basic open set V in C is of the form $V = \prod_{i \in I} V_i$, where all but finitely many $V_i = X_i$ and the rest of the V_i 's, say, for $i \in \{r_1, r_2, \dots, r_n\}$, are non trivial nonempty open subsets of X_i . We claim $V = \bigcap_{i \in I} \pi_i^{-1}(V_i)$.

For, $x = (x_i)_{i \in I} \in V \Leftrightarrow x_i \in V_i$, for all $i \in I \Leftrightarrow \pi_i(x) \in V_i$, for all $i \in I \Leftrightarrow x \in \pi_i^{-1}(V_i)$, for all $i \in I \Leftrightarrow x \in \bigcap_{i \in I} \pi_i^{-1}(V_i)$ and thus $V = \bigcap_{i=1}^n \pi^{-1}(V_{r_i})$. Hence any basic open set P in X is of the form $P = X \cap (\bigcap_{i=1}^n \pi^{-1}(V_{r_i}))$. Now $a = (a_i)_{i \in I} \in P$ implies that $a \in \pi_i^{-1}(V_i)$, for all $i = r_1, r_2, \dots, r_n$ implies that $\pi_i(a) \in V_i$, for all $i = r_1, r_2, \dots, r_n$ implies that $a_i \in V_i, \forall i = r_1, r_2, \dots, r_n$. Now let $k \geq \sup\{r_j \mid 1 \leq j \leq n\}$. Clearly, $\phi_{ik}^{-1}(V_i)$ is open in X_k , for all $i = r_1, r_2, \dots, r_n$. Now $\phi_{ik} \phi_k(a) = \phi_i(a) = a_i$, for all $i = r_1, r_2, \dots, r_n$ implies that $\phi_k(a) \in \phi_{ik}^{-1}(a_i) \subseteq \phi_{ik}^{-1}(V_i)$, for all $i = r_1, r_2, \dots, r_n$ implies that $\phi_k(a) \in \bigcap_{i=r_1}^{r_n} \phi_{ik}^{-1}(V_i)$. Let us call $U = \bigcap_{i=r_1}^{r_n} \phi_{ik}^{-1}(V_i)$. Then U is open in X_k implies that $a \in \phi_k^{-1}(U)$, where $\phi_k^{-1}(U)$ is an open set in X . Now let $b = (b_i)_{i \in I} \in \phi_k^{-1}(U)$. Thus $b_k = \phi_k(b) \in U$. Now $b_i = \phi_i(b) = \phi_{ik} \phi_k(b) = \phi_{ik}(b_k) \in \phi_k(\phi_{ik}^{-1}(V_i)) \subseteq V_i$, for all $i = r_1, r_2, \dots, r_n$, so $b \in \phi_i^{-1}(V_i)$, for all $i = r_1, r_2, \dots, r_n$ and hence $b \in P$. Therefore $\phi_k^{-1}(U) = P$, where U is open in X_k . Hence our claim follows.

- (2) Let $x = (x_i)_{i \in I} \in \bar{A}$. We claim that $x_i \in \bar{A}_i$ for all $i \in I$. Let U_i be an open set containing $x_i \in X_i$. Hence $\phi_i^{-1}(U_i)$ is open in X and contains x . Thus $\phi_i^{-1}(U_i) \cap A \neq \emptyset$. Let $a = (a_i)_{i \in I} \in \phi_i^{-1}(U_i) \cap A$. Then $a_i = \phi_i(a) \in U_i \cap \phi_i(A) \subseteq U_i \cap A_i$. So $x_i \in \bar{A}_i$. Hence $\phi_i(x) \in \bar{A}_i$, for all $i \in I$ so

$x \in \phi_i^{-1}(\overline{A_i})$, for all $i \in I$ and thus $x \in \bigcap_{i \in I} \phi_i^{-1}(\overline{A_i})$. Also, alternatively, one might note that $\bigcap_{i \in I} \phi^{-1}(\overline{A_i})$ is a closed set containing A . Thus $\overline{A} \subseteq \bigcap_{i \in I} \phi_i^{-1}(\overline{A_i})$. On the other hand let us now take $(y_i)_{i \in I} = y \in \bigcap_{i \in I} \phi_i^{-1}(\overline{A_i})$ so that $y_i = \phi_i(y) \in \overline{A_i}$, for all $i \in I$. Now let U be open in X such that $y \in U$. Then there exists some $k \in I$ such that for some open set U_k in X_k , $y \in \phi_k^{-1}(U_k) \subseteq U$. Hence U_k is an open set containing y_k . Thus $U_k \cap A_k \neq \emptyset$. Let $a_k \in U_k \cap A_k$. Hence there exists some a in A such that $\phi_k(a) = a_k$ and thus $a \in \phi^{-1}(U_k) \subseteq U$ implying that $a \in A \cap U$. Hence $y \in \overline{A}$ giving us our claim.

For the second part of the proof let us first see that $(\overline{A_i}, \phi_{ij}|_{\overline{A_i}})$ forms an inverse system. For $\phi_{ij}(\overline{A_j}) \subseteq \overline{\phi_{ij}(A_j)} = \overline{\phi_{ij}\phi_j(A)} = \overline{\phi_i(A)} = \overline{A_i}$. Clearly, $\phi_{ij}|_{\overline{A_i}}$ is well defined and thus $(\overline{A_i}, \phi_{ij}|_{\overline{A_i}})_{i \in I, i \leq j}$ forms an inverse system. Let $B = \varprojlim_{i \in I} \overline{A_i}$ and let $\phi_i|_B: B \rightarrow \overline{A_i}$ be the corresponding canonical map for all $i \in I$. Now $a = (a_i)_{i \in I} \in B \Leftrightarrow a_i = \phi_i(a) \in \overline{A_i} \Leftrightarrow a \in \phi_i^{-1}(\overline{A_i})$, for all $i \in I \Leftrightarrow a \in \bigcap_{i \in I} \phi_i^{-1}(\overline{A_i}) = \overline{A}$. Hence $B = \varprojlim_{i \in I} \overline{A_i} = \overline{A}$

- (3) Let $x = (x_i)_{i \in I} \in X$ and let $\phi_i^{-1}(U_i)$ be a basic open set in X containing x for some $i \in I$. Then $x_i = \phi_i(x) \in U_i \subseteq X_i$. Thus there is an $a = (a_i)_{i \in I} \in A$, such that $a_i = \phi_i(a) = x_i$. Hence $a \in \phi^{-1}(x_i) \subseteq \phi_i^{-1}(U_i)$. So $a \in A \cap \phi_i^{-1}(U_i)$. Thus $x \in \overline{A}$. So $X = \overline{A}$, giving us our claim that A is dense in X .
- (4) Let us first take f to be continuous. As each ϕ_i is continuous too, so is $\phi_i \circ f, \forall i \in I$. Conversely, let $\phi_i \circ f$ be continuous for all i in I . Now let $\phi_i^{-1}(U_i)$ be a basic open set in X for some i in I . Now $f^{-1}(\phi_i^{-1}(U_i)) = (\phi_i \circ f)^{-1}(U_i)$ is open in Y . Hence f is continuous.

□

Next we specialize to compact Hausdorff spaces.

3.2. Let (X_i, ϕ_{ij}) be an inverse system of non-empty compact Hausdorff spaces indexed by a directed set I . Then the inverse limit $X = \varprojlim X_i$ has the following properties:

- (1) X is a non-empty compact Hausdorff space.
- (2) $\phi_i(X) = \bigcap_{j \geq i} \phi_{ij}(X_j)$ for each $i \in I$.
- (3) If A, B are disjoint closed subsets of X , then there exists $i \in I$ such that $\phi_i(A), \phi_i(B)$ are disjoint closed subsets of X_i .
- (4) If Y is a discrete space and $f: X \rightarrow Y$ is a continuous map, then f factors through some X_k ; i.e., for some $k \in I$ there is a continuous map $h: X_k \rightarrow Y$ such that $f = h\phi_k$.

PROOF. For (1) and (2) see, for example, [6] [Propositions 1.1.5 and 1.1.6].

$$(3) \bigcap_{i \in I} \phi_i^{-1}(C_i) = \overline{A} \cap \overline{B} = A \cap B = \emptyset.$$

(4) The image $Y_0 = f(X)$ is a compact subspace of a discrete space, and thus is finite. The sets $B_y = f^{-1}(y)$ for $y \in Y_0$ are disjoint closed subsets of X . By (3) there exists $i \in I$ such that the images $\phi_i(B_y), y \in Y_0$, are disjoint closed subsets of X_i . Since X_i is a compact Hausdorff space, there exist disjoint open subsets V_y in X_i , for $y \in Y_0$, such that each $\phi_i(B_y) \subseteq V_y$. Let $C = X_i \setminus \bigcup_{y \in Y_0} V_y$, a closed subset of X_i . Now $\phi_i(X) = \bigcup_{y \in Y_0} \phi_i(B_y) \subseteq \bigcup_{y \in Y_0} V_y$, so by (2),

$$C \cap \left(\bigcap_{j \geq i} \phi_{ij}(X_j) \right) = C \cap \phi_i(X) = \emptyset.$$

However X_i is compact. So there exists finitely many $j_1, \dots, j_n \geq i$ such that $C \cap \phi_{ij_1}(X_{j_1}) \cap \dots \cap \phi_{ij_n}(X_{j_n}) = \emptyset$. Choose $k \geq j_1, \dots, j_n$. Then $\phi_{ik}(X_k) \subseteq \phi_{ij_1}(X_{j_1}) \cap \dots \cap \phi_{ij_n}(X_{j_n}) \subseteq X_i \setminus C = \bigcup_{y \in Y_0} V_y$. Thus the sets $W_y = \phi_{ik}^{-1}(V_y)$, for $y \in Y_0$, partition X_k into open sets. Hence the map $h: X_k \rightarrow Y$ that maps each W_y to y is continuous.

Furthermore

$$(h\phi_k)(B_y) \subseteq h(\phi_{ik}^{-1}(\phi_i(B_y))) \subseteq h(\phi_{ik}^{-1}(V_y)) = h(W_y) = y$$

and thus $h\phi_k = f$. □

3.3. *The following conditions are equivalent for any compact Hausdorff space X :*

- (1) *X is totally disconnected;*
- (2) *the clopen subsets of X form a basis for its topology;*
- (3) *$\bigcap \{R \mid R \text{ is a co-discrete equivalence relation on } X\}$ is equal to the diagonal of $X \times X$;*
- (4) *X is Hausdorff cofinite space, when endowed with the unique uniform structure;*
- (5) *X is the inverse limit of an inverse system (X_i, ϕ_{ij}) of finite discrete spaces.*

We believe condition (3) in this list is a new observation. The other conditions are already known. The next lemma is used in proof of 3.3.

LEMMA 3.4. *Let X be a compact Hausdorff space and let $x \in X$. Then the intersection of all clopen subsets of X that contain x is equal to the component of x .*

PROOF. Let $P = \{A_i \mid i \in I\}$ be the collection of clopen subsets of X that contain x . Let C be the connected component of X containing x . Now for all $A_i \in P, x \in A_i \cap C$. Also $A_i \cap C$ is clopen in C . Hence $A_i \cap C = C$ as C is connected. Now $X \setminus A_i$ is open in X implies that $(X \setminus A_i) \cap C = X \cap A_i^c \cap C = C \cap A_i^c = C \cap A_i^c \cup \phi = (C \cap A_i^c) \cup (C \cap C^c) = C \cap (C^c \cup A_i^c) = C \cap (C \cap A_i)^c = C \setminus (C \cap A_i)$ is open in C . Hence, as C is connected, we have $C \setminus (C \cap A_i) = \phi$. This implies $C = C \cap A_i$ and thus $C \subseteq A_i$, for all $A_i \in P$. Hence $C \subseteq \bigcap_{A_i \in P} A_i$. Let $B = \bigcap_{A_i \in A} A_i$. Suppose that B is disconnected. Then there exists two clopen subsets B_1 and B_2 in B such that $B_1 \cap B_2 = \emptyset$ and $B_1 \cup B_2 = B$. But B is closed in X , so B_1, B_2 are closed

in X as well. X is compact and Hausdorff and thus it is normal also. Thus there are open sets U_1 and U_2 in X such that $B_1 \subseteq U_1, B_2 \subseteq U_2$ and $U_1 \cap U_2 = \emptyset$. Let $E = X \setminus U_1 \cup U_2$ and note this implies that $E \cap B = \emptyset$. Also E is closed in X . Since X is compact, using the finite intersection property, there exists a finite collection $\{A_i\}_{i=1}^n$ such that $(\bigcup_{i=1}^n A_i) \cap E = \emptyset$. Let us call $I = \bigcap_{i=1}^n A_i$. Clearly, I is a clopen subset of X and $I \subseteq X \setminus E \subseteq U_1 \cup U_2$ and this implies that $I = (I \cap U_1) \cup (I \cap U_2)$ where $(I \cap U_1) \cap (I \cap U_2) = \emptyset$. Each $(I \cap U_i)$ is open in X and hence in I for $i = 1, 2$ and thus $(I \cap U_i)$ is closed in I for $i = 1, 2$. Since $x \in I$, let us take $x \in I \cap U_1$ implying that $B \subseteq (I \cap U_1) \subseteq U_1$ and hence $B_2 \subseteq U_1 \cap U_2$, a contradiction. Similarly, $x \in I \cap U_2$ implies that $A \subseteq I \cap U_2 \subseteq U_2$ and thus $B_1 \subseteq U_1 \cap U_2$!! Hence B is connected, i.e. $B \subseteq C$ as C is the maximal connected subset of X containing x . Hence $B = C$. \square

PROOF OF 3.3. (1) \Rightarrow (2). Suppose X is totally disconnected. Let U be an open subset of X and let $x \in U$. Let $(C_\lambda \mid \lambda \in \Lambda)$ be the family of all clopen subsets of X that contain x . Then $(C_\lambda \setminus U \mid \lambda \in \Lambda)$ is a family of closed subsets of X which has empty intersection (by the lemma 2.4). Hence, by compactness, there exists finitely many $\lambda_1, \dots, \lambda_n \in \Lambda$ such that

$$(C_{\lambda_1} \setminus U) \cap \dots \cap (C_{\lambda_n} \setminus U) = \emptyset$$

Thus $C = C_{\lambda_1} \cap \dots \cap C_{\lambda_n}$ is an open subset of X such that $x \in C \subseteq U$. It follows that the clopen subsets of X form a basis for its topology.

(2) \Rightarrow (3). Suppose the clopen subsets of X form a basis for its topology and let $x, y \in X$ with $x \neq y$. Since X is also Hausdorff, there is a clopen subset C of X with $x \in C$ and $y \notin C$. The equivalence relation R that partitions X into the two equivalence classes C and $X \setminus C$ is cofinite and $(x, y) \notin R$. It follows that the intersection of all cofinite equivalence relations consists only of the diagonal in $X \times X$.

(3) \Rightarrow (4). Let $I = \{R \mid R \text{ is a co-discrete equivalence on } X\}$ and assume that the intersection of the family I is equal to the diagonal. Let W be an open neighborhood of the diagonal in $X \times X$. Then

$$\bigcap_{R \in I} (R \setminus W) = \left(\bigcap_{R \in I} R \right) \setminus W = \emptyset$$

and each $R \setminus W$ is closed in $X \times X$. So by compactness, there exist finitely many $R_1, \dots, R_n \in I$ such that $R_1 \cap \dots \cap R_n \subseteq W$. Now letting $R = R_1 \cap \dots \cap R_n$, we see that $R \in I$ and $R \subseteq W$. It follows that I is a fundamental system of entourages of X ; whence X is a cofinite space.

(4) \Rightarrow (3). This follows from X being Hausdorff. Hence for $x \neq y$ in X , there exists two basic open sets $R[x], S[y]$ of X such that $x \in R[x], y \in S[y]$ and $R[x] \cap S[y] = \emptyset$. Since both of R, S are cofinite equivalence relations over X , so is $R \cap S$. Let $T = R \cap S$. Clearly, $x \in T[x], y \in T[y]$. If possible let $t \in T[x] \cap T[y]$ so $(x, t) \in T, (y, t)$ and thus also $(t, y) \in T$. Hence $(x, y) \in T \subseteq R$, a contradiction. So $T[x] \cap T[y] = \emptyset$ and hence $(x, y) \notin T$.

(3) \Rightarrow (5). The set $I = \{R \mid R \text{ is a co-discrete equivalence on } X\}$, ordered by the opposite of inclusion, is a directed set. Consider the inverse system $(X/R, \phi_{RS})$ of finite discrete spaces indexed by I , where $\phi_{RS}: X/S \rightarrow X/R$ is the natural map for all $R \supseteq S$. The quotient maps $X \rightarrow X/R, R \in I$, determine a continuous map $f: X \rightarrow \varprojlim X/R$. Condition (3) implies that f is injective. By 3.1(3), the image $f(X)$ is dense in $\varprojlim X/R$. However, f is a closed map since it is a continuous map from a compact space to a Hausdorff space. Hence $f(X)$ is closed and f is surjective. Therefore f is a homeomorphism.

(5) \Rightarrow (1). This follows since an inverse limit of totally disconnected spaces is a totally disconnected space. □

DEFINITION 3.5 (Profinite space). A compact Hausdorff space X that satisfies the equivalent conditions of the previous result is called a *profinite space*.

We will always assume that a profinite space X is endowed with the unique uniform structure that induces its topology, and hence, by 3.3(4), X is a Hausdorff cofinite space. Thus profinite spaces are precisely the compact, Hausdorff cofinite spaces.

CHAPTER 4

Topological graphs

In this chapter we introduce *topological graphs* as topological spaces Γ with a compatible graph structure. An *orientation* of a topological graph Γ is a closed subset $E^+(\Gamma)$ consisting of exactly one edge in each pair $\{e, \bar{e}\}$.

An equivalence relation R on a graph Γ is *compatible* if the corresponding quotient space inherits a natural graph structure.

We then established that if R is any cofinite equivalence relation on a topological graph Γ , then there exists a compatible cofinite equivalence relation S on Γ such that $S \subseteq R$. In the course of the proof we notice that if Γ is a topological graph with a specified closed orientation $E^+(\Gamma)$, then for any cofinite equivalence relation R on Γ , there exists a compatible orientation-preserving cofinite equivalence relation S on Γ such that $S \subseteq R$.

4.1. Topological graphs

A *topological graph* is a topological space Γ that is partitioned into two closed subsets $V(\Gamma)$ and $E(\Gamma)$ together with two continuous functions $s, t: E(\Gamma) \rightarrow V(\Gamma)$ and a continuous function $\bar{\cdot}: E(\Gamma) \rightarrow E(\Gamma)$ satisfying the following properties: for every $e \in E(\Gamma)$,

- (1) $\bar{e} \neq e$ and $\bar{\bar{e}} = e$;
- (2) $t(\bar{e}) = s(e)$ and $s(\bar{e}) = t(e)$.

The elements of $V(\Gamma)$ are called *vertices*. An element $e \in E(\Gamma)$ is called a (*directed*) *edge* with *source* $s(e)$ and *target* $t(e)$; the edge \bar{e} is called the *reverse* or *inverse* of e .

A *map of graphs* $f: \Gamma \rightarrow \Delta$ is a function that maps vertices to vertices, edges to edges, and preserves sources, targets, and inverses of edges. Analogously, we will call a map of graphs a *graph isomorphism* if and only if it is a bijection.

An *orientation* of a topological graph Γ is a closed subset $E^+(\Gamma)$ consisting of exactly one edge in each pair $\{e, \bar{e}\}$. In this situation, setting $E^-(\Gamma) = \{e \in E(\Gamma) \mid \bar{e} \in E^+(\Gamma)\}$ we see that $E(\Gamma)$ is a disjoint union of the two closed (hence also open) subsets $E^+(\Gamma)$, $E^-(\Gamma)$.

4.1. *Let Γ be a topological graph. The following are equivalent:*

- (1) Γ admits an orientation;
- (2) there exists a continuous map of graphs from Γ to the discrete graph with a single vertex and a single edge and its inverse;
- (3) there exists a continuous map of graphs $f: \Gamma \rightarrow \Delta$ for some discrete graph Δ .

Conceivably there are topological graphs that do not admit closed orientations. However such graphs will not concern us. Therefore, unless otherwise stated, by a topological graph we will henceforth mean a topological graph that admits an orientation.

We will be interested in equivalence relations on graphs that are compatible with the graph structure:

DEFINITION 4.2 (Compatible equivalence relation). An equivalence relation R on a graph Γ is *compatible* if the following properties hold:

- (1) $R = R_V \cup R_E$ where R_V , R_E are equivalence relations on $V(\Gamma)$, $E(\Gamma)$, precisely the restriction of R ;
- (2) if $(e_1, e_2) \in R$, then $(s(e_1), s(e_2)) \in R$, $(t(e_1), t(e_2)) \in R$, and $(\bar{e}_1, \bar{e}_2) \in R$;
- (3) for all $e \in E(\Gamma)$, $(e, \bar{e}) \notin R$;

4.3. If K is a compatible equivalence relation on Γ , then there is a unique way to make Γ/K into a graph such that the canonical map $\Gamma \rightarrow \Gamma/K$ is a map of graphs. It is defined by setting $s(K[e]) = K[s(e)]$, $t(K[e]) = K[t(e)]$, and $\overline{K[e]} = K[\bar{e}]$. Conversely, if Δ is a graph and $f: \Gamma \rightarrow \Delta$ is a surjective map of graphs, then $K = f^{-1}f = \{(a, b) \in \Gamma \times \Gamma \mid f(a) = f(b)\}$ is a compatible equivalence relation on Γ and f induces an isomorphism of graphs such that $\Gamma/K \cong \Delta$.

PROOF. Let K be a compatible equivalence relation on Γ . Let $V(\Gamma/K) = \{K[v] \mid v \in V(\Gamma)\}$ and $E(\Gamma/K) = \{K[e] \mid e \in E(\Gamma)\}$. If possible, let $K[x] \in V(\Gamma/K) \cap E(\Gamma/K)$. So there exists $v \in V(\Gamma)$ and $e \in E(\Gamma)$ such that $(x, e) \in K = K_V \cup K_E = K \ni (x, v)$ which implies that $x \in V(\Gamma) \cap E(\Gamma) = \emptyset$, a contradiction.

Now let us see that $s: E(\Gamma/K) \rightarrow V(\Gamma/K)$, $t: E(\Gamma/K) \rightarrow V(\Gamma/K)$, $\bar{\cdot}: E(\Gamma/K) \rightarrow E(\Gamma/K)$ are well defined. For, $K[e_1] = K[e_2]$ implies that $(e_1, e_2) \in K$, so $(s(e_1), s(e_2)) \in K$ and thus $s(K[e_1]) = K[s(e_1)] = K[s(e_2)] = s(K[e_2])$. Likewise $K[e_1] = K[e_2]$ implies that $(e_1, e_2) \in K$ so that $(t(e_1), t(e_2)) \in K$ and thus $t(K[e_1]) = K[t(e_1)] = K[t(e_2)] = t(K[e_2])$. Also $K[e_1] = K[e_2]$ implies that $(e_1, e_2) \in K$ so that $(\bar{e}_1, \bar{e}_2) \in K$ and thus $\overline{K[e_1]} = K[\bar{e}_1] = K[\bar{e}_2] = \overline{K[e_2]}$.

If possible, let $e \in E(\Gamma)$ be such that $K[e] = \overline{K[e]}$. Then $(e, \bar{e}) \in K$, a contradiction. Also, for all $e \in E(\Gamma)$, $\overline{\overline{K[e]}} = \overline{K[\bar{e}]} = K[\bar{\bar{e}}] = K[e]$.

Now for all $e \in E(\Gamma)$, $s(K[e]) = K[s(e)] = K[t(\bar{e})] = t(K[\bar{e}]) = t(\overline{K[e]})$ and $t(K[e]) = K[t(e)] = K[s(\bar{e})] = s(K[\bar{e}]) = s(\overline{K[e]})$. Hence Γ/K is a well defined graph.

Conversely, let us take Δ to be a graph and $f: \Gamma \rightarrow \Delta$ a well defined surjective map of graphs. First note that $K = f^{-1}f$ is a compatible equivalence relation over Γ . To see this we show that $K = (f \times f)^{-1}[D(\Delta)]$. For $(x, y) \in K \Leftrightarrow f(x) = f(y) \Leftrightarrow (f(x), f(y)) \in D(\Delta) \Leftrightarrow (x, y) \in (f \times f)^{-1}[D(\Delta)]$. Since $D(\Delta)$ is an equivalence relation over Δ , K is an equivalence relation over Γ , by the Theorem A.8.

- (1) Now let $(x, y) \in K$ be such that $x \in V(\Gamma)$ and $y \in E(\Gamma)$. Hence $f(x) \in V(\Delta)$, $f(y) \in E(\Delta)$ so $f(x) \neq f(y)$, a contradiction to our choice that $(x, y) \in K$. Hence $(x, y) \in (V(\Gamma) \times V(\Gamma)) \cup (E(\Gamma) \times E(\Gamma))$, so $K = K_V \cup K_E$.
- (2) If $(e_1, e_2) \in K$ then $f(e_1) = f(e_2)$, so $s(f(e_1)) = s(f(e_2))$. Then $f(s(e_1)) = f(s(e_2))$ and thus $(s(e_1), s(e_2)) \in K$. Similarly, $(e_1, e_2) \in K$ implies that $(t(e_1), t(e_2))$ and $(\bar{e}_1, \bar{e}_2) \in K$.
- (3) If $(e, \bar{e}) \in K$, then $f(e) = f(\bar{e})$ and thus $f(e) = \overline{f(e)}$, a contradiction. Hence $(e, \bar{e}) \notin K$, for all $e \in E(\Gamma)$.

Finally, let us define $f': \Gamma/K \rightarrow \Delta$ via $f'(K[x]) = f(x)$ for all x in Γ . Clearly, f' is a well defined injection and as f is map of graphs so is f' . For all $\delta \in \Delta$, there exists $\gamma \in \Gamma$ such that $f(\gamma) = \delta$ so that $f'(K[\gamma]) = f(\gamma) = \delta$. Hence f' is a surjection as well and thus a graph isomorphism. \square

4.4. *If R_1 and R_2 are compatible equivalences on Γ , then so is $R_1 \cap R_2$.*

THEOREM 4.5. *Let R be any cofinite equivalence relation on a topological graph Γ . Then there exists a compatible cofinite equivalence relation S on Γ such that $S \subseteq R$.*

PROOF. Extend the source and target maps $s, t: E(\Gamma) \rightarrow V(\Gamma)$ to all of Γ so that they are both the identity map on $V(\Gamma)$. Then $s, t: \Gamma \rightarrow \Gamma$ are continuous maps satisfying the following properties:

- $s^2 = s$, $t^2 = t$, $st = t$, and $ts = s$;
- $s(x) = x \iff t(x) = x \iff x \in V(\Gamma)$.

Similarly, extend the edge inversion map $\bar{\cdot}: E(\Gamma) \rightarrow E(\Gamma)$ to all of Γ by also letting it be the identity map on $V(\Gamma)$. Then $\bar{\cdot}: \Gamma \rightarrow \Gamma$ is a continuous map satisfying the following conditions for all $x \in \Gamma$:

- $\overline{\bar{x}} = x$;
- $\bar{x} = x \iff x \in V(\Gamma)$;

- $s(\bar{x}) = t(x)$ and $t(\bar{x}) = s(x)$.

Now define $S_1 = \{(x, y) \in \Gamma \times \Gamma \mid (s(x), s(y)) \in R\} = (s \times s)^{-1}[R]$, $S_2 = \{(x, y) \in \Gamma \times \Gamma \mid (t(x), t(y)) \in R\} = (t \times t)^{-1}[R]$, and $S_3 = \{(x, y) \in \Gamma \times \Gamma \mid (\bar{x}, \bar{y}) \in R\} = (\bar{} \times \bar{})^{-1}[R]$. Then, by Theorem A.8, S_1, S_2, S_3 are cofinite equivalence relations on Γ . Let $S_4 = R \cap S_1 \cap S_2 \cap S_3$ and observe that

- (i) S_4 is a cofinite equivalence relation on Γ ;
- (ii) if $(e_1, e_2) \in S_4$, then $(s(e_1), s(e_2)) \in S_4$, $(t(e_1), t(e_2)) \in S_4$, and $(\bar{e}_1, \bar{e}_2) \in S_4$.

Finally, choose a closed orientation $E^+(\Gamma)$ of Γ and form the restrictions $S_V = S_4 \cap [V(\Gamma) \times V(\Gamma)]$, $S_{E^+} = S_4 \cap [E^+(\Gamma) \times E^+(\Gamma)]$, and $S_{E^-} = S_4 \cap [E^-(\Gamma) \times E^-(\Gamma)]$. Then it is easy to check that $S = S_V \cup S_{E^+} \cup S_{E^-}$ is a compatible cofinite equivalence relation on Γ and $S \subseteq R$, as required. \square

The previous proof actually shows a little more, which is worth noting. Given a closed orientation $E^+(\Gamma)$ for Γ , we say that a compatible equivalence relation R on Γ is *orientation preserving* if whenever $(e, e') \in R$ and $e \in E^+(\Gamma)$, then also $e' \in E^+(\Gamma)$. Since the equivalence relation S that we constructed in the proof of Theorem 4.5 is also orientation preserving, we proved the following stronger result.

COROLLARY 4.6. *Let Γ be a topological graph with a specified closed orientation $E^+(\Gamma)$. Then for any cofinite equivalence relation R on Γ , there exists a compatible orientation preserving cofinite equivalence relation S on Γ such that $S \subseteq R$.*

COROLLARY 4.7. *If Γ is a compact Hausdorff totally disconnected topological graph, then its compatible cofinite equivalence relations form a fundamental system of entourages for the unique uniform structure that induces the topology of Γ .*

DEFINITION 4.8 (Profinite graph). A compact Hausdorff totally disconnected topological graph Γ is called a *profinite graph*.

As for any compact Hausdorff space, we will view a profinite graph as a uniform space endowed with the unique uniformity that induces its topology. Thus, Corollary 4.7 states that the collection of all compatible cofinite equivalence relations on a profinite graph Γ form a fundamental system of entourages.

CHAPTER 5

Cofinite graphs

In order to introduce cofinite graphs, we develop the concept of uniform topological graphs. Roughly speaking, these are topological graphs with a uniform structure such that the structure maps are all uniformly continuous.

Then we defined cofinite graphs in such a way that they turn out to be uniform topological graphs whose topology is induced by the cofinite uniformity. In particular, we see that a profinite graph is also cofinite graph. It follows easily that if Γ is a cofinite graph and Z is a cofinite space, then the map $f: \Gamma \rightarrow Z$ is uniformly continuous if and only if both the restrictions $f|_{V(\Gamma)}$ and $f|_{E(\Gamma)}$ are uniformly continuous. Furthermore we establish that subgraphs of cofinite graphs are cofinite graph; the inverse limit of a family of cofinite graphs is a cofinite graph; finite uniform sums of cofinite graphs and uniform quotients of cofinite graphs are also cofinite graphs.

5.1. Cofinite graphs

By a *uniform topological graph* we mean a topological graph Γ endowed with a uniform structure that induces its topology such that Γ is the uniform sum of its uniform subspaces $V(\Gamma)$, $E(\Gamma)$ and the maps $s, t: E(\Gamma) \rightarrow V(\Gamma)$ and $\bar{\cdot}: E(\Gamma) \rightarrow E(\Gamma)$ are uniformly continuous.

5.1. *If $f: \Gamma \rightarrow \Delta$ is a uniformly continuous map of uniform topological graphs then for any compatible cofinite equivalence relation R over Δ , $(f \times f)^{-1}(R)$ is a compatible cofinite equivalence relation over Γ .*

PROOF. Let $S = (f \times f)^{-1}(R)$. We already have seen that S is a cofinite equivalence relation over Γ . So now we want to see that S is compatible as well.

- For let $(x, y) \in S$ such that $x \in V(\Gamma), y \in E(\Gamma)$. Then $f(x) \in V(\Delta)$ and $f(y) \in E(\Delta)$, a contradiction, as $(f(x), f(y)) \in R$. Hence $S = S_V \cup S_E$.
- $(e_1, e_2) \in S \Rightarrow (f(e_1), f(e_2)) \in R \Rightarrow (s(f(e_1)), s(f(e_2))) \in R \Rightarrow (f(s(e_1)), f(s(e_2))) \in R \Rightarrow (s(e_1), s(e_2)) \in S$. Similarly, $(\bar{e}_1, \bar{e}_2) \in S$ and $(t(e_1), t(e_2)) \in S$.
- If possible, for some $e \in E(\Gamma)$, let $(e, \bar{e}) \in S$. Then $(f(e), f(\bar{e})) = (f(e), \overline{f(e)}) \in R$, a contradiction. Hence our claim follows.

□

We will concentrate our attention on uniform topological graphs of the following type.

DEFINITION 5.2 (Cofinite graph). A *cofinite graph* is an abstract graph Γ endowed with a Hausdorff uniformity such that the compatible cofinite entourages of Γ form a fundamental system of entourages (i.e. every entourage of Γ contains a compatible cofinite entourage).

LEMMA 5.3. *Let Γ be a cofinite graph. Then Γ is a uniform topological graph. In particular,*

1. $V(\Gamma)$ and $E(\Gamma)$ are clopen subsets of Γ ;
2. Γ is the uniform sum of its uniform subspaces $V(\Gamma), E(\Gamma)$;
3. $s, t: E(\Gamma) \rightarrow V(\Gamma)$ and $\bar{\cdot}: E(\Gamma) \rightarrow E(\Gamma)$ are uniformly continuous maps.

PROOF. The proof of the above claim is straightforward. Let us assume Γ is a cofinite graph and $I = \{R \mid R \text{ is a compatible cofinite entourage over } \Gamma\}$ is a fundamental system for the corresponding Hausdorff uniformity. Let

$$R_V = R \cap (V(\Gamma) \times V(\Gamma)), R_E = R \cap (E(\Gamma) \times E(\Gamma))$$

. Now $\overline{V(\Gamma)} = \bigcap_{R \in I} R[V(\Gamma)] \subseteq V(\Gamma) \subseteq \overline{V(\Gamma)}$. Hence $\overline{V(\Gamma)} = V(\Gamma)$. Similarly, $\overline{E(\Gamma)} = E(\Gamma)$ gives our first claim that both $V(\Gamma)$ and $E(\Gamma)$ are clopen subsets of Γ . Since $R = R_V \cup R_E$ for all R in I we obtain the second claim, that Γ is the uniform sum of its uniform subspaces $V(\Gamma), E(\Gamma)$. Finally, $R_E \subseteq (s \times s)^{-1}[R_V]$, $R_E \subseteq (t \times t)^{-1}[R_V]$ and $R_E \subseteq (\bar{} \times \bar{})^{-1}[R_E]$ yield our third claim that s, t from $E(\Gamma)$ to $V(\Gamma)$ and $\bar{}$ from $E(\Gamma)$ to $E(\Gamma)$ are uniformly continuous maps. \square

So one can clearly conclude that

LEMMA 5.4. *Profinite graphs are precisely the compact cofinite graphs.*

Let Γ be a cofinite graph and let I be a fundamental system of compatible cofinite entourages of Γ . Then we see by Note 2.8 that

- (i) $\bigcap_{R \in I} R = D(\Gamma)$, the diagonal in $\Gamma \times \Gamma$;
- (ii) Γ is totally disconnected;
- (iii) if A is any subset of Γ , then $\overline{A} = \bigcap_{R \in I} R[A]$

$$\begin{aligned}
&= \bigcap_{R \in I} (R_V \cup R_E)[A] \\
&= (\bigcap_{R \in I} R_V[A]) \cup (\bigcap_{R \in I} R_E[A]) \\
&= \overline{A \cap V(\Gamma)} \cup \overline{A \cap E(\Gamma)}
\end{aligned}$$

The following lemma is an immediate consequence of Proposition 2.20.

LEMMA 5.5. *Let Γ be a cofinite graph and let Z be a cofinite space. Then a map $f: \Gamma \rightarrow Z$ is uniformly continuous if and only if both the restrictions $f|_{V(\Gamma)}$ and $f|_{E(\Gamma)}$ are uniformly continuous.*

As one application of this lemma, we can extend the source map $s: E(\Gamma) \rightarrow V(\Gamma)$ of a cofinite graph Γ to a map $s: \Gamma \rightarrow \Gamma$ by letting it be the identity map on $V(\Gamma)$. By Lemma 5.5, the extension $s: \Gamma \rightarrow \Gamma$ is also uniformly continuous. We can similarly extend the target and inversion maps. Thus, when it is convenient to do so, we may assume that the source, target, and inversion maps are uniformly continuous maps $s, t, \bar{\cdot}: \Gamma \rightarrow \Gamma$ whose fixed points are precisely the vertices of Γ .

5.1.1. Uniform subgraphs. Let Γ be a cofinite graph. A subgraph Σ endowed with the uniformity induced on it by Γ is called a *uniform subgraph* of Γ .

Let us observe that the subgraph Σ of a cofinite graph Γ is itself a cofinite graph, as because if R is a compatible cofinite entourage over Γ then so is $R \cap (\Sigma \times \Sigma)$ over Σ .

5.1.2. Inverse Limits of Cofinite Graphs. Turning to inverse limits of cofinite graphs, let (Γ_i, ϕ_{ij}) be an inverse system of sets indexed by a directed set I . We say that (Γ_i, ϕ_{ij}) is an *inverse system of cofinite graphs* if (i) each Γ_i is a cofinite graph, and (ii) for all $i \leq j$, $\phi_{ij}: \Gamma_j \rightarrow \Gamma_i$ is a uniformly continuous map of graphs.

As in Section 2.1.5, we endow the set $\Gamma = \varprojlim \Gamma_i$ with the coarsest uniformity such that the canonical maps $\phi_i: \Gamma \rightarrow \Gamma_i$ are uniformly continuous. Then by Proposition 2.17, Γ is a cofinite space. Furthermore, we make the following observation.

LEMMA 5.6. *The set Γ admits a unique graph structure such that the maps $\phi_i: \Gamma \rightarrow \Gamma_i$ are maps of graphs.*

PROOF. First of all, we claim that for all $i, j \in I$,

$$\phi_i^{-1}[V(\Gamma_i)] = \phi_j^{-1}[V(\Gamma_j)].$$

To see this, choose $k \in I$ such that $k \geq i$ and $k \geq j$. Then $\phi_i = \phi_{ik}\phi_k$ and $\phi_j = \phi_{jk}\phi_k$. So $\phi_i^{-1}[V(\Gamma_i)] = \phi_k^{-1}[\phi_{ik}^{-1}[V(\Gamma_i)]] = \phi_k^{-1}[V(\Gamma_k)]$ as $\phi_{ik}: \Gamma_k \rightarrow \Gamma_i$ is a map of graphs.

Similarly, $\phi_j^{-1}[V(\Gamma_j)] = \phi_k^{-1}[V(\Gamma_k)]$ and the claim follows. Now it also follows that for all $i, j \in I$,

$$\phi_i^{-1}[E(\Gamma_i)] = \phi_j^{-1}[E(\Gamma_j)].$$

For the desired graph structure on Γ , the vertex and edge sets must be the subsets satisfying:

$$V(\Gamma) = \phi_i^{-1}[V(\Gamma_i)] \quad \text{and} \quad E(\Gamma) = \phi_i^{-1}[E(\Gamma_i)]$$

for all $i \in I$.

It remains to see that there is a unique way to define the source, target, and inversion maps so that the ϕ_i are maps of graphs. We begin by extending the source, target, and inversion maps to functions $s, t, \bar{\cdot}: \Gamma_i \rightarrow \Gamma_i$, whose fixed points are precisely the vertices of Γ_i , for each $i \in I$. Then for each $i \in I$, let $s_i = s\phi_i: \Gamma \rightarrow \Gamma_i$. Note that for $i \leq j$,

$$\phi_{ij}s_j = \phi_{ij}s\phi_j = s\phi_{ij}\phi_j = s\phi_i = s_i.$$

So the family of functions $(s_i: \Gamma \rightarrow \Gamma_i)_{i \in I}$ determine a unique function $s: \Gamma \rightarrow \Gamma$ such that $\phi_i s = s_i = s\phi_i$ for all $i \in I$. Similarly, there exist unique functions $t, \bar{\cdot}: \Gamma \rightarrow \Gamma$ such that all $\phi_i t = t\phi_i$ and $\phi_i \bar{\cdot} = \bar{\cdot}\phi_i$.

It is now a simple matter to check that with these maps, Γ is a graph.

For, if possible, let $x \in V(\Gamma) \cap E(\Gamma)$. Hence for all $i \in I$, $\phi_i(x) \in V(\Gamma_i) \cap E(\Gamma_i) = \emptyset$, a contradiction.

Now we also notice that for all $x \in \Gamma$ and for all $i \in I$, $\phi_i(x) \in \Gamma_i = V(\Gamma_i) \cup E(\Gamma_i)$. Hence $x \in \phi_i^{-1}(V(\Gamma_i)) \cup \phi_i^{-1}(E(\Gamma_i)) = V(\Gamma) \cup E(\Gamma) \subseteq \Gamma$. Hence $\Gamma = V(\Gamma) \cup E(\Gamma)$.

Now, if possible, let there exist $e \in E(\Gamma)$ such that $e = \bar{e}$. Hence for all i in I , $\phi_i(e) = \phi_i(\bar{e})$ and thus $\phi_i(e) = \overline{\phi_i(e)}$ in $E(\Gamma_i)$ for all i in I , a contradiction.

Also, $\phi_i(s(\bar{e})) = s(\phi_i(\bar{e})) = s(\overline{\phi_i(e)}) = t(\phi_i(e)) = \phi_i(t(e))$ and $\phi_i(t(\bar{e})) = t(\phi_i(\bar{e})) = t(\overline{\phi_i(e)}) = s(\phi_i(e)) = \phi_i(s(e))$ for all i in I . Hence it follows that $s(\bar{e}) = t(e), t(\bar{e}) = s(e)$. □

By the *inverse limit of an inverse system* (Γ_i, ϕ_{ij}) of cofinite graphs, we will mean the set $\Gamma = \varprojlim \Gamma_i$ endowed with the unique graph structure and the coarsest uniformity such that the canonical maps $\phi_i: \Gamma \rightarrow \Gamma_i$ are uniformly continuous maps of graphs.

PROPOSITION 5.7. *Let (Γ_i, ϕ_{ij}) be an inverse system of cofinite graphs. Then the inverse limit $\Gamma = \varprojlim \Gamma_i$ is a cofinite graph.*

PROOF. It is easy to see that Γ is a Hausdorff cofinite space and a graph as well. So it remains to check that the compatible cofinite entourages of Γ form a fundamental system of entourages. Without loss of generality $U = (\bigcap_{n=1}^N (\pi_{i_n} \times \pi_{i_n})^{-1}[U_{i_n}]) \cap \Gamma$, where U_{i_n} is an entourage over Γ_{i_n} for all n . Then each Γ_{i_n} , being a cofinite graph, there exists a compatible cofinite entourage $R_{i_n} \subseteq U_{i_n}$ for all n . Clearly,

$$R = \left(\bigcap_{n=1}^N (\pi_{i_n} \times \pi_{i_n})^{-1}[R_{i_n}] \right) \cap \Gamma$$

is compatible cofinite entourage over Γ and $R \subseteq U$. Hence our claim that Γ is a cofinite graph follows. \square

An alternative representation of the source, target, and the inversion of edge map.

Let Γ be as in the above discussion. Let $x = (x_i)_{i \in I} \in \Gamma$. Let us define the source map $s: \Gamma \rightarrow \Gamma$ via $s(x) = (s(x_i))_{i \in I}$. Clearly, s is well defined and for all i in I , $\phi_i(s(x)) = s_i(x)$, as in the previous lemma. Since each $\phi_i = s_i$ and each s_i is uniformly continuous, we obtain by Corollary 2.12, $s: \Gamma \rightarrow \Gamma$ is uniformly continuous. Then, using the uniqueness of s , the source map we defined here is equal to the one we defined in the last lemma. Similarly, when convenient we will use $t: \Gamma \rightarrow \Gamma$ as $t(x) = (t(x_i))_{i \in I}$ and $\bar{\cdot}: \Gamma \rightarrow \Gamma$ as $\bar{x} = (\bar{x}_i)_{i \in I}$.

5.1.3. Uniform sum of cofinite graphs. We now apply the construction in Section 2.1.6 of uniform sum of finitely many cofinite spaces to finitely many cofinite graphs.

PROPOSITION 5.8. *The uniform sum of a finite family of cofinite graphs is a cofinite graph.*

PROOF. To begin with, let $(\Gamma_i)_{i \in I}$ be a **finite** family of cofinite graphs. The uniform sum of this family $\Gamma = \coprod_{i \in I} \Gamma_i$ has both the structure of a cofinite space and a graph. It only remains to check that Γ has a fundamental system of compatible cofinite entourages. Without loss of generality let $U = \bigcup_{i \in I} U_i$ be a cofinite entourage over Γ . Hence U_i is a cofinite entourage over Γ_i for all i . But each Γ_i is cofinite so there exists a compatible cofinite entourage $R_i \subseteq U_i$. Clearly, $R = \bigcup_{i \in I} R_i$ is a compatible cofinite entourage over Γ and $R \subseteq U$. \square

Alternatively, one may define $V(\Gamma) = \coprod_{i \in I} V(\Gamma_i)$, $E(\Gamma) = \coprod_{i \in I} E(\Gamma_i)$. Clearly, $\Gamma = V(\Gamma) \coprod E(\Gamma)$. Also let us define $s: E(\Gamma) \rightarrow V(\Gamma)$ via $s|_{E(\Gamma_i)} = s: E(\Gamma_i) \rightarrow V(\Gamma_i)$. Then, by Lemma 2.20, s is uniformly continuous, as each restriction $s|_{E(\Gamma_i)}$ is uniformly continuous. Similarly $t, \bar{}$ are uniformly continuous as well. Also we make a note of the fact that a uniform sum of uniform spaces also respects their topological structures by being the topological sum of themselves. In particular, each uniform summand is a clopen subgraph of the uniform sum graph.

5.1.4. Uniform quotient graphs. Next we apply the construction in Section 2.1.7 of uniform quotient spaces to cofinite graphs. Let Γ be a cofinite graph and let K be a compatible equivalence relation on Γ . Then the uniform quotient space $\Gamma//K$ of Γ modulo K has both the structure of a cofinite space and a graph. We show that these two structures combine to make $\Gamma//K$ into a cofinite graph, provided that it is Hausdorff.

So it remains to say that for each compatible cofinite entourage R of Γ with $K \subseteq R$, $(q \times q)[R]$ is compatible, where $q: \Gamma \rightarrow \Gamma/K$ is the quotient map. Let $(K[x], K[y]) \in (q \times q)[R]$. This implies that there exists $(u, v) \in R$ such that $q(x) = q(u)$ and $q(y) = q(v)$. But this means $(x, u), (v, y) \in K \subseteq R$, which implies that $(x, y) \in R$ and thus

$$(K[x], K[y]) \in [(q \times q)[R] \cap (V(\Gamma/K) \times V(\Gamma/K))] \dot{\cup} [(q \times q)[R] \cap (E(\Gamma/K) \times E(\Gamma/K))].$$

Also, let $(K[e_1], K[e_2]) \in (q \times q)[R]$ for some $(e_1, e_2) \in E(\Gamma) \times E(\Gamma)$. As above $(e_1, e_2) \in R$ and as R is a compatible cofinite entourage, we have $(s(e_1), s(e_2)), (t(e_1), t(e_2)), (\bar{e}_1, \bar{e}_2) \in R$ and hence

$$(s(K[e_1]), s(K[e_2])), (t(K[e_1]), t(K[e_2])), (\overline{K[e_1]}, \overline{K[e_2]}) \in (q \times q)[R]$$

. Finally, if possible, let $(K[e], \overline{K[e]}) \in (q \times q)[R]$. Then as above $(e, \bar{e}) \in R$, a contradiction. Thus our claim follows.

PROPOSITION 5.9. *Let Γ be a cofinite graph and K a compatible equivalence relation on Γ that satisfies the equivalent conditions of 2.29. Then the uniform quotient graph $\Gamma//K$ is a cofinite graph.*

CHAPTER 6

Completions of Cofinite Graphs

In this chapter we define the completion of a cofinite graph Γ to be any compact Hausdorff topological graph $\bar{\Gamma}$ that contains Γ as a dense subgraph.

We verify that the completion of a cofinite graph Γ is unique up to an isomorphism extending the identity map on Γ . Existence of completions is established in the standard way.

Next, generalizing Hartley's work [2], we were able to show that if $\bar{\Gamma}$ is the completion of a cofinite graph Γ and R is a compatible cofinite entourage of Γ , then \bar{R} is a compatible cofinite entourage of $\bar{\Gamma}$ and $\bar{R} \cap (\Gamma \times \Gamma) = R$. We have concluded this chapter by establishing the fact that the completion of a cofinite graph is also a cofinite graph, and thus, being compact, it is a profinite graph.

6.1. Completions of Cofinite Graphs

THEOREM 6.1. *Let Γ be a cofinite graph contained as a dense subgraph in a compact Hausdorff topological graph $\bar{\Gamma}$. Then given any compact Hausdorff topological graph Δ and any uniformly continuous map of graphs $\varphi: \Gamma \rightarrow \Delta$,*

- (1) $\overline{V(\Gamma)} = V(\bar{\Gamma})$ and $\overline{E(\Gamma)} = E(\bar{\Gamma})$
- (2) *there exists a unique continuous map of graphs $\bar{\varphi}: \bar{\Gamma} \rightarrow \Delta$ extending φ .*

PROOF. (1) Let $v \in V(\bar{\Gamma})$, U be an open set in $V(\bar{\Gamma})$ containing v . Since Γ is dense in $\bar{\Gamma}$, $U \cap \Gamma \neq \emptyset$. Let $w \in U \cap \Gamma$. Since $U \subseteq V(\bar{\Gamma})$, $w \in V(\Gamma)$. Thus $w \in U \cap V(\Gamma)$ and hence $V(\bar{\Gamma}) = \overline{V(\Gamma)}^{V(\bar{\Gamma})}$, the closure of $V(\Gamma)$ in $V(\bar{\Gamma})$. But $\overline{V(\Gamma)}^{V(\bar{\Gamma})} = \overline{V(\Gamma)}$, the closure of $V(\Gamma)$ in $\bar{\Gamma}$. Similarly, $\overline{E(\Gamma)} = E(\bar{\Gamma})$.

(2) Since Δ is compact, Hausdorff it is a complete uniform space as well. Then there exists a unique uniformly continuous map $\bar{\varphi}: \bar{\Gamma} \rightarrow \Delta$ such that $\bar{\varphi}|_{\Gamma} = \varphi$. So it remains to check that $\bar{\varphi}$ is a map of graphs.

Let $v \in V(\bar{\Gamma}) = \overline{V(\Gamma)}$. Then there exists a net $\{v_{\alpha}\}_{\alpha \in A} \subseteq V(\Gamma)$ such that $\lim_{\alpha \in A} v_{\alpha} = v$. Hence $\bar{\varphi}(v) = \bar{\varphi}(\lim_{\alpha \in A} v_{\alpha}) = \lim_{\alpha \in A} \bar{\varphi}(v_{\alpha}) = \lim_{\alpha \in A} \varphi(v_{\alpha}) \in V(\Delta)$, as φ is a map of graphs and $V(\Delta)$ is closed in Δ . Similarly, one can show that for all e in $E(\bar{\Gamma})$, $\bar{\varphi}(e) \in E(\Delta)$.

Let $e \in E(\bar{\Gamma}) = \overline{E(\Gamma)}$. Then there exists a net $\{e_{\alpha}\}_{\alpha \in A} \subseteq E(\Gamma)$ such that $\lim_{\alpha \in A} e_{\alpha} = e$.

So, $s(\bar{\varphi}(e)) = s(\bar{\varphi}(\lim_{\alpha \in A} e_{\alpha})) = s(\lim_{\alpha \in A} \bar{\varphi}(e_{\alpha})) = s(\lim_{\alpha \in A} \varphi(e_{\alpha})) = \lim_{\alpha \in A} s(\varphi(e_{\alpha})) = \lim_{\alpha \in A} \varphi(s(e_{\alpha})) = \lim_{\alpha \in A} \bar{\varphi}(s(e_{\alpha})) = \bar{\varphi}(\lim_{\alpha \in A} s(e_{\alpha})) = \bar{\varphi}(s(\lim_{\alpha \in A} e_{\alpha})) = \bar{\varphi}(s(e))$. Similarly, $t(\bar{\varphi}(e)) = \bar{\varphi}(t(e))$.

Now $\overline{\bar{\varphi}(e)} = \overline{\bar{\varphi}(\lim_{\alpha \in A} e_{\alpha})} = \overline{\lim_{\alpha \in A} \bar{\varphi}(e_{\alpha})} = \overline{\lim_{\alpha \in A} \varphi(e_{\alpha})} = \lim_{\alpha \in A} \overline{\varphi(e_{\alpha})} = \lim_{\alpha \in A} \varphi(\overline{e_{\alpha}}) = \lim_{\alpha \in A} \bar{\varphi}(\overline{e_{\alpha}}) = \bar{\varphi}(\lim_{\alpha \in A} \overline{e_{\alpha}}) = \bar{\varphi}(\overline{\lim_{\alpha \in A} e_{\alpha}}) = \bar{\varphi}(\overline{e})$.

Thus $\bar{\varphi}$ is a map of graphs. □

COROLLARY 6.2. *As in the previous theorem $\bar{\varphi}(\bar{\Gamma}) = \overline{\varphi(\Gamma)}$.*

PROOF. The closure of Γ is $\bar{\Gamma}$ and $\bar{\varphi}$ is continuous. So $\bar{\varphi}(\bar{\Gamma}) = \overline{\bar{\varphi}(\Gamma)} = \overline{\varphi(\Gamma)}$.

On the other hand, since $\bar{\Gamma}$ is compact and $\bar{\varphi}$ is uniformly continuous, $\bar{\varphi}(\bar{\Gamma})$ is a compact subset of the Hausdorff space Δ and hence is closed. Now $\varphi(\Gamma) = \bar{\varphi}(\Gamma) \subseteq \bar{\varphi}(\bar{\Gamma})$. Thus $\overline{\varphi(\Gamma)} \subseteq \overline{\bar{\varphi}(\bar{\Gamma})} = \bar{\varphi}(\bar{\Gamma})$. □

In light of Theorem 6.1 we make the following definition.

DEFINITION 6.3 (Completion). Let Γ be a cofinite graph. Then any compact Hausdorff topological graph $\bar{\Gamma}$ that contains Γ as a dense subgraph is called a *completion* of Γ .

COROLLARY 6.4 (Uniqueness of completions). *The completion of a cofinite graph Γ is unique up to an isomorphism extending the identity map on Γ .*

PROOF. If possible, let Γ_i be two completions of a cofinite graph Γ , for $i = 0, 1$. Then the following diagram commutes for unique choices of uniformly continuous maps of graphs $f_{i+1}: \Gamma_i \rightarrow \Gamma_{i+1}$, for $i = 0, 1 \pmod 2$, where id_Γ is the identity map on Γ and i_{Γ_i} is the canonical inclusion map, for $i = 0, 1$.

$$\begin{array}{ccccccc}
 \Gamma_0 & \xrightarrow{f_1} & \Gamma_1 & \xrightarrow{f_0} & \Gamma_0 & \xrightarrow{f_1} & \Gamma_1 \\
 \uparrow i_{\Gamma_0} & & \uparrow i_{\Gamma_1} & & \uparrow i_{\Gamma_0} & & \uparrow i_{\Gamma_1} \\
 \Gamma & \xrightarrow{id_\Gamma} & \Gamma & \xrightarrow{id_\Gamma} & \Gamma & \xrightarrow{id_\Gamma} & \Gamma
 \end{array}$$

But then we also have the following commutative diagrams.

$$\begin{array}{ccc}
 \Gamma_0 & \xrightarrow{id_{\Gamma_0}} & \Gamma_0 \\
 \uparrow i_{\Gamma_0} & & \uparrow i_{\Gamma_0} \\
 \Gamma & \xrightarrow{id_\Gamma} & \Gamma
 \end{array}
 \qquad
 \begin{array}{ccc}
 \Gamma_1 & \xrightarrow{id_{\Gamma_1}} & \Gamma_1 \\
 \uparrow i_{\Gamma_1} & & \uparrow i_{\Gamma_1} \\
 \Gamma & \xrightarrow{id_\Gamma} & \Gamma
 \end{array}$$

where id_{Γ_i} is the identity map, for $i = 1, 2$.

Thus by Theorem 6.1, $f_0 \circ f_1 = id_{\Gamma_0}$ and $f_1 \circ f_0 = id_{\Gamma_1}$. Hence f_0 and f_1 are inverses of each other. \square

THEOREM 6.5 (Existence of completions). *Let Γ be a cofinite graph and let I be a fundamental system of compatible cofinite entourages of Γ , directed by the reverse inclusion. Then the inverse limit $\hat{\Gamma} = \varprojlim \Gamma/R$ ($R \in I$) is a compact Hausdorff topological graph and the natural map $\Gamma \rightarrow \hat{\Gamma}$ embeds Γ as a dense subgraph of $\hat{\Gamma}$.*

PROOF. Let us first see that I being a fundamental system of compatible cofinite entourages of Γ , directed by the reverse inclusion forms a directed set. This follows as the intersection of two compatible cofinite entourages is also a compatible cofinite entourage.

Let us now see that the uniform quotient graphs Γ/R forms an inverse system of finite discrete cofinite graphs, for all $R \in I$. Let $R \leq S$ in I . Thus $S \subseteq R$. Let us define $\varphi_{RS}: \Gamma/S \rightarrow \Gamma/R$ via $\varphi_{RS}(S[x]) = R[x]$, for all $x \in \Gamma$. Now, $S[x] = S[y]$ implies that $(x, y) \in S \subseteq R$ and thus $R[x] = R[y]$. Hence φ_{RS} is well defined. Now $S[v] \in V(\Gamma/S)$ implies that $v \in V(\Gamma)$ so that $R[v] \in V(\Gamma/R)$. Similarly, if $S[e] \in E(\Gamma/S)$ then we have $R[e] \in E(\Gamma/R)$. Also, $S[e] \in E(\Gamma/S)$ implies that $s(\varphi_{RS}(S[e])) = s(R[e]) = R[s(e)] = \varphi_{RS}(S[s(e)]) = \varphi_{RS}(s(S[e]))$. Similarly, $t(\varphi_{RS}(S[e])) = \varphi_{RS}(t(S[e]))$ and $\overline{\varphi_{RS}(S[e])} = \varphi_{RS}(\overline{S[e]})$. Thus φ_{RS} is a map of graphs and since both $\Gamma/S, \Gamma/R$ are discrete, φ_{RS} is uniformly continuous as well. Now for $R \leq S \leq T$ in I , $\varphi_{RS}(\varphi_{ST}(T[x])) = \varphi_{RS}(S[x]) = R[x] = \varphi_{RT}(T[x])$, for all $x \in X$. If $R = S$ in I , then $\varphi_{RS}(S[x]) = R[x] = S[x] = id_{\Gamma/S}(S[x])$, for all $x \in X$. Hence $(\Gamma/R, \varphi_{RS})_{R \leq S \in I}$ forms an inverse system of discrete cofinite graphs. Hence $\widehat{\Gamma} = \varprojlim_{R \in I} \Gamma/R$ exists.

Let us now see that Γ is densely embedded in $\widehat{\Gamma}$. Denote by $\varphi_R: \widehat{\Gamma} \rightarrow \Gamma/R$ the corresponding canonical projection map and let $\eta_R: \Gamma \rightarrow \Gamma/R$ be the canonical surjection for all R in I . Then the following diagram commutes for all $R \leq S$ in I , as $\varphi_{RS}(\eta_S(\gamma)) = \varphi_{RS}(S[\gamma]) = R[\gamma] = \eta_R(\gamma)$, for all $\gamma \in \Gamma$.

$$\begin{array}{ccc}
 & \Gamma & \\
 \eta_S \swarrow & & \searrow \eta_R \\
 \Gamma/S & \xrightarrow{\varphi_{RS}} & \Gamma/R
 \end{array}$$

Hence $(\Gamma, \eta_R)_{R \in I}$ forms a compatible system to the aforesaid inverse system of cofinite graphs. Thus there exists a uniformly continuous map of graphs $\theta: \Gamma \rightarrow \widehat{\Gamma}$ such that the following diagram commutes for all R in I .

$$\begin{array}{ccc}
 & \Gamma & \\
 \theta \swarrow & & \searrow \eta_R \\
 \widehat{\Gamma} = \varprojlim_{R \in I} & \xrightarrow{\varphi_R} & \Gamma/R
 \end{array}$$

Now let $x_1, x_2 \in \widehat{\Gamma}$ be such that $\theta(x_1) = \theta(x_2)$. So for all R in I , $R[x_1] = \eta_R(x_1) = \varphi_R(\theta(x_1)) = \varphi_R(\theta(x_2)) = \eta_R(x_2) = R[x_2]$. Thus $(x_1, x_2) \in \bigcap_{R \in I} R = D(\Gamma)$, as Γ is Hausdorff. Hence $x_1 = x_2$. So θ is injective.

So it remains to check that θ is a topological embedding. This follows from the claim that $\theta(R[x]) = \varphi_R^{-1}(\eta_R(x)) \cap \theta(\Gamma)$, for all $x \in \Gamma$ and for all $R \in I$. The above claim follows as $p \in \theta(R[x]) \Leftrightarrow$ there exists $q \in R[x] \cap \Gamma$ such that $\theta(q) = p \Leftrightarrow$ there exists $q \in \Gamma$ such that $\eta_R(q) = \eta_R(x)$ and $\theta(q) = p \Leftrightarrow$ there exists, $q \in \Gamma$ such that $\varphi_R(\theta(q)) = \eta_R(x)$ and $\theta(q) = p \Leftrightarrow \varphi_R(p) = \eta_R(x) \Leftrightarrow p \in \varphi_R^{-1}(\eta_R(x)) \cap \theta(\Gamma)$. \square

Notice that in the definition of the completion of Γ , we did not insist that $\overline{\Gamma}$ be a cofinite graph. However, it turns out that this will automatically be so. To see this, we first prove a lemma.

LEMMA 6.6. *Let $\overline{\Gamma}$ be the completion of a cofinite graph Γ and let R be a compatible cofinite entourage of Γ . Then \overline{R} is a compatible cofinite entourage of $\overline{\Gamma}$ and $\overline{R} \cap (\Gamma \times \Gamma) = R$.*

PROOF. The quotient Γ/R is a compact Hausdorff topological graph and the quotient map $\eta_R: \Gamma \rightarrow \Gamma/R$ is uniformly continuous. So by Theorem 6.1, η_R extends to a continuous map of graphs $\overline{\eta}_R: \overline{\Gamma} \rightarrow \Gamma/R$. Using Corollary 6.2 and as η_R is surjective, $\overline{\eta}_R(\overline{\Gamma}) = \overline{\eta_R(\Gamma)} = \overline{\Gamma/R} = \Gamma/R$. Thus $\overline{\eta}_R$ is surjective as well. Since $D(\Gamma/R)$ is a compatible cofinite entourage over Γ/R and $(\overline{\eta}_R \times \overline{\eta}_R)^{-1}[D(\Gamma/R)] = \overline{\eta}_R^{-1} \overline{\eta}_R$ we see

that $\bar{\eta}_R^{-1}\bar{\eta}_R$ is a compatible cofinite equivalence relation over $\bar{\Gamma}$ and thus endowed with the quotient topology $\bar{\Gamma}/\bar{\eta}_R^{-1}\bar{\eta}_R$ is a discrete quotient graph of $\bar{\Gamma}$ and we claim that the map $\bar{\eta}_R$ determines an isomorphism of topological graphs $\Psi: \bar{\Gamma}/\bar{\eta}_R^{-1}\bar{\eta}_R \rightarrow \Gamma/R$. Let us define $\Psi(\bar{\eta}_R^{-1}\bar{\eta}_R[x]) = \bar{\eta}_R[x]$ for all x in $\bar{\Gamma}$. If $\bar{\eta}_R^{-1}\bar{\eta}_R[x] = \bar{\eta}_R^{-1}\bar{\eta}_R[y]$ then $(x, y) \in \bar{\eta}_R^{-1}\bar{\eta}_R$ so that $\bar{\eta}_R(x) = \bar{\eta}_R(y)$. Hence Ψ is a well defined injection. As $\bar{\eta}_R$ is a surjective map of graphs so is Ψ . Since both $\bar{\Gamma}/\bar{\eta}_R^{-1}\bar{\eta}_R, \Gamma/R$ are discrete topological graphs, both Ψ, Ψ^{-1} are uniformly continuous and our claim that Ψ is an isomorphism of topological graphs follows.

Since $\bar{\eta}_R^{-1}\bar{\eta}_R \cap (\Gamma \times \Gamma) = \eta_R^{-1}\eta_R = R$. It now suffices to show that $\bar{\eta}_R^{-1}\bar{\eta}_R = \bar{R}$.

First note that $R = \eta_R^{-1}\eta_R \subset \bar{\eta}_R^{-1}\bar{\eta}_R$ and that $\bar{\eta}_R^{-1}\bar{\eta}_R$ is closed in $\bar{\Gamma} \times \bar{\Gamma}$ as Γ/R is finite and discrete and thus $D(\Gamma/R)$ is a clopen subset of $\Gamma/R \times \Gamma/R$; whence $\bar{R} \subseteq \bar{\eta}_R^{-1}\bar{\eta}_R$. Conversely, let $z \in \bar{\eta}_R^{-1}\bar{\eta}_R$ and let V be a neighborhood of z in $\bar{\Gamma} \times \bar{\Gamma}$. Then $V \cap \bar{\eta}_R^{-1}\bar{\eta}_R$ is also a neighborhood of z . However, $\Gamma \times \Gamma$ is dense in $\bar{\Gamma} \times \bar{\Gamma}$, so

$$V \cap R = V \cap \bar{\eta}_R^{-1}\bar{\eta}_R \cap (\Gamma \times \Gamma) \neq \emptyset.$$

Therefore $z \in \bar{R}$ and $\bar{\eta}_R^{-1}\bar{\eta}_R \subseteq \bar{R}$. The claim, and hence the lemma, follows. \square

THEOREM 6.7. *Let Γ be a cofinite graph and let I be the filter base of all compatible cofinite entourages of Γ . Then the completion $\bar{\Gamma}$ is also a cofinite graph and $\{\bar{R} \mid R \in I\}$ is the filter base of all compatible cofinite entourages of $\bar{\Gamma}$.*

PROOF. We will first see that $\{\bar{R} \mid R \in I\}$ forms the filter base of all compatible cofinite entourages of $\bar{\Gamma}$.

For let \bar{R}, \bar{S} be the compatible cofinite entourages over $\bar{\Gamma}$ for R, S in I . Then there is $T \in I$ such that $T \subseteq R \cap S$. Now $\bar{T} \subseteq \overline{R \cap S} \subseteq \bar{R} \cap \bar{S}$.

Now let K be any compatible cofinite entourage over $\bar{\Gamma}$. Then $K \cap (\Gamma \times \Gamma)$ is a compatible cofinite entourage over Γ . Hence there exists some R in I , such that $R = K \cap (\Gamma \times \Gamma)$. Since K is open in $\bar{\Gamma} \times \bar{\Gamma}$, any open set U in K is also open in

$\bar{\Gamma} \times \bar{\Gamma}$. Now for all $(x, y) \in K$ and $U \in \eta(x, y)$ in K , $U \cap (\Gamma \times \Gamma) \neq \emptyset$ as $\Gamma \times \Gamma$ is dense in $\bar{\Gamma} \times \bar{\Gamma}$. Hence $U \cap (K \cap (\Gamma \times \Gamma)) = U \cap R \neq \emptyset$ and thus R is dense in K . It follows that $\bar{R} = \bar{K} = K$. Hence $\{\bar{R} \mid R \in I\}$ forms the filter base of all compatible cofinite entourages over $\bar{\Gamma}$.

It remains to show that $\{\bar{R} \mid R \in I\}$ is a fundamental system of entourages of $\bar{\Gamma}$. For this purpose let W be any entourage of $\bar{\Gamma}$. We may assume that W is closed in $\bar{\Gamma} \times \bar{\Gamma}$, as the closed entourages form a fundamental system of entourages. Since $W \cap (\Gamma \times \Gamma)$ is an entourage of Γ and Γ is a cofinite graph, there exists $R \in I$ such that $R \subseteq W \cap (\Gamma \times \Gamma)$. Now $\bar{R} \subseteq \bar{W} = W$ and we see that every entourage of $\bar{\Gamma}$ contains a member of the set $\{\bar{R} \mid R \in I\}$, as required. \square

It follows from Theorem 6.7 that the completion of a cofinite graph is a profinite graph.

CHAPTER 7

Connected Cofinite Graphs

Cofinite connectedness of a cofinite graph is discussed in this chapter. A cofinite graph Γ is *cofinitely connected* if for each compatible cofinite equivalence relation R on Γ , the quotient graph Γ/R is path connected.

Similar to the standard connectedness arguments for finite graphs or general topological spaces, many analogous properties hold for cofinite connectedness as well. For instance, we establish that the following statements are equivalent for any cofinite graph Γ :

- (1) Γ is cofinitely connected;
- (2) Γ is not the uniform sum of two disjoint nonempty subgraphs.
- (3) $\widehat{\Gamma}$, the profinite completion of Γ is also cofinitely connected.

7.1. Connected Cofinite Graphs

A *path* in a graph Γ is a finite string of edges $p = e_1 \cdots e_n \in E(\Gamma)^*$ such that $t(e_i) = s(e_{i+1})$ for $1 \leq i \leq n - 1$. The *source* and *target* of this path p are the vertices $s(p) = s(e_1)$ and $t(p) = t(e_n)$. We say that Γ is *path connected* if there is a path in Γ joining any two vertices.

DEFINITION 7.1. A cofinite graph Γ is *cofinitely connected* if for each compatible cofinite equivalence relation R on Γ , the quotient graph Γ/R is path connected.

PROPOSITION 7.2. *The following statements are equivalent for any cofinite graph Γ :*

- (1) Γ is cofinitely connected;

(2) Γ is not the uniform sum of two disjoint nonempty subgraphs. We then note that by Corollary 2.19 if Γ is a profinite graph then we can restate the condition as Γ **is not the disjoint union of two nonempty closed subgraphs**.

(3) the completion $\bar{\Gamma}$ of Γ is cofinitely connected.

PROOF. (1) \Rightarrow (2): If possible, let us assume that Γ is the uniform sum of two disjoint subgraphs Γ_1 and Γ_2 . Let R_{Γ_1} be a compatible cofinite entourage over Γ_1 and S_{Γ_2} be another compatible cofinite entourage over Γ_2 . Then $W = R_{\Gamma_1} \cup S_{\Gamma_2}$ is a compatible cofinite entourage over Γ . Moreover Γ/W is not path connected, a contradiction.

(2) \Rightarrow (3): If possible, let us assume that $\bar{\Gamma}$ is not cofinitely connected. Hence there exists a compatible cofinite entourage W over $\bar{\Gamma}$ such that $\bar{\Gamma}/W$ is not path connected.

Let Σ be a path connected component of $\bar{\Gamma}/W$. Hence Σ is a subgraph of $\bar{\Gamma}/W$ and thus $(\bar{\Gamma}/W) \setminus \Sigma$ is a subgraph of $\bar{\Gamma}/W$ as well. Let $\Gamma_1 = \varphi^{-1}(\Sigma)$ and $\Gamma_2 = \varphi^{-1}(\bar{\Gamma} \setminus \Sigma)$, where $\varphi: \bar{\Gamma} \rightarrow \bar{\Gamma}/W$ is the canonical quotient map. Then Γ_1, Γ_2 are closed subgraphs of $\bar{\Gamma}$ such that $\bar{\Gamma}$ is equal to the disjoint union of two closed subgraphs of $\bar{\Gamma}$ and then by Corollary 2.19 $\bar{\Gamma}$ is equal to the uniform sum of two disjoint subgraphs of $\bar{\Gamma}$, a contradiction.

(3) \Rightarrow (1): If possible assume that Γ is not cofinitely connected. Then there exists a cofinite entourage R over Γ such that Γ/R is not path connected. But, by Lemma 6.6, \bar{R} is a compatible cofinite entourage over $\bar{\Gamma}$ such that Γ/R is graph isomorphic to $\bar{\Gamma}/\bar{R}$. Hence $\bar{\Gamma}/\bar{R}$ is not path connected as well, a contradiction. \square

Many of the properties of connectedness of topological spaces have analogs for cofinite connectedness. Next we list a few of them.

PROPOSITION 7.3. *Let Γ be a cofinite graph and let Σ be a uniform subgraph.*

- (1) *If Σ is path connected, then it is also cofinitely connected.*
- (2) *If Σ is cofinitely connected, then so is the cofinite subgraph $\bar{\Sigma}$.*
- (3) *If Σ is cofinitely connected and $f: \Gamma \rightarrow \Delta$ a uniformly continuous map of graphs, then $f(\Sigma)$ is also cofinitely connected (as a cofinite subgraph of Δ).*

PROOF. Note that Σ is also a cofinite graph

- (1) If Σ is path connected then any quotient graph of Σ is path connected as well and thus our claim follows.
- (2) We will first see that $\bar{\Sigma} = \overline{V(\Sigma) \cup E(\Sigma)} = \overline{V(\Sigma)} \cup \overline{E(\Sigma)} = V(\bar{\Sigma}) \cup E(\bar{\Sigma})$ and thus is a cofinite subgraph of Γ as well. Now, if possible suppose $\bar{\Sigma} = \Sigma_1 \coprod \Sigma_2$, where Σ_1, Σ_2 are two disjoint nonempty cofinite subgraphs of $\bar{\Sigma}$. Then $\Sigma_1 \cap \Sigma, \Sigma_2 \cap \Sigma$ are two disjoint connected cofinite subgraphs of Σ . Let R_1, R_2 be two compatible cofinite entourage over $\Sigma_1 \cap \Sigma, \Sigma_2 \cap \Sigma$ respectively. Then there exist two compatible cofinite entourages \tilde{R}_1, \tilde{R}_2 over Σ_1, Σ_2 respectively such that $R_1 \supseteq \tilde{R}_1 \cap (\Sigma \times \Sigma)$ and $R_2 \supseteq \tilde{R}_2 \cap (\Sigma \times \Sigma)$. But as $\tilde{R}_1 \cup \tilde{R}_2$ is a compatible cofinite entourage over $\bar{\Sigma}$, then $(\tilde{R}_1 \cup \tilde{R}_2) \cap (\Sigma \times \Sigma) = \tilde{R}_1 \cap (\Sigma \times \Sigma) \cup \tilde{R}_2 \cap (\Sigma \times \Sigma) \subseteq R_1 \cup R_2$. So $R_1 \cup R_2$ is a compatible entourage over Σ . Hence $\Sigma = (\Sigma_1 \cap \Sigma) \coprod (\Sigma_2 \cap \Sigma)$. Now suppose $\Sigma_1 \cap \Sigma = \emptyset$. Then $\Sigma \subseteq \Sigma_2$. However Σ_2 is closed in $\bar{\Sigma}$ and hence closed in Γ . Thus $\bar{\Sigma} \subseteq \Sigma_2$ and therefore $\Sigma_1 = \emptyset$, a contradiction. Thus $\bar{\Sigma}$ is cofinitely connected.
- (3) Let S be a compatible cofinite entourage over $f(\Sigma)$. Then as $f|_{\Sigma}: \Sigma \rightarrow f(\Sigma)$ is uniformly continuous there is a compatible cofinite entourage R over Σ such that $R \subseteq (f \times f)^{-1}[S]$. Let us define $g: \Sigma/R \rightarrow f(\Sigma)/S$ via $g(R[a]) = S[f(a)]$, for all $a \in \Sigma$. Now if $R[a] = R[b]$, then $(a, b) \in R$. Hence $(f(a), f(b)) \in S$ which implies that $S[f(a)] = S[f(b)]$. Therefore g is well defined and as f is a map of graphs and both of $\Sigma/R, f(\Sigma)/S$ are

discrete, g is a surjective uniformly continuous map of graphs. Since Σ/R is path connected then so is $g(\Sigma/R) = f(\Sigma)/S$.

□

CHAPTER 8

Cofinite Groups and their Cayley Graphs

As an immediate consequence of the properties of cofinite connectedness discussed in the previous chapter, in this chapter we talked about the cofinite connectedness of Cayley graphs of cofinite groups. We first observe that Cayley graphs of cofinite groups are cofinite graphs. Then we obtained the following generalized characterization of the connected Cayley graphs of cofinite groups:

Let G be a cofinite group and let $\Gamma = \Gamma(G, X)$ be the Cayley graph. Then X generates G (topologically) if and only if Γ is cofinitely connected.

8.1. Cofinite Groups and their Cayley Graphs

DEFINITION 8.1. *Let G be an abstract group and $X = \{*\} \dot{\cup} E(X)$ be an abstract graph such that there is a map of sets $\alpha: X \rightarrow G$ with $\alpha(*) = 1_G$, $(\alpha(e))^{-1} = \alpha(\bar{e})$, for all $e \in E(X)$. Then the Cayley Graph $\Gamma(G, X)$ is defined as follows:*

- (1) $V(\Gamma(G, X)) = G \times \{*\}$, $E(\Gamma(G, X)) = G \times E(X)$.
- (2) $s(g, e) = (g, *)$, $t(g, e) = (g\alpha(e), *)$, $\overline{(g, e)} = (g\alpha(e), \bar{e})$.

Thus it follows that

- (1) $\Gamma(G, X) = V(\Gamma(G, X)) \dot{\cup} E(\Gamma(G, X))$.
- (2) $s, t, \bar{}$ are well defined and $t(\overline{(g, e)}) = t(g\alpha(e), \bar{e}) = (g\alpha(e)\alpha(\bar{e}), *) = (g\alpha(e)\alpha(e)^{-1}, *) = (g, *) = s(g, e)$; $s(\overline{(g, e)}) = s(g\alpha(e), \bar{e}) = (g\alpha(e), *) = t(g, e)$.

(3) If possible, let $(g, e) = \overline{(g, e)} = ((g\alpha(e), \bar{e}))$ and thus $e = \bar{e}$, a contradiction.

Finally, $\overline{\overline{(g, e)}} = \overline{(g\alpha(e), \bar{e})} = (g\alpha(e)\alpha(\bar{e}), \bar{\bar{e}}) = (g\alpha(e)\alpha(e)^{-1}, e) = (g, e)$.

Hence $\Gamma(G, X)$ is indeed a graph.

We say that $\alpha: X \rightarrow G$ generates G algebraically if $\langle \alpha(X) \rangle = G$. Equivalently, $\alpha: X \rightarrow G$ generates G algebraically if the unique extension to $\alpha: E(X)^* \rightarrow G$ is onto.

LEMMA 8.2. *The Cayley graph $\Gamma(G, X)$ is path connected if and only if $\alpha: X \rightarrow G$ generates G algebraically.*

PROOF. Let us first assume that $\Gamma(G, X)$ is path connected. Let $g \in G$. Then there exists a path, say, $p = (1_G, e_1)(g_2, e_2)(g_3, e_3)\dots(g_n, e_n)$ such that $s(p) = s(1_G, e_1) = (1_G, *)$ and $t(p) = t(g_n, e_n) = (g_n\alpha(e_n), *) = (g, *)$. But $t(1_G, e_1) = (1_G\alpha(e_1), *) = (\alpha(e_1), *) = s(g_2, e_2) = (g_2, *)$. Similarly, $t(g_2, e_2) = (g_2\alpha(e_2), *) = (\alpha(e_1)\alpha(e_2), *) = s(g_3, e_3) = (g_3, *)$. Proceeding inductively one gets $(g, *) = t(g_n, e_n) = (\alpha(e_1)\alpha(e_2)\dots\alpha(e_{n-1})\alpha(e_n), *)$. Hence $g = \alpha(e_1)\alpha(e_2)\dots\alpha(e_{n-1})\alpha(e_n)$ and thus $\langle \alpha(X) \rangle = G$.

Conversely, let us assume $\langle \alpha(X) \rangle = G$. Let $(g_1, *), (g_2, *)$ are two entries in $V(\Gamma(G, X))$. Then $g_1^{-1}g_2 \in G = \langle \alpha(X) \rangle$. So there exists a subset $\{e_i\}_{i=1}^n$ of $E(X)$ such that $g_1^{-1}g_2 = \alpha(e_1)\alpha(e_2)\dots\alpha(e_n)$. Hence $g_2 = g_1\alpha(e_1)\alpha(e_2)\dots\alpha(e_n)$. Thus

$$(g_1, e_1)(g_1\alpha(e_1), e_2)(g_1\alpha(e_1)\alpha(e_2), e_3)\dots(g_1\alpha(e_1)\dots\alpha(e_{n-1}), \alpha(e_n))$$

is a path in $\Gamma(G, X)$ such that it initiates at $(g_1, *)$ and terminates at $(g_1\alpha(e_1)\alpha(e_2)\dots\alpha(e_n), *) = (g_2, *)$. Thus $\Gamma(G, X)$ is path connected. \square

DEFINITION 8.3. *Let G be a cofinite group and $X = \{*\} \dot{\cup} E(X)$ be a cofinite graph such that there is a uniform continuous map of spaces $\alpha: X \rightarrow G$ with $\alpha(*) =$*

1_G , $(\alpha(e))^{-1} = \alpha(\bar{e})$, for all $e \in E(X)$. Then the cofinite Cayley Graph $\Gamma(G, X)$ is defined as follows:

- (1) $V(\Gamma(G, X)) = G \times \{*\}$, $E(\Gamma(G, X)) = G \times E(X)$.
- (2) $s(g, e) = (g, *)$, $t(g, e) = (g\alpha(e), *)$, $\overline{(g, e)} = (g\alpha(e), \bar{e})$.

$\Gamma(G, X)$ is endowed with the product uniform topological structure obtained from

$$G \times X = G \times V(\Gamma(G, X)) \dot{\cup} G \times E(\Gamma(G, X))$$

We have already seen that $\Gamma(G, X)$ is an abstract graph. Also being the product of Hausdorff, cofinite spaces, $\Gamma(G, X)$ is a Hausdorff, cofinite space as well. So in order to check that $\Gamma(G, X)$ is a cofinite graph it remains to prove that the compatible cofinite entourages over $\Gamma(G, X)$ forms a fundamental system of entourages. So it suffices to show that the family of cofinite entourages of the form $R \times S$, where R is a cofinite congruence over G and S is a compatible cofinite entourage over X such that $(\alpha \times \alpha)[S] \subseteq R$ forms a fundamental system of entourages.

To establish the above claim let us first see that the cofinite entourages of the form $R \times S$ are indeed compatible.

- (1) Let $((x, y), (p, q)) \in R \times S$. So $(x, p) \in R \subseteq G \times G$ and $(y, q) \in S$. Thus either $(y, q) \in S_V$ or $(y, q) \in S_E$ which implies that $y = * = q$ or $(y, q) \in S_E$. Hence $(x, y), (p, q) \in V(\Gamma(G, X))$ or $(x, y), (p, q) \in E(\Gamma(G, X))$. Hence $R \times S \subseteq (R \times S)_V \dot{\cup} (R \times S)_E$. The other direction of the inclusion follows more immediately.
- (2) Let $((g_1, e_1), (g_2, e_2)) \in R \times S$. Then $(g_1, g_2) \in R$ and $(e_1, e_2) \in S$. This implies that $(\alpha \times \alpha)(e_1, e_2) = (\alpha(e_1), \alpha(e_2)) \in R$ and $(\bar{e}_1, \bar{e}_2) \in S$. Hence $(g_1\alpha(e_1), g_2\alpha(e_2)) \in R$, which implies that $((g_1, *), (g_2, *)), ((g_1\alpha(e_1), *), (g_2\alpha(e_2), *)), ((g_1\alpha(e_1), \bar{e}_1), (g_2\alpha(e_2), \bar{e}_2)) \in R \times S$. Hence $(s(g_1, e_1), s(g_2, e_2)), (t(g_1, e_1), t(g_2, e_2)), (\overline{(g_1, e_1)}, \overline{(g_2, e_2)}) \in R \times S$.

(3) If possible let $((\overline{g_1}, \overline{e_1}), (g_1, e_1)) \in R \times S$. Thus $((g_1\alpha(e_1), \overline{e_1}), (g_1, e_1)) \in R \times S$.

Thus $(\overline{e_1}, e_1) \in S$, a contradiction.

Now let $R \times T$ be any cofinite entourage over $G \times X$. Note that since α is uniformly continuous and R is a cofinite congruence over G , T is a cofinite entourage over X , $(\alpha \times \alpha)^{-1}[R] \cap T$ is a cofinite entourage over X and $(\alpha \times \alpha)[(\alpha \times \alpha)^{-1}[R] \cap T] \subseteq R$. So in particular one can take S to be a compatible cofinite entourage over X such that $S \subseteq (\alpha \times \alpha)^{-1}[R] \cap T$. Then $(\alpha \times \alpha)[S] \subseteq R$ and $R \times S \subseteq R \times T$. This proves that $\Gamma(G, X)$ is a cofinite graph.

We say that $\alpha: X \rightarrow G$ generates G topologically if $\langle \overline{\alpha(X)} \rangle = G$.

THEOREM 8.4. *Let $\Gamma = \Gamma(G, X)$ be the cofinite Cayley graph. $\alpha: X \rightarrow G$ generates G topologically iff Γ is cofinitely connected.*

PROOF. Let us first suppose that $\alpha: X \rightarrow G$ topologically generates G and let T be a compatible cofinite entourage over Γ , say $T = R \times S$ where R is a cofinite congruence over G and S is a compatible cofinite entourage over X where $S \subseteq (\alpha \times \alpha)^{-1}[R]$. Let us define $\alpha_{RS}: X/S \rightarrow G/R$ via $\alpha_{RS}(S[x]) = R[\alpha(x)]$. Clearly, α_{RS} is well defined and $\alpha_{RS}(S[*]) = R[1_G]$, $\alpha_{RS}(\overline{S[e]}) = R[\alpha(\overline{e})] = R[(\alpha(e))^{-1}] = (R[\alpha(e)])^{-1} = (\alpha_{RS}(S[e]))^{-1}$, for all $S[e] \in E(X/S)$. Let us now see that $\Gamma/T \cong \Gamma(G/R, X/S)$. Define $\theta: \Gamma/T \rightarrow \Gamma(G/R, X/S)$ via $\theta(T[(g, x)]) = (R[g], S[x])$ for all x in X and all g in G . The map is a well defined injection as the following holds true. $T[(h, y)] = T[(g, x)] \Leftrightarrow ((h, y), (g, x)) \in T \Leftrightarrow (h, g) \in R, (y, x) \in S \Leftrightarrow R[h] = R[g], S[x] = S[y] \Leftrightarrow (R[h], S[y]) = (R[g], S[x])$. Also for all $(R[g], S[x]) \in \Gamma(G/R, X/S)$, there exists $T[(g, x)] \in \Gamma/T$ such that $\theta(T[(g, x)]) = (R[g], S[x])$. Moreover it can easily be seen that θ is a map of graphs as $\theta(T[(g, *)]) = (R[g], S[*]) \in V(\Gamma(G/R, X/S))$, $\theta(T[(g, e)]) = (R[g], S[e]) \in E(\Gamma(G/R, X/S))$. For all $(T[(g, e)]) \in E(\Gamma/T)$,

$$\theta(s(T[(g, e)])) = \theta(T[s(g, e)]) = \theta(T[(g, *)]) = (R[g], S[*]) = s(R[g], S[e]),$$

$$\theta(t(T[(g, e)])) = \theta(T[t(g, e)]) = \theta(T[(g\alpha(e), *)]) = (R[g\alpha(e)], S[*])$$

$$= (R[g]R[\alpha(e)], S[*]) = (R[g]\alpha_{RS}(S[e]), S[*]) = t(R[g], S[e]),$$

$$\theta(\overline{(T[(g, e)]})) = \theta(T[\overline{(g, e)}]) = \theta(T[(g\alpha(e), \bar{e})]) = (R[g\alpha(e)], S[\bar{e}])$$

$$= (R[g]R[\alpha(e)], S[\bar{e}]) = (R[g]\alpha_{RS}(S[e]), S[\bar{e}]) = (R[g]\alpha_{RS}(S[e]), \overline{S[e]})$$

$$= \overline{(R[g], S[e])}.$$

Since $\Gamma/T, \Gamma(G/R, X/S)$ are discrete cofinite graphs, our claim follows.

Now we wish to prove that $\langle \alpha_{RS}(X/S) \rangle = G/R$. Let $R[g] \in G/R$. Then as $\overline{\langle \alpha(X) \rangle} = G$, we have $R[g] \cap \langle \alpha(X) \rangle \neq \emptyset$. Let $a \in R[g] \cap \langle \alpha(X) \rangle$. So, $R[g] = R[a]$. Also, since $a \in \langle \alpha(X) \rangle$, $a = \alpha(e_1)\alpha(e_2) \cdots \alpha(e_n)$, for some $e_1, e_2, \dots, e_n \in E(X)$. Hence $R[a] = R[\alpha(e_1)]R[\alpha(e_2)] \cdots R[\alpha(e_n)] = \alpha_{RS}(S[e_1])\alpha_{RS}(S[e_2]) \cdots \alpha_{RS}(S[e_n])$. Thus $R[g] = R[a] \in \langle \alpha_{RS}(X/S) \rangle$. Therefore $\langle \alpha_{RS}(X/S) \rangle = G/R$ and consequently, $\Gamma/T = \Gamma(G/R, X/S)$ is path connected. Hence Γ is cofinitely connected.

Conversely, let us now take Γ to be cofinitely connected. We want to show that $\overline{\langle \alpha(X) \rangle} = G$. So we intend to show that for any g in G and any open set $R[g]$ on G , $R[g] \cap \langle \alpha(X) \rangle \neq \emptyset$. We can form a compatible cofinite entourage $T = R \times S$ where S is a compatible cofinite entourage over X and $S \subseteq (\alpha \times \alpha)^{-1}[R]$. As earlier we can form the Cayley graph $\Gamma/T = \Gamma(G/R, X/S)$ and as Γ is cofinitely connected, Γ/T and therefore $\Gamma(G/R, X/S)$, is path connected. This implies $\langle \alpha_{RS}(X/S) \rangle = G/R$. So there is e_1, e_2, \dots, e_n in $E(X)$ such that $\alpha_{RS}(S[e_1])\alpha_{RS}(S[e_2]) \cdots \alpha_{RS}(S[e_n]) = R[g]$. Thus we obtain $\alpha(e_1)\alpha(e_2) \cdots \alpha(e_n) \in R[g]$ which implies that $\langle \alpha(X) \rangle \cap R[g] \neq \emptyset$ and thus $\overline{\langle \alpha(X) \rangle} = G$. Hence $\alpha: X \rightarrow G$ topologically generates G . \square

CHAPTER 9

Groups Acting on Cofinite Graphs

Our final chapter is concerned with group actions on cofinite graphs.

A group G is said to act uniformly equicontinuously over a cofinite graph Γ if and only if for each entourage W over Γ there exists an entourage V over Γ such that for all g in G , $(g \times g)[V] \subseteq W$. In this case the group action induces a (Hausdorff) cofinite uniformity over G if and only if the aforesaid action is faithful.

We say that a group G acts on a cofinite graph Γ residually freely, if there exists a fundamental system of G -invariant compatible cofinite entourages R over Γ such that the induced group action of G/N_R over Γ/R is a free action, where N_R is the Kernel of the action of G on Γ/R .

Suppose that G is a group acting faithfully and uniformly equicontinuously on a cofinite graph Γ , then the action $G \times \Gamma \rightarrow \Gamma$ is uniformly continuous. Also in that case \widehat{G} acts on $\widehat{\Gamma}$ uniformly equicontinuously.

9.1. Groups Acting on Cofinite Graphs

Let G be a group and Γ be a cofinite graph. We say that the group G acts over Γ if and only if

- (1) For all x in Γ , for all g in G , $g.x$ is in Γ
- (2) For all x in Γ , for all g_1, g_2 in G , $g_1.(g_2.x) = (g_1g_2).x$
- (3) For all x in Γ , $1.x = x$
- (4) For all v in $V(\Gamma)$, for all g in G , $g.v$ is in $V(\Gamma)$ and for all e in $E(\Gamma)$, for all g in G , $g.e$ is in $E(\Gamma)$.
- (5) For all e in $E(\Gamma)$, for all g in G , $g.s(e) = s(g.e)$, $g.t(e) = t(g.e)$, $g.(\bar{e}) = \overline{g.e}$

(6) There exists a G -invariant orientation $E^+(\Gamma)$ of Γ .

Note that the aforesaid group action restricted to a singleton group element $g \in G$ can be treated as a well defined map of graphs, $\Gamma \rightarrow \Gamma$ taking $x \mapsto g.x$.

DEFINITION 9.1. *A group G is said to act uniformly equicontinuously over a cofinite graph Γ , if and only if for each entourage W over Γ there exists an entourage V over Γ such that for all g in G , $(g \times g)[V]$ is a subset of W .*

LEMMA 9.2. *If G acts uniformly equicontinuously over a cofinite graph Γ , then there exists a fundamental system of entourages consisting of G -invariant compatible cofinite entourages over Γ , i.e. for any entourage U over Γ there exists a compatible cofinite entourage R over Γ such that for all $g \in G$, $(g \times g)[R] \subseteq R \subseteq U$.*

PROOF. Let U be any cofinite entourage over Γ . Then as G acts uniformly equicontinuously over Γ , there exists a compatible cofinite entourage S over Γ such that for all $g \in G$, $(g \times g)[S] \subseteq U$. Choose a G -invariant orientation $E^+(\Gamma)$ of Γ . As in the proof of Theorem 4.5, without loss of generality, we can assume that our compatible equivalence relation S on Γ is *orientation preserving* i.e. whenever $(e, e') \in R$ and $e \in E^+(\Gamma)$, then also $e' \in E^+(\Gamma)$. Clearly, $S \subseteq \cup_{g \in G} (g \times g)[S] \subseteq U$. Now if $S_0 = \cup_{g \in G} (g \times g)[S]$ and $T = \langle S_0 \rangle$, note that $S \subseteq T \subseteq U$. Since for all $h \in G$, $(h \times h)[S_0] = S_0$ and $S_0^{-1} = S_0$ it follows that T is in the transitive closure of S_0 .

Let $(x, y) \in T$. Then there exists a finite sequence x_0, x_1, \dots, x_n such that $(x_i, x_{i+1}) \in S_0$, for all $i = 0, 1, 2, \dots, n-1$ and $x = x_0, y = x_n$. Hence $(gx_i, gx_{i+1}) \in S_0$, for all $i = 0, 1, 2, \dots, n-1$, for all $g \in G$. Thus $(gx_0, gx_n) = (gx, gy) \in T$, for all $g \in G$. Hence for all $g \in G$, $(g \times g)[T] \subseteq T$ and our claim that T is a G -invariant cofinite entourage, follows.

It remains to check that T is compatible. Let $(x, y) \in T$. If $(x, y) \in S_0$, then there is $(t, s) \in S = S_V \cup S_E$ and $g \in G$ such that $(gt, gs) = (x, y)$. Without loss of

generality let $(t, s) \in S_V$. Then $(t, s) \in V(\Gamma) \times V(\Gamma)$ which implies that $(x, y) \in T_V$. Now let $(x, y) \in T \setminus S_0$. Then there exists a finite sequence x_0, x_1, \dots, x_n such that $(x_i, x_{i+1}) \in S_0$, for all $i = 0, 1, 2, \dots, n-1$ and $x = x_0, y = x_n$. Hence by the previous argument if $(x_0, x_1) \in T_V$ then $(x_i, x_{i+1}) \in T_V$, for all $i = 1, 2, \dots, n-1$. Thus $(x, y) \in T_V$. If $(x_0, x_1) \in T_E$ then $(x_i, x_{i+1}) \in T_E$, for all $i = 1, 2, \dots, n-1$, which implies $(x, y) \in T_E$.

Let $(e_1, e_2) \in T$. If $(x, y) \in S_0$, then there is $(p, q) \in S$ and $g \in G$ such that $(gp, gq) = (e_1, e_2)$. Then $(s(p), s(q)) \in S$. So $(s(e_1), s(e_2)) = (gs(p), gs(q)) \in (g \times g)[S] \subseteq S_0$ so that $(s(e_1), s(e_2)) \in T$. Now let $(e_1, e_2) \in T \setminus S_0$. Then there exists a finite sequence x_0, x_1, \dots, x_n such that $(x_i, x_{i+1}) \in S_0, \forall i = 0, 1, 2, \dots, n-1$ and $e_1 = x_0, e_2 = x_n$. Hence by the previous argument $(s(x_i), s(x_{i+1})) \in T, \forall i = 0, 1, 2, \dots, n-1$ and thus $(s(e_1), s(e_2)) \in T$. Similarly, $(t(e_1), t(e_2)) \in T$ and $(\bar{e}_1, \bar{e}_2) \in T$.

Finally, to show that for any $e \in E^+(\Gamma), (e, \bar{e}) \in T$ it suffices to note that T is orientation preserving. Alternatively, if possible let $(e, \bar{e}) \in T$. If $(e, \bar{e}) \in S_0$, then there is $(p, q) \in S$ and $g \in G$ such that $(gp, gq) = (e, \bar{e})$. Then $\bar{e} = \overline{gp} = g\bar{p} = gq$ which implies that $\bar{p} = q$, so $(p, \bar{p}) \in S$, a contradiction. Now let $(e, \bar{e}) \in T \setminus S_0$. Then there exists a finite sequence x_0, x_1, \dots, x_n such that $(x_i, x_{i+1}) \in S_0$, for all $i = 0, 1, 2, \dots, n-1$ and $e = x_0, \bar{e} = x_n$. Now let there is $(p, q) \in S$ and $g \in G$ such that $(gp, gq) = (x_0, x_1)$. Without loss of generality we may assume $(p, q) \in E^+(\Gamma) \times E^+(\Gamma)$. Then $(gp, gq) = (x_0, x_1) \in E^+(\Gamma) \times E^+(\Gamma)$. Hence $(x_i, x_{i+1}) \in E^+(\Gamma) \times E^+(\Gamma)$, for all $i = 1, 2, \dots, n-1$ which implies that $(e, \bar{e}) \in E^+(\Gamma) \times E^+(\Gamma)$, a contradiction. Our claim follows. \square

DEFINITION 9.3. *We say a group G acts on a cofinite space Γ faithfully, if for all g in $G \setminus \{1\}$ there exists x in Γ such that gx is not equal to x in Γ .*

LEMMA 9.4. *Let G acts on a cofinite graph Γ uniformly equicontinuously. Then G acts on Γ/R and G/N_R acts on Γ/R as well, where R is a G -invariant compatible*

cofinite entourage over Γ . If $\{R \mid R \in I\}$ is a fundamental system of G -invariant compatible cofinite entourages over Γ , then $\{N_R \mid R \in I\}$ forms a fundamental system of cofinite congruences for some uniformity over G .

PROOF. Let R be a G -invariant compatible cofinite entourage over Γ . Let us define $G \times \Gamma/R \rightarrow \Gamma/R$ via $g.R[x] = R[g.x]$, for all $g \in G$, for all $x \in \Gamma$. Now let $R[x] = R[y]$ so $(x, y) \in R$ which implies that $(g.x, g.y) \in R$. Then $R[g.x] = R[g.y]$. Hence the induced group action is well defined.

Let us now consider the group action $G/N_R \times \Gamma/R \rightarrow \Gamma/R$, defined via $N_R[g].R[x] = R[g.x]$, for all $x \in \Gamma$, for all $g \in G$. Now let $(N_R[g], R[x]) = (N_R[h], R[y])$ which implies that $(g, h) \in N_R, (x, y) \in R$. Then $(g.x, h.x) \in R$, as $h^{-1} \in G, (h^{-1}g.x, h^{-1}h.x) \in R$. So $(h^{-1}g.x, y) \in R$. Thus $(g.x, h.y) \in R$ which implies that $R[g.x] = R[h.y]$. Hence the induced group action is well defined.

Let us now show that N_R is an equivalence relation over G , for all G -invariant compatible cofinite entourage R over Γ .

- (1) for all $g \in G$, for all $x \in \Gamma, (g.x, g.x) \in R$. Hence $(g, g) \in N_R$, for all $g \in G$ which implies that $D(G) \subseteq N_R$.
- (2) Now $(h, g) \in N_R^{-1} \Leftrightarrow (g, h) \in N_R \Leftrightarrow (g.x, h.x) \in R$, for all $x \in \Gamma$. Thus $(g.x, h.x) \in R \Leftrightarrow (h.x, g.x) \in R$, for all $x \in \Gamma$. Hence $(h.x, g.x) \in R \Leftrightarrow (h, g) \in N_R$. Thus $N_R^{-1} = N_R$.
- (3) Let $(g, h), (h, k) \in N_R$. This implies $(g.x, h.x), (h.x, k.x) \in R, \forall x \in \Gamma$. Hence $(g.x, k.x) \in R$, for all $x \in \Gamma$. So $(g, k) \in N_R$ which implies that $N_R^2 \subseteq N_R$.

Also we now check that N_R is a congruence over G .

For, let us take $(g_1, g_2), (g_3, g_4) \in N_R$.

Then for all $x \in \Gamma, (g_1.x, g_2.x), (g_3.x, g_4.x) \in R$; for all $x \in \Gamma, g_3.x \in \Gamma$ and so $(g_1g_3.x, g_2g_3.x) \in R$ and $(g_2g_3.x, g_2g_4.x) \in R$, since R is G -invariant. Thus

$(g_1g_3.x, g_2g_4.x) \in R$, for all $x \in \Gamma$ so that $(g_1g_3, g_2g_4) \in N_R$. Thus our claim follows.

Let us now show that G/N_R is finite.

Let $g: \Gamma/R \rightarrow \Gamma/R$ be defined by $g(R[x]) = R[g.x]$. Now, $R[x] = R[y] \Leftrightarrow (x, y) \in R \Leftrightarrow (g.x, g.y) \in R \Leftrightarrow R[g.x] = R[g.y]$. Hence the map g is a well defined injection. Now for all $R[x] \in \Gamma/R$ there exists $g^{-1}R[x] \in \Gamma/R$ such that $g(g^{-1}R[x]) = R[x]$. Hence $g \in \text{Sym}(\Gamma/R)$. Now let us define a map $\theta: G/N_R \rightarrow \text{Sym}(\Gamma/R)$ via $\theta(N_R[g]) = g$. Now $N_R[g_1] = N_R[g_2] \Leftrightarrow (g_1, g_2) \in N_R \Leftrightarrow (g_1.x, g_2.x) \in R$ for all $x \in \Gamma$. Hence $(g_1.x, g_2.x) \in R \Leftrightarrow R[g_1.x] = R[g_2.x], \Leftrightarrow g_1(R[x]) = g_2(R[x]) \Leftrightarrow g_1 = g_2$ in $\text{Sym}(\Gamma/R)$. Hence θ is a well defined injection. Thus $|G/N_R| \leq |\text{Sym}(\Gamma/R)| < \infty$ as $|\Gamma/R| < \infty$.

So, next we will like to show that $\{N_R \mid R \in I\}$ forms a fundamental system of cofinite congruences over G .

- (1) $D(G) \subseteq N_R$, for all $R \in I$, as N_R is reflexive.
- (2) Now for some $R, S \in I$, $(g_1, g_2) \in N_R \cap N_S \Leftrightarrow (g_1.x, g_2.x) \in R \cap S$, for all $x \in \Gamma \Leftrightarrow (g_1, g_2) \in N_{R \cap S}$. Thus $N_R \cap N_S = N_{R \cap S}$.
- (3) For all $N_R, N_R^2 = N_R$, as N_R is transitive.
- (4) For all $N_R, N_R^{-1} = N_R$, as N_R is symmetric.

Hence our claim follows. □

DEFINITION 9.5. *We say that a group G acts on a cofinite graph Γ residually freely, if there exists a fundamental system of G -invariant compatible cofinite entourages R over Γ such that the induced group action of G/N_R over Γ/R is a free action.*

LEMMA 9.6. *$N_R[1]$ is a finite index normal subgroup of G and $G/N_R[1]$ is isomorphic with G/N_R . More generally, if N is a congruence on G , then $N[1]$ is a normal subgroup of G and $G/N[1] \cong G/N$.*

PROOF. Let us first see that $N_R[1] \triangleleft_f G$ for all G -invariant compatible cofinite entourage R over Γ .

Let $g, h \in N_R[1]$. This implies $(1, g) \in N_R$ and hence $(g, 1), (1, h) \in N_R$. Thus $(g, h) \in N_R$. This implies $(g.x, h.x) \in R$, for all $x \in \Gamma$ and so $(x, g^{-1}h.x) \in R$, for all $x \in \Gamma$. Hence, $(1, g^{-1}h) \in N_R$ and thus $g^{-1}h \in N_R[1]$. So, $N_R[1] \leq G$. For all $g \in G$, for all $x \in \Gamma, g.x \in \Gamma$. Hence for all $k \in N_R[1], (x, k.x) \in R$, hence $(k.x, x) \in R$. Thus $(kg.x, g.x) \in R$ and $(g^{-1}kg.x, g^{-1}g.x) = (g^{-1}kg.x, x) \in R$. Hence $(g^{-1}kg, 1) \in N_R$. So, $g^{-1}kg \in N_R[1]$ and thus $N_R[1] \triangleleft G$.

Now let us define $\eta: G/N_R[1] \rightarrow G/N_R$ via $\eta(gN_R[1]) = N_R[g]$. Then, $gN_R[1] = hN_R[1] \Leftrightarrow h^{-1}g \in N_R[1] \Leftrightarrow (1, h^{-1}g) \in N_R \Leftrightarrow (x, h^{-1}g.x) \in R \Leftrightarrow (h.x, g.x) \in R \Leftrightarrow (h, g) \in N_R \Leftrightarrow N_R[h] = N_R[g]$, for all x in Γ . Thus η is a well defined injection and hence $|G/N_R[1]| \leq |G/N_R| < \infty$. Hence $N_R[1] \triangleleft_f G$.

Let us check that G/N_R is a group. For, let $N_R[g_i] \in G/N_R, i = 1, 2$. Then $N_R[g_1]N_R[g_2] = N_R[g_1g_2] \in G/N_R$.

Let $N_R[g_i]$ in G/N_R , for $i = 1, 2, 3$. Then

$$\begin{aligned} (N_R[g_1]N_R[g_2])N_R[g_3] &= N_R[g_1g_2]N_R[g_3] = N_R[g_1g_2g_3] \\ &= N_R[g_1]N_R[g_2g_3] = N_R[g_1](N_R[g_2]N_R[g_3]) \end{aligned}$$

For all $N_R[g] \in G/N_R$, there exists $N_R[1]$ in G/N_R , such that

$$N_R[1]N_R[g] = N_R[g] = N_R[g]N_R[1]$$

For all $N_R[g]$ in G/N_R , there exists $N_R[g^{-1}]$ in G/N_R , such that

$$N_R[g^{-1}]N_R[g] = N_R[g^{-1}g] = N_R[1] = N_R[gg^{-1}] = N_R[g]N_R[g^{-1}]$$

Hence our claim.

Now let us define $\zeta: G/N_R[1] \rightarrow G/N_R$ via $\zeta(gN_R[1]) = N_R[g]$. Then for g_1, g_2 in G , $g_1N_R[1] = g_2N_R[1] \Leftrightarrow g_2^{-1}g_1 \in N_R[1] \Leftrightarrow (1, g_2^{-1}g_1) \in N_R \Leftrightarrow (x, g_2^{-1}g_1.x) \in R \Leftrightarrow (g_2.x, g_1.x) \in R \Leftrightarrow (g_2, g_1) \in N_R \Leftrightarrow N_R[g_2] = N_R[g_1]$. Hence ζ is a well defined injection. Also for all $N_R[g] \in G/N_R$, there exists $gN_R[1] \in G/N_R[1]$ such that $\zeta(gN_R[1]) = N_R[g]$. Thus ζ is surjective as well. Also for $g_1N_R[1], g_2N_R[1] \in G/N_R[1]$, we have

$$\begin{aligned} \zeta(g_1N_R[1]g_2N_R[1]) &= \zeta(g_1g_2N_R[1]) = N_R[g_1g_2] \\ &= N_R[g_1]N_R[g_2] = \zeta(g_1N_R[1])\zeta(g_2N_R[1]). \end{aligned}$$

Hence ζ is a group homomorphism and thus a group isomorphism. Also, both $G/N_R[1], G/N_R$, are finite discrete topological groups, so ζ is an isomorphism of uniform cofinite groups as well. \square

LEMMA 9.7. *The induced uniform topology over G as in Lemma 9.4 is Hausdorff if and only if G acts faithfully over Γ .*

PROOF. Let us first assume that G acts faithfully over Γ . Now let $g \neq h$ in G . Then $h^{-1}g \neq 1$. So there exists $x \in \Gamma$ such that $h^{-1}g.x \neq x$ implying that $g.x \neq h.x$. Then there exists a G -invariant compatible cofinite entourage R over Γ such that $(g.x, h.x) \notin R$, as Γ is Hausdorff. Hence $(g, h) \notin N_R$. Thus G is Hausdorff.

Conversely, let us assume that G is Hausdorff and let $g \neq 1$ in G . Then there exists some G -invariant compatible cofinite entourage R over Γ such that $(1, g) \notin N_R$.

Hence there exists $x \in \Gamma$ such that $(x, g.x) \notin R$. Hence $R[x] \neq R[g.x]$ so that $x \neq g.x$. Our claim follows. \square

LEMMA 9.8. *Suppose that G is a group acting uniformly equicontinuously on a cofinite graph Γ and give G the induced uniformity as in Lemma 9.4. Then the action $G \times \Gamma \rightarrow \Gamma$ is uniformly continuous.*

PROOF. Let R be a G -invariant cofinite entourage over Γ . Now let $((g, x), (h, y)) \in N_R \times R$, i.e. $(g, h) \in N_R, (x, y) \in R$. Since $x \in \Gamma, (gx, hx) \in R$ which implies that $(h^{-1}gx, x) \in R$. We have $(h^{-1}gx, y) \in R$ and hence $(gx, hy) \in R$. Thus our claim follows. \square

Now if $R \leq S$ in I , then $S \subseteq R$. Let $(g_1, g_2) \in N_S$. Then $(g_1x, g_2x) \in S$, for all $x \in \Gamma$ and hence $(g_1x, g_2x) \in R$, for all $x \in \Gamma$ which implies $(g_1, g_2) \in N_R$. Thus $N_S \subseteq N_R$. For all $R \leq S$, in I , let us define $\psi_{RS}: G/N_S \rightarrow G/N_R$ via $\psi_{RS}(N_S[g]) = N_R[g]$. Then ψ_{RS} is a well defined uniformly continuous group isomorphism, as each of $G/N_R, G/N_S$ are finite discrete groups. If $R = S$, then $\psi_{RR} = id_{G/N_R}$. And if $R \leq S \leq T$, then $\psi_{RS}\psi_{ST} = \psi_{RT}$. Then $\{G/N_R \mid R \in I, \psi_{RS}, R \leq S \in I\}$, forms an inverse system of finite discrete groups. Let $\hat{\Gamma} = \varprojlim_{R \in I} \Gamma/R$ and $\hat{G} = \varprojlim_{R \in I} G/N_R$, where $\psi_R: \hat{G} \rightarrow G/N_R$ is the corresponding canonical projection map.

Now if I_1, I_2 are two fundamental systems of G -invariant cofinite entourages over Γ , clearly I_1, I_2 will form fundamental systems of cofinite congruences, for two induced uniformities, over G . Now let N_{R_1} be a cofinite congruence over G for some $R_1 \in I_1$. Then there exists a R_2 , cofinite entourage over Γ , such that $R_2 \in I_2$ and $R_2 \subseteq R_1$. Hence $N_{R_2} \subseteq N_{R_1}$. Now let N_{S_2} be a cofinite congruence over G for some $S_2 \in I_2$. Then there exists S_1 , cofinite entourage over Γ , such that $S_1 \in I_1$ and $S_1 \subseteq S_2$. Hence $N_{S_1} \subseteq N_{S_2}$. Thus any cofinite congruence corresponding to the directed set I_1 is a cofinite congruence corresponding to the directed set I_2 and vice versa. Thus the two induced uniform structures over G are equivalent and so the completion of G with

respect to the induced uniformity, from the cofinite graph Γ , is unique up to both algebraic and topological isomorphism.

THEOREM 9.9. *If G acts on Γ , as in Lemma 9.4, faithfully then \widehat{G} acts on $\widehat{\Gamma}$ uniformly equicontinuously.*

PROOF. The group G acts on Γ uniformly equicontinuously. We fix a G -invariant orientation $E^+(\Gamma)$ of Γ . By Lemma 9.8 the action is uniformly continuous as well. Let $\chi: G \times \Gamma \rightarrow \Gamma$ be this group action. Now since Γ is topologically embedded in $\widehat{\Gamma}$ by the inclusion map, say, i , the map $i \circ \chi: G \times \Gamma \rightarrow \widehat{\Gamma}$ is a uniformly continuous. Then there exists a unique uniformly continuous map $\widehat{\chi}: \widehat{G} \times \widehat{\Gamma} \rightarrow \widehat{\Gamma}$ that extends χ . We claim that $\widehat{\chi}$ is the required group action.

We can take $\widehat{\Gamma} = \varprojlim \Gamma/R$ and $\widehat{G} = \varprojlim G/N_R$, where R runs throughout all G -invariant compatible cofinite entourages of Γ that are orientation preserving. Then $\widehat{G} \times \widehat{\Gamma} = \varprojlim (G/N_R \times \Gamma/R)$ and $G \times \Gamma$ is defined coordinatewise via $(N_R[g_R])_R \cdot (R[x_R])_R = (R[g_R \cdot x_R])_R$.

If possible let, $((N_R[g_R])_R, (R[x_R])_R) = ((N_R[h_R])_R, (R[y_R])_R)$.

So, $N_R[g_R] = N_R[h_R]$ and $R[x_R] = R[y_R]$, for all $R \in I$, $(g_R, h_R) \in N_R$, $(x_R, y_R) \in R$. This implies that $(g_R \cdot x_R, h_R \cdot x_R) \in R$ which further ensures that $(h_R^{-1} g_R \cdot x_R, x_R) \in R$. Then $(h_R^{-1} g_R \cdot x_R, y_R) \in R$ and $(g_R \cdot x_R, h_R \cdot y_R) \in R$. Hence $(R[g_R \cdot x_R])_R = (R[h_R \cdot y_R])_R$. So, the action is well defined.

Let $g = (N_R[g_R])_R, h = (N_R[h_R])_R \in \widehat{G}, x = (R[x_R])_R \in \widehat{\Gamma}$. Now $h \cdot (g \cdot x) = h \cdot (R[g_R \cdot x_R])_R = (R[h_R g_R \cdot x_R])_R = (N_R[h_R g_R])_R \cdot x = (hg) \cdot x$. Hence the action is associative.

Now $(N_R[1])_R \cdot (R[x_R])_R = (R[1x_R])_R = (R[x_R])_R$. Furthermore for all $v = (R[v_R])_R \in V(\widehat{\Gamma})$ and for all $g = (N_R[g_R])_R \in \widehat{G}, g \cdot v = (R[g_R \cdot v_R])_R \in V(\widehat{\Gamma})$ as each $g_R \cdot v_R \in V(\Gamma)$. Similarly, for all $e = (R[e_R])_R \in E(\widehat{\Gamma})$ and for all $g = (N_R[g_R])_R$ in $\widehat{G}, g \cdot e = (R[g_R e_R])_R$ in $E(\widehat{\Gamma})$.

For all $e = (R[e_R])_R \in E(\widehat{\Gamma})$, for all $g = (N_R[g_R])_R \in \widehat{G}$, we have $s(g.e) = s((R[g_R e_R])_R)$ and so $(R[g_R s(e_R)])_R = (g.(R[s(e_R)]))_R = g.s(e)$. Hence the properties $t(g.e) = g.t(e)$ and $\overline{g.e} = g.\overline{e}$ follow similarly.

Finally, let $E^+(\widehat{\Gamma})$ consists of all the edges $(R[e_R])_R$, where $e_R \in E^+(\Gamma)$. Since each R is orientation preserving, it follows that $E^+(\widehat{\Gamma})$ is an orientation of $\widehat{\Gamma}$. Since $E^+(\Gamma)$ is G -invariant, we see that $E^+(\widehat{\Gamma})$ is $\widehat{\Gamma}$ -invariant.

Hence this is a well defined group action. Also for all $g \in G$, and $x \in \Gamma$, $(N_R[g])_R.(R[x])_R = (R[g.x])_R = g.x \in \Gamma$. Thus the restriction of this group action agrees with the group action χ .

Now $\{R \mid R \in I\}, \{N_R \mid R \in I\}$ is a fundamental system of cofinite entourages over Γ , is a fundamental system of cofinite congruences over G . Hence $\{\overline{R} \mid R \in I\}$ is a fundamental system of cofinite entourages over $\widehat{\Gamma}$ and $\{\overline{N_R} \mid R \in I\}$ is a fundamental system of cofinite congruences over \widehat{G} respectively.

Let us now see that the aforesaid group action is uniformly continuous. For let us consider the group action $G/N_R \times \Gamma/R \rightarrow \Gamma/R$ defined via $N_R[g]R[x] = R[g.x]$, which is uniformly continuous as both $G/N_R \times \Gamma/R$ and Γ/R are finite discrete uniform topological spaces. Hence the group action, $\widehat{G} \times \widehat{\Gamma} \rightarrow \widehat{\Gamma}$ is uniformly continuous. Thus the aforesaid group action is our choice of $\widehat{\chi}$, by the uniqueness of $\widehat{\chi}$.

So the restriction of the aforesaid action $\{\widehat{g}\} \times \widehat{\Gamma} \rightarrow \widehat{\Gamma}$ is a uniformly continuous map of graphs, for all $\widehat{g} \in \widehat{G}$.

We check that for all $(x, y) \in R$ and for all $\widehat{g} \in \widehat{G}$, $(\widehat{g}.x, \widehat{g}.y) \in \overline{R}$. For, let $\widehat{g} = (N_R[g_R])_R \in \widehat{G}$ and for $x, y \in \Gamma$, $((R[x])_R, (R[y])_R) \in R$. Now $\overline{R}[(R[g_R.x])_R] = \overline{R}[g_R.x] = \overline{R}[g_R.y] = \overline{R}[(R[g_R.y])_R]$.

So, $((N_R[g_R])_R.(R[x])_R, (N_R[g_R])_R.(R[y])_R) \in \overline{R}$. This implies that $(\widehat{g} \times \widehat{g})[R] \subseteq \overline{R}$. Thus for all $\widehat{g} \in \widehat{G}$, $(\widehat{g} \times \widehat{g})[\overline{R}] \subseteq \overline{\widehat{g} \times \widehat{g}[R]} \subseteq \overline{\overline{R}} = \overline{R}$. Hence \overline{R} is \widehat{G} invariant. \square

Thus $\Phi_1 = \{N_{\overline{R}} \mid R \in I\}$ and $\Phi_2 = \{\overline{N_R} \mid R \in I\}$ form fundamental systems of cofinite congruences over \widehat{G} . Let $\tau_{\Phi_1}, \tau_{\Phi_2}$ be the topologies induced by Φ_1, Φ_2 respectively.

THEOREM 9.10. *The uniformities on \widehat{G} obtained by Φ_1 and Φ_2 are equivalent.*

PROOF. Let us first show that $N_{\overline{R}} \cap G \times G = N_R$.

For, let $(g, h) \in N_R$. Then for all $x \in \Gamma$, $(g.x, h.x) \in R \subseteq \overline{R}$. Now let $(R[x_R])_R \in \widehat{\Gamma}$. Then $\overline{R}[g(R[x_R])_R] = \overline{R}[g.x_R] = \overline{R}[h.x_R] = \overline{R}[h(R[x_R])_R]$ which implies that $(g, h) \in N_{\overline{R}} \cap G \times G$. Thus, $N_R \subseteq N_{\overline{R}} \cap G \times G$. Again, if $(g, h) \in N_{\overline{R}} \cap G \times G$, then for all $x \in \Gamma \subseteq \widehat{\Gamma}$, and so $(g.x, h.x) \in \overline{R} \cap \Gamma \times \Gamma = R$ and this implies $(g, h) \in N_R$. Our claim follows.

Then as uniform subgraphs $(G, \tau_{\Phi_1}) \cong (G, \tau_{\Phi_2})$, both algebraically and topologically, their corresponding completions $(\widehat{G}, \tau_{\Phi_1}) \cong (\widehat{G}, \tau_{\Phi_2})$, both algebraically and topologically.

Since for all $S \in I$, $\psi_S: G \rightarrow G/N_S$ is a uniform continuous group homomorphism and G/N_S is discrete, there exists a unique uniform continuous extension of ψ_S , namely, $\widehat{\psi}_S: \widehat{G} \rightarrow G/N_S$. Let us define $\lambda_S: \widehat{G} \rightarrow G/N_S$ via $\lambda_S(g) = N_S[g_S]$, where $g = (N_R[g_R])_R$. Now let $g = (N_R[g_R])_R, h = (N_R[h_R])_R \in \widehat{G}$ be such that $g = h$ which implies that $N_S[g_S] = N_S[h_S]$ and hence λ_S is well defined. Now let $(g, h) \in N_{\overline{S}}$. First of all $N_{\overline{S}}[g_S] = N_{\overline{S}}[g] = N_{\overline{S}}[h] = N_{\overline{S}}[h_S]$. So, $(g_S, h_S) \in N_{\overline{S}} \cap G \times G = N_S$. Hence $N_S[g_S] = N_S[h_S]$ which implies that $\lambda_S(g) = \lambda_S(h)$, so $(\lambda_S(g), \lambda_S(h)) \in D(G/N_R)$. Thus $N_{\overline{S}} \subseteq (\lambda_S \times \lambda_S)^{-1}D(G/N_R)$. Hence λ_S is uniformly continuous. Now for all $g, h \in \widehat{G}$, $\lambda_S(gh) = N_S[g_S h_S] = N_S[g_S]N_S[h_S] = \lambda_S(g)\lambda_S(h)$ and for all $g \in G$, $\lambda_S(g) = \lambda_S((N_R[g])_R) = N_S[g] = \psi_S(g)$. Thus λ_S is an well defined uniformly continuous group homomorphism that extends ψ_S . Then by the uniqueness of the extension, $\widehat{\psi}_S = \lambda_S$.

Now $N_{\overline{S}}$ is a closed subspace of \widehat{G} , then $\overline{N_{\overline{S}} \cap G \times G} = \overline{N_S}$ which implies that $\overline{N_S} \subseteq \overline{N_{\overline{S}}} = N_{\overline{S}}$

Let us define $\theta: \widehat{G}/N_{\overline{S}} \rightarrow G/N_S$ via $\theta(N_{\overline{S}}[g]) = N_S[g_S]$, where $g = (N_R[g_R])_R$. Now $N_{\overline{S}}[g] = N_{\overline{S}}[h]$ in $\widehat{G}/N_{\overline{S}}$ will imply $(g_S, h_S) \in N_{\overline{S}}$ and this implies for all $x \in X$, $(g_S x, h_S x) \in \overline{S} \cap \Gamma \times \Gamma = S$. Thus $(g_S, h_S) \in N_S$. Then $\theta(N_{\overline{S}}[g]) = N_S[g_S] = N_S[h_S] = \theta(N_{\overline{S}}[h])$. Hence θ is well defined. On the other hand let's take $N_{\overline{S}}[g], N_{\overline{S}}[h]$ be such that $\theta(N_{\overline{S}}[g]) = \theta(N_{\overline{S}}[h])$. Thus $N_S[g_S] = N_S[h_S]$ so that $(g_S, h_S) \in N_S \subseteq N_{\overline{S}}$. Hence $N_{\overline{S}}[g] = N_{\overline{S}}[g_S] = N_{\overline{S}}[h_S] = N_{\overline{S}}[h]$. So, θ is injective as well. Also for all $N_S[g] \in G/N_S$ there exists $N_{\overline{S}}[g] \in \widehat{G}/N_{\overline{S}}$ such that $\theta(N_{\overline{S}}[g]) = N_S[g]$. So θ is surjective. Finally, $\theta(N_{\overline{S}}[g]N_{\overline{S}}[h]) = \theta(N_{\overline{S}}[gh]) = N_S[g_S h_S] = N_S[g_S]N_S[h_S] = \theta(N_{\overline{S}}[g])\theta(N_{\overline{S}}[h])$. So θ is an well defined group isomorphism, both algebraically and topologically. Hence $\widehat{G}/N_{\overline{S}} \cong G/N_S \cong \widehat{G}/\overline{N_S}$ which implies that $|\widehat{G}/N_{\overline{S}}[1]| = |\widehat{G}/\overline{N_S}[1]|$. But since $\overline{N_S} \subseteq N_{\overline{S}}$ one obtains $\overline{N_S}[1] \leq N_{\overline{S}}[1] \leq \widehat{G}$ and thus $|\widehat{G}/N_{\overline{S}}[1]| |N_{\overline{S}}[1] : \overline{N_S}[1]| = |\widehat{G}/\overline{N_S}[1]|$. Hence $|N_{\overline{S}}[1] : \overline{N_S}[1]| = 1$ which implies that $N_{\overline{S}}[1] = \overline{N_S}[1]$ and thus $N_{\overline{S}} = \overline{N_S}$ as each of them are congruences. Thus our claim. \square

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APPENDIX A

Binary Relations

For the readers convenience, this appendix contains some of the basic theory of binary relations and uniform structures on topological spaces. Many proofs are omitted. See [1] and [3] for more details. Also we review Hartley treatment of cofinite groups; see [2] for more details.

1.1. Binary relations

Let X and Y be sets and let $R \subseteq X \times Y$. Such a subset R is called a *binary relation* from X to Y . For any $x \in X$, we write $R[x] = \{y \in Y \mid (x, y) \in R\}$. More generally, for any subset A of X , let $R[A] = \bigcup\{R[a] \mid a \in A\}$.

The *inverse* of a binary relation $R \subseteq X \times Y$ is the binary relation $R^{-1} \subseteq Y \times X$ given by $R^{-1} = \{(y, x) \mid (x, y) \in R\}$. The *composition* of binary relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ is the binary relation $SR \subseteq X \times Z$ given by

$$SR = \{(x, z) \mid \text{there exists } y \in Y \text{ such that } (x, y) \in R \text{ and } (y, z) \in S\}.$$

The *diagonal* $D(X) = \{(x, x) \mid x \in X\}$ in $X \times X$ is the “equality” binary relation on X . For any relation $R \subseteq X \times Y$, note that the compositions $RD(X) = R$ and $D(Y)R = R$.

A.1. *Composition of binary relations is an associative operation: if $R_i \subseteq X_i \times X_{i+1}$ is a binary relation for $i = 1, 2, 3$ then*

$$(R_3R_2)R_1 = R_3(R_2R_1).$$

PROOF. Let (x_1, x_4) be in $(R_3R_2)R_1$. Then there exists $x_2 \in X_2$ such that $(x_1, x_2) \in R_1$ and $(x_2, x_4) \in R_3R_2$. So there exists $x_3 \in X_3$ such that $(x_2, x_3) \in R_2$ and $(x_3, x_4) \in R_3$. This implies $(x_1, x_3) \in R_2R_1$ and $(x_3, x_4) \in R_3$ then $(x_1, x_4) \in R_3(R_2R_1)$. On the other hand let $(y_1, y_4) \in R_3(R_2R_1)$. Then there exists $y_3 \in X_3$ such that $(y_1, y_3) \in R_2R_1$ and $(y_3, y_4) \in R_3$. Thus there exists $y_2 \in X_2$ such that $(y_1, y_2) \in R_1$ and $(y_2, y_3) \in R_2$ So $(y_2, y_4) \in R_3R_2$ and $(y_1, y_2) \in R_1$ and hence $(y_1, y_4) \in (R_3R_2)R_1$. \square

A.2. Let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be binary relations. Then

(1) $(SR)^{-1} = R^{-1}S^{-1}$.

(2) for any subset A of X , we have that $(SR)[A] = S[R[A]]$.

PROOF. The proofs of the above note is pretty straight forward.

(1) $(z, x) \in (SR)^{-1} \Leftrightarrow (x, z) \in SR \Leftrightarrow \exists y \in Y$ such that $(x, y) \in R$ and $(y, z) \in S \Leftrightarrow$ for some $y \in Y, (z, y) \in S^{-1}$ and $(y, x) \in R^{-1} \Leftrightarrow (z, x) \in R^{-1}S^{-1}$.

(2) Let $z \in (SR)[A]$. Then z is in $\bigcup_{a \in A} (SR)[a]$, so there exists $a' \in A$, such that $(a', z) \in SR$. This implies there exists $y \in Y$ such that $(a', y) \in R$ and $(y, z) \in S$. Hence $y \in R[a'] \subseteq \bigcup_{a \in A} R[a] = R[A]$ and thus $z \in S[y] \subseteq \bigcup_{t \in R[A]} S[t] = S[R[A]]$. Conversely, let $z \in S[R[A]]$. Then $z \in \bigcup_{t \in R[A]} S[t]$. Then there exists $t' \in R[A]$ such that $z \in S[t']$. Hence $(t', z) \in S$ and $(a'', t') \in R$, for some $a'' \in A$, so $(a'', z) \in SR$ and thus $z \in SR[a''] \subseteq \bigcup_{a \in A} SR[a] = SR[A]$. \square

Let $T_i \subseteq X_i \times Y_i$ be a binary relation for $i = 1, 2$. Then we denote by $T_1 \times T_2$ the binary relation from $X_1 \times X_2$ to $Y_1 \times Y_2$ consisting of all pairs $((x_1, x_2), (y_1, y_2))$ such that $(x_i, y_i) \in T_i$ for $i = 1, 2$.

A.3. If $R \subseteq X_1 \times X_2$ is a binary relation, then $S = (T_1 \times T_2)[R]$ is a binary relation from Y_1 to Y_2 and

$$S = (T_1 \times T_2)[R] = \bigcup \{T_1[x_1] \times T_2[x_2] \mid (x_1, x_2) \in R\} = T_2 R T_1^{-1}$$

$$\begin{array}{ccc} X_1 & \xrightarrow{R} & X_2 \\ T_1 \downarrow & & \downarrow T_2 \\ Y_1 & \xrightarrow{S} & Y_2 \end{array}$$

PROOF. The above claims hold as

$$(t, s) \in S$$

$$\Rightarrow (t, s) \in \bigcup_{(x_1, x_2) \in R} T_1 \times T_2[(x_1, x_2)]$$

$$\Rightarrow ((x_1, x_2), (t, s)) \in T_1 \times T_2$$

$$\Rightarrow (x_1, t) \in T_1, (x_2, s) \in T_2$$

$$\Rightarrow (t, s) \in T_1[x_1] \times T_2[x_2]$$

$$\Rightarrow (t, s) \in \bigcup_{(x_1, x_2) \in R} T_1[x_1] \times T_2[x_2].$$

$$\text{Conversely, } (t, s) \in \bigcup_{(x_1, x_2) \in R} T_1[x_1] \times T_2[x_2]$$

$$\Rightarrow \exists (y_1, y_2) \in R \text{ such that } (t, s) \in T_1[y_1] \times T_2[y_2]$$

$$\Rightarrow ((y_1, y_2), (t, s)) \in T_1 \times T_2$$

$$\Rightarrow (t, s) \in T_1 \times T_2[(y_1, y_2)] \subseteq S.$$

$$\text{Also, } (t, s) \in \bigcup_{(x_1, x_2) \in R} T_1[x_1] \times T_2[x_2]$$

$$\Leftrightarrow \exists (x'_1, x'_2) \in R \text{ such that } (t, s) \in T_1[x'_1] \times T_2[x'_2]$$

$$\Leftrightarrow \exists x'_1 \in X_1 \text{ and } x'_2 \in X_2, \text{ such that } (x'_1, t) \in T_1, (x'_2, s) \in T_2, (x'_1, x'_2) \in R$$

$$\Leftrightarrow (t, x'_1) \in (T_1)^{-1}, (x'_2, s) \in T_2, (x'_1, x'_2) \in R$$

$$\Leftrightarrow (t, x'_2) \in R(T_1)^{-1} \text{ and } (x'_2, s) \in T_2 \text{ for some } x'_2 \in X_2$$

$$\Leftrightarrow (t, s) \in T_2 R(T_1)^{-1}$$

□

Equivalence relations. Let X be a set. A *binary relation* on X is a subset $R \subseteq X \times X$. A binary relation R on X is called an *equivalence relation* if it satisfies three properties:

- (1) Reflexive: R contains the diagonal $D(X) = \{(x, x) \mid x \in X\}$.
- (2) Symmetric: $R^{-1} = R$.
- (3) Transitive: $R^2 \subseteq R$.

It follows that if R is an equivalence relation, then $R^2 = R$. To see this suppose $(x, y) \in R$. Then this implies that $(x, y), (y, y) \in R$ by reflexivity of R and thus $(x, y) \in R^2$ by transitivity of R .

A.4. Let $(R_i \mid i \in I)$ be a family of equivalence relations on a set X . Then the intersection $\bigcap_{i \in I} R_i$ is also an equivalence relation on X .

It follows that every relation S on X is contained in a unique smallest equivalence relation—namely, the intersection of all equivalence relations that contain S . We denote it by $\langle S \rangle$ and call it the *equivalence relation generated by S* .

A.5. Let R_1 and R_2 be equivalence relations on a set X . Then R_1R_2 is an equivalence relation if and only if $R_2R_1 = R_1R_2$. In this case, $R_2R_1 = \langle R_1 \cup R_2 \rangle = R_1R_2$.

PROOF. Let us first assume R_1R_2 is an equivalence relation on X . Now $(x, y) \in (R_2R_1)$

$$\Leftrightarrow \exists z \in X \text{ such that } (x, z) \in R_1 \text{ and } (z, y) \in R_2$$

$$\Leftrightarrow \exists z \in X, \text{ such that } (z, x) \in R_1 \text{ and } (y, z) \in R_2$$

$$\Leftrightarrow (y, x) \in R_1R_2$$

$$\Leftrightarrow (x, y) \in R_1R_2, \text{ since } (R_1R_2)^{-1} = R_1R_2.$$

Conversely, let us take $R_2R_1 = R_1R_2$. Then (x, x) is in $R_1 \cap R_2$, for all $x \in X$

$$\Rightarrow (x, x) \in R_2R_1 \text{ and thus } D(X) \subseteq R_2R_1.$$

$$\text{Also, } (R_2R_1)^{-1} = R_1^{-1}R_2^{-1} = R_1R_2 = R_2R_1.$$

Finally, $(R_2R_1)^2 = R_2(R_1R_2)R_1 = R_2(R_2R_1)R_1 = (R_2)^2(R_1)^2 \subseteq R_2R_1$. Hence R_2R_1 is an equivalence relation.

Let's now check that $R_2R_1 = \langle R_1 \cup R_2 \rangle$.

Let (x, y) be in R_2R_1 . Then there exists $z \in X$ such that $(x, z) \in R_1$ and $(z, y) \in R_2$. Then $(x, z), (z, y) \in R_1 \cup R_2$ and so $(x, y) \in \langle R_1 \cup R_2 \rangle$. So, $R_2R_1 \subseteq \langle R_1 \cup R_2 \rangle$.

On the other hand since R_2R_1 is an equivalence relation that contains both of R_2 and R_1 , one can see that R_2R_1 contains $R_2 \cup R_1$. But $\langle R_1 \cup R_2 \rangle$ is the smallest equivalence relation containing $R_2 \cup R_1$. So we get $\langle R_1 \cup R_2 \rangle \subseteq R_2R_1$. Hence $R_2R_1 = \langle R_1 \cup R_2 \rangle$. \square

A.6 (Modular Law). *Let R, R_1 , and R_2 be equivalence relations on a set X such that $R \subseteq R_1$. Then $R(R_1 \cap R_2) = R_1 \cap RR_2$.*

PROOF. Suppose $(x, y) \in R(R_1 \cap R_2)$. This implies that there exists $z \in X$ such that $(x, z) \in R_1$ and $(x, z) \in R_2$; also $(z, y) \in R \subseteq R_1$. Hence $(x, y) \in R_1$ and $(x, y) \in RR_2$ implying that $(x, y) \in R_1 \cap RR_2$. On the other hand assume that $(p, q) \in R_1 \cap RR_2$. So there exists $r \in X$ such that $(p, r) \in R_2$ and $(r, q) \in R \subseteq R_1$. Also $(p, q) \in R_1$. Since $(p, q), (q, r) \in R_1$ it follows that $(p, r) \in R_1 \cap R_2$. Hence $(p, q) \in R(R_1 \cap R_2)$. \square

A.7. *Let X, Y be two sets and $f: X \rightarrow Y$ be a function of sets. Let $f = \{(x, y) \in X \times Y \mid f(x) = y\}$ and $f^{-1} = \{(y, x) \in Y \times X \mid f(x) = y\}$. Then the following properties are true,*

- $(f \times f)[S] = fSf^{-1}$, for all relations S on X and $(f \times f)^{-1}[R] = f^{-1}Rf$, for all relations R over Y . (This is a particular case of A.3)
- If $K_f = \{(x_1, x_2) \in X \times X \mid f(x_1) = f(x_2)\}$ then $K_f = (f^{-1})f = (f \times f)^{-1}[D(Y)]$ is an equivalence relation.

- $f(f^{-1}) \subseteq D(Y)$

PROOF. For, $(y_1, y_2) \in (f \times f)[S] \Leftrightarrow \exists(x_1, x_2) \in S$ such that $f(x_1) = y_1$ and $f(x_2) = y_2 \Leftrightarrow (y_1, x_1) \in f^{-1}, (x_1, x_2) \in S, (x_2, y_2) \in f \Leftrightarrow (y_1, y_2) \in fSf^{-1}$.

Now $(x_1, x_2) \in (f \times f)^{-1}[R] \Leftrightarrow \exists(y_1, y_2) \in R$ such that $f(x_1) = y_1$ and $f(x_2) = y_2 \Leftrightarrow (x_1, y_1) \in f, (y_1, y_2) \in R, (y_2, x_2) \in f^{-1} \Leftrightarrow (x_1, x_2) \in f^{-1}Rf$.

Finally, $(x, f(x)) \in f$ and $(f(x), x) \in f^{-1}$ which implies that $(x, x) \in (f^{-1})f$. So $D(X) \subseteq f^{-1}f$. Now $(f \times f)^{-1}[D(Y)] = f^{-1}(D(Y)f) = f^{-1}f$. Also clearly, $(x_1, x_2) \in (f^{-1})f \Leftrightarrow \exists y \in Y$ such that $(x_1, y) \in f$ and $(y, x_2) \in f^{-1} \Leftrightarrow f(x_1) = y = f(x_2) \Leftrightarrow (x_1, x_2) \in K_f$.

Now $(y_1, y_2) \in f(f^{-1})$ which implies that there exists $x \in X$, such that $(y_1, x) \in f^{-1}$ and $(x, y_2) \in f$. So $y_1 = f(x) = y_2$. This implies that $(y_1, y_2) \in D(Y)$. \square

THEOREM A.8 (Correspondence Theorem). *If $f: X \rightarrow Y$ is a map, then $(f \times f)^{-1}[R]$ is an equivalence relation on X , for all equivalence relations R over Y and if f is a surjection, also $(f \times f)[S]$ is an equivalence relation on Y , for all equivalence relations S over X , that contain K_f . Moreover if R and S have finitely many equivalence classes in Y and X respectively then $(f \times f)^{-1}[R]$ and $(f \times f)[S]$ have finitely many equivalence classes, in X and Y , respectively.*

PROOF. For all $x \in X, (x, x) \in (f \times f)^{-1}[R]$ as $(f(x), f(x)) \in D(Y) \subseteq R$. Also $((f \times f)^{-1}[R])^{-1} = (f^{-1}Rf)^{-1} = f^{-1}R^{-1}(f^{-1})^{-1} = f^{-1}Rf = (f \times f)^{-1}[R]$ and $((f \times f)^{-1}[R])^2 = f^{-1}Rff^{-1}Rf \subseteq f^{-1}RD(Y)Rf = f^{-1}R^2f = f^{-1}Rf = (f \times f)^{-1}[R]$. Hence $(f \times f)^{-1}[R]$ is an equivalence relation over X .

Next let, $|Y/R| < \infty$. Let us define ζ from $X/(f \times f)^{-1}[R]$ to Y/R via

$$\zeta((f \times f)^{-1}[R][x]) = R[f(x)].$$

We have $(f \times f)^{-1}[R][x] = (f \times f)^{-1}[R][y] \Leftrightarrow (x, y) \in (f \times f)^{-1}[R] \Leftrightarrow (f(x), f(y)) \in R \Leftrightarrow R[f(x)] = R[f(y)]$. Hence ζ is an well defined injection and

thus

$$|X/(f \times f)^{-1}[R]| \leq |Y/R| < \infty.$$

Now if f is surjection, then for all $y \in Y$, there exists $x \in X$ such that $f(x) = y$. So $(y, y) \in (f \times f)[S]$, for all $y \in Y$, since $(x, x) \in S$, for all $x \in X$. Next $((f \times f)[S])^{-1} = (fSf^{-1})^{-1} = (f^{-1})^{-1}S^{-1}f^{-1} = fSf^{-1} = (f \times f)[S]$. Lastly, $((f \times f)[S])^2 = fSf^{-1}fSf^{-1} = fSK_fSf^{-1} \subseteq fSSSf^{-1} = fS^3f^{-1} = fSf^{-1} = (f \times f)[S]$. Hence $(f \times f)[S]$ is an equivalence relation over Y .

Now let $|X/S| < \infty$. Let us define a map χ from X/S to $Y/(f \times f)[S]$ via

$$\chi(S[x]) = (f \times f)[S][f(x)]$$

Now $S[x] = S[y]$ implies that $(x, y) \in S$. So $(f(x), f(y)) \in (f \times f)[S]$, then $(f \times f)[S][f(x)] = (f \times f)[S][f(y)]$. Thus χ is a well defined map and for all $(f \times f)[S][y] \in Y/(f \times f)[S]$, there exists $x \in X$ such that $f(x) = y$ and thus $\chi(S[x]) = (f \times f)[S][f(x)] = (f \times f)[S][y]$. Hence χ is a surjection. Thus

$$|Y/(f \times f)[S]| = |Im(\chi)| \leq |X/S| < \infty$$

Hence our claim. □

APPENDIX B

Uniform Topological Spaces

DEFINITION B.1. A uniform space (X, Φ) is a set X along with a non-empty family Φ of subsets of $X \times X$, that satisfy the following:

- (1) If $U \in \Phi$ then U contains the diagonal $D(X) = \{(x, x) \in X \times X \mid x \in X\}$ of X .
- (2) If $U \in \Phi$ and $V \subseteq X \times X$ such that $U \subseteq V$, then $V \in \Phi$.
- (3) If $U \in \Phi$, $V \in \Phi$ then $U \cap V \in \Phi$.
- (4) If $U \in \Phi$, then there exists $V \in \Phi$ such that for all $(x, y), (y, z) \in V, (x, z) \in U$.
- (5) If $U \in \Phi$ then $U^{-1} = \{(y, x) \in X \times X \mid (x, y) \in U\} \in \Phi$.

Φ is called the uniform structure or uniformity of X and its elements are called entourages. We define $U[x] = \{y \in X \mid (x, y) \in U\}$, for all $U \in \Phi$ and $x \in X$. If $(x, y) \in U$ we say that x and y are U -close. If $A \subseteq X$ is such that for all $(x, y) \in A \times A, x, y$ are U -close we say that A is U -small. An entourage will be symmetric iff $U = U^{-1}$.

A **fundamental system** of entourages for a uniformity Φ over X is a subfamily $\Psi \subseteq \Phi$ such that for all $U \in \Phi$, there exists $V \in \Psi$ with $V \subseteq U$. So given any fundamental system of entourages, $\Psi, \Phi = \{U \subseteq X \times X \mid V \subseteq U, \text{ for some } V \in \Psi\}$ is the uniform system corresponding to or generated by Ψ .

Every uniform space has a fundamental system of symmetric entourages.

Topology on Uniform Spaces. Every uniform space (X, Φ) becomes a topological space by defining a subset O of X to be open if and only if for every x in

O there exists an entourage $V \in \Phi$ such that $x \in V[x] \subseteq O$. In this topology the neighborhood filter of a point x is $\{V[x] \mid V \in \Phi\}$.

The topology defined by a uniform structure is said to be induced by the uniformity. A uniform structure on a topological space is compatible with the topology if the topology induced by the uniformity coincides with the original topology.

Let (X, Φ) be a uniform space. Then for any non empty subset A of X , $\Phi_A = \{U \cap A \times A \mid U \in \Phi\}$ forms a uniformity over A and the subspace topology on A inherited from (X, Φ) agrees with the topology over A induced by Φ_A

Uniformizable Space: A topological space is called is uniformizable iff there is a uniform structure that induces the same topology on the space.

B.2. *A topological space is uniformizable iff it is completely regular.*

So in particular every compact Hausdorff space is uniformizable.

Moreover a continuous function $f: X \rightarrow Y$, where (X, Φ_X) , (Y, Φ_Y) , are uniform spaces, that pulls back entourages into entourages is called a **Uniformly Continuous Function**, i.e. for all $V \in \Phi_Y$, there exists $U \in \Phi_X$ such that $U \subseteq (f \times f)^{-1}[V]$.

DEFINITION B.3. *A map between uniform spaces is called an **isomorphism of uniform spaces** if both f and f^{-1} are uniformly continuous bijections.*

We state the following well known results without proof.

B.4. *A composition of uniformly continuous functions is uniformly continuous. Let $f: (X, \Phi_X) \rightarrow (Y, \Phi_Y)$ be a uniformly continuous function and let $\tau_{\Phi_X}, \tau_{\Phi_Y}$ be the topologies induced by Φ_X, Φ_Y respectively. Then $f: (X, \tau_{\Phi_X}) \rightarrow (Y, \tau_{\Phi_Y})$ is a continuous map. Moreover if f is an isomorphism of uniform spaces then f is a homeomorphism.*

B.5. *For a uniformizable space the following are equivalent:*

- (1) X is T_0
- (2) X is T_1
- (3) X is T_2
- (4) X is $T_{3\frac{1}{2}}$
- (5) For any compatible uniformity Φ over X , we have $\bigcap\{U \mid U \in \Phi\} = D(X)$

B.6. Let X be a topological space. Let O be open in $X \times X$. Then for all $x \in X$, $O[x] = \{y \in X \mid (x, y) \in O\}$ is open in X .

B.7. Let X be a compact Hausdorff topological space, then $\Phi = \{U \subseteq X \times X \mid D(X) \subseteq O \subseteq U \text{ for some open set } O \subseteq X \times X\}$ is the unique uniformity over X which admits the original topology on X .

B.8. Let $f: X \rightarrow Y$ be a continuous map where both X, Y are compact Hausdorff spaces. Then f is uniformly continuous with respect to the unique induced uniformities over X and Y , which admits the corresponding topologies.

Completeness.

DEFINITION B.9. **Cauchy Filter:** For an uniform space (X, Φ) a Cauchy filter Γ is a filter over X such that for all $U \in \Phi$ there exists $A \in \Gamma$ such that $A \times A \subseteq U$. A Cauchy filter is called minimal if it contains no smaller Cauchy filter other than itself. We will say that a Cauchy Filter Γ converges to a point x in X iff for all $U \in \Phi$ there exists $F \in \Gamma$, such that $F \subseteq U[x]$. Hence $U[x] \in \Gamma$, thus the neighborhood base of x in (X, Φ) , say, $\eta_x = \{U[x] \mid U \in \Phi\} \subseteq \Gamma$.

A uniform space (X, Φ) is **complete** if and only if every Cauchy filter over it converges to some x in X .

B.10. *If a filter Γ over a uniform space (X, Φ) , converges to some point $x \in X$ then Γ is a Cauchy filter.*

B.11. *Every Cauchy filter over a uniform space (X, Φ) contains a unique minimal Cauchy filter.*

B.12. *A compact Hausdorff space X is complete.*

B.13. *Let (X, Φ) be a uniform space. Then the neighborhood filter η_x of x in X is a Cauchy filter that converges to x . Let A be a dense subset of X and x in X . Then $\Gamma_x = \{U \cap A \mid U \in \eta_x\}$ is a Cauchy filter in (A, Φ_A) and if $a \in A$, then Γ_a converges to a in A . Let $f: (X, \Phi) \rightarrow (Y, \Psi)$ be a uniformly continuous map and Γ a Cauchy filter over X . Then $\Gamma^f = \{V \subseteq Y \mid \text{there exists } U \in \Gamma \text{ with } f(U) \subseteq V\}$ is a Cauchy filter over Y . Moreover Γ^f converges to $f(x)$, whenever Γ converges to x , in X .*

B.14. *Let (X, Φ) be a uniform space and (Y, Ψ) be a complete Hausdorff uniform space. Let A be a dense subset of X . If $f: A \rightarrow Y$ is a uniformly continuous map then f can be uniquely extended to a uniformly continuous map $\hat{f}: X \rightarrow Y$ such that $\hat{f} \circ i_A = f$, where $i_A: A \rightarrow X$ is the canonical inclusion map.*

A topological space which can be made into a complete uniform space, so that the corresponding uniformity induces the original topology, is called a completely uniformizable space.

Hausdorff Completion. Let (X, Φ_X) be a Hausdorff uniform space, where τ_{Φ_X} is the corresponding uniform topology. Let $Y = \{\Gamma \mid \Gamma \text{ is a minimal Cauchy filter on } X\}$. Since for all x in X , η_x is in Y , $Y \neq \emptyset$.

Now let $C_V = \{(\Gamma, \Lambda) \mid \Gamma, \Lambda \in Y \text{ and } \exists F \in \Gamma \cap \Lambda, \text{ such that } F \times F \subseteq V, \text{ where } V \in \Phi_X \text{ is symmetric}\}$.

Let $\Phi_Y = \{U \subseteq Y \times Y \mid C_V \subseteq U \text{ where } V \in \Phi_X \text{ is symmetric}\}$. Then Φ_Y is a uniformity over Y and if τ_{Φ_Y} is the topology on Y induced by Φ_Y , then (Y, τ_{Φ_Y}) is Hausdorff. Furthermore (Y, Φ_Y) is a complete uniform space.

Now $i: X \rightarrow Y$ defined via $i(x) = \eta_x$, for all $x \in X$ is an embedding and is uniformly continuous as well. Also $i(X)$ is dense in (Y, Φ_Y) .

Let (Z, Φ_Z) be a complete Hausdorff uniform space and $f: X \rightarrow Z$ be a uniformly continuous map. Then there exists a unique uniformly continuous map $\hat{f}: Y \rightarrow Z$ such that $\hat{f} \circ i = f$.

Hence together, taking account of all of the above discussion we can lump it into a single very important result:

B.15. *Let (X, Φ) be a Hausdorff uniform space and (Z, Φ_Z) be a complete Hausdorff uniform space. Then there exists a complete uniform Hausdorff space Y such that X is densely embedded in Y and for any uniformly continuous map $f: X \rightarrow Z$, there exists a unique $\hat{f}: Y \rightarrow Z$ such that \hat{f} is uniformly continuous and extends f to Y .*

APPENDIX C

Profinite and Residually Finite Groups

Let G be a residually finite group and let Σ be a separating filter base of normal subgroups of finite index in G ,

By this we mean that Σ is a set of normal subgroups of finite index in G satisfying,

- (1) For each pair $N_1, N_2 \in \Sigma$, there exists a member N_3 of Σ such that N_3 is a subset of $N_1 \cap N_2$.
- (2) $\bigcap \{N \mid N \in \Sigma\} = 1$.

Then G can be given, in exactly one way, the structure of a Hausdorff topological group in which Σ is a basis of neighborhoods of the identity. The resulting topological group is referred to as a cofinite group. Thus a cofinite group is simply a topological group in which the identity has a basis of neighborhoods consisting of normal subgroups of finite index, which will necessarily be both open and closed. A topology on a group G determined by a set Σ of normal subgroups of finite index in G , satisfying (1) and (2) above is referred to as the cofinite topology. And when the resulting topological group is profinite we will call it a profinite topological group. Any compact topological group H containing a cofinite group G as a dense subgroup of itself is called a completion of G .

For the following results we refer to [2].

C.1. *Let G be a cofinite group contained as a dense subgroup of a topological group \widehat{G} , Let H be any compact topological group, and let $\phi: G \rightarrow H$ be a continuous homomorphism.*

Then:

- (1) ϕ can be extended uniquely to a continuous map $\bar{\phi}: \widehat{G} \rightarrow H$.
- (2) $\bar{\phi}$ is a group homomorphism.
- (3) $\bar{\phi}(\widehat{G}) = \overline{\phi(G)}$
- (4) $\bar{\phi}$ is injective iff ϕ is an algebraic and topological embedding. In that case, $\bar{\phi}$ determines an algebraic and topological isomorphism between \widehat{G} and $\bar{\phi}(\widehat{G})$.

C.2. Let G be an abstract residually finite group and let τ_1, τ_2 be cofinite topologies on G . Suppose that $\tau_1 \geq \tau_2$. Let G_i be a completion of G in the topology τ_i , where $i \in \{1, 2\}$. Then the identity map id_G on G extends uniquely to a continuous epimorphism $\bar{i}: G_1 \rightarrow G_2$. Furthermore \bar{i} is an isomorphism iff $\tau_1 = \tau_2$.

C.3. The completion of a residually finite group G is unique up to an isomorphism extending the identity map on G .

We will call the completion of the residually finite group G with respect to its finest cofinite topology the UNIVERSAL COMPLETION.

C.4. Let G be an abstract residually finite group, let \widehat{G} be a universal completion of G , and let \overline{G} be a completion of G in an arbitrary cofinite topology. Then there is a continuous epimorphism $\widehat{G} \rightarrow \overline{G}$ extending the identity map on G .

C.5. (Existence): Let G be a cofinite group and let N be a separating filter base of open normal subgroups of finite index in G . Then the profinite group $\widehat{G} = \varprojlim_{K \in N} G/K$ is a compact cofinite group and the natural map $G \rightarrow \widehat{G}$ that sends $x \in G$ to the element whose component in G/K is xK , embeds G as a dense subgroup of \widehat{G} .

C.6. Let G be a dense cofinite subgroup of a topological group H and let N be the set of all open normal subgroups of G . Then

- i. H is cofinite and $\{\overline{K} \mid K \in N\}$ is the set of all open normal subgroups of H .

ii. If $K \in N$, then $\overline{K} \cap G = K$.

C.7. Let G be a cofinite group. Then

i. If H is a subgroup of G , then H is cofinite (with the induced topology)

ii. If K is a topological group and $\phi: G \rightarrow K$ a continuous epimorphism, then K is cofinite and every open normal subgroup of K is the image of an open normal subgroup of G .