EXAMINING EVIDENCE OF METACOGNITION BY PRESERVICE SECONDARY MATHEMATICS TEACHERS WHILE SOLVING TASKS SITUATED IN THE SECONDARY CURRICULUM

by

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ABSTRACT

The study's goal was to examine the metacognitive practices and problem solving by preservice secondary mathematics teachers within secondary core content areas. The effort to analyze and describe the processes led to qualitative methods, a multiple case study. Transcripts of problem solving sessions and interviews were coded for the presence of multiple strategies, descriptive language, and use of metacognitive stages. From this coding, detailed accounts of the participants' problem solving were developed and analyzed.

Findings revealed conceptual and procedural obstacles existed when participants sought to find an efficient solution to some of the problem solving tasks. Half of the participants struggled with using algebraic symbolization to represent a relationship and all had difficulty describing the variability that existed within sets of data. Both of these skills are integral components of the current secondary curriculum. The analysis also revealed findings concerning the preservice teachers' mathematical practices. A study of the stages of problem solving suggested that the participants' had limited persistence in problem solving and restricted attention to reflection on the processes used. Similarly, an analysis of their language demonstrated incomplete development of concept definition and ineffective use of terminology in a mathematical context.

DEDICATION

This document and all that led to it is dedicated to my dad, Robert N. Stickney, Sr., who taught me it is never too late to dream, to learn, or to have goals.

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CHAPTER I:

INTRODUCTION

The preparation of secondary mathematics teachers is a concern for both colleges of education and their university mathematics departments particularly in light of the constant bombardment by the media over the status of our youth's ability, or inability, in mathematics. Even though the situation feels immediate, our country has been in angst about our children's mathematical ability at least since the 1950s when our national pride took a blow with the launching of Russia's Sputnik.

What has developed over the last decades is awareness or self-consciousness by mathematics teacher educators and others involved in mathematics education that student performance has not been at the level that is expected of the educational system in the United States. In 1989, the National Council of Teachers of Mathematics (NCTM) attempted to address this by introducing what became known as *reform mathematics* in the publication *Curriculum and Evaluation Standards for School Mathematics* (1989). A key element of the *Standards* and the subsequent publications by NCTM (1989, 1991, 1995, 2000) was the concept of equity of opportunity for all students to learn mathematics to the best of their ability. This has led to a redefining of the teacher's role and realignment of the goals of teacher training programs to focus more on problem solving, communication, reasoning, and making connections across mathematics in an effort to support learning by every student (NCTM, 1991, 2000).

With a goal to define how mathematics learning by K-12 students in the United States should appear, the National Governors Association Center for Best Practices and Council of

Chief State School Officers developed the *Common Core State Standards Initiative* (CCSSI) in 2010. It was an effort to provide guidelines for the introduction of language and mathematics concepts along with defining student practices that would support learning in the K-12 curriculum. The mathematics objectives were framed in reform mathematics and endorsed by NCTM. Included with the grade by grade listing of objectives was the *Standards for Mathematical Practices*, a set of eight behaviors believed to support the development of a mathematical disposition (CCSSI, 2010). In the field of mathematics teacher preparation, this initiative draws attention to the content knowledge with which preservice secondary mathematics teachers (PSMT) should be familiar and the behaviors supporting the development of mathematical knowledge.

To reference teacher knowledge in more theoretical and general terms, it has been a challenge for mathematics teacher educators to define mathematics knowledge for teaching (MKT). This component within the larger topic of teacher preparation has been described as mathematical understanding that is composed of factual knowledge based within a conceptual framework enabling its retrieval for application (Bransford, Brown, & Cocking, 2000; Kahan, Cooper, & Bethea, 2003). There have been large studies, such as the National Center for Research on Teacher Education's (1988) *Teacher Education and Learning to Teach Study* (TELT) and the National Science Foundation's (NSF) *Research and Development Initiatives Applied to Teacher Education* (RADIATE), which yielded much information about preservice secondary mathematics teachers in the United States. These studies in many ways framed teacher educators' perception of preservice teachers' mathematical content knowledge.

This study approached an exploration of PSMTs' knowledge from a different perspective. It sought to examine the problem solving processes of six PSMTs in depth with regard to both procedural ability (based in content knowledge) and mathematical practice through the lens of metacognition. The framework used for the initial inspection of the data was based on Yimer and Ellerton's (2010) *Cognitive and Metacognitive Categories*. Using this framework, which was founded in a study of metacognition with PSMTs, allowed both an analysis of the participants' mathematical solutions and of how they progressed through the problem-solving situation.

By necessity, much of the information gathered was in the form of transcriptions derived from *think-alouds*, writings, and interviews as the PSMTs articulated mathematical task solutions and reflected on episodes of decision-making and definition. These artifacts were a rich resource through which to examine PSMTs' declarative knowledge in the form of language use. This component of metacognition is relevant for two reasons. First, as a mathematician, using terminology indicates appropriate mathematical practices (CCSSI, 2010). Second, as a teacher, being able to communicate effectively is a critical element of fostering the essential sociomathematical norm of discourse (NCTM, 2000).

This qualitative study originally was to be a study of the existence of metacognitive components exhibited during the solving of three tasks; one based each in algebra, geometry, and data analysis; by participants from a small set of PSMTs. It became apparent as the study developed that the cognitive and articulation abilities of the participants were also relevant elements to explore. So, the study evolved into the development of six mathematical profiles, originally framed through cognitive/metacognitive stages, then expanded to include discussion of the participants' cognition and language.

Metacognition and Its Position in this Study

In an effort to define and structure this study, the components of metacognition acted as a focus. During the process of attempting to describe PSMTs' adeptness in solving tasks based in the secondary curriculum, it became clear that the components of metacognition, though not always directly stated, are intimately interwoven in the literature related to mathematics education. This can be seen in the CCSSI's *Standards of Mathematical Practice* (2010). For example, the first standard, "make sense of problems and persevere in solving them," contains the idea of employing the stages of *regulation of cognition*. The goal of this short section is to lay the foundation for using the concept of metacognition by giving a brief overview of the concept, its relationship to cognition, and an outcome of its development, appropriate expression of mathematical concepts and processes.

Flavell (1981) described metacognition as "cognition about cognition" (p. 37). The relationship between cognition and metacognition may be described as intertwined. A succinct way of differentiating between the two processes was given by the mathematics researchers Garofalo and Lester (1985): "One way of viewing the relationship between them is that cognition is involved in doing, whereas metacognition is involved in choosing and planning what to do and monitoring what is being done" (p. 164). Metacognition as a higher-level process was explicated by Artzt and Armour-Thomas (1992) who stated, "… metacognition includes reflection on cognitive activities as well as decisions to modify these activities at any time or place during a given cognitive enterprise" (p. 139).

This study focuses on two dimensions of metacognition: *knowledge of cognition* and *regulation of cognition* (Brown & Palincsar, 1982; Garofalo & Lester, 1985; Jacobs & Paris, 1987; Artzt & Armour-Thomas, 1992; Bruning, Schraw, Norby, & Ronning, 2009). These two components may be seen in the scenario of a student approaching a task of organizing a set of data to determine if the set represents a function and, if so, what type. As students are examining the data, they may be recalling what they know about the concept of function (e.g., the conditions for the data to denote a function, the types of functions with which he is familiar and so forth). This evaluation of the student's self-knowledge is considered part of *knowledge of cognition*. As the student proceeds to organize the data, they will perhaps make decisions about the type of function, the best way to present the data, or evaluate the results to determine if the decisions should be modified which are episodes of *regulation of cognition*.

Knowledge of cognition focuses on what the individual understands about his own understanding. There are three components: 1) how familiar an individual is with the processes by which he or she learns mathematics; 2) what mathematical procedures are known and how they are connected; and 3) knowledge about strategies for problem solving. Flavell (1979) and Garofalo and Lester (1985) included individuals' beliefs about personal performance, their mathematical abilities in relationship to peers, and their beliefs about mathematics.

Regulation of cognition is related to the self-management or self-monitoring during learning and problem solving. It has three components: planning, regulation, and evaluation (Jacobs & Paris, 1987; Kluwe, 1987). Planning involves selecting strategies and supporting resources. It also includes such elements of the problem solving process as goal setting, recall of relevant prior knowledge, and time management. Regulation involves making predictions, pausing to reevaluate (de-bugging), and determining if alternative strategies are necessary. Evaluation is determining the reasonableness of the result, considering initial predictions, integrating new knowledge into prior structures (Jacobs & Paris, 1987; Garofalo & Lester, 1985). It is from this basic structure that Garafalo and Lester (1987) and, later, Yimer and Ellerton (2010) developed stages of cognitive/metacognitive activity.

The importance of self-monitoring (e.g., using metacognitive skills) was recognized by the National Research Council (2005) in *How Students Learn: Mathematics in the Classroom* as the third principle-supporting student learning. This publication which included the topics of history, science and mathematics presented three general principles: 1) engaging prior learning; 2) understanding the role of knowing facts and the underlying concepts; and 3) supporting the development of the skills of self-monitoring. The authors wrote that an important goal for teachers is to "help students develop the ability to take control of their own learning, consciously define learning goals, and monitor their progress in achieving them" (NCR, 2005, p. 10).

The generation of qualitative data in the form of transcriptions and writings allowed an exploration of the language that preservice secondary mathematics teachers' use. These artifacts gave some insight into the *inner speech* that preservice secondary mathematics teachers have developed to solve problems and the process that is used to translate this personal speech into a *communicative* form (Vygotsky, 1986). Responses during the interview process provided some understanding not only of the preservice teachers' use of self-monitoring but their use of mathematical language that is an integral part of developing the sociomathematical norm of discourse in the mathematics classroom.

Some consider discourse an essential norm, or practice, for any classroom. To distinguish discourse in the mathematics classroom as a sociomathematical norm, rather than a basic classroom norm, is to imply the nature of the discourse at times has qualities particular to the situative nature of the math classroom. These qualities lie in expressing mathematical differences and using mathematically sophisticated, efficient, and elegant language while engaged in conjecture, explanation, and justification (Yachel & Cobb, 1996).

The National Council of Teachers of Mathematics (NCTM) in their publication *Principles and Standards of School Mathematics* (2000) proposed that the idea of using communication (discourse and writing) is an essential component of classroom practice, a means to respectfully share ideas and clarify understanding. Talking about mathematics allows concepts to “become objects of reflection, refinement, discussion, and amendment” (p. 60). The discussion continued with “such activity also helps students develop a language for expressing mathematical ideas and an appreciation of the need for precision in that language” (p. 60).

Mathematical argumentation involves specific ways in which terminology is used. By the middle grades, children should have experienced mathematical definitions and understand their roles in supporting an argument. As they progress through the grades, this should become more sophisticated (NCTM, 2000). It is the job of the teacher to support the use of progressively mature use of mathematical language.

The qualitative nature of this study into the metacognition of PSMTs while working mathematical tasks allowed a deep consideration of the participants’ solutions and mathematical practices. The process of finding solutions was examined for strategic barriers to optimal results. Also, the language and vocabulary use in the think-alouds and interviews were analyzed for precision and concept understanding. Both are extensions of the original purpose of simply determining if metacognition existed in PSMTs while doing secondary mathematics.

Statement of the Problem

There is a need for mathematics teacher educators (MTE) to understand the problem solving practices of preservice teachers particularly in regard to the most current guidelines for the K-12 mathematics curriculum. As the *Common Core State Standards* (CCSS) and *Standards for Mathematical Practice* (CCSSI, 2010) become more prevalent in educational practice, MTEs must give attention to the development of these standards in their preservice teachers.

The CCSS has embedded in its content the development of mathematical application by students as a way to express and solve problems based in reality. Also, the Standards of

Mathematical Practice emphasizes perseverance and flexibility in the mathematics applied to these situations. Since PSMTs may not have experienced this type of mathematics in their own high school experience, it is necessary to explore their mathematical abilities and habits in problem solving of tasks situated in the high school curriculum in order to discover areas that may need attention. This study is one effort to examine current PSMTs' abilities to solve tasks situated in the CCSS.

Rationale of the Study

The rationale for exploring the problem solving of PSMTs is based in the necessity of the mathematics education community addressing factors implicit in the preservice teacher situation. One of those factors is the mental distance PSMTs have developed from the secondary curriculum because of expansion of time and experiences in more abstract mathematics. Another is the need to meet current expectations implied by the writings of organizations such as the National Council of Teachers of Mathematics (NCTM) and the National Research Council (NRC), and by frameworks produced by the American Statistical Association and the *CCSSI* (2010).

NCTM described the type of knowledge needed to teach mathematics as: "To be effective, teachers must know and understand deeply the mathematics they are teaching and be able to draw on that knowledge with flexibility in their teaching tasks" (NCTM, 2000, p. 17). But, PSMTs may not have experienced some of the concepts since they were in secondary school. Cooney (1999) expressed this in: "often preservice teachers have a poor understanding of school mathematics – having last studied it as teenagers with all the immaturity that implies" (p. 165). Also, at that time, their experiences may not have been in the form of rich, concept-forming mathematics which resulted in a type of knowledge that Ball (1990) suggested was "thin and rule bound".

The content standards of number and operation, algebra, geometry, and data analysis and probability were introduced by NCTM in the *Curriculum and Evaluation Standards for School Mathematics* (1989), repeating them in the influential document *Principles and Standards for School Mathematics* (2000). The development and discussion of these standards provided a means by which to explore different, but interconnected, topics in mathematics. With the acknowledgement of data analysis and probability as a standard, the American Statistical Association addressed objectives for K-12 mathematics education and an introductory college course in *Guidelines for Assessment and Instruction in Statistics Education* (GAISE) *Report* (2005). Using the *Principles and Standards for School Mathematics* (2000) as its frame, the GAISE Report was an effort to explicate a conceptual framework for school mathematics, not replacing the NCTM document but enhancing the data analysis and probability standard (Franklin & Garfield, 2006). In 2010, the CCSSI established the Common Core State Standards (CCSS) also organized around these standards in an effort to define and provide a national framework for the teaching of mathematics in the United States. This framework identifies mathematical objectives for each grade implying that teachers must be prepared to master the objectives to support student understanding.

One critical element of supporting student learning was given by the NRC in *How Students Learn* (2005). This document emphasized using components of metacognition, in particular, encouraging the habit of 'debugging.' The concept of metacognition was repeated throughout the mathematics section of this document clarifying what it looks like and its value in the teaching/learning of all levels of K-12 mathematics. Teacher support of student self-monitoring (i.e., regulation of cognition) was one of its guiding principles.

In developing a means to examine the use of metacognitive skills by PSMTs, previous studies with K-12 students were reviewed. It was found that qualitative methods such as think-alouds, writings and interviews generate valuable researchable data (Pugalee, 2001, 2004; Artzt & Armour, 1992; Young, 2010) for the development of themes surrounding these concepts. This study was an attempt to capture skills that may be influenced by previous experiences, personal self-evaluation and motivation as a learner, all of which lend them to qualitative investigation.

That metacognitive knowledge is emphasized in mathematics education literature supports the relevancy of exploring its existence in PSMTs. Assessing episodes of decision-making (Schoenfeld, 1981) during mathematical tasks gave insight into PSMTs' knowledge and mathematical processes. Through the activity of solving problems; one each from algebra, geometry, and data analysis, using think-alouds, writing, and interviewing, examples of the PSMTs' mathematical understanding and metacognitive processes were analyzed along with their form of expressing themselves through language.

Purpose of the Study

The purpose of this study was to explore the cognitive/metacognitive processes and language exhibited through think-alouds, writing, interviews, and observation in a subset of preservice secondary mathematics teachers from two university settings. In this research, cognition was defined in relationship to procedural flexibility. Metacognition was defined as a PSMT's knowledge and regulation of mathematical problem solving. Language was defined as the efficient use of terminology and explanation specific to mathematics.

Research Questions

This study was designed to examine metacognitive and language skills of preservice secondary mathematics teachers while working mathematical tasks embedded within the secondary (7-12) subject matter. The following overarching question for this qualitative study has evolved through reference to the literature surrounding this topic and structured with reference to Creswell (2007): What is the relationship of preservice secondary mathematics teachers' problem solving abilities and articulation of metacognition while finding solutions to secondary mathematical tasks? Following Creswell's suggestion, the following subquestions about the behaviors of the preservice secondary mathematics teachers were formed:

(1) How are the components of metacognition revealed by preservice secondary mathematics teachers through *think-alouds* and *writing* while solving typical secondary mathematical tasks;

(2) How are PSMTs' responses different or similar while solving tasks within the three categories, algebra, geometric, and data analysis; and

(3) What level of precision is used by PSMTs in vocabulary use and concept explanation when expressing their processes during self-talks and question response?

Significance of the Study

There have been few published studies of PSMTs with the research focus of metacognition. Two areas of interest in this research have been founded in determining patterns of problem solving using non-routine problems and the effect of metacognition/reflection within the context of teaching (Yimer & Ellerton, 2006, 2010; Demircioğlu, Argün & Bulut, 2010; Artzt and Armour-Thomas, 1992, 1998). But, there has not been much research into how this group of college students uses skills of self-appraisal or self-monitoring while engaged in mathematical tasks related to the mathematics they will be teaching or many studies of how they express themselves, i.e. whether they use mathematical language with efficiency or precision.

The infusion of metacognitive concepts into the secondary mathematics curriculum has been determined to enhance learning. This can be found in studies by Delclos and Harrington (1991), Kramarski, Mevarech, and Lieberman (2001), and Kramarski, Mevarech and Arami (2002). Since metacognitive processes are known to become more refined as a person progresses in age (Van der Stel, Veenman, Deelen, & Haenen, 2010), it is relevant to explore metacognitive development in the group that will be responsible for instilling these practices in the future, preservice teachers.

This study used qualitative coding through the lens of metacognition to understand PSMTs' knowledge of selected concepts in algebra (specifically, function), geometry, and data analysis, their mathematical practices and their means of expressing their knowledge. Mathematics teachers are expected to support metacognitive skills in their students as teachers (NCR, 2005) and it has been found that use of metacognitive skills in the teaching process supports effective teaching (Artzt & Armour-Thomas, 1998). Both of these rationales form a strong argument to methods teachers that the ability to use and express these skills is significant.

Limitations and Delimitations

The limitations of this study are embedded within the context of qualitative research and the necessity of using available resources. The participants were chosen for convenience. The six PSMTs were drawn from two institutions in the immediate area to which I had access. I was completely dependent on the participants continuing to participate through the required four meetings and responding in an open and honest manner.

The interaction between the participants and their interaction with me was challenging. The possibility that they may talk among themselves about the tasks was addressed by appealing to their notion of trying to maintain the integrity of the research. As developing professionals, they appeared to understand this and there was little indication that there was much discussion. Also, I attempted to minimize the possible opportunities for discussion by scheduling them in such a way that they would not have a time to compare methods and answers.

Also, there was the possibility that the interview process would mediate the participants' tendency to articulate their processes. I do not think there was any way to prevent this. Several of the participants did begin to anticipate the routine of the meetings and would volunteer explanation of taped episodes. These moments of natural reflection enriched my understanding of their processes.

Finally, the essence of the study was dependent on my interpretations of verbal and written responses of personal thought processes, which inherently may be affected by my personal lens. As mathematical profiles were developed, I was guided by two ideas: that my understanding of the participants' problem solving process is (1) limited by their ability to articulate their processes and (2) by my ability to interpret them (Jacob & Lester, 1987).

The delimitations were that the problems selected for solving were taken from topics in the secondary curriculum, particularly the CCSS. This was purposely done by selecting tasks related to topics in the CCSS and designed using Stein, Smith, Henningsen, and Silver's *Mathematical Tasks Framework* (2009). The goal was to understand what the thought processes were around tasks based on material that the PSMTs may teach.

Theoretical Assumptions and Positionality of the Researcher

Some individuals are attracted to the field of mathematics because of a perceived absolutist philosophy. By that, it is meant that mathematics is based on a set of absolute truths and certain knowledge. But, this denies that every mathematical system has a foundation on agreed upon assumptions. Ernest (2010) gave a compelling argument for a social constructivist philosophy of mathematics claiming that this body of knowledge is a social construction, going so far as to imply it is a cultural product with the symbolization and language that mathematicians use being cultural artifacts. It is based on the epistemological premise that the concepts of mathematics derive from the sense making of individuals in their direct relationship to the physical world; therefore, these meaning are negotiated within the culture they are derived and are constantly changing. This perspective of mathematics recognizes the role of individuals and groups of individuals who have developed and synthesized this body of knowledge. It also acknowledges the constantly evolving nature of mathematical knowledge.

As a teacher of mathematics courses over several years, this philosophy of knowledge as evolving resonates deeply. This does not mean that certain structures, definitions, or procedures are unstable. But, what one understands may be made deeper or extended not only in the field of mathematics but the teaching of mathematics through experiences in and exposure to new ideas.

Definitions of Key Terms

In the process of analyzing terminology for this dissertation, objectifying specific terms was found complex. In an effort to clarify this particular discussion, the following definitions will be used.

Cognition - mathematical understanding, which is composed of factual knowledge, based within a conceptual framework enabling its retrieval for application (Bransford, Brown, & Cocking, 2000; Kahan, Cooper, & Bethea, 2003)

Discourse – a respectful exchange of ideas through talk involving conjecture, argumentation, and justification of mathematical solutions and concepts

Knowledge of cognition – self-appraisal of subject knowledge (Jacobs & Paris, 1987)

Metacognition - reflection on cognitive activities as well as decisions to modify these activities at any time or place during a given cognitive enterprise (Artzt & Armour-Thomas, 1992)

Pedagogical norm or Norm – accepted classroom behaviors established through teacher practices

Preservice secondary mathematics teacher (PSMT) – a student enrolled in a university program designed to support certification in grades 7-12 mathematics teaching

Problem solving – the cognitive processes through which an individual reaches a resolution

Psychological tools – constructions by a society to support concept formation and communication; in mathematics, graphs, symbols, and other forms of organizing information and supporting mathematical work may be considered such tools

Regulation of cognition – self-monitoring during problem solving (Jacobs & Paris, 1987)

Sociomathematical norm – a pedagogical norm that has qualities particular to the situative nature of the math classroom

Speaking with meaning – a response to a mathematical problem which is conceptually based, is supported by logical argumentation, and refers to the context of the problem (Clark, Moore, & Carlson, 2008)

Think-aloud – a qualitative method of data collection during problem solving which involves the participant describing in words the actions taken

Chapter Summary

The concept of reflection is recognized as an integral component of learning to teach by educational psychologists and is emphasized by mathematics teacher educators (Artzt, 1999; Artzt & Armour-Thomas, 1998; Schön, 1987; Piaget, 2001; Silverman & Thompson, 2008; Simon, Tzur, Heinz, & Kinzel, 2004). But, reflection may be considered an abstract process without a focus of attention and a means of interpretation. Also, the results of reflection are unknown unless they are expressed in words or writings (Jacobs & Paris, 1987).

The form for reflection in this study will be provided through the concept of metacognition. In the literature, the terms reflection and metacognition sometimes appear interchangeable or the term metacognitive reflection is used. Since metacognition has form, (i.e. knowledge of cognition and regulation of cognition), it provides a means by which to explore the solving of mathematical tasks and the expression of concepts. For example, a researcher may describe an individual's understanding of function or why a relationship was represented in a particular way using the individual's articulation of his own thoughts.

The focus for this particular study was a set of tasks embedded in the mathematics that is taught in grades 7-12 which relates to the certification that is granted by the two participating institutions. Returning to secondary topics has been noted as relevant by researchers (Usiskin, Perissini, Marchisotto, & Stanley, 2003; Cooney, 1999). Among the reasons for this return to material learned in secondary are the possible lack of exposure to the concepts and procedures for several years (Cooney, 1999) and the possible disconnection between the structure of the mathematics they have experienced in the college classroom and how mathematics is presented in the secondary setting (Ferrini-Mundy & Findell, 2010).

The interpretation of the metacognitive reflection on mathematical tasks was through the examination of the language the participants will use. Pugalee (2004) found that the utterances during think-aloud experiences and writings on the same tasks contain different elements. An expansion on his concept was to examine the discourse between an interviewer and the participant to determine if the participant (as a preservice teacher) recognizes her own metacognitive processes which support problem solving and expresses the elements of the sociomathematical norm of discourse (i.e. mathematical vocabulary used in an efficient manner to justify solutions).

The next chapter elaborates on the research literature associated with the ideas presented here. To set the presentation, the first topic is a broad description of what is accepted as true about preservice teachers with regard to mathematics. Following this introduction is a discussion of the role of mathematical representations in the construction of cognitive structures and the intertwined relationship of these structures with the problem solving and thought processes of metacognition. Finally, the relationship between thought and language is discussed with reference to Vygotsky's (1986) theory of egocentric and communicative speech as a foundation for the development of meaningful discourse. There have been a limited number of published studies on preservice secondary mathematics teachers with respect to metacognition. Also, much of the research on reflection and language has been centered in elementary mathematics, which justifies the necessity of expanding the research on reflection with preservice secondary mathematics teachers.

CHAPTER II:

REVIEW OF THE LITERATURE

Introduction

The underlying premise of this study is that reflection is an essential element of learning mathematics and improving one's ability to teach mathematics. Reflection, or reflection-in-action, was described by Schön (1983) as the articulation of the thoughts behind an individual's actions. This description contains the components of actions, thoughts, and articulation that frame the proposed research of this study. In support of this idea, the following topics will be reviewed: a selection of the research which has formed current beliefs about preservice secondary mathematics teachers' content knowledge, the theoretical basis of cognitive development with an emphasis on representations as the means by which learners symbolize and act upon mathematical tasks, the interrelationship of metacognition (a form of reflection) with cognition, and the relevance of language in the articulation of mathematical concepts.

For each topic, an overview of supporting theory will be presented and then the discussion will be focused specifically on mathematics. Shulman's (1986) description of the knowledge necessary to teach will be used support a discussion of mathematical content knowledge for teaching and current ideas of preservice teachers' abilities. The foundation for building content knowledge through symbolization will be based in cognitive development and Piaget's (1953) idea of schemata. Then, the relationship of cognitive development, or cognition, with metacognition will be explicated with reference to the work of Flavell (1979), Brown (1987), and other educational psychologists. These two concepts have been described as intertwined or symbiotic in the 1970s and 1980s and have remained a topic of interest into present day. A more recent description of cognitive and metacognitive processes is given in the writings by Yimer and Ellerton (2006, 2010). Finally, the topic of language and the relationships of various forms of communication will be introduced borrowing from Vygotsky's (1986) concepts of egocentric and communicative forms then progressing into the idea of *speaking with meaning* (Clark et al, 2008).

Philosophically, these topics may be tied together through a perspective of mathematics as a body of knowledge that is a social construction, a set of agreed upon assumptions (Ernest, 2010). Ernest considered that "mathematics rests on natural language and that mathematical symbolism is a refinement and extension of written language" (p. 4). He argued that the concepts of mathematics derive from the sense making of individuals in their direct relationship to the physical world with discourse as an integral part of that sense making and forms the refinement of concepts (Ernest, 2010).

In the following review of the literature, the topics will be supported with information from theoretical sources and information from research on several levels of mathematics: K-12, preservice elementary education teachers, preservice secondary mathematics teachers (PSMT), and teacher educators. There will be an attempt to clarify the participants of each research project. To understand the relevance of experiences using metacognition and discourse, the discussion begins with findings in research identifying particular needs of PSMTs that may be resolved with focused reflection or metacognitive reflection.

Teacher Knowledge

Shulman (1986) gave structure and vocabulary to teacher educators with his categorization of teacher knowledge: content, pedagogical content and curricular knowledge. He described content knowledge as the particular understandings that a teacher has within a topic. Pedagogical content knowledge was referred to as the teacher's knowledge of students and teaching methods in the context of her subject. Lastly, curricular knowledge pertained to the teacher's knowledge about the range and types of programs designed to teach a subject along with the awareness about the characteristics of the programs that make them appropriate in different teaching situations. In his explanation of content knowledge, Shulman (1986) wrote, "The teacher need not only understand *that* something is so; the teacher must further understand *why* it is so…" (p. 9).

This description of deep subject understanding was given practicality in the National Council of Teachers of Mathematics' publication, *Principles and Standards for School Mathematics* (2000), when the authors wrote, "To be effective, teachers must know and understand deeply the mathematics they are teaching and be able to draw on that knowledge with flexibility in their teaching tasks" (p. 17). This characterization of knowledge by NCTM may be understood a little more clearly through contrast between being able to do procedures in mathematics verses being able to do the procedures with understanding of the concepts underlying the procedures. Skemp in "Relational Understanding and Instrumental Understanding" (1976) differentiated between ways of knowing mathematics from a rote understanding and a relational understanding. The first, he called *rules without reason* (p. 2). The second, Skemp described as knowing both *what to do and why* (p. 2). The terms *procedural* for the first type and *conceptual* for the second type have served as common terms in the mathematics education literature. The idea of content knowledge of mathematics has been defined, or refined, from Skemp's description of mathematical understanding to include the ideas within a topic, the procedures, the underlying concepts, and relationships, or connections, within the topic and between topics (Ball, 1990; Skemp, 1976).

The type of content knowledge that mathematics teachers must know is complex. It not only includes procedural and conceptual knowledge of mathematical tasks and the reasoning behind them, but the interconnectedness of concepts. Hiebert and Lefavre (1986) characterized the complexity in *Conceptual and Procedural in Mathematics: An Introductory Analysis* in the following way: Procedural knowledge consists of "the formal language or symbol representation system" (p. 6) along with the "rules, algorithms, or procedures used to solve mathematical tasks" (p. 6). Conceptual knowledge is "rich in relationships and thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information" (p. 3-4). The symbiotic relationship between procedural and conceptual knowledge was described by Rittle-Johnson, Siegler, and Alibali (2001) as interrelated: as one increases, it influences the other to increase. In their study of fifth graders (n=74), the researchers found that procedural experiences, which included multiple representations within a topic, support the formation of conceptual understanding and, in turn, concept development acted as organizational structures for generalizing procedures or methods of strategic problem solving. The intervening factor to prompt increase in knowledge was problem representation. As conceptual understanding increased, the ability to represent a problem increased. As problem representation increased, so did procedural knowledge.

Descriptions of PSMTs' Content Knowledge

One of the largest studies of Shulman's category of content knowledge in mathematics was conducted by the National Center for Research on Teacher Education at Michigan State University, the *Teacher Education and Learning to Teach Study* (TELT). This was a longitudinal study of 252 prospective teachers. The students, 217 elementary education majors and 35 mathematics majors who planned to teach secondary, were recruited when they entered teacher education programs at Dartmouth College, University of Florida, Illinois State University, Michigan State University, and Norfolk State University. The study used both qualitative and quantitative research methodologies to develop a description of preservice teachers' understanding of mathematics. Research was often based around a classroom scenario. After examining the responses to a division of fractions problem and other problems, Ball (1990) concluded that the understandings that prospective elementary and secondary mathematics teachers bring with them from their precollege and college experiences were "thin and rule bound," revealing the lack of a mental network linking relationships. When the students were pressed for explanations associated with problem solving, they had a specific rule for each one demonstrating a lack of connections to unifying concepts. This was common to both elementary and secondary preservice teachers. The secondary teacher candidates just remembered the rules better (Ball, 1990).

Cooney's (1999) primary goal in an analysis of the Research and Development Initiatives Applied to Teacher Education (RADIATE) Project, was to explore the beliefs that preservice secondary teachers and teachers in the field have developed; but, in his observations, he found that despite taking advanced courses in mathematics, many preservice teachers did not have a good understanding of the mathematics that they would be teaching. The RADIATE project followed PSMTs for three years of their program in an effort to explore their experiences in mathematics and education. Cooney found that it was critical that PSMTs experience mathematics in a situation that encouraged them to reflect not only on the mathematics but their beliefs about what it means to do mathematics.

Connection with Secondary Students

A major task of teaching is to understand the perceptions that secondary students have of mathematics, their abilities, and concepts. This begins by ascertaining and building on prior knowledge (NCR, 2005). When an educator with fairly sophisticated networks of mathematics knowledge evaluates a student's abilities, it is usually from a domain-centric, symbol manipulation perspective. The assessment of student ability from a symbol-dominated perspective was illustrated in Nathan and Koedinger's research article *Teachers' and Researchers' Beliefs about the Development of Algebraic Reasoning* (2000) in which they examined the relationship of teacher/researcher predictions of student performance and actual performance. The student performance was determined strictly by the ability to solve a selection of problems. Six problems were presented to students at the end of a course in algebra. Mathematics teachers and mathematics teacher educators/researchers were asked to predict which problems the students would have the most difficulty. A comparison between the teacher/researcher (n=67/35) predictions and actual performance by the students demonstrated that the teachers/researchers did not predict that students would have greater difficulties solving symbolic equations than verbally expressed ones. Among the algebraic problems, symbolically expressed ones were significantly less likely to be solved correctly than verbally expressed ones (Nathan & Koedinger, 2000). Similar predictive results were found among PSMTs (Nathan & Petrosino, 2003). Nathan and Koedinger hypothesized that the verbal format of some of the problems may have facilitated correct responses because it allowed the students to use less formal or intuitive methods of solving the problem, such as *guess and test* or *unwinding*. The researchers found that about 70% of the algebraic word problems elicited the unwinding process as did almost half of the algebraic story problems. An example of solving by unwinding is seen in the solution to: Starting with some number, multiply it by 5 and then add 22, the result is 47. The student may reverse the operations by subtracting 22, then dividing by 5. Once an answer is found, the student then reverses the process again checking for reasonableness. This allows the validation of answers with intuition rather than symbolization).

Differences in perspective. The experiences of mathematics preservice teachers, teachers, and teacher educators have shaped a perspective of the secondary curriculum that is very symbol oriented. Nathan and Petrosino (2003) described the teacher's viewpoint as having *symbol precedence*, which may not be consistent with their students' development described as having *verbal precedence*. In *How Students Learn* (NCR, 2005), the authors wrote,

> Besides providing some insight into how students think about algebraic problem solving, these studies [Nathan & Koedinger, 2000; Nathan & Petrosino, 2003] illustrate how experts in an area such as algebra may have an 'expert blind spot' for learning challenges beginners may experience. (p. 355)

This discrepancy between beliefs and actuality happens when the teacher underestimates the effort that is required by the student to understand the formal method of expressing a mathematical idea and overestimates the strength of informal ideas and processes (NCR, 2005).

Schmitt and Newby (1986) in their discussion of the role of metacognition in instructional design pointed out that a novice (secondary student) will perform slowly with deliberate actions, whereas an expert (PSMT/mathematics teacher) will enact mathematical procedures using automatic responses and little reflection. The mathematical symbolic processes have been used often, so that they have become automated (Gagné, 1983, in Schmitt & Newby, 1986). The failure of teachers to consider how their mathematics knowledge is different from the students' was found as elemental to teacher development by Tzur (2001). He found that teacher development may stall until the differences are recognized.

The failure to recognize the student's different perspective of mathematical processes and the teacher's use of automated methods may result in the teacher giving little attention to connecting formal and informal reasoning. This can be circumvented by the teacher supporting reflective processes such as class discussions and comparisons of different solution methods used by the students and transforming informal methods to formal ones. These processes not only validate the students' work but also remind teachers/researchers that students do not always think in the same form that they do (NCR, 2005). *How Students Learn* proposed that teachers use an instructional context, which the authors call a *bridging context* (p. 324, 359), to connect a student's numerical understandings with a visual one, and then symbolic one through language and a link to everyday experience. The goal is not to discourage or leave unused formal mathematics terminology or function notation, but to use reflective methods as a means of linking and building on what the student already knows and methods he already uses to build conceptual understanding of ideas and to facilitate organization of knowledge.

Development of secondary math concepts. This sampling of research supports conversations about PSMTs' experiences in their development of conceptual understanding in secondary topics. The PSMTs may not perceive connections between their college coursework and the content that they may be teaching. Also, the coursework shapes their idea of how mathematical tasks should be solved. With respect to the mathematical coursework, speculation about the quantity and nature of the required mathematics coursework has been ongoing since Begle (1979) proposed that PSMTs' coursework focus more on the mathematics that they will be teaching. There is assumption that more advanced courses by an instructor leads to better student outcomes. This idea is limited in that Monk's (1994) exploration of coursework found that teacher subject knowledge of mathematics (implied by the number of courses in which a teacher enrolled) was beneficial when the students' prior knowledge was high. But, there existed only marginal advantages as the number of advanced courses increased over five.

The nature of the mathematics courses that PSMTs take was addressed by Ferrini-Mundy and Findell (2010) in their statement "the mathematics courses which most prospective teachers take address content that does not appear on the surface to relate to the secondary curriculum" (p. 33). The basis of Ferrini-Mundy and Findell's comment may be in the differing goals/work of academic mathematics at the college level and the reform mathematics of K-12 school mathematics. Moreira and David (2008) characterized mathematics content courses as being "developed independent of the specific demands of future professional school practice: the aim is basically to promote an internalization of the "principles" and the "structure" of the discipline, together with values, techniques, methods, conceptions and ways of thinking that are proper to the work of academic mathematics" (p. 24). Whereas, the demands of teaching school mathematics require the ability to develop mathematics from a conceptual basis, e.g. construct mathematical knowledge (Moreira & David, 2008; Beswick, 2012; Burton, 2002). Ferrini-Mundy and Findell (2010) enumerated upon the skills PSMTs should have that may not be addressed within their mathematics program:

> finding logic in someone else's argument or the meaning in someone else's representation; deciding which of several mathematical ideas has the most promise, and what to emphasize; making and explaining connections among mathematical ideas; situating a mathematical idea in a broader mathematical context choosing representations that are mathematically profitable; and maintain essential features of a mathematical idea while simplifying other aspects to help students understand the idea. (Ferrini-Mundy & Findell, 2010, p. 34)

The authors point out these facets of mathematics may not be directly addressed by professors in college courses but are essential elements of reform mathematics, the current philosophy of K-12 mathematical pedagogy. The significance in K-12 teacher development is embodied in the oftquoted statement by Hiebert and Carpenter (1992), "there are direct parallels between the way a teacher is taught and the instruction they implement in their classroom as a result" (p. 90).

Standards for teaching mathematics. This discussion illustrates the proposition by researchers (e.g., Silverman & Thompson, 2008; Kahan et al., 2003) that mathematical content knowledge is important but not sufficient to teach (i.e. that more than being able to perform an operation is necessary). To answer the question of what is sufficient for teaching Schoenfeld and Kilpatrick (2008) in their coauthored chapter titled *Toward a Theory of Proficiency in Teaching Mathematics* merged concepts of mathematics instruction to develop a structure enumerating the aspects of teaching mathematics necessary for proficiency. The list included not only understanding the mathematical concepts but also understanding students, crafting a supportive microenvironment, and supporting themselves as learners (see Table 1).

Table 1

A Provisional Framework for Proficiency in Teaching Mathematics

Knowing school mathematics in depth and breadth
Knowing students as thinkers
Knowing students as learners
Crafting and managing learning environments
Developing classroom norms and supporting classroom discourse as part of "teaching for understanding"
Building relationships that support learning
Reflecting on one's practice

Note. Adapted from "Toward a Theory of Proficiency in Teaching Mathematics" by A. H. Schoenfeld and J. Kilpatrick, 2008, In T. Wood (Series ed.) & D. Tirosh (Vol. ed.). *International handbook of mathematics teacher education*. Vol. 2. *Tools and Processes in Mathematics Teacher Education*, Sense Publishers, p. 2.

From this listing, it is clear that supporting PSMTs' knowledge for teaching is a complex endeavor. Focusing only on their mathematical content knowledge, not only is it necessary to assure that their education supports a type of knowledge that has "depth and breadth" (Schoenfeld & Kilpatrick, 2008) but also it is also incumbent on educators that they develop practices within their students that will support their knowledge and their future teaching. The *Common Core State Standards Initiative* (2010) composed a list of these mathematical behaviors titled *Standards for Mathematical Practice.*

Table 2

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning

Note. Adapted from *Common Core State Standards Initiative* (CCSSI). (2010). Common Core State Standards for Mathematics. Washington, DC: The National Governors Association Center for Best Practices and Council of Chief State School Officers. http://www.courstandards.org/assests/CCSSE_Math%20Stancards.pdf.

The CCSSI acknowledged developing these standards by drawing from two sources: 1) the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections (NCTM, 2000); and, 2) the National Research Council's *Adding It Up* (2001) which enumerated a set of strands for mathematical proficiency – adaptive reasoning, strategic competence, conceptual understanding, procedural fluency, and productive disposition. The listing serves as a reminder for the development of these behaviors in PSMTs. The rationale is two-fold. The first is to support the preservice teachers' own learning/problem solving. The second is more complex. For secondary students to develop these practices, they must see then enacted and explicitly modeled.

The characterization of the knowledge that preservice teachers need is that it should be more than rule bound. It should be substantive and be based in sound practice. Mathematics professors and mathematics teacher educators draw their understanding of what and how PSMTs understand mathematics very often from large studies and the results from exams. The case study may be an alternative way to develop a description of PSMTs' knowledge and problem solving skills. It may draw a picture of the interaction of cognition, metacognition, and language in which to understand the depth of knowledge and the practices supporting its accumulation within some PSMTs. The following is a discussion of cognition with an explication of its elements that are particularly relevant to this project.

Cognition

Piaget (1953) formed our concept of knowledge development as being constructed through a network of connected concepts building on itself in the form of schemata (or schema in singular). In mathematics, these schemata are structured and supported through the development of mathematical psychological tools, often called representations. The following discussion gives a brief description of cognitive development through the construction of and connections between these tools.

Cognitive Structures

The basic configuration for cognitive learning theory is explained through the ideas of perceptual differentiation, cognitive generalization, and cognitive restructuring. In simple terms, with reference to experiences in mathematics, differentiation occurs as a teacher provides experiences that help a student understand the difference between the expression 2 x $4n =$ and 2 x $n = 4$. They both involve the same symbols but should elicit different mental actions, one being an expression and the other an equation. This ability is one of discerning the specific aspects of the symbols within a particular context. A student may develop cognitive generalization after experiencing several equations and developing a particular approach to solving them. He forms a general concept from individual cases by finding patterns and grouping similar cases together. This re-grouping may help the student to restructure prior knowledge in a more precise manner developing networks of ideas within and across concepts (Bigge & Shermis, 2004).

Developing networks. Mathematics develops through a network construction, or the production of schema, as the student forms mental relationships. These relationships are prompted through mathematical experiences and understanding grows as these networks organize and reorganize (Bruning et al., 2009; Crowley & Tall, 1999; Thomas, 2004). One way that Hiebert and Carpenter (1992) described schemata was as a *web* or *network*:

> A mathematical idea or procedure or fact is understood if it is part of an internal network. More specifically, the mathematics is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and strength of connections. A mathematical idea, procedure, or fact is understood thoroughly if it is linked to existing networks with stronger or more numerous connections. (p. 67)

Bernard and Tall (1997) contributed to this analogy by developing the idea of a cognitive unit as a node within the structure linked to other units by the web. A cognitive unit may be all the mental pictures or images of the concept whether graphical, symbolic, tabular, or in the form of definition. If an individual develops an intimate connection between the images, understanding that they all represent the same entity, then the individual will begin to use all the images as one concept instead of many – a restructuring of knowledge (Crowley & Tall, 1999).

Representations or psychological math tools. The individual’s images of a concept, for example, a linear function, may be in the form of a graph, an equation, or a table. These images are cultural artifacts (Ernest, 2010), sociocultural and cognitive refinements developed over time, that act as psychological tools which help describe and facilitate mathematical activity (Kozulin, 1998; Kinard & Kozulin, 2008; Radford & Puig, 2007). Kinard and Kozulin (2008) wrote, representations are "…symbolic devices and schemes that have been developed through sociocultural needs to facilitate mathematical activity that, when internalized, become students' inner mathematical psychological tools" (p. 3). From this perspective, mathematical acquisition is not so much the result of an individual's maturation but the ability to apply these cultural tools (Kinard & Kozulin, 2008). It is not possible to recreate the development of representations (i.e., graphs, symbolization, etc.) because of their evolutionary nature and the multitude of influences (Radford & Puig, 2007). Therefore, the role of the teacher in assisting students to acquire mathematical representational growth is significant.

The role of representations is recognized by the National Council of Teachers of Mathematics. NCTM (2000) in *Principles and Standards for School Mathematics* (the *Standards*) designates multiple representations as one of the standards that should be addressed at every grade level. The authors write, "When students gain access to mathematical representations and the ideas they represent, they have a set of tools that significantly expands their capacity to think mathematically" (NCTM, 2000, p. 67). The *Standards* define a representation as both the process of creating or capturing a mathematical concept or relationship and the object itself.

> Representations should be treated as essential elements in supporting students' understanding of mathematical concepts and relationships; in communicating mathematical approaches, arguments, and understandings to one's self and to others; in recognizing connections among related mathematical concepts; and in applying mathematics to realistic problem situations through modeling. (NCTM, 2000, p. 67)

Mathematical representations may help students make the underlying abstraction of a concept more concrete. For example, examining a graph of a function may reinforce the idea of domain and range. “Different representations often illuminate different aspects of a complex concept or relationship” (NCTM, 2000, p. 69).

Driscoll (1999) wrote in *Fostering Algebraic Thinking: A Guide for Teachers Grades 6-10*, “The learner who can, for a particular mathematical problem, move fluidly among different mathematical representations has access to a perspective on the mathematics in the problem that is greater than the perspective any one representation can provide” (p. 141). For a teacher to support this type of conceptual understanding there must be connections between the students’ previous knowledge and the construction and subsequent use of various representations. Driscoll (1999) described this passage from arithmetic into algebraic symbolization with:

> Therefore, one reason it is important to emphasize for middle and secondary students the linking of algebraic representations is to expand or perhaps reorient some of the conceptions they may have developed in elementary school, in particular, to advance their conceptions of equation and function toward being more process-based and less action–focused. (p. 145)

Very often students do not perceive mathematical representations as tools to express and build mathematical concepts but as isolated skills, discrete bits of content. For example, a student may be able to graph a line using intercept and slope but not connect the act of graphing as the means to model a real world relationship. Kinard and Kozulin (2008) wrote, “The problem in current mathematics instruction is that these devices [representations in the form of equations, graphs, etc] are perceived by students as pieces of information or content rather than as “tools” or “instruments” to be used to organize and construct mathematical knowledge and understanding” (p. 3). This prevents them from building the connected structures necessary to support transfer and create new meanings (Kinard & Kozulin, 2008). An example of this type of teaching can be seen in Schoenfeld’s (1988) *When Good Teaching Leads to Bad Results: The Disasters of ‘Well-Taught’ Mathematics Courses.* He visited a highly successful geometry classroom (as defined by test scores) and found that the instructor waited until the end of the course to introduce geometric constructions. Geometric constructions are visual representations of shapes and other figures in two and three dimensions. In Schoenfeld’s analysis, the students were deprived of the rich meaning that constructions bring to geometry in their role of supporting and developing relationships and proofs inherent in the subject matter.

Role of representations. The role of mathematical notation, graphs, tables, and language are two-fold: to support concept formation and to lay the foundation for strategies of problem solving. It is natural to assume that there is a relationship between the cognitive structures the teacher has developed and the versatility of teaching practices. If the mathematics being taught is understood in a well-structured manner, the teacher is able to be responsive to student prior knowledge, take cues from questions, and support generalizations within and across exercises with ease (Livingston & Borko, 1990; NCTM, 2000). Lloyd and Wilson (1998) supported these findings in a case study of an inservice teacher. Using what they called a card sort in which different representations (graphs, functions, word problems, and tables) of linear, quadratic, rational and exponential functions were printed on cards, the researchers found a positive relationship between the teacher’s ability to organize and connect the images of function and his performance in the classroom (Lloyd & Wilson, 1998).

The purpose of the above activity was two-fold: to examine the teacher’s connections across the concept of function (as it is covered in the secondary classroom) and to determine if there was a relationship between the teacher’s knowledge and his students’ accomplishments. The initial activity of sorting did not have the purpose of asking the teacher to solve any problems. Instead it had the intent of observing the teacher’s cognitive structures of representation. But, the process most assuredly prompted the teacher to reflect on his knowledge of function. This action of reflection on knowledge has become an integral part of teacher education. It is prompted by the desire of mathematics teacher educators and researchers to discover a means to support preservice and inservice teachers in their concept development (Piaget, 1976, 2001; Simon et al., 2004; Thompson, Carlson & Silverman, 2007; Silverman &Thompson, 2008; Artzt, 1999).

Representations within problem solving. The relevance of representations in problem solving was explicated by G. Pólya in *How to Solve It* (1945). In his framework, he developed a four-step process by which to approach a task: understand the problem, devise a plan, carry out the plan, and look back. In the first two steps, the problem is translated from either a physical or written situation into a symbolic representation. The representation may be in the form of a numeric expression or table, a drawing or graph, a verbalization or writing, or algebraic symbolization. The translation has the purpose of transforming the problem so that strategies (cued from the representation) could be used to find a solution to the problem. Then, once the strategies are employed, the solution is transformed back into the original form so that it may be understood and reasonableness may be evaluated (Pólya, 1945; Nathan & Koedinger, 2000). The strength of the problem solving process is determined by the use of representations as the foundation of mathematical strategies and connections to prior experiences (the employment of schemata).

The connections across topics and within topics, along with the identification of the use of representations, may provide a means by which to discuss the concept development of PSMTs. As seen in Crowley and Tall's (1999) work, a student may encounter a barrier to progress by an incomplete understanding of how the interaction of these components of concept development. Connections and representations as the basis of strategies became objects of interest during coding for this project.

Metacognition

The development of knowledge, or cognition, is a constructive process that involves building a web of concepts, or schemata. But, the process entails an individual to think about and organize the structures used to express mathematics. These thoughts, or reflections on mathematical representations and their use, are called metacognition. The significance of the development of metacognition as a form of reflection during problem solving is expressed by Bruning et al. (2009) in:

> One of the most important educational implications of metacognitive research has been the growing awareness that knowledge and skill are only part of cognitive growth. Although knowledge and skills are important, students' learning strategies and their ability to reflect on what they have learned - to think critically - may be even more important. (p. 7)

Metacognition as a means of self-reflection by the individual learner and reflection on mathematical activity is described in the following definition by Flavell (1976), one of the first educational psychologists that explored the concept:

> Metacognition refers to one's knowledge concerning one's own cognitive processes and products or anything related to them, e.g., the learning-relevant properties of information or data….Metacognition refers, among other things, to the active monitoring and consequent regulation and orchestration of these processes in relation to the cognitive objects on which they bear, usually in the service of some concrete goal or objective. (p. 232)

In other words, it is a person's knowledge of how he or she learns. According to Lloyd and Wilson (1998), an example of metacognition in the context of teaching would be teachers expressing their decision-making process and their relationship with that process. When this occurs, the teaching activity moves into the realm of exploring metacognitive processes. Metacognition contains two dimensions: *knowledge of cognition* and *regulation of cognition*

(Brown & Palincsar, 1982). In mathematics and mathematics teaching, the first, *knowledge of cognition*, is related to what students understand about the processes by which they learn mathematics, what mathematical processes are known, how they are connected, and knowledge about strategies for problem solving. Flavell (1976) along with Garofalo and Lester (1985) included students' beliefs about their performance and about mathematics. The second component of metacognition, *regulation of cognition*, is related to the self-management or self-monitoring during learning and problem solving. It has three components: planning, regulation, and evaluation (Brown & Palincsar, 1982; Garofalo & Lester, 1985; Jacobs & Paris, 1987; Kluwe, 1987; Artzt & Armour-Thomas, 1992; Bruning et al., 2009).

Consider the following, if a student is given two sets of data perhaps representing test scores for two classes and is asked to organize the data so that the two sets can be compared, the student typically will mentally review what he knows about organizing data, what he remembers about measures of central tendency and dispersion, and internally measure his level of confidence in completing the activity. This is *knowledge of cognition*. As he is organizing the data and doing the necessary computations, there will be moments or episodes (Schoenfeld, 1981) at which decisions will be made or when computations are checked for reasonableness. Also, at the conclusion, the student will make sure the question is answered using the appropriate descriptive data and displayed with appropriate graphs. These episodes of decision-making are the components of *regulation of cognition*. An organization of this topic is found in Figure 1.

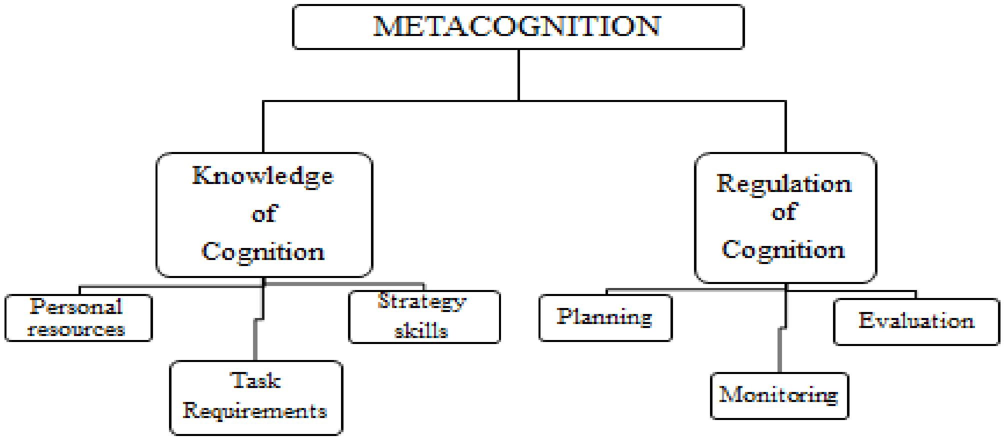


Figure 1. A diagram of the components of metacognition. Adapted from M. Schmitt, M., & Newby, T (1986). Metacognition: Relevance to instructional design. *Journal of Instructional Development*, *9*, 29.

This topic is continued below with a brief description of findings regarding the development of metacognition and explanations of both knowledge and regulation of cognition.

Metacognitive Development

Brown, Bransford, Ferrara, and Campione (1983) were able to demonstrate that metacognitive abilities develop slowly throughout the school years. The development was not limited to the elementary classroom as was commonly assumed. It continued into adolescence and adulthood. The condition that must exist for these skills to mature is to provide an environment that supports reflection on the introduction of new input regarding a belief. The reflection is a determination of the effect of the input to verify or discount that belief (Kuhn, in Carr, 2010).

Kuhn (1999) found that there was some evidence of the development of metacognitive awareness in preschoolers. During play, certain ones may be seen to plan and monitor their activities. As children enter school and are exposed to a supportive learning environment, they begin to develop additional metacognitive skills (Kuhn, 1999). An example of this in mathematics can be found in Carr and Jessup's (1995) research that found that if elementary students were able to explain why and when to use a particular strategy, the children were better at solving problems. This concept of developing metacognitive skills was supported by Veenman, Wilhelm, and Beishuizen (2004) in their findings that the accumulation of metacognitive skills followed a linear growth pattern from about age nine to adult with more abstract forms, such as evaluation evolving later. The researchers derived this information through the application of a discovery-learning task across multiple age groups.

Of particular interest because of the age of the participants in this study is the presentation in Schoenfeld's (1987), *What is all the Fuss about Metacognition?* He asked a group of college-age students to self-question during problem solving: What am I doing? Why? Is it helping? He found that the questioning and the development of self-explanation supported positive outcomes. Another finding of this study was high and low functioning students were differentiated by the high functioning ones spending considerable time in engagement and transformation. The low functioning ones jumped straight into trial and error attempts to solving the problem. These two findings related closely to knowledge of cognition and regulation of cognition.

Knowledge of Cognition

In reviewing the literature, there appeared to be two methods of discussing knowledge of cognition. One focused on the object of reflection, whether person, task, or strategy; and, the other focused on the form of reflection, whether declarative, procedural, or conditional. The following attempts to explicate each area and is followed by Schmitt and Newby's (1986) figure that relates these two classifications of knowledge of cognition.

Flavell (1976) along with Garofalo and Lester (1985) described knowledge of cognition as being subdivided into three parts: *person*, *task*, and *strategies*. By *person*, the researchers included what the problem solver believes about himself as a learner, how he learns, what he believes about mathematics, and what his position as a learner of mathematics is. *Task* is the knowledge the problem solver has about the particular task he is attempting, what the expectations are, and how difficult the problem is. The *strategy* category included the knowledge of cognitive procedures with which to solve the problem along with which would be most efficient in carrying out the tasks (Flavell, 1987; Garofalo & Lester, 1985).

A second perspective on the knowledge of cognition was given by Brown (1987) and Jacobs and Paris (1987). They limited the concept of metacognition to "…any knowledge about cognitive states or processes that can be shared between individuals" (Jacobs & Paris, 1987, p. 258). In other words, metacognition was limited to what can be made public. The authors developed three subcategories: *declarative, procedural*, and *conditional* knowledge. Declarative knowledge referred to what a problem solver knows about the context of the problem, his encounters with similar problems and what factors will affect performance. Procedural knowledge was defined as knowing how to use cognitive strategies to solve a problem, such as solving equations or finding a mean. The final subcategory, conditional knowledge, was awareness of when and why to use particular techniques of problem solving effectively (Schmitt & Newby, 1986; Jacobs & Paris, 1987; Bruning et al., 2009). Schmitt and Newby (1986) developed a model to show the relationship between person, task, and strategy with declarative, procedural, and conditional knowledge (see Figure 2).

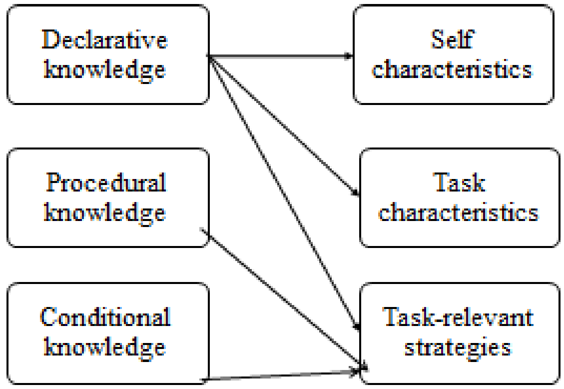


Figure 2. The relationship between requisite types of knowledge and objects of knowledge. Adapted from Schmitt, C., & Newby, T. (1986). Metacognition: Relevance to instruction design. *Journal of Instructional Development*, *9,* 30.

In Figure 2, Schmitt and Newby (1986) illustrate the relationship Brown and Palincsar's (1982) concept of declarative knowledge as being the articulation of self, task, and strategy awareness. Procedural and conditional knowledge relate to knowing how and when to employ strategies. Declarative knowledge became particularly relevant to this study since I was reliant on the articulation of the problem solving by the participants. The term is used in the study to draw attention to the words the participants employed in their descriptions and terminology.

Knowledge of cognition refers to the ability of an individual to reflect on the substance of what he knows. It may pertain to himself as a learner, the characteristics of a task, or the strategies he knows with which to complete the task. *How* he knows this information may be declarative (the ability to articulate the knowledge), procedural (how he goes about the solving) or conditional (under what circumstances he will use a particular procedure). The thought processes that are employed when proceeding through problem solving are a second dimension of metacognition called regulation of cognition.

Regulation of Cognition

Regulation of cognition is a sequence of processes to ensure success in a cognitive activity, generally subdivided into planning, monitoring, and evaluation. Planning involves analyzing the problem or activity for the purpose of invoking memory, selecting strategies and determining if additional resources are necessary. It also includes such elements of the problem solving process as goal setting, recall of relevant prior knowledge, and time management. Regulation involves making predictions, pausing to reevaluate, and de-bugging, choosing alternative strategies. Evaluation is determining the reasonableness of the result, considering initial predictions, and integrating new knowledge into prior structures (Jacobs & Paris, 1987; Garofalo & Lester, 1985).

When reviewing regulation of cognition, one may make two assumptions. One is that it is simply a rewording of Pólya's (1945) method in *How You Solve It*. The distinction may be made by Pólya addressed what the individual should do; whereas, metacognition addresses the decisions and thoughts that guide that process (Garofalo & Lester, 1985; Artzt & Armour-Thomas, 1992). Second, readers may presume that the process is linear. This is not the case. Artzt and Armour (1992), Pugalee (2001, 2004), Yimer and Ellerton (2006), and Young (2010) all found that a better description of the process was one in which successful learners move in between the stages, returning to previous ones if necessary using the incidents of metacognition to direct them to reconsider their processes. The learner may find that after evaluating an answer for reasonableness and finding it inconsistent with the original question he may return to the planning stage. This activity is consistent with *de-bugging* (NRC, 2005) during learning and the problem solving process.

Cognition and metacognition are so intertwined that it is difficult to describe when one ends and the other begins (Artzt & Armour-Thomas, 1992). It is easier to describe when metacognition is not occurring. It does not occur when automatic skills are being employed (i.e., finding a mean from a set of 10 numbers, solving a simple equation after having extensive experience, etc). This is considered rote strategy use (Jacobs & Paris, 1987). Metacognitive strategies or skills are used when deciding that the mean is appropriate to describe a set of given data or which type of equation is the most appropriate method to represent a relationship between two variables. Cognitive skills are used to achieve a goal, whereas metacognitive strategies happen before or after a cognitive activity/process in problem solving. They particularly become activated when a cognitive process fails [i.e., during de-bugging (Roberts & Erdos, 1993)]. For example, in the previous scenarios: What if the median was more appropriate to describe the data or the type of equation chosen did not fit the pattern of the values?

The complexity of measuring or determining the use of metacognitive skills has been approached by the use of questionnaires (Kramarski et al., 2001; Kramarski et al., 2002; Young, 2010) and the employment of think-alouds (Van der Stel et al., 2010; Demircioğlu, Argün, & Bulut, 2010; Young, 2010). Kramarski et al. studied the effect of a mathematics program using group work and metacognitive instruction with 13-14 year old students (n=182). The students were subdivided into a control group using traditional teaching strategies, a group that used cooperative learning, and a group that used group learning and received metacognitive training. The participants were evaluated using a pre-test, post-test, and metacognitive questionnaire. It was shown that the third group that received the training outperformed both the other groups. Van der Stel et al. used the think-aloud method to compare a limited number of two groups of students, 13-14 year olds (n=29) and 14-15 year olds (n=30), by recording instances of spontaneous use of metacognitive activity. Their purpose was to examine the relationship between maturity and the use of metacognitive activity in mathematics. They found that as the student ages the role of metacognition in mathematical activity increases. In Young's research, she compared methods. It appeared that her conclusion was that a questionnaire using self-assessment was less reliable than the think-aloud process.

This sampling of research suggests that the method used depends on the structure of the study. Since this study has the qualitative purpose of developing a deep understanding of a few participants, it would be appropriate to use the think-aloud process to explore instances of cognitive and metacognitive activity. To structure and support this idea, there have been frameworks developed to serve as guides.

Cognitive/metacognitive Frameworks

To help identify incidences of metacognitive activity in mathematics, Garofalo and Lester (1985) proposed one of the first frameworks structured with the components: orientation, organization, execution, and verification. Their purpose was to provide a tool for examining metacognition during mathematical performance. It was designed to a guide research tasks, support interview processes, and organize analysis/interpret findings.

More recently, Yimer and Ellerton (2006, 2010) developed a problem-solving model similar to the one by Garofalo and Lester (1985) that they believe defines the stages of cognitive actions and metacognitive decision-making. The categories include: engagement, transformation-formulation, implementation, evaluation, and internalization (see Appendix D). As with the Garafalo and Lester's framework, the authors acknowledge the movement back and forth between the categories. This model is particularly suited for the research proposed in this study with PSMTs because of the last reflective step, internalization. The process of internalization in which the PSMT reflects back over the entire problem solving process relevant especially in light of the CCSSI's *Standards for Mathematical Practice* (2010). This reflection on the process rather than the answer to determine if there exists a different or more elegant (Yackel & Cobb, 1996) solution is of specific interest.

Studies using a cognitive/metacognitive framework. Garafalo and Lester's framework was employed successfully by Pugalee (2001, 2004) in his studies of secondary students in the United States and by Demircioğlu et al. (2010) while exploring the metacognitive processes of preservice mathematics teachers in Turkey. Both used the qualitative methods of *think-alouds* and coding for evidence of the type of metacognitive strategies their subjects used during problem solving. Young (2010) used similar methods for gathering information on secondary participants by mapping their movement between preparation, performance, and evaluation during problem solving. One of the findings by Young and Demircioğlu et al. (2010) was that there was not a relationship between the level of metacognitive awareness and academic achievement as measured by GPA. But, Pugalee (2001, 2004) and Young (2010) did find a relationship between metacognitive skills and a program of study that employs strategies to enhance these skills which implies that metacognitive skills may be developed in more mature individuals. These researchers' findings are of particular interest to this study because of their work with more mature students (i.e., secondary and college).

Yimer and Ellerton (2006) explored the metacognition of preservice teachers using non-routine problems in their effort to determine the extent of the metacognitive processes present. They found the existence of metacognition to different extents in their participants and were able to elaborate on their results to compile the framework used in this study. The relevance of using the framework is apparent in that it was developed from working with PSMTs. Their findings were used in this study to explicate the metacognitive stages of PSMTs while working tasks based in the mathematics they will be teaching.

Relevance to teaching mathematics. The importance of self-monitoring (e.g. using metacognitive skills) was recognized by the National Research Council (2005) in *How Students Learn* in their third principle for the support of student learning. The principles included engaging prior learning, understanding the role of knowing facts and the underlying concepts, and *supporting the development of the skills of self-monitoring*. Also, a stated goal of teachers/teacher educators by the National Research Council is to "help students develop the ability to take control of their own learning, consciously define learning goals, and monitor their progress in achieving them" (NCR, 2005, p. 10).

More recently, the CCSSI (2010) developed the *Standards for Mathematical Practice*, which, though not explicitly stated, contains elements of metacognition. This can be seen in the first standard," make sense of problems and preserve." Essentially this standard proposes that students engage with the problem through analyzing what is given, form conjectures and transform the information into a structure that can be used, apply the structure, changing course if necessary (debug), check answers possibly by using a different means, and evaluate their answers. This process closely parallels the cognitive and metacognitive frameworks to be employed in this study. Other standards, such as "using tools strategically" and "attending to precision" in the use of terminology, are also pertinence in this exploration of PSMTs' problem solving.

It follows that for teachers to instill these practices in their students then they must first have the skills themselves and recognize the importance of incorporating them into their practice. What one may recognize is that not all mathematical activity requires metacognitive skills or strategies. If a teacher assigns a set of exercises, all similar, after scaffolding several during lecture, she should not expect the student to employ deep thought processes. Tasks that sustain higher order thinking are generally open-ended and designed to support consistent pressure for justification of answers. These types of problems are ones that Stein et al. (2009) categorized as *procedures with connections* or *doing mathematics*. The authors suggested that teachers understand their goals when selecting problems. Then, if a teacher has a lesson goal of supporting higher order learning, then she should choose tasks that necessitate conjecture and explanation. In designing tasks for this study, particular attention was given to the structure of mathematical tasks classified as *doing mathematics* and the standards set forth by the CCSSI (2010).

Supporting mathematics teachers' reflection. Brown (1987) suggested that regulation of cognition skills may not be easily recognized by adults in that many of their processes may have become automated or developed without conscious reflection. In mathematics, since it is a constructed body of knowledge, a preservice teacher may have used procedures, such as solving an equation, several years without connecting the process with the properties of equality and number. Also, within her coursework, the PSMT may not have the opportunity to explore topics that are assumed to be elementary (part of the K-12 curriculum). This may require providing the opportunities to unpack concepts through reflection on the mathematical concept.

In their effort to support explanation development with K-8 teachers, Charalambous, Hill, and Ball (2011) found that it was possible for the preservice teachers to use metacognitive reflection to help unpack mathematical knowledge and improve upon their declarative abilities. Steele and Widman (1997) in their action research in elementary mathematics methods classes discovered another result of reflection. The authors found that during the process of reflection on elementary concepts preservice elementary teachers developed a new appreciation of the complexity that may be inherent in concepts that they conceived of as simple allowing them to perceive the necessity of developing these skills.

In the field of statistics, Makar and Confrey (2002, 2005) used a similar immersion technique with a group of preservice mathematics and science teachers. In a preservice teacher education class, the participants were engaged in a statistical inquiry with testing data. The researchers found use of standard and non-standard statistical terminology in their descriptions of the data. But, the non-standard terminology did express understanding of the concepts of variation between sets of data. This is significant in that these studies support the concept that secondary preservice teachers may benefit from similar experiences of metacognitive reflection since they must decompose their own automatic processes to support their future students' learning.

Carr's (2010) argument in *The Importance of Metacognition for Conceptual Change and Strategy Use in Mathematics* was that the development of *declarative* metacognitive knowledge was a key to supporting conceptual knowledge. Noted earlier, declarative knowledge has a role in the learner's self-perception, his understanding surrounding a task, and the selection of strategy to solve the task. Carr (2010) explicated this concept as the ability a person has to express problem solving and their knowledge of mathematics. This quality of declarative knowledge, i.e. being able to talk about mathematics, supports procedural knowledge and strategy building (Carr, 2010). Major hindrances to its development are poor vocabulary skills (Carr, 2010; Pappas, Ginesburg, & Jiang, 2003) or a struggle with the ability to reflect on mathematical activity in a meaningful manner (Brown, 1987; Bruning et al., 2009). The way that metacognition may be clearly assessed is through producing a scenario that pushes for explanation and justification. This requires an understanding of the relationship between thought, language, and the mathematics classroom.

Mathematical Language

Yinger (1987) described the teaching process as one of conversations between teacher, student, and mathematical situation that is characterized by flexible and responsive reasoning by the teacher. Perhaps he described the teaching process as a conversation to portray the activity as an interaction between teacher and student within the context of mathematical subject matter. But, conversation implies an exchange through language. One may describe the role of language in the sense-making processes that occur within and between persons; one may discuss it with respect to its structure, the terminology and symbolism that have evolved over thousands of years; or, one may examine the use of discourse as a means to develop what it means to think mathematically.

Usage of Speech

In a very rudimentary fashion, one may connect the development of declarative knowledge, i.e. to communicate about mathematics, with Vygotsky's theories surrounding speech, thought, and writing. In *Thought and Language* (1986), Vygotsky distinguished between speech for personal understanding (egocentric) and speech for interaction with others (communicative or social). Egocentric speech may be described as internal communication, which occurs out loud when a child is young, then becomes more internalized. It becomes particularly activated in moments of discontinuity or dissonance of ideas. Communicative or social speech is used to interact with others. It is learned within the sociocultural environment of the child. Finally, the role of writing is one, which extends the child's sense-making further in that it requires a deeper analysis. The relationship between egocentric speech, communicative speech, and writing was expressed by Vygotsky (1986) with:

> In speaking, he [a child] is hardly conscious of the sounds he pronounces and quite unconscious of the mental operations he performs. Written language demands conscious work because its relation to inner speech is different from that of oral speech. (p. 182)

Language and Symbolism

Returning to the philosophical perspective stated in the introduction of this chapter, every mathematical system has a foundation of agreed upon assumptions (Ernest, 2010). Ernest gave an argument that mathematics is a social construction, going so far as to imply it is cultural product and the symbolism that represents mathematical work cultural artifacts. He wrote, “Thus mathematics is a branch of knowledge which is indissolubly connected with other knowledge, through the web of language” (p. 4). He perceived dialogue as an integral part of sense-making and refining of mathematical concepts: “mathematics rests on natural language and that mathematical symbolism is a refinement and extension of written language” (p. 4). Radford and Puig (2007) agreed that symbolism and syntax may be an extension of language but to some extent the symbolism has evolved into an entity of its own. Their perspective, through an historical review, found that early manipulations of unknown quantities were geometric, handled through transformation. It was in the late Middle Ages/beginning of the Renaissance that there began a shift to quantity and algebra became considered a generalize arithmetic. The authors wrote “…the human voice starts fading away…With this move, a new kind of conception of symbols and a new form of production of meaning start to become elaborated” (p. 151).

Returning to the two examples of mathematical notation in the discussion of cognitive development, 2 x $4n =$ and 2 x $n = 4$, it may be difficult for some novice mathematicians to distinguish the differences, beyond the order of the symbols, in the two. In this form, both are abstractions of relationships that have no context. Those individuals that are familiar with mathematics and its accepted symbolism immediately begin to notice the differences between the expression and simple equation. Radford and Puig (2007) suggested that many students encounter difficulties in moving beyond an arithmetic situation because of a struggle with the differences illustrated by these two examples, the meaning of the signs, the syntax, and the embedded abstraction.

Metalinguistic awareness. This difficulty with symbolization may be attributed to low levels of metalinguistic awareness in mathematics. Metalinguistic awareness involves the ability to analyze structures in verbal or numerical form, make choices about representation, and manipulate expressions. This requires making explicit the connection between the relationships of quantities and the symbolization used to represent those relationships. It also requires differentiation in the way relationships are expressed symbolically and the specification in the use of mathematical vocabulary particularly for those that struggle with the language used in the classroom, Standard English (MacGregor & Price, 1999).

In their research with secondary students, MacGregor and Price (1999) discussed two of the components of metalinguistics, symbol awareness and syntax awareness. They found parallels between basic metalinguistic components in language and the symbol processing which is necessary for higher-level mathematics study. Symbol awareness was described as being aware that symbols (numerals, letters, and operation symbols) are used to express real world situations but can then be used in a detached manner. They can be manipulated and rearranged to simplify expressions. Also, expressions like $(x + 2)$ can be considered a single unit. Syntax awareness was defined as the recognition of syntactic (sentence) structure and the implied meaning of the structure. For example, $2x = 12$ implies $x = 6$. It is not appropriate to write $2x =$

12 = 6; and, 2x = is an open-ended expression requiring more information before having a numerical value. The researchers concluded, “It is possible that students’ poor understanding of symbols and syntax in algebra reflects inadequate symbol and syntax awareness in ordinary language” (p. 456). Furthermore, MacGregor and Price (1999) found that “students with low levels of syntax awareness in ordinary language will make errors in algebra items because they misinterpret the original question, because they misuse algebraic syntax or both” (p. 461). This suggestion is supported by evidence that students who struggle with the language used in secondary classrooms (i.e., those from a low socioeconomic status, African-American, Hispanic, and/or Native American) are underrepresented in more challenging mathematics courses (Schoenfeld, 2002; Tate, 1997; Ladson-Billings, 1997; Rousseau & Tate, 2003). They have a higher dropout rate and perform lower on mathematics competency tests like the National Assessment of Education Progress (NAEP) evaluation (Schoenfeld, 2002). Students that exhibit high language skills but not algebra may have confusion over the fact that algebra has grammatical rules and organization. MacGregor and Price (1999) developed the conclusion that “For students with low metalinguistic awareness, however, supporting their language development may be a better approach” (p. 462).

Tevebaugh (1998) discussing the limited-English-proficiency (LEP) student wrote that the language needs of LEP students are largely ignored because mathematics is seen as a subject that does not require English proficiency. This is a myth. Not only do students have to learn such terminology as denominator, numerator, ratio, they also have to learn the mathematical syntax and logic related to their achievement level. This proposal by Tevebaugh may be generalized to any minority group that does not have the standard vocabulary and language patterns of the middle class White majority.

Verbal precision. According to Adams (2003), some of the factors that complicate mathematics vocabulary proficiency for students are: 1) definitions-transforming informal definitions to formal, for example a square which is a four sided figure that has all the sides the same length to a regular quadrilateral with properties that the diagonals are congruent and perpendicular; 2) multiple meanings-‘base’ is the bag you touch in baseball; in math it may be the side a geometric figure sits on or the value raised to a power in an exponential expression; 3) homophones-plane in geometry and a plane you fly in; and 4) symbols-multiple types, parentheses, braces, brackets, all of which provide context, focus and organization

The value of clarifying vocabulary in the teacher education program was noted by Gay (2008) in an article entitled *Helping Teachers Connect Vocabulary and Conceptual Understanding.* In her experience as a mathematics teacher educator, she noticed that student teachers used some mathematical vocabulary without specificity. An example Gay recounted was “graph this expression” (p. 218). Typically, one graphs a function and evaluates or simplifies an expression. The need for this precision in vocabulary was emphasized by the author in, “In particular, my preservice teachers needed to be aware of how their use of vocabulary contributes directly to students’ understanding or misunderstanding…” (pp. 218-219).

Discourse. This discussion of metalinguistics circles back to the earlier discussion on the findings of Nathan and Koedinger (2000) along with Nathan and Petrosino (2003) of the differences between PSMT’s and teacher’s symbol precedence view of mathematics versus the possible student’s verbal precedence view, plus the necessity of bridging the two. One way of resolving this problem is making discourse, the use of language to form a basis for conjecture, justification, and argumentation, a sociomathematical norm. Yackel and Cobb (1996) refer to sociomathematical norms as “normative understandings of what counts as mathematically different, mathematically sophisticated, mathematically efficient, and mathematically elegant in a classroom" (p. 461). These are modeled and constructed through discourse within the classroom's microenvironment by the teacher. They assert that developing these sociomathematical norms support higher-level cognitive activity (i.e., metacognition) in the form of student development of acceptable mathematical explanations and justifications of processes.

Walshaw and Anthony (2007) supported this idea with, "Classroom mathematical discourse plays a central role in shaping mathematical capability and disposition" (p. 765). Engagement in discourse, the respectful exchange of ideas, can be a complicated process for it requires the participants, students and teacher, to think, speak, and engage with others. Its purpose is to develop language and mathematical understanding. In fact, it is a bridging mechanism between students' intuitive thought and formalized mathematical thought (Walshaw & Anthony, 2007, 2008; NCR, 2005). Walshaw and Anthony (2007) described the role of discourse, "Through students' purposeful involvement in discourse, through listening respectfully to other student's ideas, through arguing and defending their own positions, and through receiving and providing a critique of ideas, students enhance their own knowledge and develop their mathematical identities" (p. 772).

NCTM (2000) emphasized the need for communication as an essential component of learning to share and clarify meanings. The activities of the interchange of ideas through groupwork, writing, and whole class discussion help concepts "become objects of reflection, refinement, discussion, and amendment" (p. 60). NCTM (2000) advocated attention and use of language in the mathematics classroom in "such activity also helps students develop a language for expressing mathematical ideas and an appreciation of the need for precision in that language" (p. 60). By the middle grades, students should understand the role of definition and, as they progress, their language and writing should become increasingly sophisticated. The purpose is for reflection and the consolidation of meaning. These ideas are repeated in the *Standards for Mathematical Practice* (CCSSI, 2010) in the first and third standards. Students should be able to express relationships between different representations of problems and be able to construct arguments to support their thinking and critique the thinking of others.

PSMT and teacher discourse. Supporting the use of meaningful discourse in preservice teachers and teachers has been found difficult. Brendefur and Frykholm (2000) took particular interest in the discourse styles of secondary mathematics interns. In their research analysis, they examined the discourse patterns of two interns as to whether they used uni-directional, contributive, reflective, or instructional methods. Uni-directional was telling; contributive was interaction with the purposes of assistance; reflective used conversations objectifying the activity; and, instructional was student-teacher conversations involving argumentation and justification. They found that for one case, it was difficult to move past the lower levels of discourse. For the other, the intern attempted to move into the more difficult levels of sustaining meaningful conversations. The researchers realized that the interns were affected by their beliefs about instruction and discourse in the classroom, "novice mathematics teachers that have been exposed to years of traditional instruction may be socialized to this manner of teaching; therefore, it may be very difficult to … help them develop different conceptualizations of mathematics teaching" (Brown & Borko, 1992, p. 145, in Brendefur & Frykholm, 2000).

Clark et al. (2008) found that it was very challenging to prompt quality discourse, or what they classified as *speaking with meaning,* with experienced teachers also. In their research with science and mathematics teachers in a professional learning community, the researchers found the quality of speech within mathematical discussions lacking. The researchers defined the concept as: "*Speaking with meaning* implies that responses are conceptually based, conclusions are supported by a mathematical argument and explanations include reference to the quantities in the problem context" (p. 298). Clark et al. (2008) supplied the following example to contrast the difference between a simple explanation and speaking with meaning: A photo of width 8 inches and height 10 inches is enlarged to a width of 32 inches. What is the new height?

Table 3

Example of Procedural Explanation vs Speaking with Meaning

Student: The answer is 40. Teacher: How did you get that? Student: Well, 32 divided by 8 is 4 and 4 times 10 is 40. Teacher: OK, good.	Student: The answer is 40 Teacher: How did you get that? Student: Because of the relationship of height and width … if the width is increased by 4 times, the height would be increased by four times as well. Since the ratio of width to height in the original photo is 4 to 5, then the width is increased by a factor of 4, the length must be increased by a factor of 4. This will assure that when I set up the ratio of the new photo, the ratio of width to length will still be 4 to 5. Teacher: OK, good.

Note: Adapted from Clark, P., Moore, K., & Carlson, M. (2008). Documenting the Emergence of "Speaking with Meaning" as a Sociomathematical Norm in Professional Learning Community. *The Journal of Mathematical Behavior, 27*, 298.

To explain their findings, Clark et al. (2007) differentiated between individuals with a procedural or conceptual orientation. A procedural orientation in teachers and students is exhibited when a solution is derived without reference to context. The emphasis is only on the getting a correct answer. If the person is prompted for a deeper explanation, usually the response focuses on the procedures. An individual with a conceptual orientation refers to the problem's context and explanations reveal a broad scheme of ideas (Thompson, Philipp, Thompson, & Boyd, 1994, in Clark, et al., 2008). Clark et al. continued with, "Speaking with meaning draws heavily from the notion of a conceptual orientation, which places a strong emphasis on attempting to communicate meaningfully with others" (p. 298). The authors expressed the belief that both those with a procedural or a conceptual orientation can speak in this manner, but "that individuals for whom speaking with meaning becomes normative are likely acquiring a conceptual orientation in their mathematics communication patterns" (p. 309). This idea is strongly related to Carr's (2010) premise that developing declarative knowledge supports conceptual development. Clark et al. (2007) continued with " In order for an individual to make these judgments and consider how not only themselves, but also others make sense of explanations, an individual must shift from only giving or listening to explanations to making the explanations themselves objects of reflection" (p. 299).

Chapter Summary

To discover an individual's understanding of a concept or process, not only does the researcher require an examination of the written work but an exploration of the individual's thoughts. This is not always possible. The use of the metacognitive ideas knowledge of cognition and regulation of cognition provide a means by which a few processes by a small group of individuals may be investigated.

The relationship between metacognition, content knowledge, and language (a pedagogical skill) has been explored in various combinations in mathematics and statistics. The connection between metacognition and content knowledge has been richly explored in the elementary grades (Carr, 2010). In secondary school, Van der Stel et al. (2010) assessed the acquisition of metacognitive skills in mathematics between students aged 13 and 15. They found an increase in use and quality over those years. Schneider and Artelt (2010) summarized the findings of several studies with, "…metacognitive knowledge and…use of learning strategies predict mathematics performance in primary and secondary school settings even after differences in intellectual abilities have been taken into account" (p. 158-9). In college age students, Schoenfeld (1987) explicated how self-questioning can facilitate problem solving.

The use of metacognition among preservice teachers and teachers often are coached in the frame of reflection on mathematics. The two studies mentioned, Charalambous et al. (2011) and Steele and Widman (1997), both describe the effort of instructors to help elementary preservice teachers reflect on topics in mathematics. These studies report that the unpacking of concepts was accomplished and many participants considered it a positive experience. Similarly, Makar and Confrey (2001, 2005) found that immersion in concepts of statistics supported understanding.

Metacognition in preservice secondary mathematics teachers has been approached from two directions, as a frame for examining teaching practices and as a means to create a framework to describe problem solving. Artzt and Armour-Thomas (1998) proposed the use of metacognition as a meaningful method of examining teacher practices. The next year, Artzt (1999) developed a structure to support preservice teachers' reflection on their practice. The second use of metacognitive principles was by Yimer and Ellerton (2006, 2010) as they explored the extent in which PSMTs used metacognition in non-routine problems, problems in which there is not a typical way to work them. From that information, they developed a cognitive/metacognitive framework similar to an earlier one developed by Garafalo and Lester (1987).

In the initial exploration of the literature for this study on the metacognitive processes of PSMTs, it was found that two techniques have been generally employed, questionnaires/surveys and think-aloud sessions. Kramarski et al (2001, 2002) followed up problem solving with a series of questions about the strategies used; whereas, Pugalee (2001, 2004) used the think-aloud process and writing. Young (2010) found in her comparison of the two techniques that the survey method was insufficient to accurately describe metacognitive abilities. Also, Yimer and Ellerton (2008, 2010) were successful in establishing their framework through the more descriptive method of interpreting think-alouds. These findings led to the decision to use the qualitative methods of think-alouds and writing for this study.

With the current interest in the CCSSI's grade level and practices standards, it is relevant to consider the understandings and processes held by PSMTs in the context of problem solving. In the next chapter, there will be a plan presented to examine the cognitive and metacognitive processes of PSMTs during problem solving of tasks that are part of the secondary objectives. Briefly, the plan was to request PSMTs to solve three open-ended tasks, one each in algebra, geometry, and data analysis. Their efforts were video recorded and coded for instances of metacognitive elements. Then, the group as a whole was examined to determine if there are common metacognitive processes. Then, the perspective changed to determine if there are common themes within the types of problems.

CHAPTER III:

METHODOLOGY

This chapter describes the research design used to study the interplay of cognitive and metacognitive behaviors by preservice secondary mathematics teachers (PSMT) during problem solving. Published research on metacognition of PSMTs in mathematics (Demircioğlu et al., 2010; Yimer & Ellerton, 2006, 2010) almost exclusively focused on problem solving non-routine tasks for evidence of metacognitive behaviors. This study departed from that format by using mathematical tasks developed from concepts embedded in the curriculum of grades 7-12. The task solutions were used to explore PSMTs' metacognitive practices, knowledge of the content that they will be teaching, and the precision of the participants' mathematical language used in the problem solving process. The organizing structure for this investigation was the use of the cognitive-metacognitive categories for problem solving developed by Yimer and Ellerton (2010), which was introduced in the topic of metacognition in the literature review and is also found in Appendix D. The content of this chapter contains a rationale for using a qualitative design along with an explanation for the choice of a case study approach. Also included are the research questions with descriptions of the research design, participants, data collection, data analysis, and the timeline for the study.

Overview of the Study

In the desire to develop a rich description of PSMTs' problem solving, this project used a qualitative approach. Its purpose was to define the metacognitive processes used during problem solving, contrast mathematical processes across topics of mathematics, and realize the levels of language precision. Data sources included video recordings of think-aloud problem solutions of mathematics tasks, writings describing the derivation of those same solutions, semi-structured face-to-face interviews, and researcher memos. An inductive process of coding to analyze cognitive and metacognitive activities was utilized with additional attention to mathematical strategies and language use. Of particular interest with language was the use of vocabulary specific to the mathematical tasks, reference to underlying concepts, and support of conclusions through logical mathematical reasoning based in the problem's context (Clark et al., 2008).

The exploration of cognitive-metacognitive processes was centered on three mathematical tasks, one each in algebra, geometry, and data analysis. Since the only means by which a researcher may determine these processes is through the participants' articulation of problem solving (Jacobs & Paris, 1987), the participants were asked to solve each problem first through the process of a think-aloud, then through writing. To further explore the participants' work, a follow-up interview with the problem solver was conducted using the digital video recording of the think-aloud as an object on which to explicate perceived moments of unarticulated metacognition or to clarify vocabulary/language use. It was anticipated that this process would generate information about PSMTs' moments of automated problem solving and language use. Furthermore, it was conceivable that the PSMTs benefited by developing some self-awareness of their thinking while watching and analyzing their problem solving.

The following overarching question for this qualitative study has evolved through reference to the literature surrounding this topic and structured with reference to Creswell (2007): What is the relationship of preservice secondary mathematics teachers' problem solving abilities and articulation of metacognition while finding solutions to secondary mathematical tasks? Following Creswell's suggestion, the following subquestions about the behaviors of the preservice secondary mathematics teachers were formed:

(1) How are the components of metacognition revealed by preservice secondary mathematics teachers through *think-alouds* and *writing* while solving typical secondary mathematical tasks;

(2) How are PSMTs' responses different or similar while solving tasks within the three categories, algebra, geometry, and data analysis; and

(3) What level of precision is used by PSMTs in vocabulary use and concept explanation when expressing their processes during self-talks and question response?

Theoretical Framework

There is a difference between the knowledge that a teacher must have of mathematics and other users of mathematics. This proposition was recognized by Shulman (1986) in his description of the categories of knowledge a teacher should have. Shulman wrote, "The teacher need not only understand that something is so; the teacher must further understand *why* it is so …" (p. 9). The question is how an individual transforms mathematical knowledge into structures that support teaching and foster a student's learning and understanding of mathematical skills and concepts.

Reflection on problem solving, with respect to subject matter or teacher practices, has been proposed as an essential element necessary for transforming mathematical content knowledge and pedagogical practices into useable forms for teaching (Schön, 1983; Silverman & Thompson, 2008; Artzt, 1999; Artzt & Armour-Thomas, 1998). In the 1970s, the concept of metacognition as a form of reflection during problem solving provided a structure by which to gain insight into the thought processes of students (Flavell, 1979, 1987; Brown, 1987; Kluwe, 1987). Frameworks for the study of metacognition in mathematics have been developed

(Garofalo & Lester, 1985; Artzt & Armour-Thomas, 1992; Yimer & Ellerton, 2010) which provide a means to examine an individual's metacognitive skill existence and level. It also provides a framework in which, a researcher may turn the problem solvers attention from the problem to the method of solution, transforming the process of finding a solution into an object of reflection. This use of metacognitive processes is not limited to individual task solutions. This idea of teaching mathematics as a problem-solving situation was proposed by Artzt and Armour-Thomas (1998). They found evidence of the important role that metacognitive processes have on pedagogical practices.

One component of metacognitive reflection is declarative knowledge, which is the ability to talk about mathematics with regard to personal resources, tasks, and procedures. Carr (2010) proposed that developing declarative knowledge influences conceptual understanding. In turn, Clark et al. (2008) contended that conceptual knowledge supports speaking with meaning, a pedagogical skill for effective teaching. This emphasizes the close relationship between being able to *talk* about mathematics and understanding mathematics. A low level of declarative knowledge with a struggle to use appropriate mathematical vocabulary and language has been associated with individuals from a low socioeconomic level or English as a second language (Carr, 2010). This may also be evident in those that have not experienced activities that involve reflection in such a way to develop declarative knowledge about a particular concept in mathematics. Support for the idea of providing reflective opportunities for preservice teachers may be found in the writing of Silverman and Thompson (2008). They proposed that it is necessary to construct reflective experiences for preservice teachers to transform their mathematical content knowledge and pedagogical skills into forms for teaching. The following research design is centered on the concept of metacognition and a framework to explore the understanding of topics from the secondary mathematics curriculum and make the PSMTs' cognitive and metacognitive activities an object of reflection.

Research Methods

Qualitative research has procedures that Creswell (2007) described as "inductive, emerging, and shaped by the researcher's experience in collecting and analyzing the data" (p. 19). This characterization allows the researcher to modify her plan or pursue avenues that were not originally apparent. The emergent quality of this type of research and the researcher's view of social and physical reality align closely with these attributes. Corbin and Strauss (2008) described qualitative research as having the ability to, "… get at the inner experience of participants, to determine how meanings are formed through and in culture, and to discover rather than test variables" (p. 12). The intimate nature of metacognition lends itself to this type of research. The design attempts to discover and express the uniqueness of each participant's cognitive and metacognitive practices during problem solving along with each participant's ability to verbally expression himself.

The study is centered on evidence of metacognitive practices found in participants' mathematical problem solving, a fundamental component of learning and teaching mathematics (NCTM, 2000). In an effort to understand the cognitive and metacognitive processes of PSMTs, data in the form of verbal and written explanations of mathematical tasks with transcriptions of follow-up interviews were gathered. Through an examination of the data there was an attempt to describe and develop possible themes regarding the presence of metacognitive processes during problem solving. Also, since the participants were asked to solve tasks from three strands of the secondary mathematics curriculum, algebra, geometry, and data analysis, the data were reviewed for themes within and between the strands. The limited information through research about the relationship between problem solving, cognitive-metacognitive processes, and level of speaking with PSMTs lent itself to a desire to obtain the most information possible by keeping the process fluid and open to change (Creswell, 2007).

A Case Study Approach

The case study, a methodology of qualitative research, is defined by Creswell (2007) as "the study of an issue explored through one or more cases within a bounded system (i.e. a setting, a context)" (p. 73). He described the research design as one in which the researcher uses multiple sources of information in an effort to clarify the case or illuminate themes embedded in the case and/or across cases. This form of research has the purpose of providing information to enlighten a particular phenomenon which in this study is PSMTs' cognitive and metacognitive processes during problem solving.

A case is defined as instances of the phenomenon (Gall et al, p. 447). For this study, the individual cases are the PSMTs' problem solving described through the three mathematical events. The primary goal was to develop a rich description of the metacognitive thought processes each had developed to support problem solving. Secondly, a cross case analysis was developed for general themes of metacognition, cognition, and language within each strand of mathematical task, algebra, geometry, and data analysis. The cases were bounded by the participants' problem solving experiences in their secondary and college education and the point at which the participants were in their certification program. All were enrolled in the semester before internship (Gall, Gall, & Borg, 2007; Creswell, 2007).

The unit of analysis for this study, the aspect of the phenomenon studied, was the presence and use of metacognitive aspects of problem solving (Gall et al, 2007). The six cases within the unit of analysis were examined for the use of cognitive/metacognitive techniques to support their problem solving across the three tasks. The focus was on the movement among the cognitive/metacognitive categories (Appendix C), the use of representations giving evidence of strategies, and the participants' use of language to communicate their thoughts and procedures. These codes were not emergent but reliant on the established theory presented in Chapter 2 as suggested by Yin (2009, p. 130). There was an attempt to "treat each individual case study as a separate study" with the goal of "aggregating findings across the studies" (Yin, 2009, p. 156).

This research methodology was consistent with the purpose of this study. Reiterating the description above, the phenomenon to be studied was the PSMTs' problem solving of mathematical tasks taken from the curriculum of grades 7-12. The plan used multiple cases, six PSMTs prior to their internship. And, the cognitive-metacognitive processes of solving the series of mathematical tasks were analyzed through an interpretation of the PSMTs' verbal and written explanations. The study proceeded in the qualitative tradition using methods to explore the individual's construction of knowledge. The tradition allowed for fluidity with tolerance for change in themes or research questions if the situation called for them, but remaining consistent with the nature of the study and the lens through which I chose to view the data.

Research Design

The research used a multiple-case study approach. The case study was chosen in response to the form of the questions. Yin (2009) advocated this particular type of study to answer questions framed with "how" or "why." He suggested that the case study should be used to describe "a contemporary phenomenon in depth" (p. 18). In this case, the phenomenon is preservice teachers' ability to problem solve. Though it is not unique to this time period, it is one of ongoing interest. The advantages of using a multiple-case study is expressed by Yin in, "The analysis is likely to be easier and the findings likely to be more robust than having only a single case" (p. 156). Since it was my goal to explore the cognitive and metacognitive activities of six PSMTs working individually, each PSMT representing one case, this methodology fits well with the goals. Often research has focused on cooperative group problem solving (Artzt & Armour-Thomas, 1992; Kramarski et al., 2003), but in this study, the focus was on the individual participant in an effort to provide an in-depth description of the problem solving activity in the single case and to look for patterns of behavior within and across cases.

Setting

The study involved participants at two institutions, one a large research university with an enrollment of 30,000 students and the other a state university with an on-campus enrollment comprised of residents and commuters of approximately 2300. The larger institution draws students throughout the United States with a minority population of about 12%. The unconditional college entrance score is 23 on the ACT. The smaller institution, which has historically been considered a teachers college, draws students primarily from a region, which is traditionally rural with a low socioeconomic status. The student body has a large minority population, about 55%. Unconditional entrance to the smaller university is set at an ACT score of 20. Both universities are located in the same state in the southeastern region of the United States. The two institutions were chosen because of my ability to access the pool of possible participants and, also, because of my desire to develop a "robust" description (Yin, 2009) of preservice secondary mathematics teachers' problem solving.

The two institutions have similar mathematics requirements for a Bachelors of Science in Mathematics with certification, but there are differences. The mathematics coursework for both institutions includes a three semester calculus series, linear algebra, differential equations, introduction to advanced mathematics, and abstract algebra. The programs do differ in the following ways, which include: (1) Both institutions require a statistics course. At the research institution this course is provided by the College of Business. At the regional institution, the course is taught in the Department of Mathematics. (2) The research institution requires their PSMTs take a sequence of two capstone classes taught in the Department of Mathematics designed to connect the mathematics taught at the university to the secondary program; whereas, the regional institution requires a college geometry course and a course titled Technology for Teachers which acts as a capstone course with similar objectives as the larger institution's courses. It is designed to make the connections through the medium of technology and inquiry learning. One major difference that the participants will have experienced is in the substance of the capstone geometry. The course offered at the larger institution is largely based in the standards outlined in the CCSS. The geometry course at the regional institution is primarily a traditional Euclidean proof-based course. (3) Finally, at the smaller institution, the students are required to enroll in a mathematics seminar course and at least two credit hours in the developmental mathematics laboratory, which provides one-on-one tutoring.

The coursework provided in the Colleges of Education differ as well. The larger institution requires a three-hour clinically-based field experience course and three mathematics methods courses focused on technology, curriculum/assessment, and unit planning design. With the limited number of students at the smaller institution, it offers one mathematics methods course with an additional sequence of four one-hour field courses.

Participants

The participants were drawn from a population of PSMTs that were at the same point in their academic career, seniors, enrolled in their last semester of courses before entering their internship. I was not an instructor of the PSMTs at either institution but was able to collaborate with faculty members teaching the student populations for recruiting purposes.

Participants for the study were recruited from the pool of students enrolled in courses titled Teaching Secondary School Mathematics at the research institution and College Geometry at the smaller institution. Even though these classes are not equivalent, they are both populated with PSMTs within the last semester before their internship. The research project was explained to all potential participants along with the compensation for their time (i.e. $15.00 per session). The possible participants were contacted to determine who may be interested in contributing their knowledge and time to the study. Six participants were sought from the pool of volunteers, three participants from each institution.

At the initial meeting, an informal self-assessment (see Appendix A) was asked of those willing to participate. On the one page questionnaire, the volunteers were asked to rate their knowledge of mathematical skills taught in grades 7-12. On a scale of 1 to 10, 10 representing high confidence in their skills, they self-rated and justified the value chosen. The activity has validity in Brown's (1987) contention that being able to reflect on one's own abilities is a fundamental skill in support of metacognition. Also, I had used this technique in previous studies as a rudimentary means to appraise students' ability to reflect on their mathematical skills.

Since the purpose of this study was to build a deep understanding of PSMTs' problem solving, the participants were chosen from a variety of self-rated scores. From the larger institution, after contacting all that initially indicated willingness to participate, only three were consistent in communication and ready to commit the necessary time. They represented different ratings, 6, 7, and 9. Those three continued to follow through with their commitment. From the smaller institution, three willing participants were chosen for their diversity. The diversity was determined by gender and self-rating on the screening questionnaire. This set of participants rated 7, 8, and 9. (All the scores were from 6-9.) At this point, individual conferences were set through email to review the requirements of the study and negotiate schedules for meetings.

The process of information gathering involved multiple forms of data: recordings and transcriptions of think-alouds, writings, and transcriptions of interviews along with researcher memos. These sources supplied the means for analysis, which included coding, developing themes about the cognitive-metacognitive performance and language within cases and across cases. The following provides details about the plan.

Protection of Privacy/Confidentiality

This study was in compliance with the Institution Review Board (IRB) protocol as set forth by both institutions of higher education. In consideration of the elements of privacy and confidentiality, IRB compliance approval was ascertained. I consider preservice teachers a vulnerable population. With that in mind, I assured the potential participants through informed consent that there were no repercussions if they chose not to contribute their expertise to the study or, at some point in time, decided to withdraw from the study. If a participant did chose to withdraw either by word or any other method, the data collected would be destroyed. The structure of the project was presented to the PSMTs in a straight forward manner and all questions were answered whenever they arose during the research process (Yin, 2011). Fortunately, once the participants began the project, each completed the required series of meetings.

Confidentiality was preserved through being transparent about who will have access to the data gathered and the use of pseudonyms. Participants concerns that their responses may influence other's perception of their mathematics skills were reassured. Every attempt was made to maintain anonymity. The identity of the participants was only definitely known by the researcher and they were assured that their current professors would not be informed of their participation or results. Also, during videotaping, the camera was focused on the participant's work, limiting exposure as much as possible that would lead to identification (Gall et al., 2007). Finally, all artifacts were secured in the researcher's office under lock and key. Approximately three years after completion of the project, they will be destroyed.

In an effort to safeguard the participants' privacy, the focus of interviews was on the objects/artifacts produced during problem solving with occasional reference to the teaching of the tasks. The only personal information requested was demographics during the screening process. Also, there was every attempt made to be transparent about the structure of the research and research questions (Stake, 2010). In the process of gathering data and expressing the results, the participants were given pseudonyms: Andy, Bev, Cynthia, Danni, Elizabeth, and Fran. The names were chosen randomly except the first letter of each name-supported organization.

Data Collection

Data collection included four meetings with each participant. The original plan for the meetings was first meeting would incorporated a review of the objectives of the study, the requirements for the participant, and a video recorded think-aloud experience in algebra followed by a writing of the problem solving process. The second session would involve an interview in which the algebra session was reviewed and questions were posed to explore the participant's thinking processes occurred. During the same session the participant was to solve a geometry problem. The third session was a follow up regarding the geometry problem solving activity and then the participant was to solve a data analysis problem. The last session was a follow up interview of the data analysis problem solving session and a debriefing. Each problem solving session was allotted 30 minutes and each interview 30 minutes. Table 4 shows a summarization of the sessions noting the number of days between each meeting and the time spent on each task. Interviews were more open-ended and not specifically timed.

Table 4

Summary of the Data Collection Process

	Session 1	Session 2	Session 3	Session 4
Andy	Introduction to project; Algebra task (20:25)	2 days later: Interview of Alg.; Geometry task (12:02)	3 days later: Interview of Geo.; Data Analysis task (9:08)	2 days later: Interview of Data Analysis task
Bev	Introduction to the project; Algebra task (17:36)	7 days later: Interview of Alg.; Geometry task (8:38)	5 days later: Interview of Geo.; Data Analysis task (10:59)	2 days later: Interview of Data Analysis task
Cynthia	Introduction to the project; Algebra task (32:09)	7 days later: Interview of Alg.; Geometry task (41:00)	5 days later: Interview of Geo.; Data Analysis task (9:20)	2 days later: Interview of Data Analysis task
Danni	Introduction to the project; Algebra task (13:07)	2 days later: Interview of Alg.; Geometry task (6:05)	19 days later: Interview of Geo.; Data Analysis task (6:42)	2 days later: Interview of Data Analysis task
Elizabeth	Introduction to the project; Algebra task (18:56)	5 days later: Interview of Alg.; Geometry task (5:56)	2 days later: Interview of Geo.; Data Analysis task (8:54)	12 days later: Interview of Data Analysis task
Fran	Introduction to the project; Algebra task (21:55)	2 days later: Interview of Alg.; Data Analysis task (10:33)	12 days later: Interview of Data Analysis; Geometry task (11:42)	2 days later: Interview of Geometry task

Note. This table is developed to illustrate the time span between incidents of data collection.

The original plan was to have the four meetings with each participant over a period of about two weeks with only about 2-5 days in between the problem solving session and the interview. Also, the tasks were originally planned to be presented in the order algebra, geometry, and then data analysis. This had to be modified. Schedules had to be adjusted due to illness, availability to start, and pressures from the participants' coursework. The result was revisions in meetings times and, in an effort to control for discussion between the participants about the tasks, the order was adjusted. If more than one participant was meeting on a day, they were scheduled relatively close together and given the same task. That way, if they discussed the task, it would be after completion. Also, they were asked not to discuss the tasks with the rationale that it may skew the results.

The *think-aloud* has successfully been used as a means of investigating cognitive and metacognitive aspects of problem solving in mathematics research (Artzt & Armour-Thomas, 1992; Young, 2010; Demircioğlu et al., 2010; Yimer & Ellerton, 2006, 2010). Ericsson and Simon (1984) described this technique as one in which the participant is asked to verbalize his thoughts and procedures as he progresses through to his solution; in other words, speaking aloud any of the ideas that come to mind as he solves the task. A simple protocol based on the elements of Ericsson and Simon's (1984) work was used for each problem. Its purpose was to remind the participant of the think-aloud technique; introduce the type of problem; and orient him to the available tools (see Appendix B).

During each problem solving session, the participants were asked as a second step to *write* their method of solving the task. The rationale for this step was to introduce an intermediate process in which the participants would transform their previous actions into an object of reflection (Vygotsky, 1986). Since one purpose of this study was to examine the use of language (i.e., vocabulary, context, etc.), this provided an opportunity to look at the connections between the spoken and written word (Pugalee, 2001, 2004).

The last step to each mathematical task was a follow-up interview. The participant and I reviewed the video recorded think-aloud for clarification of episodes of decision-making and member checking by the participant. I attempted to *decenter* (Piaget, 1955, in Carlson, Bowling, Moore, & Ortiz, 2007), or try to adopt a view of the problem solving that was the participants', to listen and follow their reasoning. Prior to the viewing, I noted moments in the video, which were pertinent to the questions of the study focusing on the problem solving and language use in problem solving. Intermittently, I paused the recording to ask questions and to listen to the participants' volunteered explanations in an effort to explicate relationships or concepts within the mathematical task.

Setting for Data Collection

Collection of data was by necessity at two different locations. At the larger institution, arrangements were made to use unoccupied classrooms or conference rooms. There were basically three locations used. At the smaller institutions, all meetings were held at one location, an available office.

The problem solving and interviews at both locations were video recorded and recorded using a Livescribe smartpen. The smartpen, when used with the appropriate paper, served as a recorder of the spoken word and as a writing tool. The advantage of the pen with the paper was that the writing and spoken word were coordinated. If the pen touched the paper at a particular position, it would pick up the sound that was recorded at that point. Using both instruments ensured the recording of all interaction between the participant and the task during problem solving and between the participant and me during the interview process. Effort was made during the recordings to preserve privacy. The camera was focused on the work area only.

At all locations the participants were provided with the smartpen, pencils, straight-edge, paper, and two types of graphing calculator. The calculators were the TI-83+ and Casio 9780-GB+. Both of these calculators had graphing, function fitting, and statistics abilities beyond the normal ability to do calculations. All participants appeared comfortable with these calculators and familiar with their capabilities.

Design of the Mathematical Tasks

The mathematical tasks were written with reference to the Task Analysis Guide developed by Stein et al. (2009). The authors espouse that the task must fit the goals established by the teacher (in this case, researcher). A fundamental factor in determining the characteristics of a task is the cognitive demand. Cognitive demand is defined as “the level of thinking required of students in order to successfully engage with and solve the task” (Stein et al., 2009, p. 1). Four levels of task were proposed and include: a) *memorization*; b) *procedures without connections to understanding, meaning, or concepts*; c) *procedures with connections to understanding, meaning, or concepts*; and d) *doing mathematics*. For the purposes of this study, tasks were selected in the category with highest cognitive demand, *doing mathematics*. The rationale for this is the open-ended nature of the tasks that allow for a variety of levels of response.

According to Stein and colleagues (2009), the characteristics of tasks designated *doing mathematics* include: (1) a lack of examples or instructions on how to approach the problem, the problem solving approach is open-ended possibly not algorithmic; (2) a student exploring the task through various strategies to understand the relationships within the task; (3) demands of regulation of cognition in the form of self-monitoring; and (4) recall of relevant procedural knowledge that may not have been recently used. Furthermore, they note that the task may inspire anxiety because of its unknown nature (Stein et al., 2009). This type of task prompts the type of cognitive and metacognitive behaviors that are integral to this research project.

Selection of the Mathematical Tasks

This project proposed using three tasks: one based in each algebra, geometry, and data analysis. To select the three, multiple resources were reviewed and discussed. Some of the problems reviewed were similar to those on college entrance exams, released problems by the National Assessment of Educational Progress (NAEP, http://nces.ed.gov/nationsreportcard/mathematics), and suggestions from mathematics professors and mathematics educators. The final selection of the problems was based on a consensus that they preserved the standard of doing mathematics (Stein et al., 2009) and could be referenced to the Common Core State Standards in Mathematics (http://www.corestandards.org/the-standards/mathematics) and the Guidelines for Assessment and Instruction in Statistics Education (GAISE) Report. The rationale for choosing each mathematics problem is explained below:

Algebra. The task selected for this topic was a series of three figures that followed a pattern of development. The participant was asked to describe the pattern and then use the pattern to answer an additional question (see Question 1, Appendix C). This task illustrated two components of the Common Core State Standards for high school mathematics: interpreting functions and building functions with particular reference to sequences. Interpreting functions includes understanding the definition of function and using appropriate notation. Students should be able to analyze functions using different representations and build functions that model a relationship between two quantities. Sequences as a type of function is specifically expressed. Students should be able to write sequences both recursively and with explicit formulas.

This task was chosen because it has the elements described above from the CCSS and the characteristics of *doing mathematics*. It is open-ended with no particular method of finding the pattern suggested. Also, it was selected because of the multiple ways it may be approached and the different levels of completion. The participant may be able to produce the next figure, produce a description of the pattern, or produce the nt^h expression. The construction of each figure may be determined in various manners. One person may see the third figure as $(3 \times 5) + 2$ while another person may see it as: $3^2 + (2 \times 4)$. (See Appendix F)

Geometry. The geometry task involved uses rigid motion and dilation to produce a similarity transformation to place one quadrilateral onto a larger one (see Question 2, Appendix C). This task incorporated objectives under the heading of congruence in the CCSS's high school geometry standards. The participant was asked to experiment with transformations in the coordinate plane, exhibit an understanding of congruence through rigid motions, and show an understanding of similarity transformation through the use of dilation.

This task also had multiple ways of achieving the goal of defining a similarity transformation from one figure onto another (see Question 2, Appendix C). The participant was asked to describe the necessary transformations, making appropriate notations. There were as many solutions as there were participants (see Appendix F for one solution).

Data analysis. For this task two sets of data were presented. They were based in a realistic scenario. The goal was to elicit a comparison of the data beyond the typical computation of a mean (see Question 3, Appendix C). The CCSS expresses this objective in high school statistics and probability as the ability to interpret quantitative data on a measurement variable and to use appropriate statistics to describe the distribution of two data sets, interpreting the differences. This task also addresses the interpretative aspects of variability within data analysis expressed in the GAISE Report (2004). Similar to the previous two tasks, this one may also be approached at different levels. As the GAISE Report suggests, a study of the variability of the data may be a simple computing of the range, a comparison of the data to the mean, or a more organized discussion of the data's variability within the groups and between groups.

The sets of data had a common mean. But, the dispersion was very different. This task should draw out the need to have multiple ways to describe data. Finally, the participant was asked to apply what was noticed about the data bringing out the applicability of statistics. Again, this task is related to the qualities of *doing mathematics* in that it requires self-monitoring to determine if the data is interpreted in a meaningful way and it requires recall of descriptive data terminology (see Appendix F).

Analysis of Research Data

Creswell (2007) and Yin (2009) provide plans for the data analysis of multiple cases. The first step is to conduct an analysis within the cases to produce a rich description of each, and then perform a thematic analysis across the cases. Similar to this construction, the data for this study, transcriptions of the think-aloud sessions and interviews along with the work in mathematics, were coded using three categories: (1) the movement of the participant through the categories of cognitive and metacognitive stages (see Appendix D) which corresponds to the metacognitive component called regulation of cognition; (2) the use of numeric, visual, written, and symbolic representations which gives evidence of mathematical strategies; and (3) the use of math talk or declarative ability which was primarily present during the stages of transformation and implementation. An example of the coding is given below in Table 5. After this initial coding that was developed by reading and re-reading the data, a description of each problem solving session was developed for a total of 18 depictions of PSMTs' work in mathematics. The goal of this first effort was to produce a fulsome description of each of the participant's solving process. The results are presented in Chapter IV.

Table 5

An example of Coding and Memos for Math Talk, Strategies, and Stages

1. Okay, so first I want to look and see how many tiles are in each figure.	**ENGAGEMENT**
2. So, in the first figure, there are one, two, three, four, five. There are five tiles [writes 1st – 5 tiles]. In the second there are one, two, three, [counts to ten; writes 2nd – 10 tiles beneath the first writing]. And in the third there are one, two, three, [counts to seventeen and makes the entry on the table she is developing]. [Touches each tile while counting.]	
3. Okay now I want to see what they are doing to change the pattern.	**TRANSFORMATION**
4. Umm … It looks like they are adding a row and a column. [Writes "adding a row and a column".] Okay.	-strategy/table -strategy/writing for reflection
5. Okay I don't really want to focus on the one … at the bottom left and on the top right because each of those has those two. So, I kind of just want to focus on the … number of the rectangle.	-strategy/connection to geometry -strategy/focusing on the area
6. So, in the first one there's three, in the second there would eight, and in the third one it would be fifteen. [Writes in a column 3, 8, 15.]	-strategy/table
7. So, let's see …. One has one column and three rows in the first figure. And then the second figure has two columns and four rows. And, the third figure, it has three columns and five rows.	
[While speaking, develops another table in the form '1 column 3 rows'.]	Procedural Knowledge: -using a table to extend a pattern
8. So, when I want to find the twentieth figure, I know that there will be twenty columns because in the first figure there is one column, in the second figure there is two columns, and in the third there is three columns. [Pointing at the table developed in line 7.]	**IMPLEMENTATION**

Note. The above example of coding is from the transcription of Danni's work on the algebra task. The underlined type indicates incidences of her declarative talk. Strategies evidenced by the use of representations/connections across topics and the movement through cognitive and metacognitive categories are noted in the right-hand memos.

After the problem solving descriptions were written, I returned to the data to construct an organization by which the research questions could be addressed for each participant and then for each task. Referring to Yin (2009), word tables were developed with a strong influence from my mathematics background. Two tables were formed. One was structured by decomposing the tasks into components that formed the rows and the participants formed the columns. This table concentrated on the participants' ability to solve the tasks, their answers. The second table was more complex in that it focused on the elements of the initial codes. The rows addressed the use of the stages of cognition/metacognition, representations, and descriptive language. Again, the columns were headed by the participants. The cells contain examples from the transcriptions and problem solving. These two tables contain a condensing of the data from which final analyses for Chapter V were made (see Appendix G).

This analysis was written in two sections. For the participants, a perspective of their problem solving was developed across the three tasks, algebra, geometry, and data analysis. Their progression through the stages of cognitive and metacognitive categories, their use of representations, and their use of language were examined with respect to the data. Finally, an assessment of the PSMTs' performance across the category of task was developed. Its purpose was to form the basis through which responses to the research questions could be developed. These responses are presented in the final chapter and framed with respect to the research problem, developing an idea of PSMTs' problem solving of the mathematics they will be teaching.

Trustworthiness/Validity

The extent to which the methods and procedures of a project ensures validity is a key to the value of the findings. Yin (2011) addressed this in: "A valid study is one that has properly collected and interpreted its data, so that the conclusions accurately reflect and represent the real world (or laboratory) that was studied" (p. 78). This study supports characteristics of validity by providing rich descriptions of data from multiple sources. The design of the study provides at least three sources of information and artifacts: video recordings of think-aloud problem solving, writings explicating the problem solving process, and transcriptions of interviews in which the participants are asked to review the video recordings for confirmation of their actions and thought processes. Along with these artifacts, the researcher will memo for nuances that may be missed in the various recordings and writings. Hopefully, this will provide a *thick description* (Geertz, 1973) of the participants' problem solving processes and language use during the mathematical tasks.

The elements of rich data, respondent validation, and triangulation are included in this proposal to support validity and trustworthiness. The data will be drawn from three experiences (i.e., video recording, writing, and interview) with each PSMT for each of three types of mathematics tasks. During the interview process, the participant will be asked to confirm her actions and verbalizations while problem solving. This data should provide a deep description of each case's cognitive-metacognitive activity. The researcher will in her analysis and presentation of the data supply participant dialogue as examples of findings. In particular, she will be examining the data for those incidents, which provide information significant to developing a description of participants' metacognitive processes and problem solving within and between the mathematical topics (algebra, geometry, and data analysis).

Chapter Summary

This chapter has provided methods for examining the problem solving processes of PSMTs while engaged with tasks related to the secondary curriculum as well as the rationale for choosing a qualitative research design of case study. The questions are centered on two foci. The first is the PSMT as a problem solver and the second is the problem solving processes employed while working on tasks taken from three different strands of the Common Core Standards for Mathematics. Data collected in the form of words, either verbalizations or writings, and visual representations were analyzed to discern the ability of the PSMTs in articulation of their thinking and the problem solving process.

CHAPTER IV:

RESULTS

The role of the previous chapter was to present the method by which data was gathered on the cognitive and metacognitive processes that PSMTs use to problem solve tasks based in the secondary curriculum. This chapter continues the explanation of the research process by presenting the results of the project's design. For each participant, Andy, Bev, Cynthia, Danni, Elizabeth, and Fran, three narratives were created. They are based in an examination of the data gathered during the think-aloud sessions of the three tasks in algebra, geometry, and data analysis along with excerpts from the interviews that followed. A summarization of findings concerning each participant's method of problem solving may be found in the initial table of Appendix G.

Also for reference purposes, there are problem solutions to the tasks in Appendix F. These solutions are not the only strategies by which the participants may have approached the tasks, but they acted as a guide and a solution referent. The examples of solutions are organized using Pólya's (1945) framework for problem solving: understanding the problem, devising a plan, carrying out the plan, and looking back. The rationale for presentation in this form is that Pólya's process would be the one used by the problem solver.

It was through an inspection of the problem solving using a lens of metacognition that the focus turned from a study of the problem solution to a study of the problem solver. The profiles begin with a) a description of the participant's self-evaluation from the initial questionnaire; b) an account of my experiences with each; c) a description of each problem solving episode oriented through the stages of Regulation of Cognition (engagement, transformation, implementation, evaluation, and internalization); d) a summarization pertaining to the expression of Knowledge of Cognition (their knowledge about the task and strategies); and e) the use of mathematical tools for obtaining solutions. Times spent in each stage of Regulation of Cognition are given with reference primarily from the starting point and rounded to the nearest half of a minute or minute. The mathematical profiles begin with Andy's below.

Andy

The two populations of students from which the participants were selected contained a total of three males seeking state certification in secondary mathematics. Andy was the only volunteer. On the initial questionnaire volunteers were asked to rate their confidence in their secondary mathematics skills on a scale from 1 to 10 (10 being very confident). Andy rated himself with a score of nine writing, "In high school, my friends would come to me for help on problems that they did not understand." His confidence seemed to be validated by others who sought his help in problem solving. This external validation appeared significant and his words suggest that others saw him as being more capable in math than they.

In that initial contact, the requirements for the study were explained. These included participation in four meetings during which each participant would be asked to work on three tasks (i.e., based in algebra, geometry, and data analysis) with a follow-up interview for each. There was also the request that the meetings be fairly close together in order to capture recalled thoughts that may have occurred during and after the problem solving session. After contacting Andy to participate, we negotiated a schedule. The schedule allowed us to meet consistently over two weeks with two to four days between each problem solving session. He was at each meeting promptly and arrived ready to engage in each task. As per my request, Andy tried to express his thoughts and procedures as much as he was able in both verbal and written form. His problem solving is described here.

Algebra Task

In the stage of engagement, which is the initial sense-making of the task, Andy let me know he was reading the problem and becoming oriented to the structure of the presentation of the task along with the method of recording his work (using the Livescribe smartpen). After 2½ minutes, he moved into the transformation stage, exploring the information to develop an initial conjecture or plan of action. His initial step was to develop an informal listing of the total tiles in each figure. His notes can be seen in Figure 3.

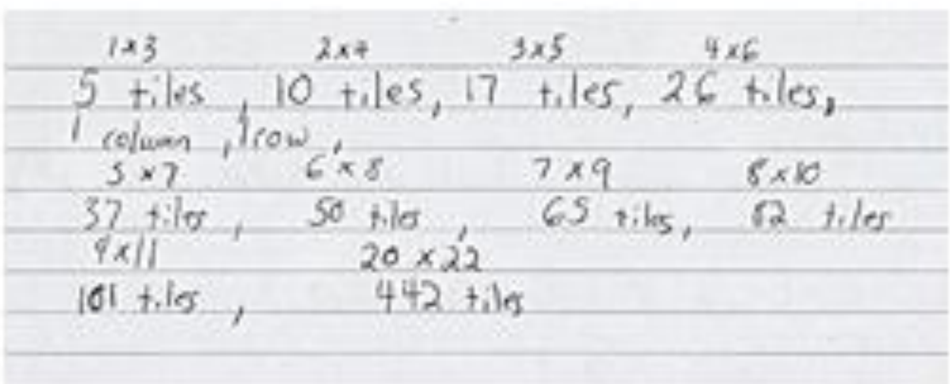


Figure 3. Andy’s informal table illustrating his process of determining the total tiles in the 20th figure.

Andy organized the visual information he derived from the figures into quantities and noted the change between the figures. He wrote, “One column, one row” and verbalized, “From the first figure to the second figure it adds a column…and a row to the middle where there (sic) are two [tiles] on the outside.” This shows that he decomposed the figure into the rectangle determined by the number of columns and rows, plus the two external tiles. Andy found the next figure (4^{th}) mathematically, without drawing the figure, even though the visual presentation of the previous figures seemed to have a strong influence on his findings. He expressed his declarative knowledge about the task in,

> What I am doing is taking, multiplying the row by the column and the two outside ones at the beginning and the end just kind of leaving them off. So, that is going to be one times three which will be three plus two is five.

While verbalizing this, Andy pointed at the first figure, showing the relationship between numbers and shape by using his hands. After six minutes had passed from the initial engagement, Andy moved into the stage of implementation, which is acting on plans. His talk changed from a focus on the elements of the task to a focus on the procedure he had developed. He wrote the numeric relationship as the number of columns times the number of rows above the listing of the first through fourth figures then continued the pattern using numeric statements for the fifth through eighth figures and totals. When Andy computed the total for the eighth figure, he paused and checked the pattern working from the eighth backwards to the first, then computed the ninth. At this point, 8½ minutes of the problem solving process had passed.

Andy recognized the relationship between the number of the figure and the number of the columns. Once Andy computed the ninth total, he summarized his finding verbally,

> So, by the pattern, it is just columns that would be whatever figure you are on, and the rows would be plus two and then you multiply those two together, then add two tiles on the outside; so, the twentieth figure would be twenty by twenty-two plus two.

It appeared that he had a very clear understanding of the relationship between the figure number and total tiles, but did not formalize it using algebraic symbolization (see Figure 3).

After computing the number of tiles in the 20^{th} figure and ten minutes into the task, Andy paused for a short period of time reviewing his immediate work. By this action, he appeared to briefly experience the stage of evaluation, judging the appropriateness of his actions. Then, he moved quickly to the task of finding the figure with at least 10,000 tiles. After a brief period of engagement in which Andy re-read the question, he stated, “I am just going to make up some numbers and get them in there.” The rationale for this method was described in the interview as, “I really had no idea how to do it. I was just doing stuff.” He used a method called guess and check with the support of a calculator to develop an answer with dimensions 100 by 102. This result was produced quickly through three trials. From his description of his actions, the method was to divide 10,000 by a number, then judge if the divisor and quotient were close in value. The divisor represented the dimension of column or row and the quotient being the second dimension of the figure. After finding 100 x 100, Andy proposed that a figure with at least 10,000 tiles would have dimensions of 100 x 102. This answer described one figure in the solution interval but not the least. He did not directly identify his solution as the 100^{th} figure. He paused briefly. Then, he ended the session after 14 minutes.

Researcher Observations. Andy primarily relied on numeric expressions. Examining his written work revealed it was exclusively in that form. One point of inquiry during the follow-up interview two days later was to determine the roles of other mathematical representations, such as writing, drawing, and algebraic symbolization, and their influence on his process.

It was interesting that the figures had a strong visual role in Andy’s problem solving. During the follow-up interview, he revealed that during that initial stage of engagement he was seeking familiarity to this type of problem in previous experiences. When asked if he had encountered a similar problem, he said, “I’ve seen one kind of like it. It wasn’t the exact problem but it was something kind of like this…It gave you figures.” This revealed that his cognitive visualizations supported his method by triggering a memory of solving a similar problem. Also, the decomposition of the figures led to his method of finding the total tiles in the twentieth figure. Contrary to this visual influence, he did not draw any figures nor did he verbalize any connection to the figures' geometric compositions. When questioned about why he did not draw, he was not very responsive. Though he did not state it, Andy's approach suggested that he considered the mental visualization as sufficient to move him to the next stage of the task solution (i.e., expressing the geometric composition as a drawing was not necessary for the task solution). Andy's relationship with visual representations was of interest in light of the research supporting the use of multiple representations of mathematical situations.

During the session, there were moments when Andy expressed his thoughts very clearly in both describing the components of the task and communicating his procedure. His declarative knowledge of the task could be seen in his analysis of the figures. He said (repeated),

> So, by the pattern, it is just columns that would be whatever figure you are on, and the rows would be plus two and then you multiply those two together and then add two tiles on the outside; so, the twentieth figure would be twenty by twenty-two plus two.

Also, when asked about writing "one column, one row", he said, "I was just trying to write it out to see the difference…because I still hadn't figured it out right here, at this point; not the pattern yet…I try [to write] when I get stumped a little bit." The data suggest that writing and language have a function and role in his problem solving.

Andy could express the relationship very clearly but did not model it using function notation. We explored this some during the follow up interview. As we reviewed the video recording, I stopped the recording at that point and asked if he had considered doing so. As I attempted to prompt the writing of the relationship in algebraic symbolism, Andy's response was minimal. I suggested that he let *n* equal the number of the figure. He then with little hesitation wrote an expression correctly: $n(n + 2) + 2 = n^2 + 2n + 2$, but not in function form such as $f(x) = x^2 + 2x + 2$ or $S(n) = n^2 + 2n + 2$.

As we continued the recording to his efforts to find the figure with at least 10,000 tiles, we had the following exchange:

R: The second question …

A: I really had no idea how to do it. I was just doing stuff.

R: You were using the old fashioned method of trial and error (i.e. guess and check). What if you had that?” [referring to $n^2 + 2n + 2$]

A: It would have been a whole lot easier, a whole lot easier.

This confirmed that Andy recognized the relevance of using function notation. However, he did not utilize this type of representation to model the relationship between the figure and total tiles.

Though Andy appeared to use the geometric idea of conservation of area in his decomposition of the figures and the sequential idea of pattern in determining the increase by a row and a column to find the number of tiles in each figure, he did not convey connections to the topics of geometry and sequence through expressing area or writing the n^{th} term. Also, in finding the figure with at least 10,000 tiles, the mathematical phrase *at least* was not interpreted by Andy. He did however demonstrate an understanding of it through his choices during trial and error, for example:

> Ten thousand … divided by 200.... That is too high. It is 50 … all right … 10,000 divided by 100 is 100; so, I figure that 100 columns by 102 rows plus two would have at least 10,000 tiles. Which that would be … a hundred times one oh two which is 10,200 plus 2 which would be 10,202.

Andy was also asked to write his procedure for finding the solutions to the task. Before beginning the writing, he was asked to add any thoughts about alternative ways to solve the problem or expound on any ideas that may come from reflection while writing. I found the writing very procedural, offering little insight. Also, when writing, Andy completely left off the method by which he found an answer to the second component of the task.

In summary, it appeared that Andy relied strongly on numeric representation to find solutions for this task. This may have been rooted in his strong number sense, which was demonstrated in how quickly he derived a solution for the request of a figure with at least 10,000 tiles. Strategically, he appeared to use visual and verbal methods of making sense of a problem but did not extend those skills in such a way that may have supported developing the model through symbolization and then finding the most correct answer to the second component of the task using that model.

Using the frame of a cognitive and metacognitive analysis of Andy's problem solving, it was possible to uncover his minimal amount of time expended in reflection on each incident of evaluation and the total lack of time devoted to the stage of internalization, a review of the entire problem solving process. He appeared satisfied once he found a numeric answer. During each step in his problem solving, he would check over his computation or reflect back over his immediate work but did not spend time looking over the task as a complete endeavor.

Geometry Task

After engaging with the task for 1½ minutes, Andy began demonstrating his knowledge about rigid motion in geometry with, "one line of reflection … would be at … y = 8." He used the provided straight edge to place the line and label it. He continued by producing the vertices of the reflected quadrilateral, naming each point using coordinate pairs [i.e., (7, 5)]. His solution may be seen in Figure 4. He used the term *reflection* easily during this process. When speaking about the vertices, he placed them saying, "D would reflect to (7, 5)." It appeared that he reflected the vertices through counting and visual orientation to the line of reflection. Later in the follow up interview, he was able to elaborate to some extent on the concept of reflection with, “Well, the points A, B, C, and D over here [pointing at the shapes on the first reflection] are the same distance away from the line…and…you know [pointing] flipped over.”

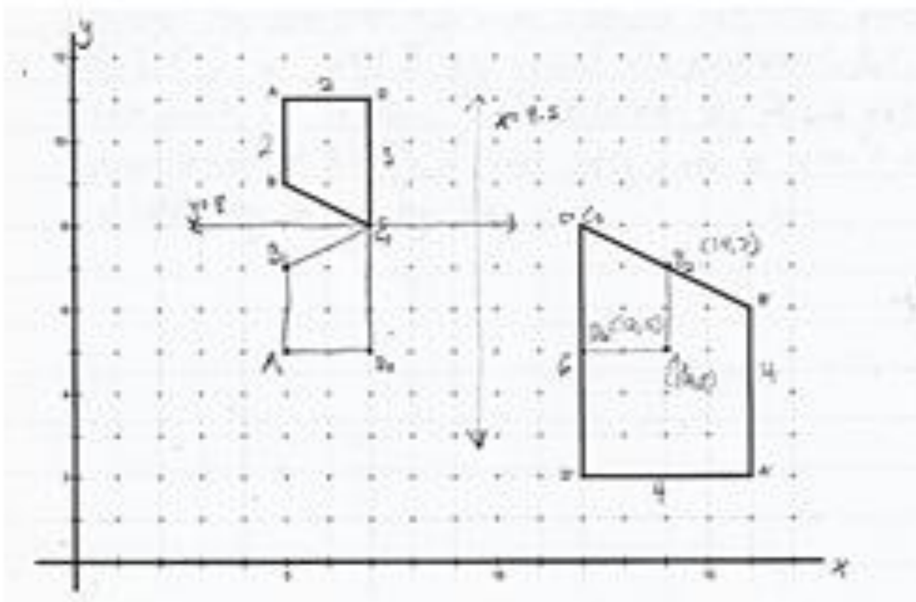


Figure 4. Andy’s solution: reflection over y = 8; reflection over x = 9.5; and dilation by a factor of two.

After the first reflection and three minutes into the task, Andy paused to evaluate his work, labeling the vertices with corresponding capital letters using subscripts, A_1, B_1, C_1, and D_1 (letting C = C_1) . He appeared to hesitate and return to the stage of engagement. In preparing for the next step in the task, he questioned, “Similarity transformation, do you want me to…make it the same … the first figure is smaller than the second one. [Do] you want me to…dilate it or anything?” This conflict was explained in the interview by,

> I was trying to think of … see I had to get it [reflected shape, $A_1B_1C_1D_1$] turned around [pointing to the final shape] and bigger. So, I was just, I was thinking how am I going to do this? Because I know that just reflecting it over is just going to make it this right here [pointing at C’, B_2, A_2 , D_2, the result of a second reflection]. How do I get it bigger?

At this point he had not decomposed the motion into two transformations.

At five minutes, Andy again moved into a stage of transformation, preparing for the dilation by mathematically determining the factor by which the smaller figure is enlarged to cover the larger one. By counting the length of the sides AD and A'D,' DC and D'C,' and finally AB and A'B,' he concluded, "I would have to dilate the original figure by two." So, he procedurally found the ratio of similar lengths and confirmed it by examining more than one set of corresponding sides.

Andy described his next two transformations with, "There is a line of reflection…at…x equals nine and a half. We reflect A_1 B_1 C_1 D_1 and dilate it by two." At this point, he sketched the line x = 9.5 and noticed that C_1 would go to C'. He paused for almost a minute; then at 8 minutes into the task, concluded, "Let's just reflect this first." It was clear at this point that he decided to decompose the motion into two transformations, reflection first, then dilation by a factor of two.

Andy implemented this plan verbalizing procedural steps of reflecting his first image over the line x = 9.5 to produce a second one with C_1 onto C.' Previously, as Andy went through this process of reflecting, he articulated the coordinates of the vertices of the new image. This time he verbalized the counting to find the points but did not say the new coordinates. He did label them and implemented notation to support the identification of corresponding parts of the quadrilaterals. In describing the final transformation, Andy said, "And, then if we…dilate by two that would get the … A', B', C', D'." Eleven minutes into the task, after a brief pause and an expression of self-doubt: "That doesn't feel right … [pauses and laughs]. That really doesn't feel right. But, I don't know…" Andy ended the problem solving session.

Researcher Observations. Andy's approach to the task contained a minimum of extraneous computation or redundancy of activity. He decomposed the activity into a sequence of rigid movements from what appeared an initial plan of two reflections. There was the one event, the second reflection and dilation, which involved a movement back into engagement and transformation. But, the conflict was resolved by taking the anticipated transformation and decomposing it into the two.

There was evidence the majority of the motions were done mechanically but supported by mathematical understanding. Confirmation of this can be found in the following excerpt from the interview three days after the problem solving session. Andy revealed his thinking and mathematical understanding about reflection and finding the similarity ratio in:

> R: Now, how do you know when you have a reflection?
>
> A: Hard questions…
>
> R: Now, how did you know that was a reflection?
>
> A: Well, the points A, B, C, and D over here [pointing at the vertices on the first reflection] are the same distance away from the line…and…you know [pointing] flipped over.
>
> R: And, how did you know when you made it bigger…what the factor was?
>
> A: Well, I noticed right here it was two on the smaller sides and over here it was four. …and, four divided by two would be two. This one's three and this whole length is six.

Andy began the explanation about reflection using a technical description of the corresponding points being equidistant from the line of reflection but then dropped back into the less rigorous way of describing the motion as the figure being flipped. Also, he described the means by which he determined the proportionality of the quadrilaterals' sides, but did not use terms such as ratio or proportion.

Andy demonstrated visual skills scanning the smaller figure and quickly developed a plan to move the smaller figure onto the larger. The visual plan had a minimum of movements, which is considered optimal in transformations. Also, his images and lines of reflection were drawn clearly so that one could follow his choices without the narrative. Add to this the fact that he labeled the lines of reflection and vertices of the images in a meaningful way demonstrated that he had a clear understanding of the requirements of the task.

His use of language also supported the idea that he had a good understanding of the task. Andy clearly declared the lines of reflection and used terminology related to reflection and dilation correctly. Andy would say, "…D would reflect over to…(7, 5)…" and "…dilate by two." Although his verbal account of his procedures during the think-aloud diminished as he progressed through the task, his mechanics were solidly demonstrated.

Andy's primary struggle was with expressing the rule for dilation. He became aware that dilation was necessary for the solution of this task even though it was not listed in the task description. Also, he understood dilation as an enlargement of a figure to an image in which the dimensions of the figure and its image are in proportion. Andy's uncertainty was rooted in expressing the rule of the dilation. This was shown in, "For some reason I want to say…I can't remember if the term would be one half or if it is two. But, I think it is two." This conflict was picked up again at the conclusion of the session when he said, "That doesn't feel right…[pauses and laughs]. That really doesn't feel right. But, I don't know…" He found the proportion of the smaller figure to the larger figure, but he failed to mention another feature of dilation, the point of projection which in this case was C.'

It was difficult to determine if Andy connected reflections with other transformations, such as rotation or translation. He did not verbalize any exploration with functions other than reflections during his cognitive and metacognitive process. So, it was not discernible how complete his understanding of the relationship between reflections, rotations, and translations was. It appeared that his initial plan involved the use of the two reflections and he did not deviate from that plan.

Andy was again asked to write about the process he used during the task. The purpose was to see if the writing process would support and evoke more substantial reflection over the task. Similar to the writing done with the algebra task, it was a simple enumeration of his step-by-step process:

> I reflected Quadrilateral ABCD at y = 8 and named the quadrilateral A_1 B_1 C_1 D_1. Then, I reflected quadrilateral A_1 B_1 C_1 D_1 at x = 9.5 and named the quadrilateral A_2 B_2 C_2 D_2. At this point, C_2 is on C'. I dilated by 2 to get the quadrilateral A'B'C'D'. I was very unsure about whether to dilate by 2 or by ½.

In summary, Andy's problem solving process for the geometry task exhibited similarities to that in the algebraic task. For instance, he moved through the stages of engagement, transformation, and implementation smoothly. Furthermore, when he encountered a difficulty or gained new insight into the task, he would return to engagement or transformation to make sense of the situation then returned to implementation to complete the task. As he worked, he would pause, review the prior step and very deliberately move on to the next. His pace appeared measured and methodical. But, as before, he demonstrated evaluation only in the form of reviewing the previous step and no internalization in the form of reviewing the entire task or the problem solving method. In the video recording and my personal observation, he was not seen returning to the beginning of his work and following his path through to the final solution, which is usually indicative of the stage of internalization.

Statistics Task

This task provided a table of values representing accumulated data regarding attendance at two theaters for one year. The data was organized by each day of the week. Also, the attendance mean for each set of data was provided. The feature of significance between the two sets was the difference in variability from the mean.

After an initial engagement with the data for one minute, Andy interpreted the mean with reference to attendance saying, “So, generally they are getting about the same…the same amount of people.” He continued by visually examining the data for each theater. He differentiated between the two sets of data by noting, “Theater A is a little spread out. Sunday, Saturday, they have…a lot of people. Wednesday and Thursday, they do not have too many people. Theater B is a little more…evenly distributed.” This characterization was accurate, but he did not make any further attempt to identify the dispersion of data using the statistical term range nor did he compute the range for the two sets of data. The data were not explored in any depth with the use of either a graph or a discussion of other measures of central tendency. This implies that Andy attempted to answer the questions of the task without a systematic approach of examining variability at the most basic level, computing the range of each set of data.

From the description of the two theaters, Andy decided that he would prefer to be the manager of Theater B. His rationale was “just because it is more evenly spread out…across each day.” His decision appeared to be made only from a visual perusal of the data. The decision was not expanded upon with perhaps a reference to personal work style, implied challenges or conditions that may exist in the two theaters.

After 3½ minutes into the task, Andy continued with the additional component of the task to remark on considerations that the data on theater attendance may imply and to give recommendations of how the problem solver may interpret the given situation as a potential manager. Andy addressed this part of the task by developing questions, rather than interpretive statements, regarding the data. He described the series of questions as those that he would possibly ask if he was interviewing for the job. His questions included

> I would inquire about Theater A's personnel during the week. Do they have more people working on the weekends? Does Theater B have the same amount of people working each day?
>
> Does Theater B show more movies during the week? How are the prices of the tickets related? Are they higher on the weekends?
>
> Do the theaters have any kind of promotional events?

The first questions focused on the number of personnel per day during the week for Theater A versus the number of personnel per day at Theater B. This would determine if there was a difference in how personnel may be assigned and if the data was being appropriately used. After the first two questions contrasted the placement of personnel between the two theaters, Andy developed four more questions to gather information to explain the differences in attendance between the two theaters. They were not as clear in creating a comparison between the two theaters. Upon completing the questions, Andy stopped the problem solving session immediately. The total time expended on the task was nine minutes.

Researcher observations. One goal of this task was to determine if the participants would interpret the mean in context and if they would move past that first interpretation to notice the difference in the variability of the data from the mean between the two sets. Part of the analysis of the data might include some consideration of other measures, like mode and median but of greater importance would be an examination of the dispersion of the data. Another goal was to see if they would use any other method of examining the data, such as a graph or software on a graphing calculator. By Andy's approach to this task, it did not appear that all of these aspects of the task were apparent.

Andy interpreted the mean clearly with reference to theater attendance, the variable through which the data was organized. This can be seen in the above statement and repeated here, "So, generally they are getting about the same…the same amount of people." During the problem solving session, he also compared the distribution of the data in a general manner. When questioned about this during the interview, the following exchange occurred:

R: Why did you choose theater B?

A: Umm, just because it was…there were about the same amount of people coming in each day.

R: Just because...

A: Theater A, you know, Sundays and Saturdays would be kind of…hectic, it seems like. Theater B would be more constant.

R: You referred to theater A as 'spread out.' What is that 'spread out' called?

A: Umm…[pause] It won't be distributive, would it?

The term he attempted to recall was range, the difference between the lowest and highest piece of data. Andy did not explore other concepts associated with measures of central tendency; measures of dispersion; nor, attempt to display the data in any other form.

This task did not elicit the level of cognitive or metacognitive activity that the algebra and geometry ones did. When questioned about the task, he said, "I was trying to…get some numbers. But, I can't really do this with [the] numbers too much. So, I was just trying to think of questions…" It was evident by his work that he did not develop a plan in the stages of engagement and transformation to mathematically compare the two sets of data resulting in use of a less rigorous visual/mental method of doing his comparison of the attendance of the two theaters.

Andy's response to this task was a departure from his previous methodical movement through the first stages of metacognition while addressing the tasks in algebra and geometry. He did not he develop a clear plan of action in the transformation stage which affected his implementation. Since he did not delve deeply into the features of the table, his approach appeared to be less defined and rigorous. As to the last stages of the cognitive/metacognitive process during this task, he continued with the same tendency to expend a minimal amount of time in evaluation, checking only the question structure and correcting grammar. There was no time expended in internalization, reviewing the process.

Bev

Bev was a unique participant in this study in that she is married and has a young child. I knew when asking her to participate that we would encounter some scheduling problems, but she was willing to work with me on finding available dates and times to meet. On the initial questionnaire through which the volunteers were asked to rate their confidence in their secondary mathematics skills on a scale from 1 to 10 (10 being very confident), Bev self-rated a score of 7 or 8 writing, “It has been a while since high school.” Her ambiguity about her skills appeared based on the perception of the amount of time since she may have experienced items based in the secondary mathematics curriculum. This implied that she may not have considered the mathematics that she has studied over the last three to four years in college as an extension, connected to and supportive of, the mathematics she will be teaching.

In the initial contact, the scheduling requirements of the study were discussed. It was agreed that for optimal results we should have four meetings closely spaced together to capture recalled thoughts during and after problem solving. We negotiated a schedule that fit around Bev’s coursework and other responsibilities. The original schedule was over two weeks. It had to be modified, but the series of appointments was completed in three weeks. The longest period without meeting was seven days. Bev’s situation was more complex than the other participants in that she was balancing work, and a child, but she arrived to each meeting ready to participate. I found that she had long periods of silence even though the concept of a think-aloud was discussed and she indicated understanding of the method of gathering data. This made it difficult at times to determine when she moved from one stage of cognition/metacognition to the next, particularly between engagement and transformation. She was more articulate about her thoughts in the interview process. So, excerpts from the interviews were added to the problem solving explanation. Her problem solving is described below.

Algebra Task

After Bev was given the algebra task, she was silent for 2½ minutes then commented, "It is hard." When prompted to articulation her thoughts with, "What are you thinking," she responded, "I don't remember this from high school." During this engagement stage, Bev was virtually motionless and silent.

After five minutes, it appeared that Bev moved into the transformation stage remarking, "I understand what it is asking me to do, I'm just…trying to figure out how." She returned to silence. Then, she questioned what the expectations were with, "Do you want me to draw what it would be like or do you want me to say it or…" The response was, "However you want to do it." In considering a method, she said, "That's a lot of squares to draw." She fell silent again. After 30 seconds, Bev could be seen moving into implementation as she went through a series of movements of picking up the pen, putting it down, picking up the calculator, and doing some calculations. She explained her thought processes in the follow up interview a week later with this exchange:

> B: I think at this point I was really thinking about how to put it on paper. And, that's when I, that's when I figured out the formula and I started punching in numbers [into the calculator] to make sure it added up right.

R: So, you had an idea of the relationship and then you were just trying numbers?

B: Uh huh. Like with this one. I put one in [the expression she developed] just to see if it came out with the same…That is the reason I jumped straight to twenty on here. I like working with a calculator rather than pen and paper.

At this point, eight minutes had passed. Moving into the stage of implementation, Bev began to express her findings with,

> The twentieth figure … is going to have twenty-one blocks across….[drawing]. Then, the middle … will be a square that is twenty blocks tall, twenty blocks wide. The bottom will have twenty-one blocks (see Figure 5).

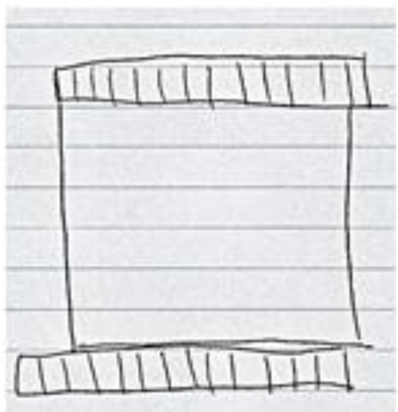


Figure 5. Bev's representation of the 20^{th} figure.

> Now …. [Bev wrote the expression $2(n + 1) + n^2$ before vocalizing.] To figure out how many blocks, it is two times (n + 1) plus n squared. So, two times 21 plus 400. Two times 21 is 42 plus 400 equals 442.

This process took approximately two minutes. During the interview, Bev was able to articulate her composition using the connections that she made to geometry and her strategy of working backwards in:

> Because each of these had one more than whatever figure it was [pointing to the top row] and so that is where the plus one [is]. And, there are two bars so that is times two and that was a perfect square in the middle.

At this point in the problem solving, ten minutes have passed. Bev reviewed the work for less than half a minute and moved on to the second component of the task.

Bev again became silent for two minutes as she determined how she would approach the second component of the task. She signaled her first idea for a solution to finding the figure with at least 10,000 tiles with, “I guess set it equal to the equation and then work it out that way.” She paused for about 30 seconds then wrote, “$2(n+1) + n^2 = 10{,}000$.” She fell silent for another two minutes. Bev then put down the pen and picked up the calculator indicating that she had changed her strategy. When asked about this in the interview, she said, “I didn’t want to work it out; so, I just plugged numbers in to be honest.”

Again, Bev was silent, working with the calculator. After one minute into this process, she said,

> Okay. The 99th figure would require 10,001 tiles and how I did that was plugging in numbers because 99 squared would be 9,801, … and when you take two times 99 plus one which is two times 100 or 200 which give you 10,001 which is the closest you can get.

While stating this, Bev was evaluating her expression for n=99, verifying the amount, and then she immediately stopped the recording. This work did not demonstrate solving an equation or inequality but using the algebraic expression to validate her guess in the mathematical method called guess and check. The problem solving session was approximately 17 minutes long. Her work can be seen in Figure 6.

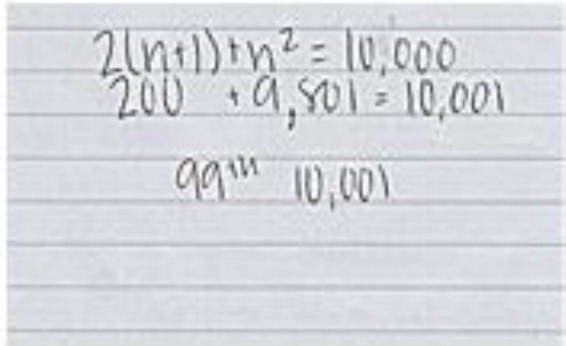


Figure 6. Bev's numeric solution to the second component of the algebra task.

Researcher observations. What was noticeable about Bev's problem solving was the strong use of visual and algebraic representations. But, during the task solution, she did not vocalize her process or write any statements summarizing findings. She did not use any numeric methods such as a table to express the relationship of the figures and the number of tiles. She only articulated findings after coming to conclusions, without vocalizing how she came to those conclusions. This had to be drawn out as much as possible within the interview.

The figures supported Bev's finding of a common decomposition and the relationship between the number of the figure and its shape. This can be seen in Bev's explanation, repeated from above:

> …each of these had one more than whatever figure it was [pointing to the top row] and so that is where the plus one [is]. And, there are two bars so that is times two and that was a perfect square in the middle.

She exhibited strong visual skills with an understanding of conservation of area and basic properties of a square. Her method of connecting directly to geometric shapes was a departure from the method used by the other participants who based their solutions on the idea of columns and rows forming a rectangle. Bev recognized the value of using algebraic symbolization, even though she did not model the relationship using function notation or use it in the second component of the task. In the interview, Bev expressed the need for a formula to find the total number of tiles in the twentieth figure. But, when questioned about this necessity, she seemed challenged to explain why it might be advantageous to develop one or how to produce one:

R: …how did you know that you needed a formula? [pause]

B: Every math problem has a formula.

R: Okay. As I was looking at your work, I wondered how did you know you needed a variable and what to assign to that variable?

B: I don't like questions like that. I don't know.

This may suggest a superficial understanding of the use for algebraic modeling. Bev produced an algebraic expression because that is what is done with an algebra task.

As already mentioned, Bev's articulation of some thoughts and processes were minimal, although the process of the think-aloud method was explained in the initial meeting and reiterated before beginning the first task. Apparently, she performed the majority of the stages of engagement and transformation mentally or on the calculator that was provided. When asked about using the calculator, Bev said she preferred it because it was easier and faster allowing her to change values in an expression quickly. Bev did not demonstrate the use of writings or tables in her responses during her problem solving except in the answer to the second component of finding the figure with at least 10,000 tiles.

In making a statement about Bev's connections across topics in mathematics, it may be relevant to comment on her use of geometry in the form of recognition of the central square within the figures and conservation of area. Her visual cognition was strong in forming this model. From her work, it would not be appropriate to conclude that she connected this task to function or sequence. Bev did find the pattern. This can be seen in her use of the strategy working numerically backward, which was employed to verify the structure of the geometric model. Other than that, there was not any mention, verbally or written, that she connected to function.

Bev was asked to write about her problem solving process. While writing, she was asked to reflect on it and add any thoughts or methods of solving that may occur during the process. As found with other participants, the writing was basically a step-by-step recounting of what she did. In the first steps of this portion of the task, Bev wrote:

> After reading the problem I began to think of a formula that would satisfy the first three figures. The formula I thought of was $2(n + 1) + n^2$. I plugged a 1 in first into the calculator to see if the formula worked. Next I plugged a 2 into the formula…

There was no explanation of how she derived the expression by finding a common decomposition. As the writing continued, the procedural structure provided little insight or new thoughts. Bev was successful in completing the entire task, finding an expression by which the total number of tiles may be found and finding the number of the figure that would have at least 10,000 tiles. But, she exhibited reluctance to express any of the concepts behind her procedures, as exampled by the response: “I don’t like questions like that. I don’t know.”

In the frame of a cognitive and metacognitive analysis of Bev’s problem solving, her instances of evaluation were very limited and her expenditure of time on internalization non-existent. She was appropriately proud that she accomplished the task. But, if she had taken the time to review her work (during the writing task or simply after completion), perhaps the complete context of the task would have occurred to her.

Geometry Task

During the initial stage of engagement, Bev reacted to the task with, “I hate quadrilaterals.” After 1½ minutes, she moved into transformation. She explored the problem with,

> It looks like…the first one, the small one ABCD, is flipped down where A and D are at the bottom and then…let me see…[using hands] then flipped around that way [motioning with hands]. Or, you could say that it is rotated at C and then flipped.

Two minutes into problem solving, Bev moved into the stage of implementation by picking up the ruler and drawing a reflected figure over the line x = 7.5. This was not the reflection she described. She acknowledged this in, “I am probably just doing this completely the opposite from what I just said.” Labeling the new image with vertices A, B, C, and D, the same as the original, she proceeded to reflect the image over y = 7.5 and again labeling A, B, C, and D. During this series of reflections, Bev did not label any lines of reflection nor did she label the corresponding vertices appropriately. These transformations may be seen in Figure 7.

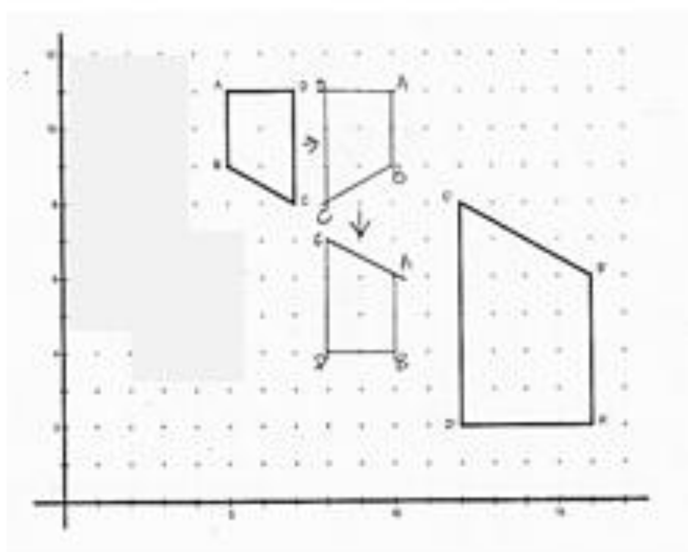


Figure 7. Bev’s initial attempt of finding a similarity transformation from ABCD to A’B’C’D’ which included a reflection and rotation.

At this point in the task solution, four minutes from the beginning, Bev stopped to draw a conclusion, “It looks like the figure is…reflected over line…CD and then rotated…well, no, that wouldn’t be rotated.” She recognized that there were flaws in the description of her task solution.

There were two: (1) Bev reflected over x = 7.5. $\overline{CD}$ = 7. (2) The other problem was her ambiguity about the second transformation that was a reflection.

Bev reconsidered her approach (an example of debugging) saying, “Well, it could be rotated around C.” She was offered a fresh task sheet, but she said it was not necessary. Her second attempt can be seen in Figure 8. At 5½ minutes, Bev began again by redrawing the original ABCD at a new position, four units to the left, on the grid. In the interview three days later, she was asked if she realized that moving the figure in that manner changed the problem. Her reply was somewhat correct with, “Well, you are still moving it with transformations.”

Bev continued by rotating the figure she placed around the point (3, 5.5), not the intended C. Again, she did not make any notations to define the transformation, in this case she could have supplied an angle of rotation; and, though she had the idea of a pivot point, she did not rotate around the one she chose (see Figure 8).

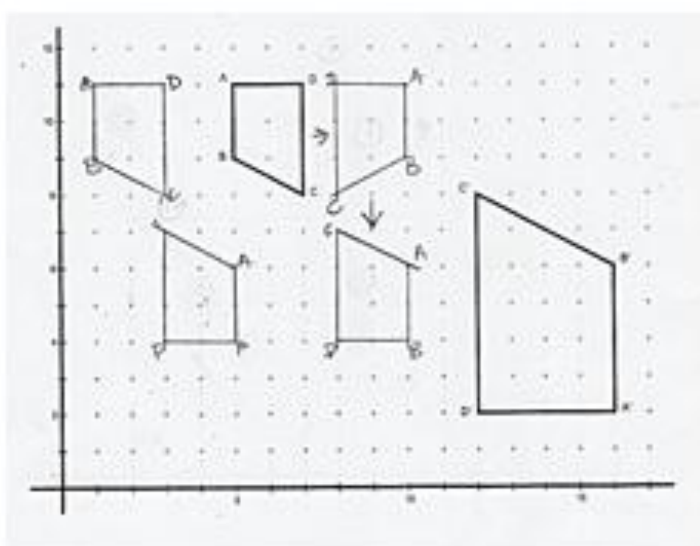


Figure 8. Bev’s second attempt from the new position which included a rotation and slide.

Then, she asked, “Does it matter that it is bigger?” and put her pencil down. Bev then made the statement, “So, ABCD is to A’B’C’D’.” When given the prompt, “What can you say about the original and the final one?" Bev responded, "Well, I know how it moved. I got two different ways that it moved actually. I probably could move another way too, but…Yeah, I'm done."

Researcher observations. This task appeared difficult for Bev. She was aware of the different transformations, as revealed in the interview, but she did not effectively use the necessary terminology or properties of the transformations during the problem solving. Bev did become aware that there were multiple ways to move ABCD onto A'B'C'D' and the images she produced were congruent. But, her knowledge was not sufficient to complete the task.

The two paths that Bev chose, two reflections or a rotation, were valid initial steps of transforming ABCD onto A'B'C'D'. With both attempts, she had the figures oriented so that a translation and dilation would move one of her images onto the final A'B'C'D'. As we were watching the video recording and dissecting her work, we stopped the recording at the point she put her pencil down and I asked, "Which bothered you most, moving it over or making it larger?" The phrasing of my question could have helped her understand that two transformations were necessary, not one. I hoped that decomposing the transformation would allow her to recognize the two that were required. Her response was, "I don't know." My concern at that point was her recall of transformations. I asked another question, "Do you remember the terms [for transformations]?" Her answer was, "Yeah, I remember reflections, slides, and rotations. To me it is flip, slide, and move however many spaces over." There was not any further elaboration.

One of the transformations necessary was dilation. Bev did not introduce the term, but she did visually recognize the relationship in size of the original two figures and stated it verbally as noted below. In the last step of her writing, she attempted to express the relationship between ABCD and A'B'C'D':

After reading the problem, I looked at the first figure to see what I could do to it to get the second figure.

At first I thought to reflect over segment CD then flip the figure vertical.

After drawing the figure and looking at it again I realized the figure could have been rotated at point C.

Quad ABCD is two times smaller than Quad A'B'C'D.'

The last sentence above shows her recognition of the change in size but she could not express it as a transformation or in a mathematically correct manner. This demonstrated a lack of precision in language since the quadrilateral ABCD is half the size of A'B'C'D' rather than "two times smaller."

Although mechanically she maintained congruent shapes during her transformations, it is not evident that Bev had the necessary terminology and procedural skills to express how the figures were produced. This can be seen in:

R: Okay, what is it called when you flip it over?

B: [pause] Okay, I forgot.

R: You flipped it once and then you flipped it again.

B: Uh huh. I flipped it twice…. Wouldn't it be a reflection?

This effort to relate the action to the term reflection indicated that a more technical description of a geometric transformation was not possible.

In contrast to the algebra task, Bev did not come to a satisfactory conclusion to this one. Evaluating the problem solving from a cognitive/metacognitive frame, one could say that the lack of cognitive structures in the form of concept understanding of the properties of transformations limited her progression through Yimer and Ellerton's (2010) stages. When questioned about her experiences with transformations during the interview, Bev admitted that she did not feel that she had an extensive foundation in her high school or college experiences. Her college geometry course was proof based and primarily in Euclidean Geometry.

Statistics Task

After a minute of engagement, Bev gave an initial reaction to this task with, "I don't really have to write anything." I encouraged her to write any response to the data down. Shortly after that, she moved into transformation as she began an interpretation of the data through a visual inspection. She described the two sets of data for Theater A and B in, "Well, B has got steady numbers. From 92 …yeah, and the lowest is 65. [Theater] A … one day you have 110, the next day you have 24, and one day you have ten." Bev did not extend this idea by discussing or computing the range as a way to compare the two sets or comparing the items of data to the mean. Moments later, she commented, "But, the average is the same." Again, there was no expansion on the statement or evidence of an effort to interpret the implications inherent in a common mean.

After the description of the two theaters and 2½ minutes into the task, Bev moved to implementation when she stated, "I guess I would prefer to work at Theater B." She explained her choice in, "The numbers are still…they are steady. So, you get about the same number of people each day. Theater A is…up and down…" There was not an expansion on this decision during the problem solving session. During the interview, Bev did elaborate saying, "I guess because these [data for attendance] with B stay steady so it would give you something to do…if you worked at Theater A on Thursday night, you would be just sitting there half the time."

Bev moved to the second question in the task, recommendations suggested by the data, without taking any notes or writing any conclusions. After a pause of 1½ minutes (five minutes into the task), she interpreted the question with, "…the question is about personnel?" She verbally compared personnel requirements for Theater A with those of Theater B, noting A would need more people on Sundays and Saturdays, Theater B would need the same amount each night. To this point in the task, Bev had not written anything down.

Finally, after 6 minutes into the task, it was necessary to prompt her to write with, “Can you write those thoughts down?” Bev began the process of writing with her first conclusion of which theater she would prefer to manage. Her process was to write, then read her conclusions:

Prefer theater B

-steady attendance numbers

-Theater A not predictable because one day you may have 100 people next day10 people.

Bev stayed consistent to her recognition of the relationship of attendance with requirements for personnel but failed to take into account the fact that the data was a summarization over a year’s time. The attendance for Theater A would be predictable. Bev was now eight minutes into the task. She continued for the next two minutes to respond to the second component of the task following the procedure related above, writing and then reading her work:

> Theater A – If the same numbers of people work each night, some nights would be slow while others busy.
>
> Theater B – You can have the same number of people working each night since the numbers are steady.

Bev appeared focused on personnel. It became apparent that she was assuming that Theater A would be manned with the same number of personnel each night, which was not a factor, given in the task. After completing these writings, Bev immediately put down her pen and stopped the session. The session lasted approximately 12 minutes.

Researcher observations. The goals of this task, to interpret the mean in context, to move past the mean to consider how the data varied with respect to the mean, and to use alternative ways to examine the data, were not apparent in Bev's problem solving of this task. That the underlying premise for the task was not familiar to her was explained in the interview two days later. She commented, "I've never seen one of those problems in math before. I mean like…Okay, they would have the table and they would ask you questions about it. You would have to find the average and all that." Given her reflection, it appeared that without specific questions stated in the problem, she was at a loss regarding what mathematics were expected in this data analysis task.

Primarily, Bev mentioned the concepts of mean and range, though range was not directly identified. She did not directly attach or interpret the mean with reference to attendance. Also, it was unclear whether Bev recognized the table as a condensing of attendance over a year. This is shown in the writing of recommendations. She wrote that attendance was unpredictable for Theater A. The data was described as accumulated over a year or typical. So, the attendance for any week should have been fairly consistent with the data given.

Her use of the concept of range can be seen in the following excerpt of the interview:

R: One of the things you mentioned was the attendance for Theater A was kind of all over the place, right?

B: Oh yeah. You can have 124 on a Saturday and then on Thursday have 10 people.

R: What is that called when you look at the difference between the highest and lowest?

B: Gosh, I don't know…What's that word?…I know there is a word for it. The range is very broad.

The last sentence demonstrated that Bev recalled the term range and could use it in context when pressed. We continued with some recall of the concept of median as "the middle." So, it appeared that she had a basic concept of measures of central tendency. But, this task did not elicit their use by Bev.

It was not apparent in Bev’s work or the video recording that anything more than a visual perusal of the data was used. Similar to other participant’s problem solving of this task, she did not develop a plan in the stages of engagement and transformation to rigorously approach the data. As she attempted to make sense of the task, she began interpreting it in relationship to her own experience, particularly during the interview. Here is an example:

> B: I figured Wednesday would have been … had the lowest numbers because that is a church day. That is one of the church days.
>
> R: Now you are associating the data to your life, right?
>
> B: Yeah, well, that is what I was thinking at first. When I was looking at it because Sunday … I would have figured 110 would be on a Friday night because on Sunday everyone’s getting ready to go to work the next.

Bev’s struggle to develop some context to the task and formulate answers can be seen in her reactions to the task and the limited scope of her responses to the second component. The process of reviewing the recording of the problem solving session and interview did provide some reflection on the task, but it was not apparent that it influenced her to move into the stages of evaluation or internalization.

Cynthia

On the initial questionnaire whereby participants were asked to rate their confidence in their secondary mathematics skills on a scale from one to ten, Cynthia self-rated a score of eight. Her rationale for the score was, “Because I have worked in the math lab as a tutor. By me being able to work there, it has allowed me to brush up on all my basic math skills and my upper level math skills as well.” The mathematics lab was open to students for study and tutoring in remedial and Precalculus algebra. As a preservice secondary mathematics teacher, Cynthia was required to participate for one semester in the one-on-one tutoring, but she continued to work in the lab for a second semester. She was employed there the semester of this study. The rating was based in her assessment of those experiences.

Cynthia understood the requirements for the study and when contacted through email agreed to meet. We set up a schedule recognizing that the meetings should be relatively close together. The plan we developed set our meetings twice a week for two weeks. Unfortunately, Cynthia became ill after the first meeting, which led to a gap of about a week between the first and second meeting. But, after that interruption, we were able to continue and complete the series. She arrived on time and demonstrated persistence in her attempts to solve the tasks through the think-aloud method. Her attempt to solve the algebra task follows.

Algebra Task

After less than a minute of engagement in which she read the description of the task, Cynthia moved into transformation. During this stage, her primary method of interpreting the pattern was counting the tiles and beginning an informal table. The table associated the number of tiles with the number of the figure. In phrases, she noted that the second was the first increased by five and the third was the second increased by seven. This became the basis for her initial conjecture on how to find the number of tiles in the 20th figure.

Two minutes into the problem solving, Cynthia appeared to move into implementation writing, "The twentieth figure would be in the shape of the original figure but it would contain at least," leaving it open-ended. She began applying the conjecture that each successive figure was the previous plus five and then plus seven. So, the total number of tiles for the fourth figure was found to be 22. She then drew a figure with 22 tiles. The drawing can be seen below in Figure 9. She failed at this time to realize that it did not visually fit the pattern of the given figures.

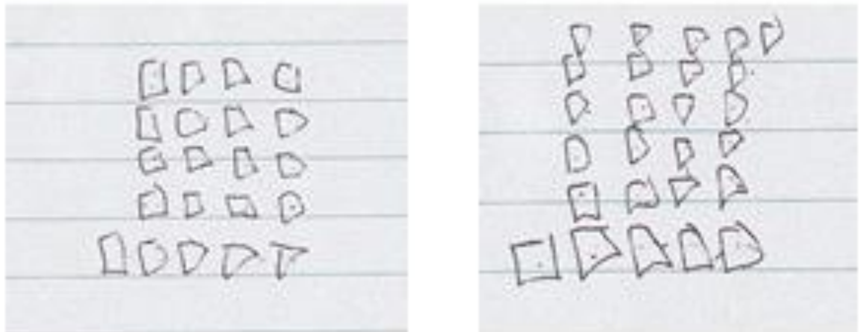


Figure 9. Cynthia's two sketches of the fourth figure. It became apparent to Cynthia that he first sketch did not fit the pattern of the figures given.

Cynthia continued the pattern of plus five, plus seven, listing the figure number and the conjectured total of tiles beneath. She found the number of tiles in the twentieth figure to be 118. Returning to her initial writing, she completed the sentence with, "but it would contain at least 118 tiles." She explained her algorithm aloud and in writing, "The way I came up [with] the amounts is b/c in figure 3 the increase by five and figure [4] the tiles increase by 7 but the figure kept the shape-outline as figure one." Then, she read the second component of the task, finding the figure with at least 10,000 tiles. This occurred nine minutes into the task solution.

The second component of the task appeared to prompt Cynthia to reassess her previous work. On the recording, she could be seen to hesitate and return to the sequence of figures and to her drawing several times. This period of evaluation and attempt at debugging lasted 4½ minutes. Her confusion was expressed in, "I feel like I am missing something." She continued in this sense making process for another five minutes and then said, "It's increasing by the odd numbers instead of by five or seven; therefore, it would be more than 118 tiles." Her drawing helped support this recognition that there was a problem with her conjecture of the pattern.

Cynthia moved back into the stage of implementation. This was 19 minutes into the task solution. She returned to her original listing and scratched out the previous values for the figures and began again adding consecutive odd numbers. Finding the terms of the pattern recursively in this manner allowed for error. Cynthia skipped the 13th term and almost considered the answer complete with finding the 17th term. She finally ended the process with the 20th figure having 401 tiles, 41 tiles less than the true amount. Cynthia wrote, “So, therefore the shapes increase by an odd amount of tiles.” Evaluating her statement, she returned to draw an appropriate sketch of the fourth figure found above in Figure 9.

Twenty-five minutes into the session, Cynthia returned to the second component of the task and attempted to find the figure that contained at least 10,000 tiles. Her first impulse was to make a list of odd numbers. She computed up to the 31st figure and then became very quiet. For 4 minutes, Cynthia continued to struggle. At that time, I asked if she wished to end the session and we agreed to do so. Cynthia worked on this task approximately 32 minutes.

Researcher observations. There appeared to be two factors that contributed to Cynthia’s struggle with this problem. One was her tendency to move quickly into implementation without expending time in engagement and transformation to plan. The second appeared to be a lack of connections to the geometry and pattern features that would support solving this task. Cynthia’s lack of familiarity with a sequence having a geometric structure was clear in her interview response to the question “Have you ever seen a problem like this before?” Her response was “No, not really…I had seen it in numbers but not in shape form.” Her drawings expressed this. It was apparent that she did not make the connection to area that other participants did. Looking at both sketches of the fourth figure, it can be seen that she used individual tiles rather than a rectangle with two additional tiles (see Figure 9).

The role of drawing had a significant impact on her problem solving. In response to reflection on the initial drawing, she changed her method for finding the total tiles in the 20th figure to one of adding consecutive odd numbers. Cynthia gave the following explanation of the role of drawing in the interview:

> I was sitting there looking at that one [the left drawing below] and I think that [inaudible] there were just twenty-two blocks and I understood that when I drew twenty-two blocks it would not make the top row with the part that is extended. And that's when I was like I am missing something.

So, the recognition that first drawing did not fit the pattern supported Cynthia in finding the correct pattern of adding consecutive odds. In fact, she returned to drawing to confirm the pattern in the fourth figure as seen in Figure 9.

Cynthia was not able to produce an algebraic expression or function to model the relationship between the figure and the total number of tiles. She did not associate the recursive pattern of consecutive odds (i.e., 3, 5, 7…) as differences with a function containing perfect squares (i.e., 1, 4, 9, 16…). Also, she was not familiar with a method called *finding finite differences* to produce the function. She was familiar with arithmetic and geometric sequences referencing them in tutoring in a math lab. In the interview, she expressed experiences with the developmental students. It is possible working with these simpler sequences may have interfered with her ability to find a relationship in this task. This possibility may be seen in the following interview excerpt:

> R: Had you ever seen a problem like this before?
>
> C: No, not really, not in that wording. I had seen it in the numbers but not in shape form. Like find the pattern of it.
>
> R: Oh, as numbers. You think that it would have been easier for you if you had seen it as 5, 10, 17…
>
> C: Yes, ma'am. I think if it had been like that. Because that is what they normally have, like when they [tutees] come in with the basic algebra in the math lab. When they have the patterns and you tell what the patterns were. I think I like numbers better than figures.

The last part of this interchange may reflect her frustration with the task in that she must have felt confident in finding the simpler arithmetic or geometric sequences that were part of the developmental program.

When Cynthia approached the second component of the task, to find the figure with at least 10,000 tiles, her frustration was evident. In the interview, she spoke about the reason for this.

> C: I think when I got more to the part of finding the ten thousand tiles it kind of got frustrating a little. And I was trying, kind of over thinking, trying to figure it. And, also, it frustrated me when I couldn't think of an equation, to come up with one. I could write it by words but I just didn't write an equation for it.
>
> R: Why were you trying to find a formula?
>
> C: Because if I could have had the formula I could have figured out what number, what number figure would have ten thousand tiles in it. How many actual tiles would have been in the 20^{th} figure.

Cynthia, similar to some of the other participants, had difficulty moving from a recursive description of the sequence to one based on the number of the term. She had developed an informal table based on the number of the figure and the total tiles but could not model the relationship using function notation.

During the session, Cynthia's verbal expression was a mixture of declarative knowledge about the task and procedure. Her written work was unique in that she used sentences to express conclusions, such as "The 20^{th} figure would be in the shape of the original shape but it would contain at least ~~118~~ ~~290~~ 407 tiles." The writing acted as communication and an object of reflection/organization.

When she was asked to write her method for working on the task, Cynthia's composition was different than the previous participants' in that it addressed the struggle she encountered,

> The problem was kind of tricky at first. Then as I continued to focus on the problem more things came to me. As I continue I got a little confused on how to find a description of the problem algebraically, also when figuring out what figure would require at least 10,000 tiles. But I feel that as I think more about the problem I feel that it will come to me (answer of course). At some point I did get a little worried as I tried to figure out the problem, and what the answer could be. I feel that there was a lot of information that was given in the question. When I first looked at the problem I was pretty calm about it, but as I began to work the problem out I could feel myself over thinking the problem. There were some simple things that I forgot about because I was trying to put more into the problem than needed.

This writing revealed that Cynthia was aware that she was "missing something" that would have supported her successful completion of the task. She understood that having a model of the relationship between figure and total tiles was a key element of the problem solving. Unfortunately, by her own words, she did not have sufficient prior experiences or did not connect with essential concepts that would have supported the development of the relationship in function notation.

Examining her work from a cognitive/metacognitive perspective, Cynthia gave an excellent example of debugging. It was during her process of reflection on a drawing that she recognized that her original conjecture, add five and then seven, was not correct. She then approached the task again and did come up with a valid conjecture, add consecutive odds. Cynthia did move through the first three stages of the cognitive/metacognitive framework: engagement, transformation, and implementation. It was also apparent that she used evaluation; otherwise, she would not have realized the error in her original conjecture. Regrettably, we agreed to end the session due to her frustration before she achieved a satisfactory solution.

Geometry Task

Cynthia began by reading the task aloud and taking one minute to understand the orientation of the two figures. In the recording, she used the pencil to repeatedly touch the vertices of ABCD and then draw her focus to the corresponding vertices of A'B'C'D.' After that engagement, she began to plan by visually and verbally connecting corresponding sides $\overline{AB}$ and $\overline{A'B}$.' She paused and then concluded, “It’s rotated…clockwise…let’s see…180.” Then she wrote at the bottom of the task sheet, “The quadrilateral was rotated 180° clockwise.” She continued into implementation of her first transformation by sketching a small set of axes centered at (6, 9) and marking an arc, rotating the figure, and labeling the corresponding vertices.

Cynthia did not immediately implement the next step of a plan. Viewing her actions on the video recording, it could be seen that she returned to the stage of transformation to determine her next step and the method to accomplish it. She engaged with her first image using her pencil to retrace it and indicate a slide as the next step. During these moments of planning, apparently, she decided the rotation around (6, 9) was not the optimal point around which to rotate. So, she experimented with a rotation around Point B. The first image was erased. Returning to her writing, she completed the sentence about rotation with, “at pt. B” and then, using her mechanism of imposed axes, rotated around B explaining, “…okay, with rotation it doesn’t change [shape], it just rotates around the point.” This can be seen in Figure 10.

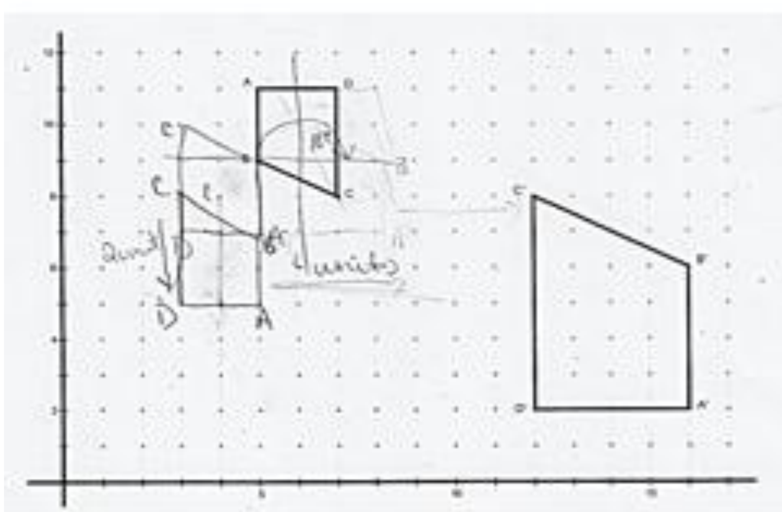


Figure 10. Cynthia’s first set of transformations: rotation around B, translation down.

Again, Cynthia returned to transformation trying to plan her next move. She considered perhaps using a reflection of some kind but returned to her first idea of using a slide. Moving again into implementation, she added to her writing, “Slide the quadrilateral ABCD downward two units.” Cynthia then drew the second image two units downward aligning C and C’ with the line y = 8; she paused and retraced her work.

Moving back into transformation, Cynthia began to plan her next move to align her image with A’B’C’D.’ She decided to use a slide four units to the right. She added to her writing, “Then slide the quad ABCD 4 units to the right.” She used the width of her image as her unit (contrary to her use of one length between the dots on the dot paper as one earlier). At this point in the problem solution, Cynthia had been working 14 minutes. For the next four minutes, she reviewed her work repeatedly. On the recording, she was seen retracing figures and adding notes to the problem sheet.

After four minutes passed without any progress, I asked Cynthia if she was confused. Her response was, “I’m not really confused. I just see, I think I see another way…and I am kind of thinking it is different than this one.” She was offered a new task sheet that she accepted and then she began to explore a second way to solve the task. At this point, 21 minutes had passed in the session.

Cynthia began by enlarging ABCD. Repeatedly counting the corresponding sides, she then implemented the action by first writing at the bottom of the second task sheet, “In order for quad ABCD to look like quad A’B’C’D’, $\overline{AB}$ and $\overline{AD}$ has to be extended 2 more units and $\overline{CD}$ has to be extended 3 more units.” Then, she proceeded to sketch the figure using this description and reproducing $\overline{BC}$ by connecting the endpoints of $\overline{AB}$ and $\overline{CD}$. Again, Cynthia had a long pause, reflecting on her work. This period of time was 6 minutes. This second attempt at a solution is found in Figure 11.

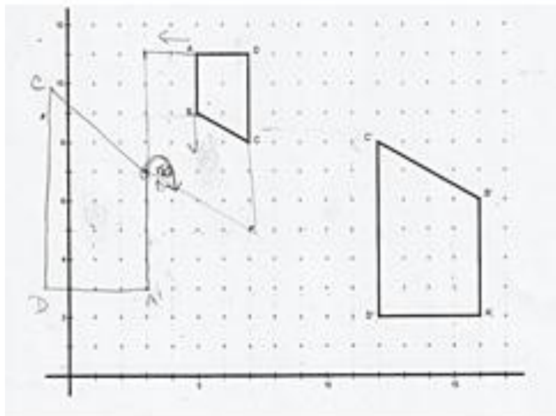


Figure 11. Cynthia's second similarity transformation: enlargement by two, rotation, and slide.

Thirty-one minutes had passed since beginning the task. When prompted, Cynthia replied, "I am trying to think of the transformations, all the transformations…It has been so long since I have thought of them." Recognizing that she was becoming fatigued, I asked, "Is there any way you can move that [image] so that it is on A'B'C'D'?" She responded with,

> What you could do is … you can rotate it around B clockwise. I think … 180…. It can't be 360 because it will be back in the same position; so, … if I move it … it is 90 or 180. I don't think 90 will quite be it. I need to be sure. It will be 180 clockwise.

She sketched the image. She added, "and you slide it to the right…nine units." She checked her slide and wrote, "Then rotate the quad ABCD a 180 degree clockwise at point B. Next move the quad ABCD 9 units to the right." She recounted for the slide and corrected her writing to 13 units (counting the slide correctly on the dot paper but not taking into account the slide down by one unit). Cynthia completed her writing with, "Therefore, quad ABCD is onto the Quad A'B'C'D'. $\overline{B'C'}$ reflects $\overline{BC}$, $\overline{A'B'}$ reflects $\overline{AB}$, segment $\overline{A'D'}$ reflects $\overline{AD}$, and $\overline{C'D'}$ reflects $\overline{CD}$."

Cynthia then rechecked her units and reviewed her conclusions ending the session after 41 minutes.

Researcher observations. Cynthia's approach to this task involved repeated movement between the stages of transformation, implementation, and evaluation rather than the development of a plan in transformation and the implementation of it in a continuous manner. When asked about this in the interview seven days later, Cynthia replied,

> I…I remember in high school some doing it, transformations, but not a whole lot. Mostly in high school we did geometry…we did just like the area and…umm…constructions, stuff like that. We never really…she taught us how to do transformations, well, she taught transformations but not a whole lot.

She confirmed familiarity with reflections and rotations in her high school experience. But, Cynthia had not encountered these concepts in her college geometry course. She also verified that she had not used dot paper in this context before.

Her mechanics for producing a rotation were to use a small set of axes centered on the pivot point. This was unique to the participants, so it became an object of interest during the interview. She was asked if that was how she was taught to do rotations. Cynthia said it was symbolization that she developed to support her understanding of rotation. The crosshairs provided a small set of axes that helped to determine the degrees of rotation, 90°, 180°, 270°, or 360°. During problem solving, Cynthia demonstrated her understanding of the effect in a rotation of 180° versus 360° in the passage, "I think…180…It can't be 360 because it will be back in the same position."

Cynthia's persistence in finding a solution to the task resulted in two attempts. She appeared to encounter a barrier in her first attempt: the enlargement of her image of ABCD. In the interview, we explored this difficulty in the following exchange:

C: I think more now I am thinking how… to get it from there to the other figure. And, I was kind of thinking … should I slide the figure or … And, I am also thinking, this figure is smaller than that one. *What is the transformation I would use to make it bigger?* That is what really got me kind of stumped. I couldn't actually remember the transformation that is used to … to grow, to stretch the figure.

R: So, what really stumped you was anticipating making the figure larger?

C: Uh hm. This is when I decided to start over.

From this exchange, it was clear that when Cynthia had to confront the enlargement of ABCD, a barrier existed. To explore the effect of vocabulary on her progress, we had the following mostly one-sided question and answer.

C: And, I was just trying to think, 'what is the transformation … what is it?'

R: So, you were seeking the name of the transformation?

C: Uh huh.

R: Was that slowing you down?

C: Uh huh.

R: Do you think just having the name for it would have helped you?

C: Kind of. It would have given me an idea of where to go next. More so than just sitting there, just trying to think on my own and … fumble through all the stuff that was going through my head. I think it would have helped me more.

R: That is interesting. It is called dilation.

C: Uh hum.

R: You looked it up, didn't you?

C: Uh hum.

Although Cynthia struggled with this term/concept, she was expressive in her declarative knowledge at other incidences in the series of geometric transformations. She was able to articulate her steps in phrases such as "rotate clockwise around point B" or "slide nine units to the right" which made her process, though complicated because of the two attempts, easy to follow. Cynthia produced two writings. Even though they are included above in the problem solving description, they are repeated here in their entirety to give a very clear picture of her solutions and the difficulties she came across. During both attempts, her method was to write first and then draw the image that was described. The writing for the first attempt (see Figure 10) at a solution was

> The quadrilateral was rotated a 180°clockwise at pt B. Slide the quad ABCD 4 units to the right.

Both of these transformations were appropriate except her designation of 4 units was based on the width of the image produced, not the units according to the dot paper. Notice that she gave the point of rotation, the degree, and a direction. It is apparent that she encountered a barrier while contemplating the next transformation that was to enlarge the figure. The writing for her second series of transformations (see Figure 11) was

> In order for quad ABCD to look like quad A'B'C'D', $\overline{AB}$and $\overline{AD}$ has to be extended 2 more units and $\overline{CD}$ has to be extended 3 more units downward. Then rotate the quad ABCD a 180° clockwise at point B. Next move the quad ABCD 13 units to the right. Therefore, quad ABCD is onto the quad A'B'C'D'. $\overline{B'C'}$ reflects $\overline{BC}$, $\overline{A'B'}$ reflects $\overline{AB}$, $\overline{A'D'}$ reflects $\overline{AD}$, and $\overline{C'D'}$ reflects $\overline{CD}$.

In this second attempt, Cynthia tackled the barrier that she hit in the previous one at the start. She enlarged the figure using a visual recognition that the line segments that composed the quadrilateral needed to double in length. So, her effect was an enlargement without using the rules underlying dilation (i.e., a point of projection and ratio of enlargement).

It is noticeable that she focused primarily on corresponding line segments rather than vertices as other participants did. She was also the only participant to indicate angles of rotation on her task sheet and to use appropriate notation for segment in her writing. It was unfortunate that she used the term *reflect* when *corresponding to* would be more correct.

What was evident in Cynthia's problem solving was her repeated movement through the stages of the cognitive/metacognitive framework. In her persistence to respond to the task requirements, she would engage with presentation, develop a plan to accomplish one transformation, perform the transformation, and then spend time evaluating what was done and engaging with the result to produce the next step.

Statistics Task

After one half of a minute in engagement indicated by a reading of the task description and a short pause, Cynthia continued into the stage of transformation noting the high attendance of Theater A on Sunday. Her interpretation of the data was continued in, "For Theater A, Wednesday and Thursday are their slowest days…which in turn…Theater B has a constant flow throughout the entire week rather than Theater A." This comment illustrated that Cynthia had acknowledged the variation in the attendance of Theater A versus the relatively more constant attendance at Theater B, but she did not make any attempt to describe the differences in the attendance in a more precise manner. There was no attempt to find the difference between the highest and lowest amounts of attendance for the two theaters or the variability with respect to the common mean. This implied that the expectations of the task, to interpret and compare the two sets of data, were not apparent to her.

After two minutes, Cynthia moved into implementation. Over the next six minutes, all of her thoughts were expressed aloud as she wrote her responses to the task. There was very little additional expression of thoughts. She selected Theater B as the one that she would choose to manage explaining why in the following writing:

> Theater B because there is a constant flow of people throughout the entire week rather than theater A where certain days there is [pause] a rush of people then on others not so many. By theater B having a constant flow of people the employees as well as the

> customers, they will not have [a feeling] of being overcrowded whereas in theater A they would have moments on certain days.

Shortly after this writing, Cynthia re-read the second component of the task concerning recommendations and wrote (while verbalizing):

> For the Theater A, some of the recommendations I will make ... on the days that they're more customers have more staff on duty rather than the regular staff.
>
> Theater B, I would say they need to promote the theater more so that they could increase on the flow of customers.

The totality of time spent to this moment in the problem-solving task was eight minutes.

After re-reading her written thoughts, Cynthia paused for a minute. Then she said, "Let's see…the means are the same." It appeared that she was thinking that there may be some implication to be made about the common mean but it was not explored or verbalized. She paused and then ended the session. Cynthia spent approximately nine minutes on the task.

Researcher observations. Cynthia's response to the data did not appear analytic in nature similar to other participants. This can be seen in her initial lack of expressing an analysis of the common mean. When pressed on her thoughts about the data and the mean during the interview two days later, Cynthia responded with

> At that point, I believe I was just … [pause] in a way just trying to make sure, make sure the numbers that I figured up were … because I noticed that they had the same mean. And, I was looking at that and like if they have the same mean then … throughout the week it is umm … how can I put it … Even though Theater A on certain days had more than Theater B, they still averaged out to have the same mean even though A may have more than B some days, A wouldn't have any on some days, B would have an average crowd.

In these comments, it appeared that she was struggling with how to express the difference between the mean and attendance at Theater A versus the mean and attendance at Theater B. She said, "B would have an average crowd."

To discuss the difference in the daily attendance between the two theaters, Cynthia could have used the concept of range. In the interview, there was an attempt to help her recall this term. At a point in the video recording when she mentioned the data varied, we stopped and I asked:

> R: Okay. So, Theater A, the numbers 'varied' more. Do you remember how you may describe this using terms from statistics, measuring the difference in the numbers?
>
> C: Difference of the numbers. Is it … I just had statistics. No, I can't recall it.

This confirmed the idea that Cynthia did not recall the terminology associated with the comparison of the two sets of data. So, this path of discussion was not pursued further.

That Cynthia's choice to be manager of Theater B was not based on an analysis of the data could be seen in the following passage from the interview:

> Yeah, I think it was because of the constant attendance that it had. And, I saw Theater A had days where it would just be slow. And, most of those days you would be bored out of your mind. And then on days that they have … the largest number, attendance, people tend to get overwhelmed when there is so much going on and you can hardly pay attention. So, that is why I choose Theater B because I know that … you can work with attendance at Theater B. You can build it as you go and you won't just get in over your head dealing with a theater with a crowd.

Cynthia's problem solving may be perceived as limited in scope in that she did not vocalize the use of data analysis techniques to differentiate between the data sets except for a vague reference to range and the data of one theater's attendance as being close to the mean. During the interview, it was difficult to delve deeply into her method of comparing the data in that it did not appear to be structured or clear.

To interpret her process in terms of a cognitive/metacognitive framework is problematic. Similar to other participants, the limited scope of Cynthia's application of data analysis concepts in the stages of engagement and transformation did not result in a rigorous plan for approaching this task. Her decisions and analysis appeared to be the result of an examination of the data that was more visual than mathematical.

Danni

On the initial questionnaire based on a scale from 1 to 10 (10 being very confident) rating the participant's confidence in her secondary mathematics skills, Danni rated herself a score of nine. She wrote,

> I feel very confident in solving high school problems. I put 9 because there may be a few things that I just can't remember. However, I feel like I could figure it out if I thought through the problem.

In the first face-to-face meeting, the requirements for the study were explained and a tentative schedule set up. The goal was to have four meetings over two weeks. Though we tried to have the four meetings fairly close together, due to end of the semester obligations and a major holiday, the schedule had to be modified. Danni was able to meet twice one week, and then completed the meetings two weeks later. We were able to resume our exchange without difficulty. Danni was conscious of my interest in verbal description and accommodated the think-aloud procedure on the tasks. She tried to express her thoughts and processes through written and verbal means. Descriptions of her problem solving are below.

Algebra Task

Danni spent only 30 seconds in the initial stage of sense-making engagement. During those first moments she found familiarity with similar problems she had experienced in a class that involved deductive reasoning. This was determined during the interview two days later. Her familiarity with the task was based in, "…I knew I needed to break, analyze each part of the pattern."

Danni's first step in organizing the information in the stage of transformation was to develop a table relating the number of the figure and the total tiles. She articulated her next step in, “now I want to see what they are doing to change the pattern…It looks like they are adding a row and a column” and wrote her thoughts. She summarized her findings verbally with:

> I don’t really want to focus on the one … at the bottom left and the top right because each of those [figures] has those two [tiles]. So, I kind of just want to focus on the … number in the rectangle.

It was clear that Danni made connections to patterns and geometry (see Figure 12).

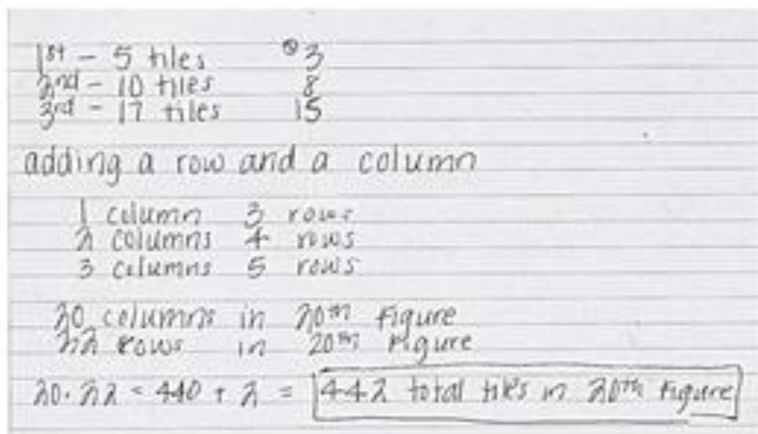


Figure 12. Danni’s tables and solution for the 20th figure.

At three minutes, Danni added the area of the rectangles to the previous table and began to build a second one relating the number of columns and the number of rows. At this point, she had moved into implementation by establishing the pattern of the number of columns and rows from the first figure through the third. Then, Danni stated, “So, I am going to assume that there are twenty columns in the twentieth figure because after each figure, it’s adding a column.” She continued filling in her table with, “when I look at the rows there are two more rows than there are columns.” Using the relationship between the number of columns and rows, Danni wrote, “22 rows in the 20th figure.” Finally, she computed the total number of tiles in the 20th figure verbalizing the process in, “So, to find how many tiles are in that center rectangle, I’m going to multiply twenty times twenty-two because that is how you find area.” Danni did the computation on the provided calculator and wrote, “20 x 22 = 440 + 2 = 442 total tiles.” The only concern about her work to this point may be the inappropriate use of the equality symbol.

Danni was now 5½ minutes from beginning the task. At this point, she changed her focus to find an expression for the n^{th} figure. She explained how she modeled the relationship in the following verbalization and writing

> For the n^{th} figure, it would have n# of columns and n + 2 # of rows. To find how many tiles, multiply n x (n + 2) and then add the 2 other tiles, those would be the tiles on the bottom right and top left that stick out. [n x (n + 2)] + 2 = total number of tiles for nth figure.

I asked Danni why she thought it was necessary to find this algebraic expression during the interview two days later. She responded, “I don’t know…I guess maybe I was just trying to…put it in more general terms and then to use that for the next part of the problem.”

At 8 minutes into the task, Danni addressed the second component, finding a figure requiring at least 10,000 tiles. After a moment of engagement, she moved directly into implementation saying, “I am just going to take a guess…” Using the calculator and the method commonly called guess and check, Danni tried letting the number of columns equal 30; therefore, the number of rows would be 32. She vocalized her procedure precisely and derived 962. She stopped to consider a different strategy; but, she changed her mind to continue with guess and check. As Danni reviewed her work to determine her next guess, she realized that she had mistaken 1,000 for the required 10,000. She decided to change her strategy to using an equation saying, “I want to see if I can do it this way, just setting my equation that I wrote to 10,000.”

This strategy failed in that Danni appeared to lose her method for solving a quadratic equation. Her work can be seen in the Figure 13.

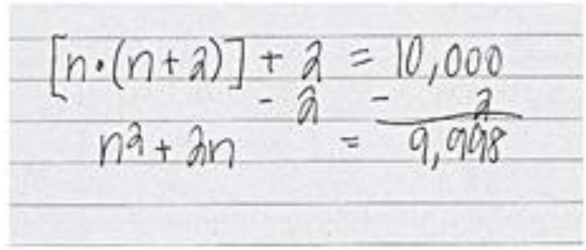


Figure 13. Danni’s attempt to solve for the least figure with 10,000 tiles using her expression.

This appeared to confuse her and she reverted back to guess and check using the calculator. This was at 10 ½ minutes into the task. Her first guess was 100 x 100 because she recognized that the product was 10,000. She got 10,200; then, she tried 98 and 100. She did not vocalize this answer. At this point, Danni declared that, “if I do anything less than a hundred…it wouldn’t be ten thousand.” This assumption led her to conclude her procedure. After restating her conclusion, she wrote, “the figure that would require at least ten thousand tiles would be any figure that had at least a hundred columns.” She stopped the recording immediately and pointed at her equation and said, “This is a mess.” The problem solving session was approximately 13 minutes.

Researcher observations. Danni demonstrated a strong capacity to organize information numerically to find the relationship between the figure and its total tiles. Her process was supported by accurate recall of relevant prior experiences. It was easy to follow her method in that she carefully articulated her process in the think-aloud and notes. The only apparent difficulty was in the second component of the task that required the use of the model she developed of the relationship.

The strategies employed by Danni included the use of numeric tables to establish relationships, writings for reflection and communication, and algebraic notation to model the relationship between the number of the figure and the total tiles. In the use of two tables, she first established the relationship between the figure number and total tiles; in the second, she found the connection between the number of columns and rows. Her ability to use tables effectively supported the ease with which she found the total number of tiles in the 20th figure. From that structure, she was able to compose an algebraic expression, substituting *n* for the number of columns/figure.

In addition to Danni's ability to organize information, her ability to articulate her thoughts and actions by vocalizing and writing was a point of interest in the interview process. She explained it in,

> R: This organization that you have, is that something you have developed or is it something [you have seen] other teachers use?
>
> D: I think it's developed since I have been learning how to become a teacher. Just making sure that I…include every step and, umm, really just explain why everything is.
>
> R: In you methods class, is that where you picked this up? Or, was it in a math class?
>
> D: Maybe just, just reading, just reading through math textbooks. I don't know. Just seeing tables done and what they say and just…watching the teachers teach and how they explain each step before going on to the next.
>
> R: Your college professors or your high school teachers?
>
> D: High school. I also think I was writing this because I wanted you to know why I did…all of what I did up here…with the how I got the twenty columns and twenty two and then the total. I was just trying to explain.

From this conversation, it was apparent that Danni was attempting to develop teaching practices and establish habits of decomposition of tasks by modeling them in written and verbal forms.

The notation Danni used to indicate some of her computations was inconsistent at times. During the interview, the compound statement 20 x 22 = 440 + 2 = 442 was examined briefly. When it was drawn to her attention, she immediately recognized the incorrect use of the equal sign. Also, even though Danni had written the relationship of the figure and total number of tiles in a sentence and algebraically, she did not transfer the knowledge to form an inequality for a solution to the second component of the task.

Danni was effective in her use of prior knowledge in geometry and sequences. But, the one tool that Danni did not use was drawing. Though Danni used the geometric concepts of finding the area of a rectangle and decomposing a shape, she did not draw a representation for any figure. As already mentioned, Danni connected this task to sequences in her use of patterns and the variable *n*, but her references were implied rather that direct. This can be seen in her writing, "For the n^{th} figure, it would have n# of columns and n + 2 # of rows."

Part of the procedure for this task was to write out the participant's process, adding any thoughts or alternative methods. Common to the other participants, Danni's writing did not give any new insights. It was well phrased and included the incident of debugging (when she thought 1,000 instead of 10,000) but no new insights.

In summary, the majority of Danni's solving process was organized and well defined. She used numeric strategies efficiently and was able to articulate her technique in words and writing. Danni's one difficulty occurred in the second component of the task, applying her findings. She changed strategies three times; found an answer; then, appeared confused stating, "This is a mess." This brought the problem solving process to an end. From the frame of the cognitive and metacognitive stages, Danni's connection to prior knowledge and strong strategies supported a smooth movement through the stages of engagement, transformation, and implementation. Similar to other participants, there was little evidence through verbalizations or action on the recordings that much attention was given to evaluation or internalization of the components of the task.

Geometry Task

Danni approached this task in a very efficient manner expending only six minutes in the problem solving session. After an engagement and transformation period of about 30 seconds, she began implementing a plan. Her work can be seen in Figure 14.

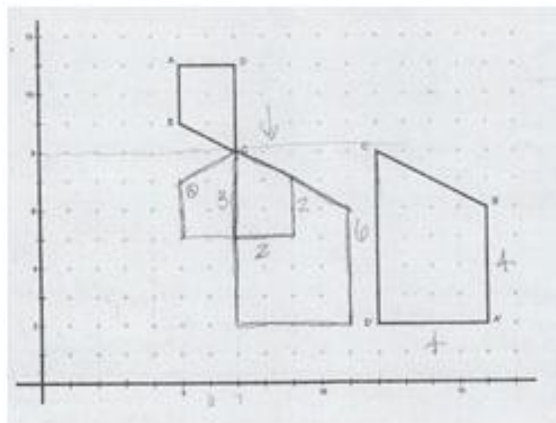


Figure 14. Danni’s similarity transformation: reflection over y = 8, x = 7, dilation by 2, and slide.

Danni used the point C as a focus saying, “I first know that I want C on top, not on the bottom. So…I want to…reflect it…over y = 8.” At this point, Danni sketched a line of reflection and began drawing the new image. (She had available a straight edge but chose not to use it.) As she sketched, Danni said, “All I am doing now is reflecting it.” From these two verbalizations and the actions on the recording, this was accomplished by counting and a visual orientation of ABCD with the line of reflection. She took a moment to review the figure then wrote down her actions on the provided pad of paper in anticipation of being asked to write her procedure once she had completed the task. In contrast to her think-aloud process on the function task, her procedural articulation during this geometry task was not detailed.

Danni continued with, “I…now…I see I need C on the left side [of the quadrilateral] and not the right. So, I am going to reflect it again over x = 7.” She wrote this down before doing the geometric transformation. This time she did not sketch a line of reflection. From reviewing the recording, her method for obtaining the second image appeared to be the same as the earlier reflection, counting and visual orientation to the previous figure. This time there was no articulation of counting or any other expression of her process. Her notation was also limited. Danni did not label the lines of reflection per the directions of the task for either reflection nor did she label the vertices of the two images she had produced.

For her last sketch, Danni decided to enlarge her second image. Her rationale for performing the transformation at this point was convenience, "Before…I translate it over…I want to make it…the same size [as A'B'C'D'] because I just…it is easier to see…without it being over there [pointing to the final figure] and having all the lines." So, clearly Danni was aware that the transformation from her second image to the final one involved two functions, dilation and translation. There was little hesitancy that indicated that she was easily able to decompose the movement.

In preparation for the dilation, Danni said, "I want to count how big it is…Does that make sense?" She proceeded to count three sets of the corresponding sides of the image and A'B'C'D' to compare their lengths. Her conclusion was, "And, so…I know that the scale factor is two…because the sides are being doubled." Danni enlarged the second image, using her term "by a scale factor of two". Extending first the vertical segment corresponding to CD, Danni completed the larger image by sketching the remaining horizontal and vertical sides then finally sketching in the oblique side.

Danni approached the final transformation with, "now I just have to…translate it over…to the right." She proceeded to add to her list of procedures then confirmed the action by counting from C to C’. At this point, Danni concluded, “This should be onto…this one” and ended the session immediately.

Researcher observations. From Danni’s actions it was clear that she quickly determined a method of transforming the quadrilateral ABCD so that it was onto A’B’C’D.’ Apparently, a plan was developed by decomposing the task’s goal into a series of transformations. Her motions on the video recording contained a minimum amount of time spent in hesitancy or second guessing. This demonstrated a familiarity with the task’s content.

There was evidence the majority of the motions were done mechanically. During the interview there was an attempt to determine her understanding of concepts and their terminology. When questioned about the first reflection, we had the following exchange,

R: How did you determine that the new figure was reflected?

D: Umm…it is the same shape but flipped over the line.

R: Yes, it is flipped. But, what do you mean by flipped?

D: It is the same shape but instead of going up, you go down. Umm…you do the…not exactly the opposite…

This demonstrated that Danni had a procedural knowledge of how to obtain a reflection, but it was not apparent that she understood the relationship of the figure, its image, and line of reflection. Another quality of Danni’s reflections was that in each, the figure and its image intersected the line of reflection, which supports a limited understanding of the concept. The term translation was used with regard to the motion required by this task without explanation. Regarding the concept of dilation, Danni was very capable of finding the scale factor for the dilation but did not use the term or express how she used its properties, such the projection point.

In addition to using less verbal description in the stage of translation, Danni used a minimum of notation. Though her work was neat and capable of being followed, there was no labeling of the lines of reflection or the vertices of her images. The labeling of lines of reflection was specifically indicated in the task instructions. This work, in contrast to her work on the algebra task, did not demonstrate a high level of detail.

From the think-aloud session and the recording, it was difficult to determine if Danni drew relationships between the transformations since there was no discussion about alternative methods of placing ABCD onto A'B'C'D'. But when we met for the follow up interview, she mentioned reflecting on the task and thinking that it might have been achieved with fewer transformations than those that she used. She proposed that she could have used a rotation instead of two reflections. The relationship that a rotation can be achieved through two reflections implied that she had some understanding of the relationship between the two concepts. This could possibly have minimized her number of transformations if there had been further adjustment to her plan.

At the conclusion of the problem solving session, Danni had already written her procedure:

Reflect across y = 8

Reflect again over x = 7

Enlarge by a scale factor of 2

Translate to the right 5 units

Before receiving it from her, I asked her to review it to add any thoughts that may have come to her. She did not modify the writing at all.

Danni's procedure through the first three stages of the cognitive/metacognitive framework was fluid. It can be seen by how she reviewed the task description, developed a plan for implementation, and proceeded to present a solution. The missing elements of Danni's solution were associated with an apparent lack of attention to detail, such as not labeling lines of reflection and images. As she worked, Danni would pause, review the step just completed, and then move on. Through a review of the recording several times, there was minimal evaluation and no perceivable review of the entire task. Nor did she consider that there may be a more efficient means by which to find a solution during the problem solving session.

Statistics Task

After an initial engagement with the data of one half of a minute, Dannie described her method as, “Okay, so right now, I’m just looking at…each day and comparing the numbers.” This implied that only a visual perusal of the data was being employed to contrast the daily attendance between the theaters, not the variance to the common mean. From this Danni developed a conjecture, “I would prefer…to be the manager of Theater B because…the attendance is more consistent.”

In her comparison of the data, Dannie did contrast the boundaries of the range for the two theaters by identifying the lowest and highest values in each set: “The lowest is 65 and the highest is at 98 [for Theater B]. So, every day is in that range. But, in Theater A, you have all the way from 10 to 124.” She did not compute the range for either set. This would have provided a basis to develop a statement contrasting the attendance of the two theaters. The rationale for her choice of Theater B was, “I think it would be easier…to have around the same amount…every day rather than a really low number then a really high number.” The decision was not expanded upon with perhaps a reference to implied challenges or conditions that may exist in the two theaters. The data were not explored in any depth with the use of either a graph or a discussion of other measures of central tendency or dispersion. Also, she did not formalize her conclusion with a written statement.

At three minutes into the task, Danni began exploring the second component of the task, making recommendations from the data. Over the next two minutes, she developed a series of statements about the attendance at the two theaters:

> Umm, if I was the manager of Theater A…then I would have…less personnel working on Thursday cause there's…and Wednesday cause there is only 24 and 10 on those days. And, have more on Sunday and Saturday…umm and on other days, but the most workers on Sunday and Saturday because those days are the busiest days.
>
> Umm…then Theater B, you have about…the same amount of workers every day. Maybe a few less on…Monday, Tuesday, Wednesday. Those are the sixties, but…you would have…about the same amount every day.

Danni paused 30 seconds and put down the pen. To this point she had not written anything or taken any notes regarding the task. She made one more comment, "I may not even open the theater that day [Theater A on Thursday] because you probably [would] almost be losing money…"

At this point, four minutes into the task, Danni began to interpret the situation through her personal experience, exploring elements that were not in the task presentation. She expressed her thoughts by relating the attendance of Theater A to that at a theater complex she attended in which there may be 10 showings of 10 movies simultaneously. It appeared that she was seeking some source from where to draw information. She explained this in the interview with,

> I am thinking…obviously looking at the numbers…if the highest number is 124 in the theater, they probably do not have ten different movies. So…but, I was thinking the movie theater where I am from has twelve. And, so, I was kind of thinking…you know ten to twelve movie theaters.

After this diversion, Danni returned to her initial conclusion, preferring Theater B, and justifying the choice by drawing attention to the "consistent amount." She made one more connection to a new aspect of theater management, food preparation, and concluded with, "So, I think it would be just more, it would be easier to manage, having that consistency."

Danni did not write any conclusions or computations during these five minutes that she was occupied by this task. I decided to prompt her to write with, “Could you please interpret those ideas in sentence form?” She responded by writing without speaking the following:

> Theater B because attendance is more consistent.
>
> If I was manager at Theater A, I would consider having the theater closed on Wednesdays and Thursdays because of the low attendance.

Immediately after concluding the writing, she ended the session.

Researcher observations. The goals of this task included determining if the participants would interpret the mean in context and if they would move past that first interpretation of a central tendency to consider the variability of the data concerning attendance for each theater and then comparing the two. This could be found by contrasting the range of each or comparing the data to the mean. Other possibilities for comparison are exploring measures like the mode or median if relevant. Another goal was to see if they would use any other method of examining the data, such as a graph or software on a graphing calculator. Danni’s attempt to make sense of this task did not involve any of these aspects.

Danni’s decision that she would prefer to manage Theater B did not include any mention of possible implications about the mean or that it was the same for the two sets of the data. In the interview two days later, she responded to this in:

> Yeah, well the reason that I said Theater B was because of the attendance. Then I said “well…both of them…” I didn’t even think about…I looked at the mean when I was doing the problem, but…I didn’t really take that into account. I was thinking about it later and thinking “Well…both, they had at least…” Well, hold on. I don’t know.

From Danni’s comments during her involvement with the task and during the interview, she did not perceive the need to address the issues around the mean. In the think-aloud, the significance of the mean representing accumulated attendance was not recognized as a way to analyze the pattern of patronage of the two theaters. Also, she did not talk about the implications based in the fact that the mean was the same for both theaters. This can be seen in the comment from above, "I looked at the mean when I was doing the problem, but…I didn't really take that into account."

Even though Danni used the term range when describing the attendance at Theater B, by saying, "every day is in that range," it was not clear that she was using the term descriptively or more technically. When asked about this in the interview, she responded, "I think so. I know range is the difference between high and low." So, Danni understood the concept, but it was not clear she was using it in a precise manner. Danni did not explore the concepts of mode or median to determine if they were relevant to this task.

This data analysis task did not appear to elicit the methodic approach or confidence in Danni that the function and geometry task did. When asked about her comfort level with the task, Danni said, "I don't know, I don't know how comfortable…I felt. I don't know…I don't feel like I knew a lot about…like what I needed to do." This was reflected in her verbalized thought processes as she moved beyond the context of the task trying to make sense of it by extending the parameters of the question into her own experiences. She did not show evidence of the development of a plan to explore the data in the usual manner, i.e. examining each descriptive concept. She used a less rigorous method of approaching this task than the others posed to her.

More evidence of her uncertainty may be seen in her written response to the task. After being prompted to write, Danni's answers were two sentences. In response to the first component to the task, she simply wrote, "Theater B because attendance is more consistent." In response to the second component, she addressed the request for any recommendations with, "If I was manager at Theater A, I would consider having the theater closed on Wednesdays and Thursdays because of the low attendance." There were no statements contrasting the two theaters' attendance to draw comparisons or justify her decisions.

Danni's response to this task was a departure from her previous methodic movement through the stages of regulation of cognition while addressing the tasks in algebra and geometry. This task appeared to bring forth a less rigorous approach to mathematics. The experience in geometry demonstrated a somewhat less precise attention to detail, but this experience appeared to not involve the stages to any substantive degree. Her movement through engagement, transformation, and implementation were superficially addressed. The fact that she immediately ended the session once the writing was completed implied that the stages of evaluation and internalization were not used.

Elizabeth

Elizabeth scored herself seven on the questionnaire administered on the initial meeting. Her rationale for the score was, "While I have learned a lot in college, I am not perfect and have much more room to improve. I enjoy middle school math better." After that initial introduction to the study, Elizabeth was contacted through email and the first face-to-face meeting was planned. During that first meeting, the requirements were repeated and a schedule negotiated. The schedule allowed us to meet three times over a week and a half. The last meeting had to be set after a major holiday, ten days later. Elizabeth was very conscientious about meeting and willing to discuss her task solutions. When reviewing the recordings, she would volunteer what she was thinking at a particular point in time without being prompted. Her method of solving the algebra task begins here.

Algebra Task

After the task was presented to Elizabeth, she proceeded to read it aloud and decompose it into a series of smaller components, underlining key phrases.

> I need to know the number, total number of tiles it contains [underline] and how they are arranged [underline] in the twentieth figure [underline]. So, there are three things I am looking for so far. Then, explain the reasoning you used to determine the information [underline]. Write a description algebraically or in words [underline] which could be used to find any figure in the pattern. So, that's going to be…that's going to have the *n* in it to replace…Okay…which figure will require at least ten thousand tiles [underline]. It's going to take some thinking.

In this engagement with the task, it is noticeable that Elizabeth connected with the notation of sequence. She explained her reasoning in the interview two days later with, "…because they [figures] were the same thing, just different. The same shape, different amounts. Sequences have a pattern to them. And, I noticed the pattern."

After 1½ minutes, Elizabeth moved into the stage of transformation by trying to make sense of the pattern she noticed. She did not immediately decompose the figures into geometric shapes. Her first approach was to count the number of tiles across the top and bottom and the number vertically. She began organizing this information in an informal table explaining her method in, "Right now, I am just trying to see what the sequence…would be." Elizabeth began to discuss the construction of the shapes as having one, two, and three rows. This revealed some confusion in terms as she mistakenly used *row* for *column*. Her attention then moved to the number of total tiles in each figure. She listed the figure numbers with the total tiles and recognized a pattern. At four minutes into the activity, Elizabeth articulated her discovery in, "So, it is always going to be whatever is in the middle plus two. Whatever is in the middle is going to add a row each time…a row and a column…each time." Her notes can be seen in Figure 15.

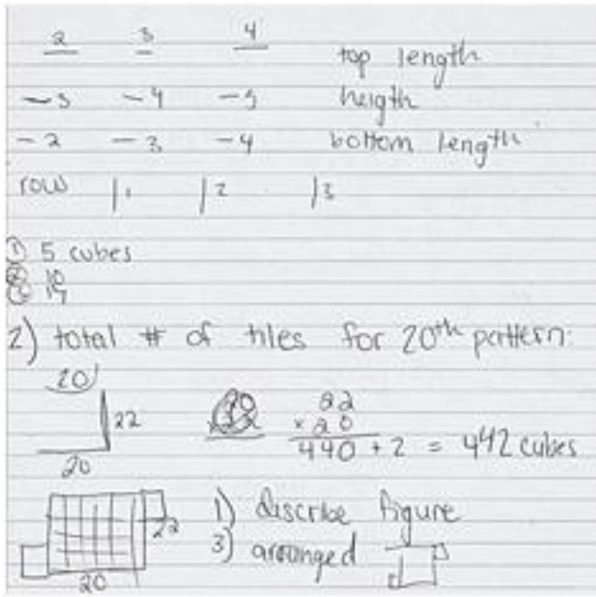


Figure 15. Elizabeth’s notes during transformation and implementation.

Elizabeth began implementation after 4½ minutes with an attempt to find the total number of tiles in the twentieth figure. She began, writing while speaking, “The total number of tiles for the twentieth pattern…” She returned to the figures on the task sheet, numbering them, and noting the number of rows [actually columns]. Her conclusion was, “it is going to have twenty rows.” Returning to the figures again, Elizabeth mentally noted the columns by rows for each figure and continued the pattern illustrating the fourth figure using a horizontal and vertical line to denote rows and columns, labeling the lines four and six. She drew a conclusion, “we are just talking about the middle columns and rows, because all you need to do is add two to that [the product].” Using the same illustration for the twentieth figure, she tried to extend the pattern to find the total tiles (seen in Figure 15). At this point, Elizabeth appeared to become confused: “Okay, so the fourth one is four by six. So, the twentieth one will be twenty by…I want to say twenty minus three…” Again, she returned to the values denoting the columns and rows finding the common difference of two and then completed her illustration placing 20 on the horizontal line and 22 on the vertical. After this, Elizabeth quickly computed the total number of tiles for the twentieth figure by stacking 22 and 20, multiplying, and adding two. Her summarization was, "it is going to have twenty columns and twenty-two rows and it's going to have one extra right here and one extra right here [drawing a somewhat more detailed figure]."

At this point ten minutes into the session, Elizabeth returned to the outline of the task she developed during engagement and reviewed her numbered items to evaluate her progress. She felt like she had achieved describing the figure. So, she moved to explaining her reasoning. This was done through writing while articulating her process:

> Figure 1 had 1 row and 3 columns. [She stopped realizing her confusion over rows and columns. Returning to the sentence, she corrected it.] Figure 1 had 3 rows and 1 column. 2^{nd} had 4 rows and 2 columns; noticed each row and column increased by one row/column; so, for 20^{th}, 20 columns, add two rows…

At this point in the process of writing, Elizabeth began to question the basis for her problem solving, confusing the change in the number of rows and columns between figures, the relationship between the rows and columns within a figure, and the role of the figure number. After struggling with these relationships for two minutes, she pulled her thoughts together and completed the writing with, "then you do the area of those and add two because the first four in the sequence didn't change."

At 14 minutes into the problem solving session, Elizabeth moved back into implementation to write the expression for the n^{th} term. She relatively easily used her verbal expression and her drawings to pull together an expression for the n^{th} term, (n + 2)(n) + 2, simplifying it to $n^2 + 2n + 2$. Above the first expression, she indicated that n + 2 represented the number of rows and the n indicated the number of columns. Also, below the expression, she encompassed the (n + 2)(n) to represent the area of the rectangle. This connected all of her interpretations of the pattern. But, it was not written using function notation. She then tried to verify this expression by finding the 5th term. Using a horizontal and vertical structure, Elizabeth labeled it with five and seven, gave it a visual evaluation, and said, “It looks correct.” She did not compute the total tiles to determine if the total fit the pattern established by the previous figures.

After 16 minutes, Elizabeth moved to the second component of the task with, “you use your algebraic representation to equal at least ten thousand.” The inequality she produced had two errors: the appropriate inequality symbol for *at least* was not used and her method for solving a quadratic failed. Her work is seen in Figure 16.

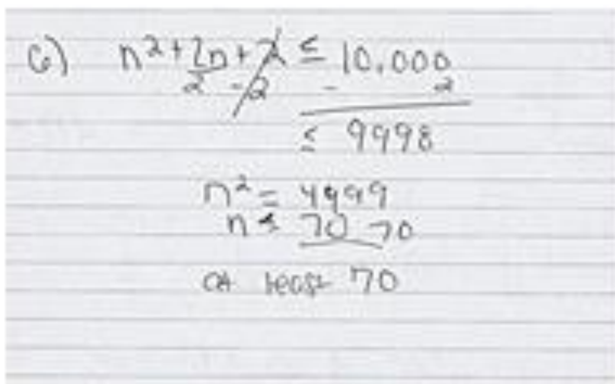


Figure 16. Elizabeth’s attempt to solve the inequality produced to answer the second component of the algebra task.

When asked during the interview if she checked her solution, Elizabeth did not recognize the contradiction between her numeric and expressed answer. She also could not immediately recall how to determine the validity of the interval.

Researcher observations. Elizabeth demonstrated strong connections with the topics of sequences and geometry along with constructive use of drawings, numeric expressions, and algebraic symbolization. In the second component, she alone attempted to use the model of the relationship between figure and total number of tiles as an inequality to solve the task (although she was not successful). Unfortunately, Elizabeth experienced moments of confusion during her problem solving based in imprecise term use and confusion in patterning.

Elizabeth began to discuss the construction of the shapes as having one, two, and three rows. Her hand motions and diagrams on the task sheet indicated confusion between row and column. This lack of precision in language complicated the articulation of her thoughts throughout the task. But, she did realize her interchange of rows for columns and finally resolved it in the writing of her procedure as noted in the quote at ten minutes into the session.

Two other moments of confusion occurred in her efforts to follow patterns. One was when Elizabeth tried to extend the pattern of the first four figures to find the number of total tiles in the twentieth figure. She was reciting the relationship of rows to columns as one by three, two by four, three by five, four by six, and then, for a moment, thought the difference between the rows and columns was three. This conflict was resolved by returning to her notes on the figures and finding again the common difference of two. This allowed her to decide that there were 20 columns and 22 rows to compute the total tiles for the twentieth figure. The second moment of confusion preceded writing the expression for the nth term. Elizabeth had examined the figures at the beginning of the task and found that there was an increase in rows and columns between the figures by one. She began to question the relationship she found between the number of rows and columns returning to this idea when attempting to write the relationship between the number of the figure and the total tiles. This conflict was also resolved after returning to the figures and her drawings, and the expression was written.

It is notable that Elizabeth recognized the implication that a model would be useful during the stage of engagement. When she was reading the task, she remarked, “So, that’s going to be … that’s going to have the *n* in it to replace.” We talked about this during the interview two days later in:

> R: How did you decide that you had to write something expressed with *n*?
>
> E: Because…umm…which could be used to define any figure, any figure is a variable. And, I use *n* whenever I am doing sequences.
>
> R: How did you realize this was a sequence?
>
> E: Umm…because they were the same thing, just different. Umm…the same shape, different amount. Sequences have a pattern to them…And, I noticed the pattern.

Recognizing the use of a variable in the initial stages of the problem solving supported her eventual production of an algebraic model to use in the second component of the task. Unfortunately, in the second component, Elizabeth failed to remember the appropriate method for solving an inequality and did not recall the symbol used with the phrase *at least*.

Elizabeth used both drawing and writing to monitor her problem solving. By noting the columns and rows on each of the figures, she was able to continue the pattern, sketching new members of the sequence. Several times she returned to these notes when she became confused. Elizabeth used writing as a method to clarify her thoughts and to reflect on her process. She described the role of writing as,

> You can actually write to…put your thoughts on paper and say it. Like if I am trying to teach it to you, I can see where I mess up. And, if I write then I can see where I am messing up. It won’t exactly make sense.

This use of multiple representations supported her problem solving and refined her thought processes.

During the problem solving of this task, Elizabeth progressed through the first four stages of the cognitive/metacognitive framework. Her engagement of the task involved decomposition of the task into subtasks and recognition of connections to sequences. The stage of transformation included exploring the figures until a pattern was found and a conjecture was developed. Her implementation resulted in an expression that could be generalized to any figure in the pattern and supported through multiple representations. Also, once she felt like it was appropriate, she returned to the task description to determine if all the subtasks were completed, evaluating her progress. The completion of the second component of the task was somewhat muddled. Whether this was due to a moment of confusion, inadequate knowledge, or cognitive overload was not clear.

Geometry Task

Elizabeth engaged with the task in the same manner as she did with the algebra task. She read the task aloud and then paused for a moment. Focusing on the task description, she re-read it aloud, underlining the term similarity transformation, and repeating the instructions to indicate any lines of reflection, slides, or rotations around a point.

After 1 minute, Elizabeth moved into transformation using the ruler that was provided and her hands to experiment with possible series of transformations to place ABCD onto A’B’C’D’. Her initial conjecture appeared to confuse two transformations, “You rotate it…you slide it over and you flipped it.” During this attempt to understand the task, she sketched an image of ABCD at the bottom of the task sheet and a partial sketch of a second image that appeared to be a rotation but mislabeled (see Figure 17).

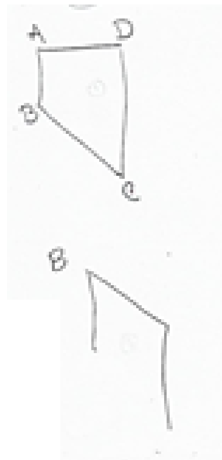


Figure 17. Elizabeth's attempt to rotate the original figure.

After producing these sketches, Elizabeth decided, "So, it is definitely not a slide then a rotation" and began to mimic geometric movements with her hands again. After a pause, she returned to the idea of using a rotation. Her apparent confusion about how to express this can be determined from a disjointed description of her rationale:

> Let's see [using hand motions]. Now, if you, if you rotated it…The C would still be here [pointing at C and C']. So, it looks like some type of rotation cause if…The point [C] would still be here, it would still slide down to the…But, and A and then D. So, yeah, it is definitely going to be a rotation.

Elizabeth ended this effort concluding that she should rotate the figure and slide it five units so that, "your C's can be on each other." She added, "And, then you would need to rotate it 90 … let's see, 180 degrees where the C is there."

After three minutes, Elizabeth returned to the task instructions confirming that she should indicate any lines of reflection, slides, or rotations around a point. She proceeded to find a point midway between C and C' and sketch a line at x = 9.5, failing to label it. But, she stopped, realizing her plan did not include a line of reflection. Returning to her plan, she stated, "it's a slide and a rotation" underlining the terms in the task instructions. Elizabeth enacted the plan by verbalizing and writing brief notes:

> You slide it over [counting again], you slide it over five [writing *slide*] to the right. So, you are going to slide from 7 to 12. And, then you are going to rotate it [writing 'rotate 180 degrees'] 180 degrees.

Though the translation to the right was denoted with a counting of units, she did not draw an image. She also did not indicate the rotation in any form. So, Elizabeth failed to illustrate her transformations instead relying on her verbal description to justify of her process.

At this point, Elizabeth appeared to notice the difference in size for the first time. By counting and comparing all the corresponding sides, she established that A'B'C'D' was double ABCD. She attempted to recall the term used to describe an enlargement like this in, "I forget what it is called whenever something…gets twice as big." She did not relate this change in the figure as a ratio or mention the role of proportion in similar figures. It was also curious that she indicated that the oblique side on ABCD was two units long and the corresponding side on A'B'C'D' as four units. She paused for a moment and wrote, "shape will double"; and declared, "that is pretty much all I can tell you about it," and reiterated her process. There was no evidence in the recording that she took time to evaluate her procedures or review from the initial point.

Researcher observations. Elizabeth's verbal solution to this task was valid with a minimum of transformations. But, her process would be impossible to follow if relying on her written work. There was only the one indication of the slide. In the transformation stage, the verbalization of her thoughts indicated some confusion. She developed a path to place ABCD onto A'B'C'D' and then dropped it to pick it up again. The fact that she did not draw the images of her transformations made interpretation of her problem solving more difficult.

Her planning was done through a visual manipulation of the quadrilateral ABCD using her hands and/or a ruler to facilitate her understanding. The enactment of these maneuvers could be seen on the video recording. During the session as she moved into implementation, Elizabeth returned to the task instructions and repeated the request for lines of reflection, degrees of rotation, and indications of slides which implies the production of images for each transformation. But, she did not produce any. The only drawings done during the problem solving session were the two images sketched at the bottom of the task sheet. One of those was not complete or appropriate for the transformation she was describing. When we explored this during the interview, she was able to produce an image of ABCD rotated and one before the dilation.

Another struggle Elizabeth appeared to encounter was connecting vocabulary to the transformations. In the problem solving session and the writing, she used the phrase *flipping it* several times. We spoke about that in the interview.

R: You are talking about flipping it. Which one of the transformations are you talking about?

E: That is a reflection …

R: How do you know you have a reflection?

E: [pause] This point C and this point C' are on the exact same line. So, if you slid it over, and you reflected it … I would have to slide it farther. It would actually have to slide to B. I guess that was another thing I wasn't thinking about. I would have had to slide it over B and then flip it and it would have been going this way. [referring to the task solution]

In this confusing passage, Elizabeth appeared to avoid giving a precise description of reflection. Later, she was able to add, "…reflection, you have to reflect about an axis and I knew I needed to find the middle of something." This indicated that she was aware that a line of reflection was necessary and that the figure and its image were equidistant from the line. The other concept with which she struggled was dilation. When asked about the concept, Elizabeth replied that she was not very familiar with the term; therefore, she had trouble expressing the change in the figures.

When Elizabeth was determining the change in size between ABCD and A'B'C'D', her description of the process was correct but lacked precision in its expression and in her computation. The verbal expression of the difference between ABCD and A'B'C'D' was ABCD was doubled. From this, it can be surmised that she was not connecting the change with a ratio, which is usually employed to express similarity. Her process to find the difference was done by relating corresponding sides and counting. A computational error occurred in Elizabeth's consideration of the oblique sides. Though it was not necessary for her to confirm that all corresponding sides had a common ratio (for a quadrilateral, three sets of sides are sufficient), she counted the two oblique sides as two and four respectively when the actual values were $\sqrt{5}$ and $2\sqrt{5}$. This arithmetic mistake was addressed in the interview process.

R: Is that [BC] actually two units long?

E: N-no. I was just … there was a dot here and a dot here. So, I was just counting two. I think I was just making sure it doubled … but I wasn't meaning that many units. I was trying to think of … if it is doubling it. But, you are right. How many, I mean how would you measure … oh yeah, the Pythagorean Theorem. Yeah.

This demonstrated that she possessed the basic knowledge to find the length but as with other concepts relied upon in this task, there appeared to be a lack of precision in her application.

At the end of the problem solving session, Elizabeth was asked to expand on the notes she produced while involved in the problem solving process. She wrote the following:

Trial & Error

I tried sliding it then flipping it, but the variables didn't match up.

> I figured out I needed to slide it where the C's matched up then rotate it 180° to be on line C'D'. Then I recognized that it was ½ the size of quadrilateral A'B'C'D' so I decided we would need to double the size.
>
> I don't know how or what it's called to 'double it.'

In summary, Elizabeth's initial stages of engagement and transformation appeared to be marked by trying out strategies and testing them to see if they led toward a solution. As we reviewed the recording, she twice voluntarily commented about her thought processes, once while in engagement and once while in transformation. The following excerpt was given by Elizabeth while we reviewed the recording during engagement:

> It takes me a while to process a question because I don't…there have been so many times in my life I have gotten…I have just read and started it and, but I did the wrong…I made it a question in my head basically. So, I have to make sure I slow…I have been trying to slow down … and really grasp what it is asking before I start answering …because I have bad test anxiety.

As we reviewed her process in transformation, she expressed her internal tensions by saying.

> …I paused and…and I thought, "is that really what it is supposed to be called?" [The term was slide.] It was my first instinct and I hate, I hate going with my first instinct and so I always second guess it and I didn't have anything…to second guess it with, so that is when I was like that is what it has to be. Yeah, so there is nothing else.

It was interesting that despite Elizabeth's self-doubt, the implementation was a very efficient series of transformations.

Statistics Task

Elizabeth followed her pattern of reading the task aloud and underlining the terms in the task description that she determined had particular relevance. In the first part of the task, she unlined *average* and *which theater would you prefer and why*. In setting criteria for answering the questions, she stated, "I would prefer it [being manager] based on higher attendance." Continuing in the stage of engagement, Elizabeth read the second question and then re-read the first.

Moving into transformation after 1 minute, Elizabeth initially thought that she had to compute the mean but noticed that the mean was given for the two theaters. From the recording, her reaction indicated that this fact may have confused her briefly. Once she recognized that it was the same for both, she did not interpret the implications of the common mean with respect to attendance.

At this point in transformation, Elizabeth moved outside the boundaries of the problem introducing the factor of income with an assumption, “If they have the same amount of money coming in, I don’t understand why it would matter how it is dispersed.” Then, she began to question her understanding of mean, median, and mode. She said, “I know mode doesn’t count…Mean is average; median is the middle.” After this reflection on definition, she refocused on mean deciding it was the relevant measure to consider.

Elizabeth continued in transformation, still trying to make sense of the task and the data presented. After three minutes into the task, she stated, “…the only reason that I would pick Theater A is because the weekends have a higher amount…I know the weekends are more expensive sometimes.” Again, she moved outside of the description of the task entering the idea of theater income from ticket prices when it was not a stated variable in the task.

At this point, Elizabeth considered the difference in the attendance per day between the two theaters in a very vague way. Her comment above on the weekend attendance for Theater A was followed by, “…Theater B is very consistent.” Then, she speculated if it mattered since the means were the same. She even went so far as to use the calculator to take the totals of the data for the two theaters, checking the means to see if they were actually equal.

After five minutes from beginning the task, Elizabeth moved into implementation writing, “I would prefer either. A has higher weekend attendance, B has more consistency. Both have the same attendance per week." She moved immediately to respond to the second part of the task, giving recommendations. Writing while speaking, she continued with, "Theater A needs to be more attractive during the week, maybe have lower prices to have higher attendance" and "Theater B…is consistent which is good, however there needs to be a way to have more people attend." Then, she stopped the recording immediately. Elizabeth expended approximately 8 minutes and 30 seconds on this task.

Researcher observations. Elizabeth expressed her method of approaching data analysis in a recitation of the terms mean, median, and mode. Although it appeared that she reviewed mental definitions of the measures of central tendency, she did not interpret them with respect to this data. Even though the mean was the focus of her consideration, she did not explore the variability of the data with respect to the mean in any depth. She also failed to mention a measure of dispersion, range, which would have been relevant in comparing the data accumulated about the two theaters.

During the problem solving when Elizabeth mentioned mode, she immediately dismissed it. In the interview when asked about why, she did not explain but returned to her focus on mean with, "I know for a fact the definition of mode is…an amount of something that reoccurs. Now, mode didn't count in what we were looking for, mean." This demonstrated that she knew the definition but did not show that she understood why the mode is not applicable with this data.

Elizabeth also did not reference range except in an ambiguous manner. The only comments comparing the dispersion of the two sets of data were "…the only reason that I would pick Theater A is because the weekends have a higher amount" and "…Theater B is very consistent." She did not compute the range for the two sets or compare the spread of the data with respect to the mean.

While exploring the task, Elizabeth focused on an interpretation of the common mean as implying that both theaters were “the same.” She extended this in, “If they have the same amount of money coming in, I don’t understand why it would matter how it is dispersed.” Introducing the variable of money appeared to complicate her interpretation of the task. In reference to Theater A, Elizabeth mentioned the cost of tickets on the weekend. But, in the interview two days later, she may have resolved this conflict. The possibility was seen in the interview comment, “Well…I was thinking…if it was a graph on attendance…that was a key that attendance needed to be taken into consideration whenever I had to…prefer one.”

The response to the second component, eliciting recommendations, was not very specific. There was a minimal mention of a marketing strategy for both but nothing about personnel or any other factors that may be addressed by examining the data. Also there was not any mention that the data was accumulated over a year, which provides some predictive possibilities. In the interview, Elizabeth did add to the recommendations with, “I didn’t exactly give a way [to increase attendance at Theater B]. I couldn’t…I couldn’t think of one. So, I guess if you have matinee prices on weekends, you will have more people on the weekends.”

The brevity in this section of the task may have been the result of an emotional response to the task. During the interview as we were viewing the recording, she explained her feelings while working on the task with, “I’m just mad. I am just thinking, there is no way this is correct but this is as good as I got. I am, I am just mad.”

Elizabeth’s response to this task contained her previously noted problem solving techniques. She spent time in the engagement stage, decomposing the task components and noting the main ideas along with expectations. From her response to the task information, Elizabeth had a method of interpreting data, which included examination of measures of central tendency, but not dispersion. Also, the stage of implementation was abbreviated as a result of frustration with the task. Since the conclusion of the task was abrupt, there was no evidence of the stages of evaluation or internalization.

Fran

Of the six participants, Fran's scored herself 6 (on a scale of 1 to 10) on high school skills, which was the lowest self-rating. She explained the rating this way, "I am fairly confident that I could answer a high school curriculum question because of my math background in high school and college but I know that there is a lot I have forgotten." In the initial contact, the requirements for the study were explained: the four meetings encompassing three tasks with follow up interviews. After contacting Fran to participate, we negotiated a schedule: twice before a major holiday and twice after. She was at each meeting promptly without reminders or coaxing. Fran understood the concept of the think-aloud and was willing to articulate her problem solving. During the interviews, she volunteered clarification of her thoughts and methods with little prompting. Her problem solving is described in the following.

Algebra Task

Fran spent a minute in engagement during which she was trying to connect to some previous experience. She encountered some conflict during this stage, which she explained in the interview with, "…looking at patterns and like figuring out the next figure was familiar but I wouldn't say that I have ever seen one similar to that." She was familiar with geometric and arithmetic sequences, but she recognized that this one did not fit the pattern of those types.

In the stage of transformation, Fran analyzed the pattern recursively, expressing her thoughts before beginning to write. As she was decomposing the figures, she articulated her thoughts with

> I notice that the pattern is … growing. Now, the two side ones on the top and the bottom, the top right and the left bottom, they are going to stay the same in each one. So, whatever our algebraic formula, I guess, will be or the number, you would add two to figure out the number of tiles.

She continued with, “It’s not going to be like a square because it is different. Because it is a … one by three, then two by four, then a three by five.” In these statements, it is possible to see connections that Fran made in her knowledge about geometric shapes and patterns. They also reveal her goal of an “algebraic formula.”

To this point in transformation, three minutes into the session, Fran had not written anything. Her next step was to produce an informal table 1 x 3, 2 x 4, 3 x 5 and labeling each numeric expression with 1, 2, and 3. She asked if she could draw and proceeded to create a sketch of the fourth figure. Her drawing was done meticulously. She added 4 x 6 to her table; then, she added 5 x 7. This can be seen in Figure 18.

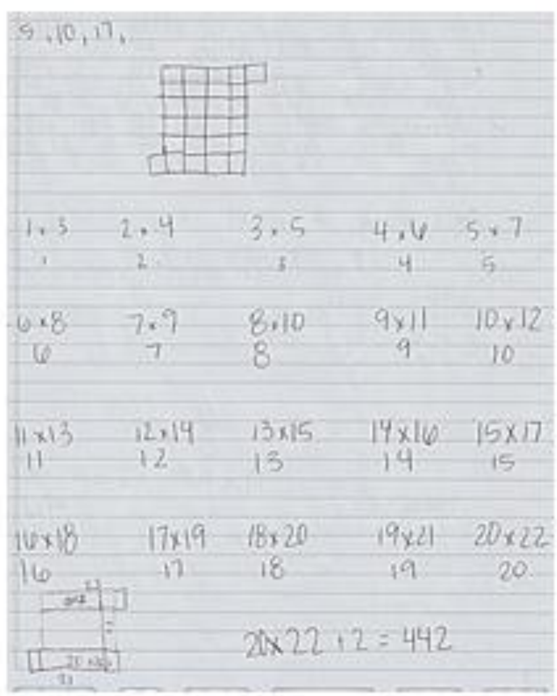


Figure 18. Fran’s informal table, sketches, and solution for the 20th figure.

At five minutes into the problem solving session, Fran moved into implementation making the decision, "I think I am going to just write the pattern out. That will be the easiest." She continued her table writing the numbers through 20 and returning to fill in the corresponding numeric expression for each. At the 13th item, Fran paused. She explained this moment of insight in the interview two days later in,

> …that was when I was writing it and I was adding each one [expression] and then I noticed that…I first said that they [columns and rows] were two apart…then I noticed that this [the first factor] went along with whatever number [figure] it was.

Once Fran had written the expression for the 20th figure, she summarized with writing, "20 x 22 + 2 = 442" and "So, the twentieth figure would have four hundred and forty-two blocks, tiles."

Fran was seven minutes into her problem solving and had produced an answer for the first component of the task but decided to return to drawing. She proceeded to produce a sketch of the 20th figure beginning with 20 tiles on the bottom row, adding the extra two tiles, and 22 tiles high. The recording revealed that she would draw a part of the figure, pause; draw another, pause, complete the figure, then label it. She softly vocalized her process. She sat back, looked over the drawing concluding with, "this is my figure…and it would have four hundred and forty-two tiles." She completed this in two minutes. The drawing can be seen in Figure 19.

In her interpretation of the task, Fran assumed that she should write a description of her work. The following was her written explanation:

> Figure grows each time by adding a column to the right and adding a row. It began with 1 column and 3 rows. The figure has a tile on the top right and bottom left in each increasing figure in the pattern. These do not grow and the top right tile changes with each increasing figure.

This was produced over three minutes with a minimum of articulation. She checked it carefully.

After this description, Fran continued writing, apparently trying to express an algebraic relationship seen in Figure 19.

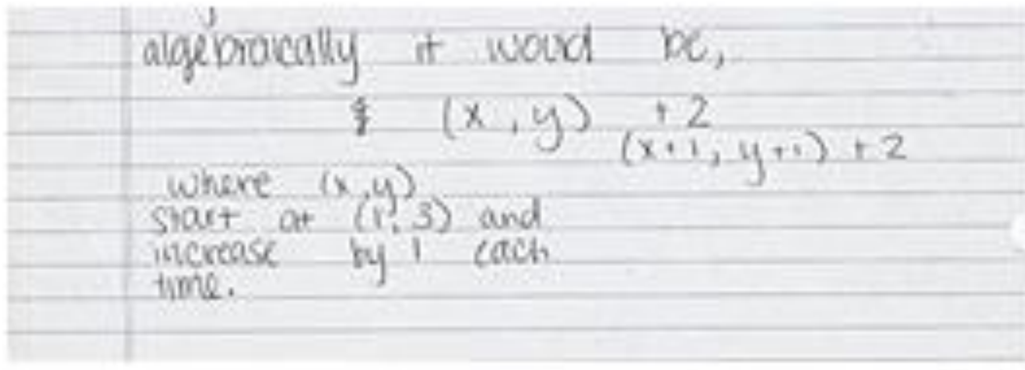


Figure 19. Fran’s attempt to produce a model for the relationship between the figure and its total tiles.

When asked about this in the interview, Fran expressed the difficulty she encountered trying to develop a model in the following way:

> I knew what it…what to say but I just didn’t know how to say it. I knew that I needed to say that the row grew by one and that the column grew by one…But, I just didn’t know, like what variable to use and what, like, form to use for this. And, so it just really stumped me. I remember thinking like “I don’t know how to write this and it being the correct formula.”

Fran was still defining the relationship by comparing the consecutive figures, or recursively. She had not moved into a way of modeling it connecting the figure with the total tiles.

At 15 minutes into the session, Fran turned to the second component of the task, finding the figure with at least 10,000 tiles. After a minute of engagement and transformation, she picked up the calculator and began punching numbers. She paused and asked if I wanted to know which ones she was using and then continued her technique of guess and check vocalizing her method in,

> …I noticed that the twentieth figure was twenty times twenty-two plus the two. So, you would have twenty on the bottom, not including the bottom left or the top right. You would have twenty on the bottom and twenty-two going up. So, we need to multiply that

> [the number of columns and rows] to get over ten thousand. So, I am going to just try different numbers. So, I get over ten thousand…well, trial and error.

She tried a series of values. After trying 100 x 102 and 98 x 100, Fran said, "I am getting close" and, concluded, "So, the smallest would have to be…one hundred times one hundred and two." This assumption was made 20 minutes into the problem-solving situation.

Fran did not end her solution at this point. She proceeded to sketch the 100th figure. While drawing, she stopped and declared,

> Nope, it's not right because if you do ninety-nine times one hundred and one you get nine thousand nine hundred and ninety nine and then you have to add two, so…you would get ten thousand one…Okay, so it would be ninety-nine right here and it would be one hundred and one right here [referring to drawing] and your plus two**.**

When asked about the clue that 100 x 102 was not the best answer, Fran replied that "…when I drew out those two [the tiles on the upper right and lower left on the 100th figure] I realized I forgot the two." She had already done the product for 99 x 101, but she had forgotten the + 2. In Fran's case the drawing process supported finding the best answer. The problem solving session ended after 22 minutes.

Researcher observations. Fran used multiple representations during her problem solving of this task. When examining her written work, she demonstrated skill in extending patterns using numeric expressions and visual representations. Also, at times, her declarative capacity to express her analysis of the task was very clear, for instance: "The figure grows each time by adding a column to the right and adding a row…The figure has a tile on the top right and bottom left in each increasing figure in the pattern." However, even though Fran characterized herself as very oriented in algebra, she was not able to develop a model for the relationship between the figure and its total tiles.

From the beginning of the session, Fran wanted to write an "algebraic formula". She declared within the first two minutes, "I always have to think of things algebraically. It is just the way my brain is. So, I am trying to figure out some kind of formula for it." But, the formula did not immediately occur to her. In the interview, we discussed this. Fran described the way she handled it as, "I thought: it's geometric and I don't know how to solve this. So, I had to try everything in the book."

She attempted to write a relationship again at the conclusion of the first component of the task. Fran commented on this struggle:

> And, I was trying so hard to get an algebraic equation. And, it is awful. I just sat and thought. "how do I do this?" and I was so embarrassed. [barely audible] I did not know how to say the column and the row. And, then I kept thinking about matrices and how you write them. And, I was trying to do that. And, then I couldn't remember it.

It appeared that the problem was assigning a variable, letting it represent either the figure number or the number of columns. She had written a strong numeric statement for the 20th figure, but did not use it to generalize the relationship.

She appeared focused on the relationship of the consecutive figures. During the interview, I suggested using the notation of sequences, *n*. She responded with,

> Well, I knew I had to multiply two numbers that were just two apart. And, so, if I had gotten that … if I done the *n* plus two in parentheses times *n* plus two, then, … I would have set it equal to ten thousand tiles.

This quick recognition of what the next step would be implied that introduction of the variable was a barrier.

During the first few minutes when Fran realized that she would have difficulty developing a model or "algebraic formula," she proceeded to employ other tools (everything in the book) with which she was familiar, primarily drawing and numeric patterns. She used these alternately with efficiency. Fran wrote the numeric pattern for the rectangle forming the main part of the three figures given. Then, she sketched the fourth figure, adding the numeric expression for it to her listing. After a moment of reflection/decision, she continued the numeric pattern to the 20th figure to successfully find the number of total tiles. Returning to her technique of drawing, Fran then drew the 20th figure for a second examination of her solution (seen at the lower left of Figure 20). Similar methods were used to find the answer to the second component of the task. Fran used a similar numeric technique, guess and check; then, she used a sketch to evaluate her answer. Her first answer for the dimensions of the rectangle, 100 x 102, was common to some of the other participants. But, as she drew a sketch of this answer, the process of drawing made her realize that 100 x 102 was not the best answer. This use of alternating representations to produce and check this task among the participants was unique to Fran.

During the session, there were moments when Fran expressed declarative knowledge about the task and her strategies distinctly. In the stage of transformation, she described how she found the pattern of the first three figures and her goal of an algebraic formula in the passage quoted above which began, "I notice that the pattern is…growing…" And, later, while working on the second part of the task she said, "So, I am going to just try different numbers…trial and error." In the interview situation, when we discussed the necessity of describing relationships, her response was

F: I really don't like descriptions. I know it is important, though.

R: Is it just hard to translate something into words?

F: Yeah, I … for me it is just easier like I said, I just want the numbers. I just want to solve it and get my little circle, box my answer and that is just it. That is just how I grew up but … I always have a hard time explaining why it is this way...

This was interesting in that she was very articulate particularly during transformation.

To summarize Fran's problem solving, she was successful in finding the best answer for the second part of this task. Her strong use of pattern in both numeric expressions and drawing supported her accomplishing it. That she desired to write an algebraic relationship for the figures and total number of tiles was apparent, but she could not extend those skills to develop the symbolization.

In examining her problem solving using the frame of a cognitive and metacognitive analysis, it appeared that Fran expended more time in evaluation than many of the other participants. She also employed alternative representations to verify her finding, which was unique. But, after reviewing the recording several times, there was no evidence that she returned to the beginning and reviewed her process, which would have been evidence of internalization.

Geometry Task

During the half of a minute that she engaged with the task, Fran's reflection on previous experiences was evident. She commented on the types of transformations with, "I just know because we just did geometry class last semester that…There are also glide reflections that are, I think they are rotations and reflections or something." Apparently, she was mentally reviewing the list of transformations that she could recall. Her next step was to clarify if the ones mentioned in the task description were the only ones to be used. I reassured her that she should use any that seemed appropriate.

Once Fran had resolved all her questions about the task, she moved into the stage of transformation. This was 1½ minutes into the session. Similar to her experience with the algebra task, she verbally expressed her process of analyzing the task. For example, from her visual examination of the task, Fran concluded, "…okay [pointing at the line y = 8] I notice that C is on the same line…[as C'], so it is going to be…it is probably going to be a rotation and then a slide." After proposing this, Fran took a moment to list the coordinates of the vertices of ABCD and A'B'C'D'. When asked why she did this in the interview twelve days later after the holiday break, Fran said, "I don't know that I necessarily thought that it was important in solving the problem. I think it was just like more information for me to have." She continued explaining her method with, "When I am solving a problem, I want to have or I want to write or figure out all of the information I can about it…I don't normally like just solve something just right off." Fran continued to explore the task by decomposing the two quadrilaterals into a square and a triangle, noticing that the area for the smaller square was four and that of the larger was sixteen. Fran also returned back to the statements defining the task with, "I just now noticed…that it is a similarity transformation, so it is a similar figure." These notes may be seen in Figure 20.

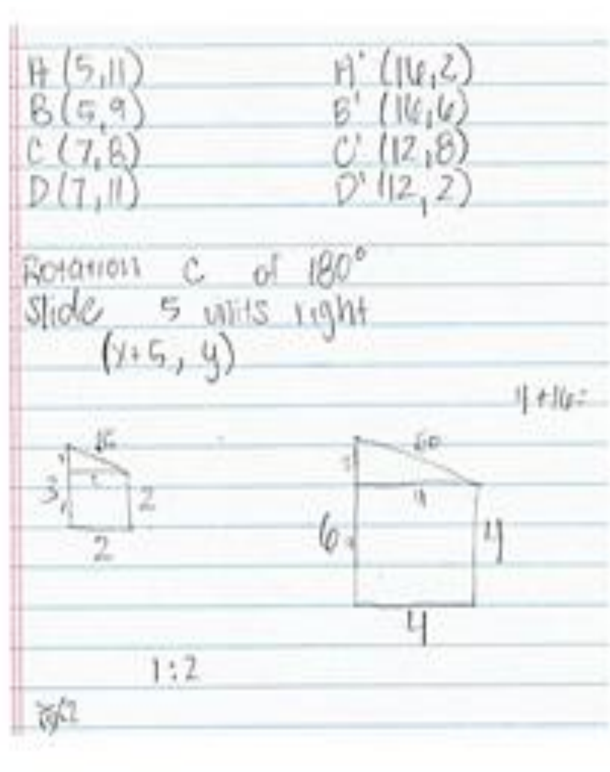


Figure 20. Fran's geometry notes during transformation include a model of the slide used in her similarity transformation and indications of her process to find the ratio of similarity.

After 4 minutes, Fran began to implement the original plan to rotate ABCD and slide it. Her solution may be seen in Figure 21. She began describing her process with, “Well, it will be rotated around point C.” This demonstrated that she understood the concept of pivot point. She proceeded to rotate the figure once 90°, and then rotate it again for another 90°. Her method appeared to be a visual orientation to the original figure and counting for the first rotation. For the second rotation, her visual focus was on the image, but she returned to the original for lengths of line segments. So, she rotated ABCD clockwise with C as the pivot point twice for a total of 180°. Fran then labeled the corresponding vertices A, B, and D; C remained C. Her statement at the completion of this was, “Okay…you have a rotation at C, around C of 180 degrees.” She made a note of the rotation under the previous ones taken of the vertices earlier.

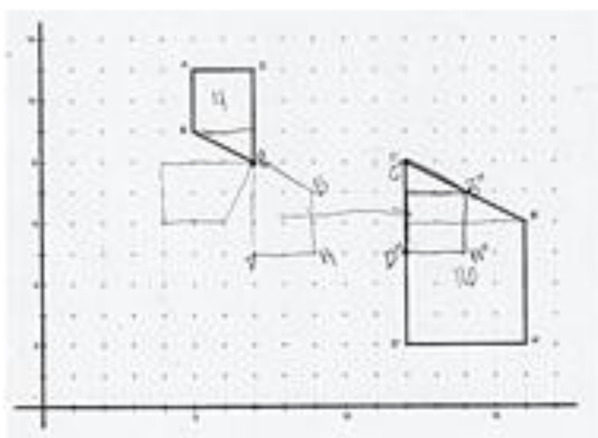


Figure 21. Fran’s similarity transformation.

Fran moved into her next transformation explaining it with,

> We are going to have a … slide … what is the …geometric word for that? Translation. So you are going to have a … slide of … five units to the right …. I think you write those like x + 5.

She wrote the note, “slide 5 units to the right” and added the ordered pair “(x + 5, y)”. From her justification of this ordered pair, it was evident that Fran was using it to write a rule for the motion. She sketched the translated figure and labeled the image A”B”C”D”, renaming C’ to C”.

At this moment in the problem solving process, Fran appeared uncertain of how to express the enlargement of the figure to A’B’C’D’. Her description of A’’B’’C’’D’’ was, “There is our new ABCD that has not grown yet.” She realized that a transformation was necessary but had difficulty expressing it. She added, “Well, the…length and the width of the square…are going to be times two.” Her first attempt was to try to write it in a coordinate pair but realized she could not.

Fran returned to the idea she recognized earlier, that the task was a similarity transformation. This led to the remark, “Okay, a similar figure…always is a ratio.” Fran began to explore this idea by sketching both of the original quadrilaterals on notebook paper (see Figure 20) and assigning lengths to the corresponding line segments by counting. She stated that she employed the Pythagorean Theorem for the oblique line segments. Then, Fran stated, “So, it grows by two…The ratio would be…one to two.” After contemplating this for a moment of evaluation, Fran ended the session after a total of 11½ minutes.

Researcher observations. After four minutes in the stages of engagement and transformation, Fran moved steadily through a sequence of transformations until the last. At that point, she appeared uncertain of how to express the final change between the figures. She struggled to remember the term dilation and its relationship to similarity. But, she was able to communicate the ratio between the two figures by actually using the term, ratio.

The transformations were expressed using a procedural understanding of rigid motion. For instance, when Fran rotated the figure around C, she was explicit about the center of rotation and how many degrees she was rotating it. She did not indicate the rotation with any notation. That she used two rotations of 90° instead of a single rotation of 180° may indicate a somewhat limited idea of the concept or it was not her intent to use a minimum of transformations. Her translation was clearly done and all three of the images were appropriate. To explore her vocabulary, I asked her the following:

R: You rotated ABCD two times. What is the relationship of these two figures to the original?

F: What do you mean? They are all the same, congruent?

R: And, when you slid it?

F: Still congruent.

R: How can you tell they are congruent?

F: They are the same…the shape…sides.

It was interesting that Fran was able to connect the concept of similarity with ratio but struggled confirming the ratio between the third image and A'B'C'D'. She expended four minutes confirming the ratio of 1:2 by redrawing the figures (see Figure 20) so that they did not overlap, counting the corresponding sides, and using the Pythagorean Theorem to find the measures of the oblique line segments. She did not recall that for similar quadrilaterals it is only necessary to determine if three sets of corresponding sides are in proportion. When asked in the interview if she recalled the term dilation, Fran said, "Oh, I didn't remember that. That is a lot better than 'grow.'"

Fran demonstrated the visual skills to complete this task but used limited notation to show how she moved through the transformations. Even though she began the task by noting the coordinates of the vertices on the original quadrilaterals, she did not label any on the task sheet. Also, usually, when labeling figures in geometry, the method is to label the vertices of the figures with capital letters but not use the same letters from image to image unless they are accompanied with subscripts. The label of A indicates a distinct point. Fran did not label the vertices of the first image and labeled the second image in the same notation as the original figure. But, she did label the third image appropriately. This indicated that she knew how to name the quadrilaterals but did not consistently do so.

Fran's declarative knowledge was exhibited in not only a procedural way, describing how she was creating a transformation, but also in a manner expressive of the relationship between the two figures. She made statements like, "it will rotate around C," a vocalization of what should be done. But, she also was able to recall terms and verbalize concepts, such as, "Okay, a similar figure…always has a ratio."

Fran was asked at the end of the problem solving session to put her ideas on paper. She was encouraged to add any thoughts she may have as she wrote. Under the notes that she had taken on the notepad (see Figure 20), she wrote the following:

> Summary:
>
> Figure ABCD is rotated by point C by 180°, then the new figure slides by 5 units to the right. The figure is then doubled. The area of the figure as well as sides increase by a multiple of 2.

As has been evidenced in all the participants' writing, there were no additional ideas given. Fran recited her process in a minimum of words. When questioned about the area doubling during the interview, the following short exchange occurred.

> R: In you final statement, you said the figure grew by two. What exactly grew by two?
>
> F: The length of the sides. I don't … I don't think the … inside … the area.

In the stage of transformation Fran compared the areas of the squares that helped compose the quadrilaterals. She found that the area of the smaller square was four and the area of the larger was 16, noting it on her exercise sheet.

If there is a difference in Fran's approach to the other participants,' it would be in the stage of transformation. This stage is described as when a person attempts to make sense of the problem, to explore and develop plans. Fran spent four minutes in this stage compared to ½ to 1½ minutes by others. In this task, she searched for as much information as she could deduce at the initiation of the task. But, in the video recording and my personal observation, she was not seen returning to the beginning of her work and following her path through to the final solution, which is usually indicative of the stage of internalization.

Statistics Task

As she approached this task, Fran engaged with the data for a half a minute then moved into the stage of transformation to make sense of the information. She was very vocal, reasoning aloud. First, she re-worded the first component of the task, personalizing it and then noted that the mean given for each set was the same. From this she formed a conjecture which was, "that kind of makes me want to say that…either one." Her justification for the comment was, "I think the mean…is the best way to look at…a performance of something like this." Immediately after that, Fran noticed that the data was obtained over a year. She interpreted this in, "their mean is the same; so, I would say both theaters are very similar." How the theaters were similar was not added. There was not any comment about the predictive properties of data being accumulated over time.

Fran continued in transformation while attempting to describe the data. Her first action was to arrange the data compiled for each theater from least to greatest. She said, "I am going to see if I can get the mode out what they have given me." She found what she called the mode for Theater A as 91 and for Theater B as 71. She vocalized both and added, "That is something to say." She actually found the median.

Fran also commented on the data over the week. She implied the concept of range but she did not explicitly discuss the concept or compute the two ranges. Her ideas was expressed in, "…it looks like…Theater A has a lot more dips, like highs and lows. And, Theater B is pretty…it is more around the mean." After four minutes, she concluded her analysis speaking and writing, "I would definitely prefer theater B because the average is the same but the mode [median] shows A much higher but the attendance is more around the same for B than A." The significance of the mode/median was not discussed.

At six minutes into the task, Fran moved to the second component of the task, recommendations for the two theaters. She centered her first impressions on marketing. Fran suggested, "I think for Theater A, since it has so many highs and lows, you could probably … even it out more if you…changed some marketing." She suggested specials on low attendance days or using coupons to entice families. But, this would only be employed on days that traditionally have low attendance. Fran considered that Theater B needed better marketing in general. In her rationale, she again referred to the lowest and highest values in the data. But, again, she did not compute the range. She even suggested firing some workers at Theater B to see if it could operate well without them. After considering personnel in Theater B, Fran returned to the data on Theater A attendance to remark on the need for more employees on the weekends. Fran's addressing of the question about recommendations involved four minutes.

While working on the recommendations, Fran made the following notes:

A: marketing more specials for low days, coupons for the low days, more [employees on] weekends

B: marketing around the town, maybe fire a few people but stay where they can meet customers' needs.

Researcher observations. In answer to addressing the underlying goals of this task, Fran did attempt to describe the data beyond the mean. But, she struggled with both recall and application of the necessary terminology for descriptive data. Similar to other participants, Fran described the range, a measure of dispersion. But, she did not compute it. Also, Fran did address variability with respect to the mean in, "Theater B is pretty…it is more around the mean." There was not a responding comparison of the data for Theater A.

Fran was very clear that she considered the mean the best way to look at the data. Also, she did notice that the mean represented data accumulated over a year. But, she did not interpret the common mean as indicating the same weekly attendance (on average). She gave the ambiguous statement, "I would say both theaters are very similar" without an explanation of how they were similar or by what quality they were similar.

Fran explored the data by arranging it and attempting to find the median, but she inaccurately called it the mode. In the interview two days later, I drew her attention to this. We had the following exchange that ended up being a review of the terminology. I contributed some of the terms in that she did not appear to be able to recall them:

F: I was thinking about the median, and then the mode, and the…what's it…I can't think of it now…It's like all together…the average, the mean…The mean, median, mode…" [pause of 30 seconds]

R: And, range.

F: And, range. But, I kind of talked about the range a little bit. But, I didn't think the median would be that important because…I just figured the mode would be the most important. But, I think I did the mode wrong, didn't I?

R: How did you do the mode?

F: I did the mode as the one…yeah, I did it totally wrong. I got mean, median, and mode wrong. Because I did the median. The mode is the difference between the highest and the lowest. I did…that is not what I did.

R: The mode actually is the most frequently occurring number in the data.

F: That is it. Okay. See, I am getting all of them confused. That is probably why I didn't use any of the other ones. And, I talked about range a little bit right there. I don't think that I called it the range though.

It appeared that she knew the procedure for median but did not connect it to the correct term. Also, with range, she knew to compare the lowest and highest pieces of data but did not attach the process to the vocabulary.

When asked about her familiarity with this type of task, Fran said that she did relate this task to similar ones that she had experienced in her statistics class but she had not expected one during this project. She explained it as follows:

F: Yes, we…I took a statistics class. And, we had some problems like this. It was really a business statistics class; so, we had more questions like this. And, so…I had had problems like this. I just wasn't expecting this. Like, us meeting.

R: Is that the reason you kind of hesitated here?

F: Uh huh. When I was looking at all the data and I was going to add it up to find the mean, you know, then I saw the mean at the bottom. I was like, 'oh…' [implying it surprised her].

But, Fran continued with the task in an attempt to describe the data.

Fran's response to this task was not a great departure from her previous movement through the stages of metacognition. Her behavior in expending a goodly proportion of her time in transformation was seen here, similar to her other problem solving experiences. Also, common to those other experiences, she verbalized her consideration of the information at more than one level. Though her terminology was confused during this task she did move beyond the initial consideration of the mean to attempt to describe the data in more detail. Fran implemented a plan to explore the data and also approached the second question by staying focused on factors that may address the attendance at the two theaters. There was evidence on the video recording that she did pause and review her writing of the conclusions, though it was short. There was no indication that she reviewed the entire task solution in an effort to make sense of the entire process.

Chapter Summary

The 18 descriptions of problem solving given in this chapter were developed with the goal of laying a foundation to support the analysis that is offered in the next chapter. With each account of the participants' work, the researcher observations were written to focus attention on the moments in the problem solving and interviews that were of particular interest to this study. The next chapter presents the data organized in two methods. One examines the data regarding the elements of metacognition being explored with relationship to the individual participants and the second one examines the elements with relationship to the mathematical task. This analysis required an organizational structure that may be viewed in Appendix G.

CHAPTER V:

DATA ANALYSIS

The previous chapter described the participants' problem solving with an attempt at maximum detail in order to provide the basis for the analysis presented in this chapter. The lens used for this analysis was metacognition, or what Andy, Bev, Cynthia, Danni, Elizabeth, and Fran revealed through their mathematics and interviews about their knowledge concerning these particular tasks in algebra, geometry, and data analysis. The information gathered from the think-alouds and interviews were organized in two tables, one describing the solutions and one supporting the research questions (see Appendix G). The tables were structured using Yin's (2009) suggestion of word tables (p. 156) as organizational tools. An analysis of each case (participant's work) was developed and then an analysis across the cases focusing on the category of task was compiled. The organization of the chapter reflects this process.

The first section is a short analysis of each participant's problem solving practices. It will be partially based on evidence taken from the participants' work and from the interview transcriptions. Examples of their language use during planning and procedure will be used to support assumptions about the depth of their declarative and procedural abilities across the tasks. The remainder of the analysis will be inference concerning whether there exists evidence supporting the presence of conceptual knowledge about the basis of the tasks. As Rittle-Johnson et al. (2001) suggested, it is not always possible to articulate this type of knowledge but it may be seen in the participants' choice of *when* they employ a mathematical strategy and knowing *why* it was necessary Also, flowcharts have been drawn to illustrate the movement among Yimer and Ellerton's (2010) categories of cognitive and metacognitive stages (engagement, transformation, implementation, evaluation, and internalization). One possible cycle is illustrated in Figure 22. It shows a progression through engagement, transformation in which a plan is developed, implementation, and evaluation. At that point, the participant may encounter a moment of debugging, returning to transformation, or proceeding to internalization which supports a return to the initial stages for a holistic look back at the problem solving process.

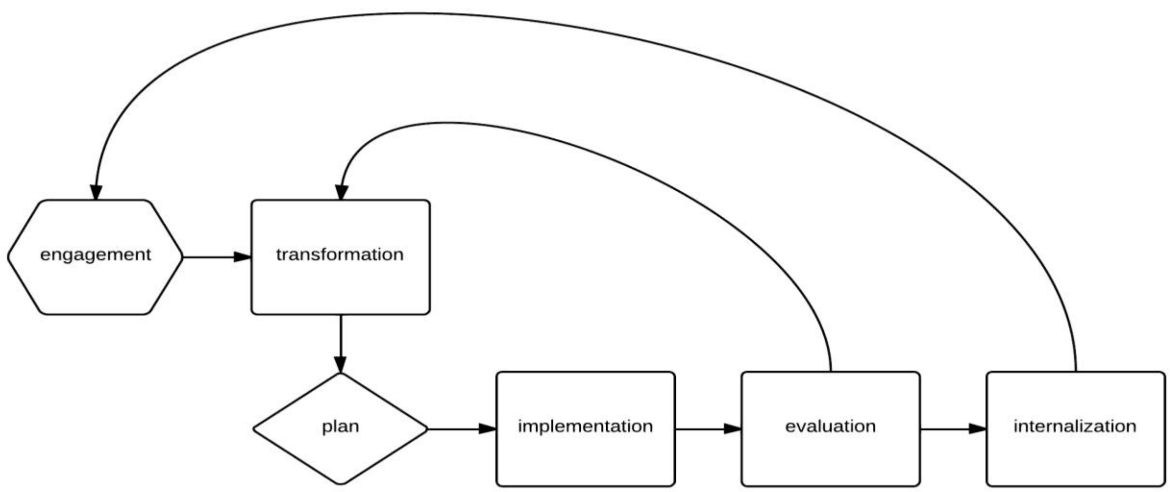


Figure 22. A Model for the Movement through the Categories of Cognition/Metacognition.

The second section is an examination of the task solutions organized by the category of task. Of particular interest was the use of representations (i.e., numeric/tables, verbal/written, sketches/graphs, symbolization) to support problem solving strategies, precision in language and mechanics, and progression through the stages of regulation of cognition. Examining these factors gave insight into the relationship between these elements of metacognition and mathematical problem solving.

Analysis of the Participants' Responses to the Tasks

The analysis of the participants' task solutions has the purpose of describing their practices, not assign a score to their ability or compare one participant to another, although general trends may be mentioned. Also, there will be a comment when a participant has a unique method of approaching a task or uses a distinct procedure while solving. This analysis is limited in that it examines only three particular tasks. Also, the participants are from two different institutions and may not have shared similar experiences. In examining their work, it was kept in mind that the participants were asked to work the problems, not scaffold solutions as if they were teaching.

Andy

Of the three tasks, algebra, geometry, and data analysis, Andy's performance was most complete in the execution of the geometric transformations. He was able to complete the algebra task to the point of finding the total tiles for the 20th figure and producing a reasonable guess for the figure with at least 10,000 tiles. For the data analysis task, Andy recognized the importance of the common mean and interpreted the implication but failed to mathematically explore and contrast the variability of the two sets of data. In contrast, his work with the geometric task demonstrated an understanding of the requirements of the task through the production of meticulous images and use of precise terminology.

Andy's incomplete solution for the algebra task was a result of his limited use of representation. His reliance on numerical patterns exclusively produced the correct answer for the number of tiles in the 20th figure. But, he did not extend the pattern to an algebraic model that would support considering any figure or any number of tiles. Why he did not was unclear for Andy was able to express his thoughts about the relationship between the figure and its structure clearly in both the stage of transformation and implementation. While exploring the given figures, he articulated the pattern,

> What I am doing is taking, multiplying the row by the column and the two outside ones at the beginning and the end just kind of leaving them off. So, that is going to be one times three [for the first figure] which will be three plus two is five.

Also, after the production of the 8th figure during implementation with his informal table, he clearly stated the rule for the function in, "So, by the pattern, it is just columns, that would be whatever figure you are on, and the rows…plus two and then you multiply those two together, then add the two tiles on the outside." At this point, he stopped using the table and moved directly into using this relationship to find the total tiles for the 20th figure. That he did not produce a model implies that Andy was not able to translate his thoughts into an algebraic model, such as $S(n) = n^2 + 2n + 2$, or did not perceive the advantage of doing so. This failure may be due to a lack of procedural knowledge in how to use/assign a variable or a conceptual barrier in not knowing *why* it would be advantageous to construct an algebraic model. In our discussion in the follow up interview, when I suggested letting *n* represent the number of the figure, he was quick to produce a function and recognize its usefulness.

The flowchart below in Figure 23 displays Andy's movement through the task. It shows the moment in implementation when he recognizes the pattern of rows and columns and then transforms his process for finding the total tiles in the 20th figure. It also displays the abrupt end to his process for finding the figure with at least 10,000 tiles. His uncertainty in his conclusion is reflected in his immediate termination of the process without evaluation.

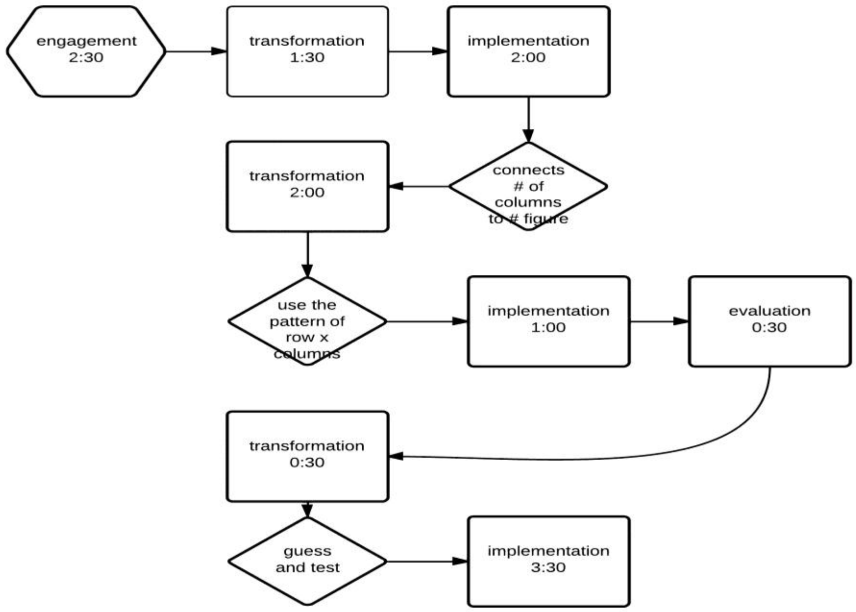


Figure 23: A Flowchart of Andy's Problem Solving of the Algebra Task.

Andy's effort on the second task, geometric transformations, was solid in that he was able to produce a series of transformations to place ABCD onto A'B'C'D' with a minimum of moves while using both appropriate language and notation. His process in this task was different from the previous one in that he did not verbally explore the task in the stage of transformation but proceeded quickly into implementation using complete thoughts like, "One line of reflection…would be at…y = 8" and "D would reflect over to…(7, 5)…" In response to the task description, Andy labeled his lines of reflection and added notation to designate the corresponding vertices of the quadrilateral's images. Watching Andy construct his images, it was apparent that he was using a numeric method of counting to place vertices. But, when pressed during the interview, he was able to give the rule for reflection relating figure, image, and line of reflection. The transformation that he struggled most over was dilation. He did recall the term (which was unique) and determined the ratio of smaller to larger figure. But, his ability to express the rule governing dilation apparently was not complete. In his work and articulation of the task, Andy appeared to have sufficient conceptual basis to understand the task requirements and develop a valid solution. Figure 23 maps Andy's procedure through the geometry task. Of note is his moment of struggle to decompose one transformation into two which forced him to return to the description of the task.

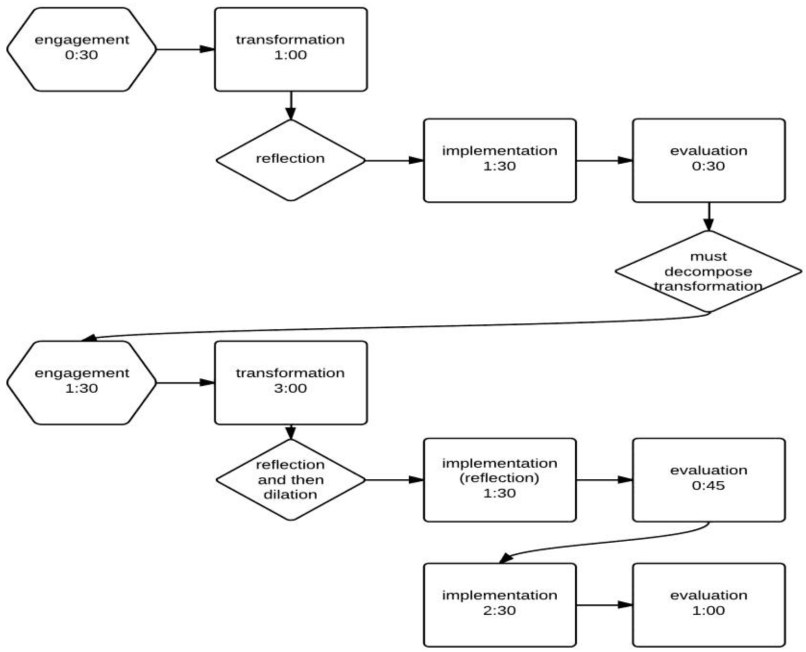


Figure 24: A Flowchart of Andy's Problem Solving of the Geometry Task.

The final task of data analysis stimulated the least cognitive/metacognitive activity in Andy which can be seen in Figure 25, a flowchart of his problem solving. He used a visual/mental examination of the data to develop his answers. During the stage of transformation, he articulated the relevance of the common mean in, "So, generally they [the theaters] are getting about the same…amount of people." His interpretation was unique among the participants. Also, he commented, "Theater A is a little spread out…Theater B is a little more…evenly distributed." Different from the previous tasks, Andy did not produce a plan to mathematically examine the data in the stage of transformation. This implied that the concept of examining the two sets through computing the range or variance did not occur to him. This can be seen in the comment, “I was trying to…get some numbers. But, I can’t really do this with numbers too much. So, I was just trying to think of questions…” The majority of the time expended on this task was used forming questions concerning the attendance at the two theaters. Andy produced several questions, some in an effort to contrast the features of the two theaters. All of the questions were focused on attendance and did not introduce elements outside the task. From his work, it was apparent that Andy did not recognize *when* to apply the concept of variability to examine each set of data and then compare the two. This limited perception of the task may be seen in Figure 25.

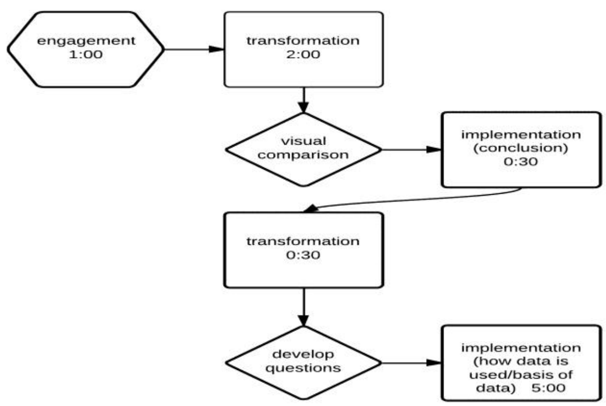


Figure 25: A Flowchart of Andy’s Problem Solving of the Data Analysis Task.

Examining Andy’s work from the direction of the stages of cognition and metacognition, it was noticeable that he spent the majority of his time in engagement, transformation, and implementation. During transformation and implementation, he attempted to express his thoughts clearly. Most of his talk was in sentence form with a minimum of confusion in terminology. In reviewing the flowcharts, it is apparent that he did not expend much time in evaluation and none in internalization.

Bev

From an examination of Bev's work, it was discernible that the algebra task was the most developed. She was not able to complete the similarity transformation. Also, it was not apparent that she understood the requirements for a satisfactory comparison of the data sets in the final task. But, Bev was able to produce an expression for the relationship in the algebra task and find the least figure with 10,000 tiles.

Bev's processes for the production of a geometric model and an algebraic expression for the relationship between the figure and the total number of tiles in the figure were difficult to determine because of the lack of descriptive talk. During engagement, transformation, and implementation the articulation of her process was negligible. Only after she had developed her solution would she explain her thoughts and actions. From observation during the problem solving session and reviewing the video recordings, it was clear that she used multiple modes of representation to support her answers. This could be determined by her visible use of the calculator, decomposition of the figures, and generalization to symbols. After producing the sketch of the 20^{th} figure and development of an algebraic expression, she was able to combine these findings and justify her conclusions in

> Because each of these [pointing to the top row] had one more than whatever figure it was and so that is the plus one. And, there are two bars so that is times two and that was a perfect square in the middle.

When approaching the second part of the task, she set the expression equal to 10,000 rather than expressing an inequality. She did not factor or use the solving function of a calculator to produce an answer but employed guess and check, using the expression, not the equation, to check her guesses. So, she recognized the value of using the algebraic symbolization. But, when pressed as to why it was important, she said, "I don't like questions like that. I do not know."

This reply suggested a limited understanding concerning the advantages of modeling relationships and the use of function notation.

The analysis of Bev's problem solving of the algebra task can be seen in Figure 26. Although there appears to be a smooth movement through the categories of cognition and metacognition, this may not have been the case. Her process may have been more complex. But, since she did not articulate her thoughts, this was not discernible. It was clear that she did not expend time in evaluation. This and her reluctance to expand on her methods imply limited reflection.

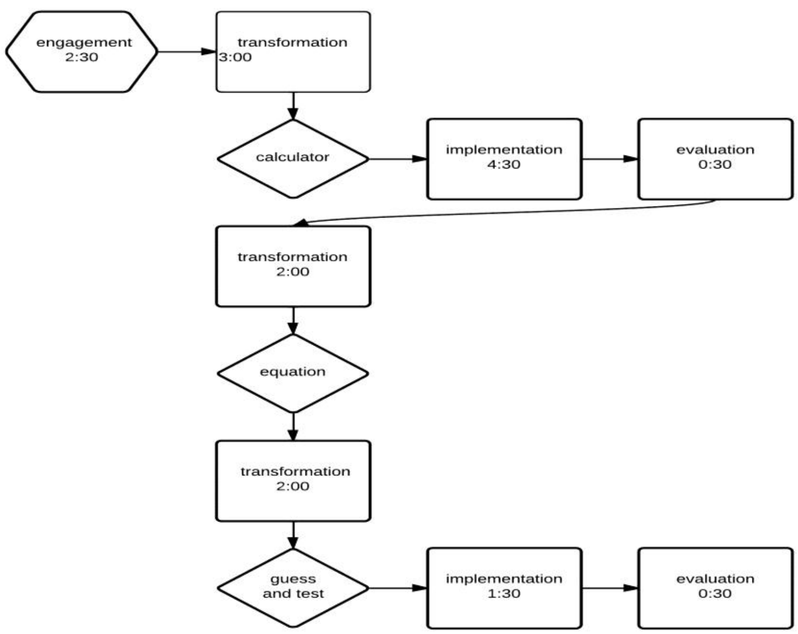


Figure 26. A Flowchart of Bev's Problem Solving of the Algebra Task.

Bev struggled with the geometric task. Her lack of precision in terminology, procedure, and notation was clear from the initiation of the task. During the planning stage of transformation, her confusion could be seen in,

> It looks like…the first one, the small one ABCD, is flipped down where A and D are at the bottom and then…let me see…[using hands] then flipped around that way [motioning with hands]. Or, you could say that it is rotated at C and then flipped.

While in implementation, her procedures lacked precision in that reflections were not conducted across the lines she stated and rotations were not produced around the pivot point indicated. There was no drawing of lines of reflection or angles of rotation as requested in the task instructions. Bev was able to recognize the change in size of the figures but used imprecise language to describe it: ABCD is "2 times smaller" than A'B'C'D'. Although Bev demonstrated persistence in her two attempts to complete the task by developing two series of moves as seen in Figure 27, it was evident the she did not have the conceptual basis on which to build the similarity transformation successfully. She had not experienced this topic in her undergraduate program and could not recall sufficient knowledge from her secondary experience to support development of a satisfactory solution.

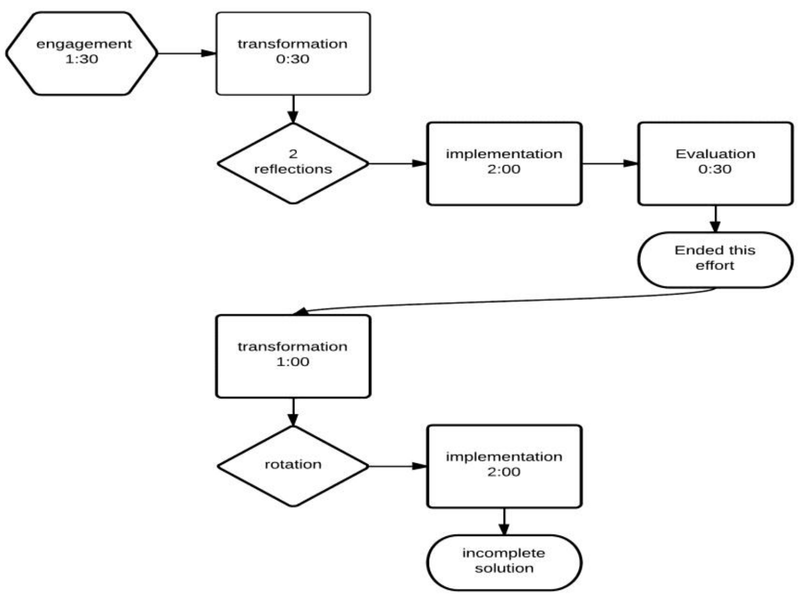


Figure 27. A Flowchart of Bev's Problem Solving of the Geometry Task.

The data analysis task did not engage Bev to the extent that the previous tasks did. She did note the highest and lowest pieces of data in each set but there was no indication that she explored the data in more depth during transformation. Essentially, there was no implementation except in the form of writing two notes justifying her choice of theater and two notes comparing personnel requirements. Evidence that this task did not engage Bev was found in that she did not address the fact that the two theaters had the same mean. Also, she did not recognize the significance that the data was accumulated over a year, which provides some attendance predictability. This was revealed in her statement about the attendance at Theater A being "unpredictable." Similar to Andy's struggle with this task, Bev did not develop a plan to mathematically compare the attendance of the two theaters and abruptly ended the task after completing her writing. Similar to Andy, she did not know to employ the concept of variability to compare the two sets of data. The limited nature of her problem solving may be seen in the following flowchart:

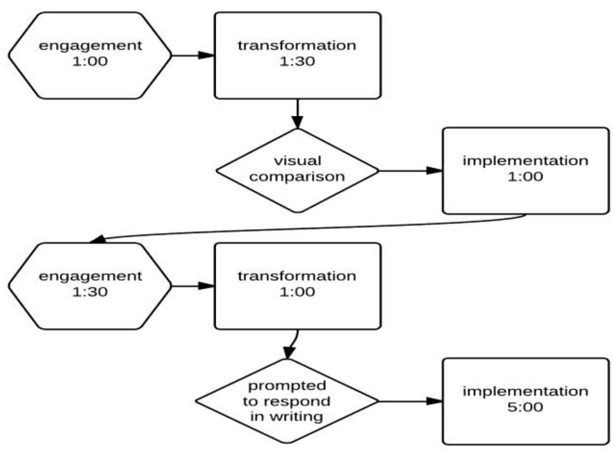


Figure 28. A Flowchart of Bev's Problem Solving of the Data Analysis Task.

In considering Bev's problem solving through the stages of Yimer and Ellerton's (2010) categories, it could be seen that she used a minimum of evaluation and no internalization. While working, she would reach a point in the task and indicate that she had done all that she could. She would stop without looking back. Even though she was reminded of the think-aloud process before each task, her articulation of her process was limited. This gave an incomplete vision of her problem solving in that she obviously had multiple strategies available as evidenced in the algebra task but there was not any explanation of why she used each.

Cynthia

Cynthia struggled with the tasks. Her approach to the algebra task was unique in that she discerned the recursive pattern of the total number of tiles was to add consecutive odds, but she was not able to interpret this in a form that would support her continued engagement with the task. Her work with the similarity transformation was characterized by persistence resulting in two paths but one was incomplete and the other had errors. And, her result with the data analysis task was similar to the other participants in that the task was not approached mathematically.

Cynthia in her effort to express some algebraic relationship in this task exhibited much persistence. Of note was her example of debugging. When she realized that it would be to her advantage to have an algebraic model of the relationship between the figure and its total tiles, she returned to her initial conjecture and rejected it which is shown in Figure 29 below as a movement from evaluation back to transformation.. By using both numerical and visual representations for the figures, Cynthia was able to recognize a recursive pattern of adding consecutive odds. But, it became clear that she did not know the pattern implied embedded perfect squares (i.e. 1, 4, 9, 16 …) and how to use this finding. Apparently, she did not have a conceptual base connecting the sketch she produced or the pattern she found to an algebraic model. Since she could not generalize the relationship, she failed to progress further.

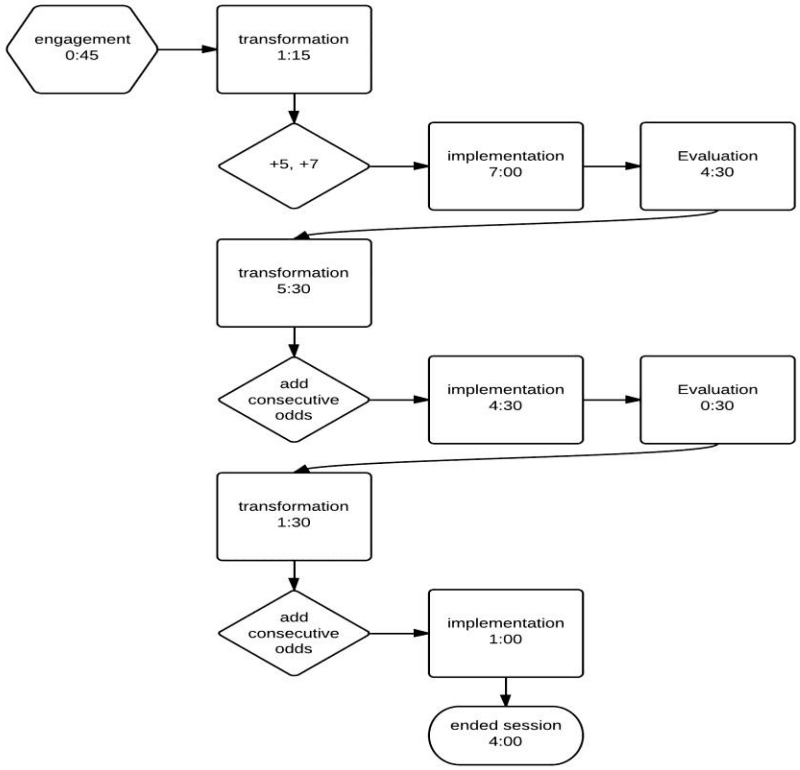
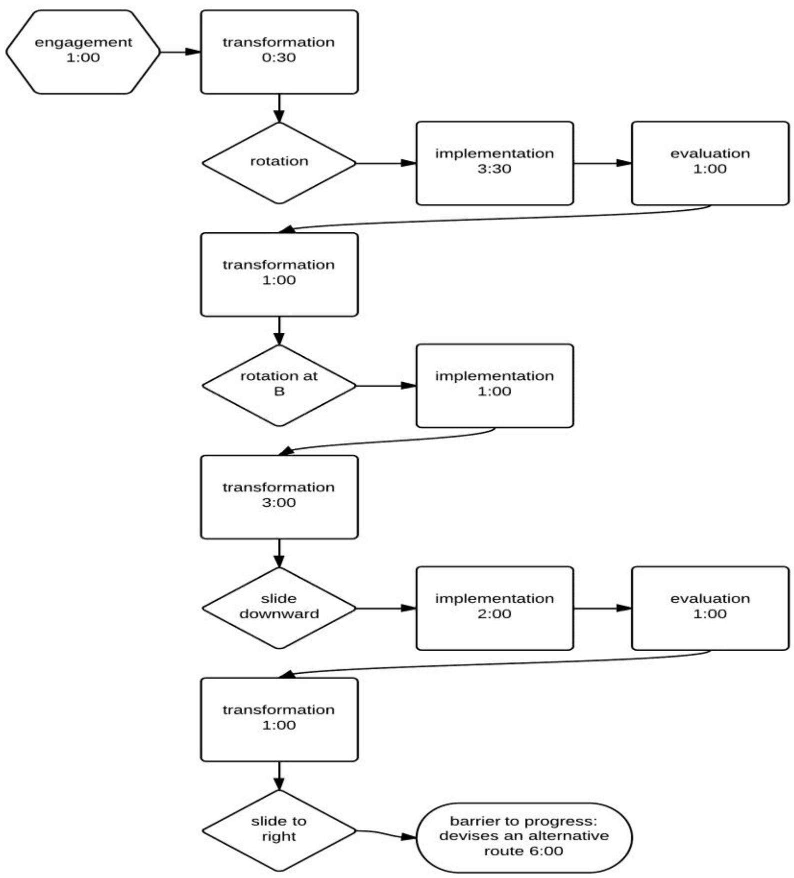


Figure 29. A Flowchart of Cynthia's Problem Solving of the Algebra Task.

During two attempts to produce a similarity transformation from ABCD onto A'B'C'D', Cynthia exhibited persistence again. Her process was to move repeatedly from transformation into implementation and back as shown in Figure 30. This could be explained in her comment, "I am trying to think of the transformations, all the transformations…It has been so long since I have thought of them." The two rigid motions that she used were rotation and slide. Her descriptions of the procedures were clear, for example, "It's rotated…clockwise… let's see…180…now, it needs to slide." One of the tools she used was a small set of axes to support her procedure of rotation. This visual representation was unique to her. During her first attempt, she encountered the need to enlarge an image. This created a problem in that she did not recall the term or concept of dilation. To resolve the barrier, she returned to the stage of transformation and devised a new route for the similarity transformation in which she increased the size of the quadrilateral at the first and then moved it. The enlargement was done by numerically extending the length of the sides, not technically using the rule for dilation. Following that, she was able to rotate and slide the enlarged image onto the A’B’C’D’ but with some technical difficulties. For this task, Cynthia was able to recall from her high school experience sufficient knowledge of the concepts of rotation and slide to use them fairly well. She was not familiar with the function dilation. The complexity and persistence in her process is visualized in the following flowchart, Figure 30.



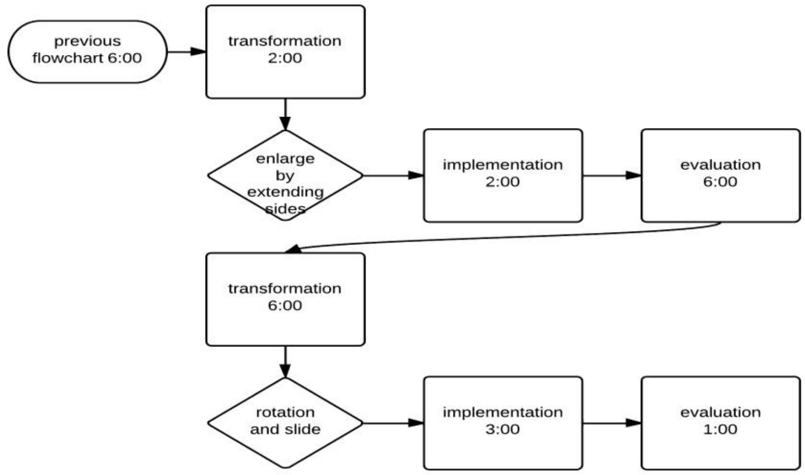


Figure 30. A Flowchart of Cynthia's Problem Solving of the Geometry Task.

Similar to the previous participants, Cynthia did not develop a plan to mathematically compare the two sets of data on theater attendance either by comparing the range of the two sets or the variance with respect to the common mean. She approached the material from a more visual/interpretive stance. Her comparison of the two sets included, "For Theater A, Wednesday and Thursday are their slowest days…Theater B has a constant flow throughout the entire week..." There was a vague reference to comparison of one theater's attendance to the mean in, "B would have an average crowd." There may be a suggestion that Cynthia was alluding to the data in set B as closer to the mean than that in set A. But, there was not any elaboration on this. At the conclusion of the task, Cynthia did pause for a moment and reflect on the implication of the mean being the same but again did not put the fact in context. So, her performance on this task was not significantly different than the other participants but as exhibited in the following flowchart, Figure 31, there was some reflection back to the task description as she sought a way to respond to the second component of the task.

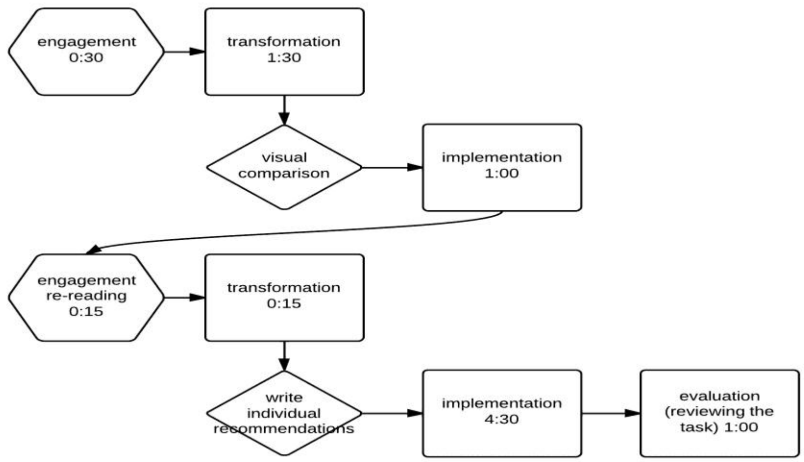


Figure 31. A Flowchart of Cynthia's Problem Solving of the Data Analysis Task.

When examining her work from the perspective of the categories of cognition and metacognition, Cynthia displayed several examples of debugging and persistence in problem solving. Her greatest hindrance was an apparent lack of experiences to support her conceptual development, particularly in function.

Danni

In the algebra task, her strength in numeric-organizational skills was particularly displayed as she found an equation for the relationship between the figure and total number of tiles it contains. But, the same detail was not apparent in her presentation of the geometry task. She accomplished the similarity transformation but her description and notation was negligible. And, finally, her work for the data analysis task was similar to the other participants in that she failed to address a way in which the two sets of data could be mathematically compared.

Danni's use of two tables relating the number of the figure with the total tiles and the number of the figure with the number of rows and columns provided a strong support to finding a numeric expression for the total tiles in the 20th figure and then a numeric/word equation to generalize the relationship, "[n(n+2)] + 2 = total number of tiles for the nth figure." This writing of the complete relationship was unique to Danni. Others had written an expression but this was the closest model to function notation. Danni's ability to express herself in sentences was seen in the stages of transformation and implementation. In transformation she summarized her explorations in, "I don't really want to focus on the one … at the bottom left and on the top right…I kind of just want to focus on the number of the rectangle." This ability to articulate was continued in implementation, for example, "So, to find how many tiles are in that center rectangle I'm going to multiply twenty times twenty-two because that is how you find area." This completeness of expression was continued throughout the task including the answer to the final inequality. She said, "So, the figure that would require at least ten thousand tiles would be any figure that had…at least a hundred columns." Danni's procedure of using exclusively a numeric strategy may have hindered her ability to find the most correct answer, 99 or greater. She used the geometric structure of the figures, but she did not recreate the structure by drawing to support her findings. During the interview, when asked why she wrote the algebraic relationship, she revealed her understanding of the task in, "I don't know…I guess maybe I was just trying to…put it in more general terms and then to use that for the next part of the problem." This statement demonstrated that she understood *why* the algebraic model was appropriate.

The following flowchart, Figure 32, illustrates Danni's sequential progress through the task. It shows her direct path through the first component of the task and her uncertainty during the second part, finding the figure with at least 10,000 tiles. She vacillated between the two approaches, using an algebraic method and guess/check. Her doubt about her final answer was marked with little evaluation or reflection.

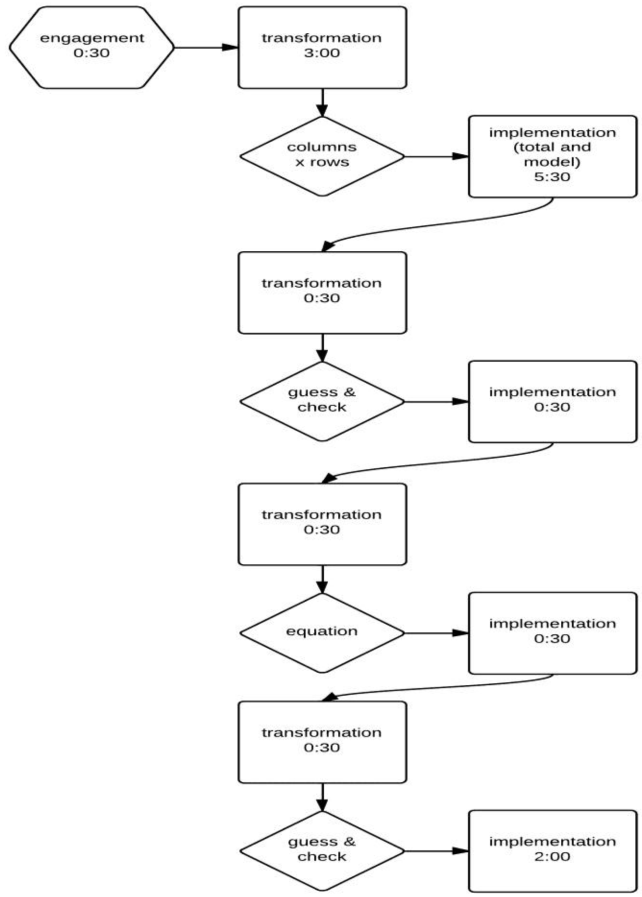


Figure 32. A Flowchart of Danni's Problem Solving of the Algebra Task.

For the geometry task, Danni found a similarity transformation that would place ABCD onto A'B'C'D' composed of three rigid motions and dilation. The attention to detail exhibited in her algebra work was not shown in this task. She did not use notation to indicate lines of reflection or corresponding vertices of the images. When articulating her plan, her talk did not suggest the underlying principles of the geometric transformation she chose, for example, "I first know that I want C on top, not on the bottom. So…I want to…reflect it …over y = 8…All I am doing now is reflecting it." She used two reflections. Her procedure with both was limited in that the images touched the implied line of reflection which is not necessary. When asked to describe a reflection, Danni replied that the figure is "flipped." On being pressed about that definition, she was not prepared to explain the relationship between the figure, image, and line of reflection. This suggested Danni had sufficient visual understanding of the concept to perform the transformation but not sufficient conceptual understanding to explicate a rule. With the dilation, she knew the necessity of a scale factor but did not recall the term dilation. Again, Danni appeared to have sufficient visual understanding to perform this task but did not have the necessary understanding to articulate a justification of the procedures involved. Similar to her solution of the algebra task, Danni's solution for the geometry task may be diagramed (Figure 33) with little evaluation of process. Her method was very sequential.

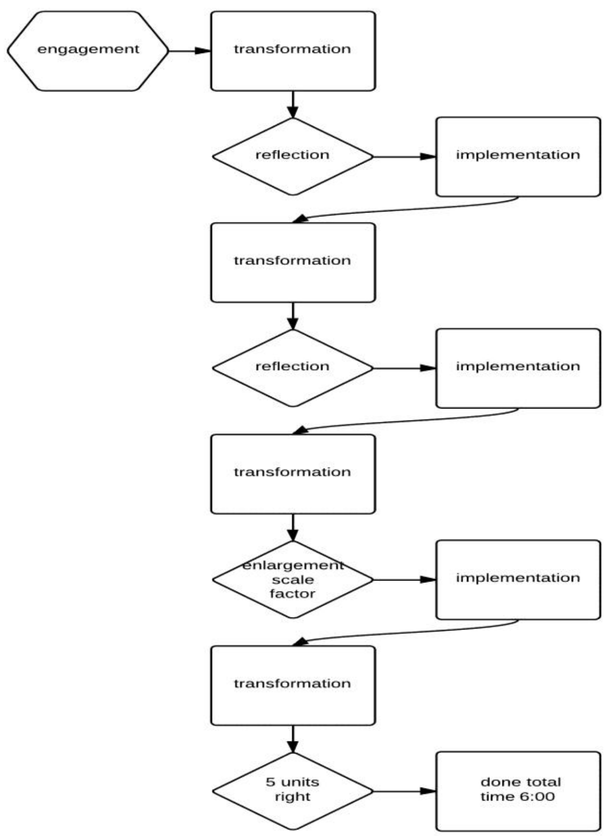


Figure 33. A Flowchart of Danni's Problem Solving of the Geometry Task.

Danni also approached the two data sets in the final task using a visual/mental process. Contrary to the previous two tasks, she did not develop a clear plan during transformation as illustrated in the flowchart below, Figure 34. During that stage, she described her process of examining the data in, “I’m just looking at…each day and comparing the numbers.” She failed to recognize the importance of the mean. This was determined in the interview when she responded to a question about the mean with, “I looked at the mean when I was doing the problem, but…I didn’t really take that [the mean was common to the two sets] into account.” In an effort to make sense of the task, Danni began to reference theaters that she attended. In the interview, she explained this in, “I don’t feel like I knew a lot about…like what I needed to do.” So, it was evident that the concept of exploring the variability of the data was not apparent to her.

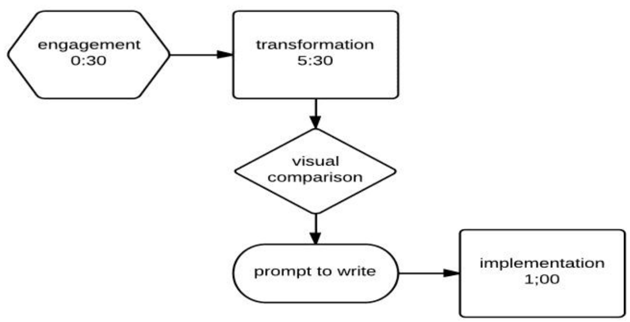


Figure 34. A Flowchart of Danni’s Problem Solving of the Data Analysis Task.

It was noticeable in Danni’s work that she did not move into the cognitive/metacognitive stage of evaluation. She may glance back over her immediate work but not for any extent of time. For the algebra and geometry task, she moved through the stages of engagement, transformation, and implementation forming a plan during transformation and performing it during implementation. She did not return to previous stages except when vacillating between two plans during the algebra task. It did not appear that she formed a plan to examine the data in the final task. She relied on a visual perusal of the data.

Elizabeth

Elizabeth's work revealed a strong understanding of procedure during the algebra task, but her grasp of the procedures particular to geometric transformations and data analysis did not appear as sound. For the algebra task, she followed a path of finding an expression for the n^{th} term of the sequence to an attempt to use the expression to solve an inequality. But, for the geometry task, though she developed a path for the similarity transformation, she could not demonstrate her knowledge by producing images. For the final task of data analysis, it was not apparent that she developed a mathematical plan to compare the data sets.

Elizabeth's use of representations in the algebra task was cyclic in nature. She connected the numeric representation of the figures in terms of rows and columns with the geometric representation of rectangular area plus two. While developing these relationships, she would review the figures, develop informal tables, draw new figures, and use writing for reflection. Her talk could be clear, for example, "Right now, I am just trying to see what the sequence…would be;" and "So, it is always going to be whatever it is in the middle plus two." It could also be muddled in that she confused the terms row with column and confused her recognition of patterns. This may be seen in the extensive time in evaluation as shown in Figure 35, a flowchart of her progress. From the beginning of the task, Elizabeth recognized that the figures represented a sequence and she should develop an algebraic model for a relationship. She also realized to use the algebraic expression in an inequality to find the least figure with 10,000 tiles. This was unique. Unfortunately, she failed to recall that the expression *at least* employed *the greater than or equal* symbol and to solve an inequality one must set it equal to zero. These errors resulted in an incorrect answer. From the initiation of the task, Elizabeth appeared to understand the conceptual basis of the task and why she was looking for an algebraic model.

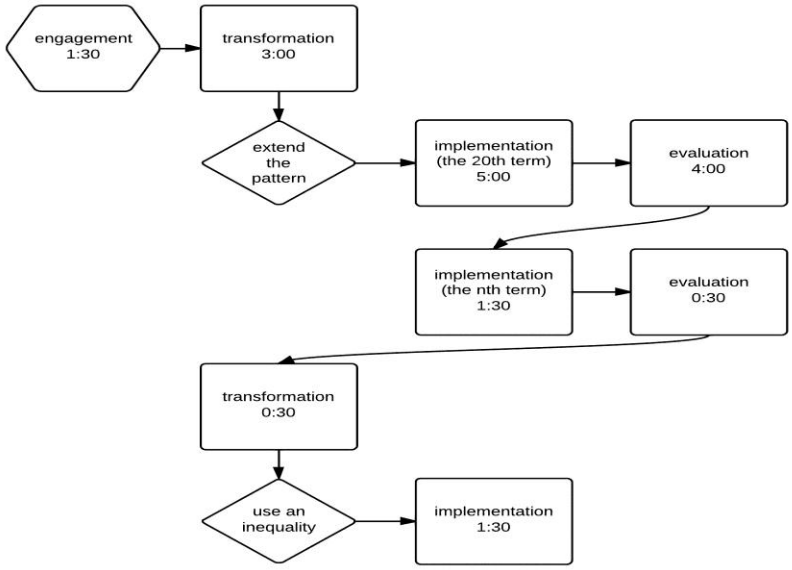


Figure 35. A Flowchart of Elizabeth's Problem Solving of the Algebra Task.

Elizabeth was able to compose a similarity transformation verbally but was not capable of creating the images to represent her plan. At the beginning, she used her hands and a ruler to determine a plan for moving ABCD onto A'B'C'D'. But, upon attempting one transformation, a rotation, she stopped trying to represent the rigid motions with images. Different from her math talk in the algebra task, she articulated in somewhat broken sentences, for example, "Let's see [using hand motions]. Now, if you, if you rotated it…The C would still be here [pointing at C and C']. So, it looks like some type of rotation cause if…" She was able to use vocabulary associated with rigid motion but failed to recall the term dilation. An example of the depth of her understanding of transformations was reflected in her response to a question about reflections in, "…reflection, you have to reflect about an axis and I knew I needed to find the middle of something." Her talk revealed some conceptual basis in understanding the transformations and their effect, but she failed to implement the concepts by demonstrating the motor ability to produce images. This was unique and is symbolized in the following flowchart, Figure 36, as an interrupted implementation of the task. This difficulty in producing images may have its basis in

Elizabeth being able to recite definitions but not sufficient experiences in reproducing images using the properties of geometric transformations.

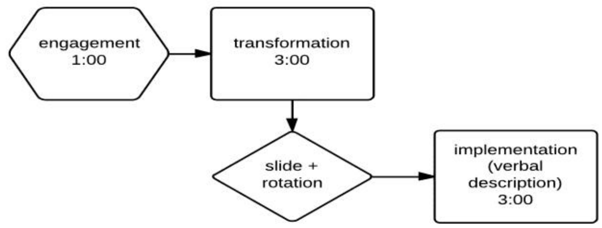


Figure 36. A Flowchart of Elizabeth's Problem Solving for the Geometry Task.

In approaching the data analysis task, Elizabeth exhibited knowledge of measures of central tendency but did not extend her consideration of the two sets of data to measures of dispersion. Though she reviewed mean, mode, and median, concentrating on mean, she did not contemplate exploring the fact that the mean was common to the two sets and the implications of that. Similar to the other participants, she did not compute the range or variance to express a mathematical difference between the two sets. Inadvertently she complicated the task by introducing the variable of ticket cost which was not relevant to the analysis. Elizabeth moved through an abbreviated cycle of engagement, transformation, and implementation without forming a plan to define the variability in the data as seen in Figure 37. She immediately terminated the problem solving session without any evaluation.

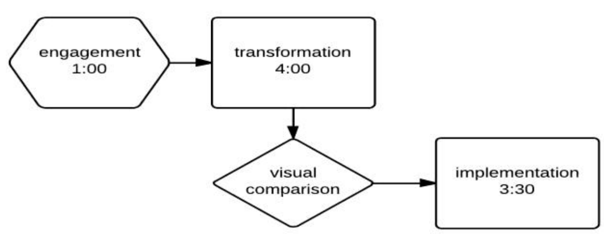


Figure 37. A Flowchart of Elizabeth's Problem Solving of the Data Analysis Task.

While working on the algebra task, Elizabeth moved through the stages of engagement, transformation, and implementation. She also expended a considerable amount of time in the stage of evaluation, which was unique. But, for the geometry task, she failed to implement the task with regard to the instructions. Although she returned to the instructions repeatedly, Elizabeth did not produce any images. During transformation with the data analysis task, a mathematical plan to compare the data sets was not developed. Her frustration could be seen in a response to the task with, “I’m mad!” Elizabeth’s movement through the cognitive and metacognitive stages may be seen in Table 9.

Fran

Although Fran failed to find a function for the relationship in the algebra task, she demonstrated persistence in her attempt to complete it. That same persistence may be seen in her work on the geometry and data analysis task. In her approach to the tasks, Fran described her problem solving method as pulling everything she could out of her bag (of skills).

Although she failed to find the “algebraic formula” she sought to describe the pattern of the figures, Fran was able to find the total tiles for the 20^{th} figure and the lowest numbered figure that would contain 10,000 tiles. Her problem solving style was cyclic in that she drew the fourth figure to confirm the numeric pattern she conjectured and then expanded the pattern. After finding the total tiles for the 20^{th} figure, she drew again to evaluate her conclusion. Fran had all the elements to write an algebraic relationship between the figure and its total tiles during transformation. This can be seen when she stated, “I notice that the pattern is…growing. Now, the two side ones on the top and bottom…they are going to stay the same in each one;” and, after finding the 13^{th} total, she added, “then I noticed that this [the first factor] went along with whatever number [figure] it was.” Fran appeared caught in a recursive definition of the sequence, which may indicate that she did not have a sufficiently strong connection to sequences or function to recall how to write the relationship by assigning a variable. Her persistence in finding a solution for the second part was evident in that she continued with the task using the numeric strategy of guess and check to find a possible answer to the least figure containing 10,000 tiles, the 100th. That Fran then took time to evaluate this answer by drawing and labeling the 100th figure led her to realize that the answer was not the most correct. So, she revised her answer to the 99th figure. Fran used a cycle of strategies to validate answers moving to evaluation, which resulted in debugging as seen in Figure 38, a flowchart of her problem solving. But, her work did reveal a limited procedural understanding of how to express the pattern embedded in this task.

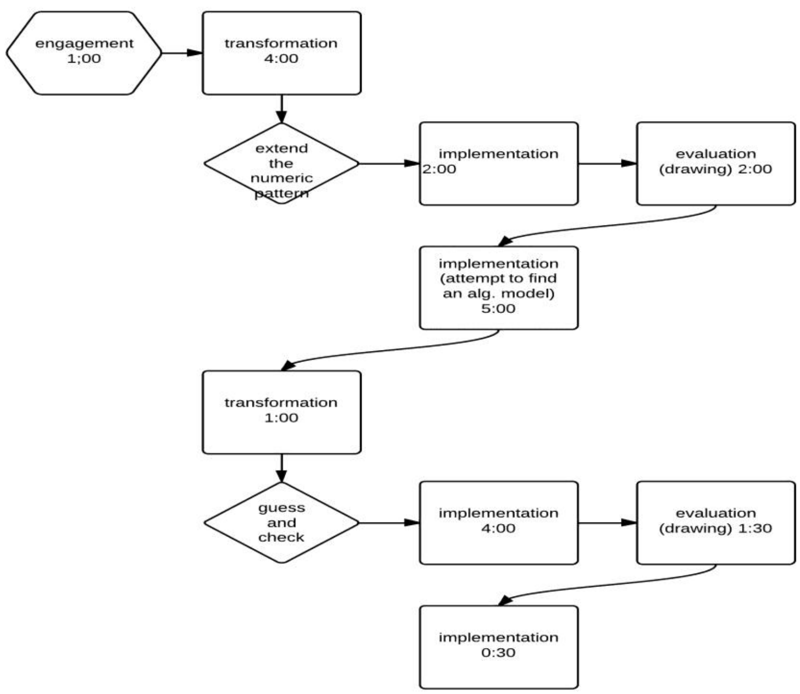


Figure 38. A Flowchart of Fran's Problem Solving of the Algebra Task.

Fran was successful in finding a similarity transformation to move ABCD onto A'B'C'D'. Her tendency to explore a task in detail before implementation was exhibited in her note-taking and verbal decomposition of the problem. Her math talk in this stage was in sentence form, for example, "…okay [pointing at the line y = 8]. I notice that C is on the same line…[as C'], so it is going to be…it is probably going to be a rotation and then a slide." The ability to specify her steps with clarity was continued into implementation as in, "Okay…you have a rotation at C, around C of 180 degrees;" and in "you are going to have a…slide of…five units to the right." The transformation that she could not recall was dilation. Fran did recognize the relationship between her images and A'B'C'D' expressed in, "Okay, a similar figure…always is a ratio…The ratio would be…one to two." In characterizing Fran's work, it would be appropriate to mention that it contained nonessential explorations. For instance, when finding the ratio between her final image and A'B'C'D', she examined all pairs of corresponding sides of her image and the quadrilateral, which was not necessary. Also, she examined the areas of the figures. But, this information did not appear to confuse her or distract her from finding a satisfactory solution to the task. Although she did not use a minimum of transformations and her notation was limited, Fran demonstrated that she understood the demands of the task.

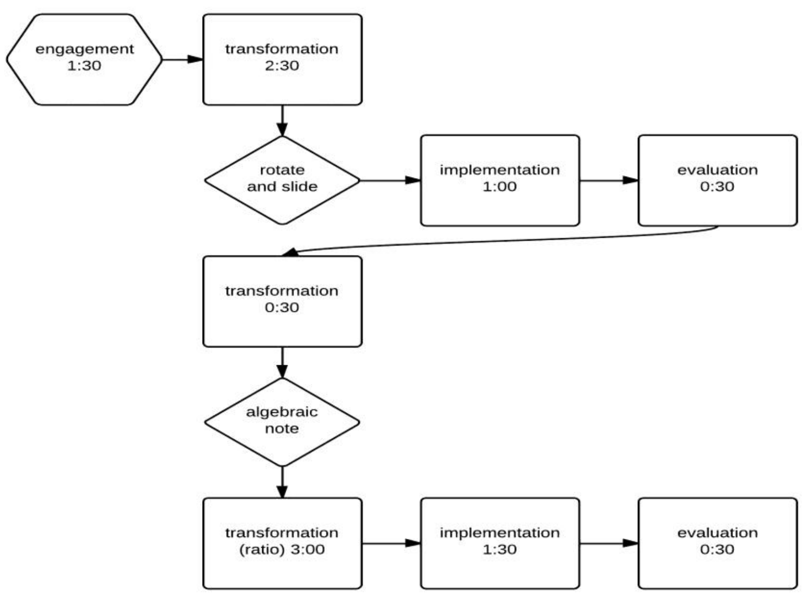


Figure 39. A Flowchart of Fran's Problem Solving of the Geometry Task.

In the above flowchart, Fran's moment of struggle to express the dilation may be seen in her return to transformation to explore the task again.

Fran's approach to the data analysis task was unique in that she did attempt a plan to explore the two sets of data to some extent. Her focus was on measures of central tendency. In portraying the mean, she stated, "their mean is the same; so, I would say both theaters are very similar." She also gave an ambiguous description of the variability of the two sets in, "…it looks like…Theater A has a lot more dips, like highs and lows. And, Theater B is pretty…it is more around the mean." This mention of one set of data being "around the mean" was only implied by one other participant. Fran did persist in an attempt to describe the data sets by looking at measures of central tendency. From Figure 40, a flowchart of her progress, it is apparent that she engaged with task repeatedly in her desire to describe the data. She organized each set and found the *median* but mistakenly identified it as the *mode*. Similar to other participants, Fran struggled with this task but she did try to describe the data staying within the parameters of the task. It appeared that she did not identify the need to determine the variability of the data sets through computing the range and/or variance.

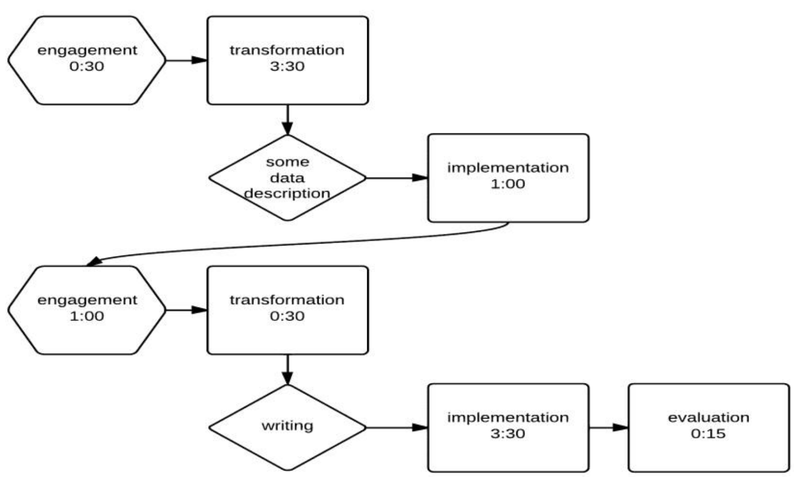


Figure 40. A Flowchart of Fran's Problem Solving of the Data Analysis Task.

Fran moved through the categories of regulation of cognition developing plans during transformation and then moving to implement them. It was also evident that she progressed into the stage of evaluation particularly during the algebra task in which she used alternative representations to judge the validity of her answers and debug. The stages of regulation of cognition were not as visible during her involvement with the data analysis task but she did attempt to describe the data in some form. Her progression through Yimer and Ellerton's (2010) may be seen in Table 10.

Summary for the Analysis of the Participants' Responses

Each participant had particular strengths. Some were focused, using a single strategy; whereas, others used multiple strategies in a cyclic method. Similarly, some terminated their problem solving once a possible answer was found and others paused to evaluate their answers. The similarities and differences in problem solving are explored more from the perspective of across the tasks in the following section.

Analysis of the Tasks

To structure this section of the analysis, each task is decomposed into two or three components. Each component was examined with attention to the differences and commonalities of the participants' problem solving. Attention was given to the ability of the participants to complete the task.

Algebra

To frame the analysis for this task, three features were inspected: (a) the method used by the participants to find the total tiles in the 20^{th} figure; (b) the model created to generalize the relationship between the figure and total tiles; and (c) the method used to find the figure with at least 10,000 tiles. This task demonstrated the most variation in problem solving techniques (see Appendix G).

Finding the total tiles in the 20th figure. All participants developed a method for finding the total tiles in the 20^{th} figure. None were exactly the same. Four of the participants, Andy, Danni, Elizabeth, and Fran, found the pattern of rows x columns plus 2 using a numeric listing. Andy, after computing the 9^{th} term, used his verbal model to compute the total tiles in the 20^{th} figure. Danni organized information gathered from the three given figures into two structured tables and moved directly into computing the total without finding totals for other figures or sketching. Elizabeth used a cyclic method between figures and computation. She would refer to the structure of the figures, find the total for one, and confirm the total by sketching the figure. She found the pattern then used it for the 20^{th} total. Fran listed numeric statements from figure 4 through 20 and then related her total for the 20^{th} figure to a sketch. The other two participants used very different methods for finding this total. Bev decomposed the figures into two rows and a square each, developing a geometric and algebraic model first and then used it to find the 20^{th} total. Cynthia found the pattern of adding consecutive odd numbers. Like Fran, she did a complete listing for the figures through the 20^{th}.

Finding a model for the function. Of the six participants, three were able to use algebraic symbolization to represent the relationship between the figure and the total tiles it contains. Each used a different method to form the symbolic generalization and only one expressed it informally as a function. Bev used the decomposition of the figures into a square and two rows of $n + 1$ to write the expression: "$n^2 + 2(n + 1)$". Elizabeth also wrote an expression, "$(n + 2)(n) + 2$" after synthesizing the pattern of rows times columns plus two and the area of the rectangle plus two. Danni referred to her numeric expression for the 20^{th} figure and developed the partially numeric, partially verbal equation "[n(n + 2)]+2 = total number of tiles for nth figure." This final one may be considered the most similar to function notation, which is traditionally used to convey the connection between two related amounts.

Andy, Cynthia, and Fran were not able to write the rule between the figure and total tiles. Both Cynthia and Fran desired to write one as evidenced in their interviews. This was not as apparent in Andy. Cynthia failed in that, as revealed in her interview, she did not know or recall the significance of finding consecutive odds as the difference of terms indicating that the terms contain perfect squares. Perhaps if she had this knowledge base, she could have continued with the task. But, this failure in conceptual knowledge essentially terminated her effort. Fran had difficulty moving beyond a recursive definition for the total tiles, even though in a review of her math talk during the task, she presented all the elements necessary to write the relationship. This was true for Andy as well. He clearly created and expressed a verbal model after he found the total tiles for the 9^{th} term. But, he did not write it in symbols and there was no evidence that he realized that the development of algebraic symbolization would benefit his final solution of the task. This implied a lack of procedural understanding of how to express the relationship using a variable or a lack of conceptual understanding of the role of algebraic symbolization.

Finding the figure with at least 10,000 tiles. Five of the participants continued with the task to attempt to find the figure with at least 10,000 tiles. Of the five, Bev and Fran found the most correct answer for the boundary of the interval, the 99^{th} figure. The group could be subdivided into two sets: those that found an algebraic expression (Bev, Danni, and Elizabeth) and those that did not (Andy and Fran). All except one used guess and test, a numeric method of solving.

Bev and Danni both initially wrote an equation setting their algebraic expression equal to 10,000 and then changed their method to guess and check. Bev found 99 as the answer by using the calculator and her equation to evaluate guesses while visually referring to her sketch. Danni found the 100^{th} figure and, relying on her visual/mental understanding of the figures, attempted to rationalize the answer with reference to a square with sides of 100. Elizabeth was the third to attempt an algebraic solution to the inequality. She set her expression less than or equal to 10,000. From there, she failed to recall the necessity of setting the inequality to zero and to factor.

Andy and Fran, neither of whom found a way of expressing the relationship between figure and total tiles, moved directly to guess and test. Andy settled on the least figure being the 100^{th} using the same rationale as Danni. Fran originally thought the answer was the 100^{th} but revised her answer once she sketched the figure.

Summary. The supportive nature of using multiple representations (i.e. numeric patterns, sketches, etc.) could be seen in the work by Bev and Fran. These two incorporated sketches into their problem solving showing that they used the connection between the geometric structure of the figures and the computation of the least figure. Bev stated the relationship connecting the geometric structure directly to her algebraic model and used it to validate her finding. Fran also employed sketching but in a different manner, in a cyclic method of evaluating her conclusions. Both Andy and Danni did not produce any sketches apparently relying on their visual/mental constructions. This appeared to limit their success in finding the most correct answer. In Cynthia's work, it was clear from her sketches, she did not develop a geometric connection to the pattern, which might have supported her problem solving.

With only three out of six participants able to infer a function from a pattern such as this implies that they also may have difficulty modeling other situations, such as those that may occur in a mathematics curriculum based in CCSS. Also, this last portion of the task revealed that the PSMTs had difficulty extending their findings into application. All except one reverted to the numeric method of guess and test to produce an answer to this component of the task even though they have had many experiences solving equations and inequalities by hand and using technology. The method of guess and test was possible in this relationship and with these values, but that may not always be the case. Also, from the participants' responses, it was not clear that any made a complete connection from pattern to sequence to function with the purpose of expressing the relationship in its most applicable form.

Geometry

This task was examined with particular interest in the use and expression of (a) rigid motions to move ABCD onto A'B'C'D' and (b) dilation to enlarge an image of ABCD. This task differed from the previous one in that instead of asking the participants to discover a function they were asked to apply functions in the form of geometric transformations. The analysis is below.

Moving ABCD using rigid motion. The accuracy and detail with which the rigid motions were performed varied greatly among the participants as did the ability to describe the geometric transformations. In performing the rigid motions, Andy performed two reflections, using the reflections to orient corresponding parts between the images and A'B'C'D' and move the figure. He marked all lines of reflection and used appropriate notation to indicate corresponding vertices. Both Fran and Danni achieved the movement, one using two rotations and a slide and the other two reflections and a slide respectively. Both used limited notation.

Cynthia persisted in the task to find two routes, one composed of a rotation and two slides and another was a rotation and slide (the second one had an error). Her notation for the movements was clear but the images were not precise. Elizabeth developed a verbal explanation for a path from ABCD to A'B'C'D' but was unable to present her work. Finally, it became apparent in her work that Bev did not have a well-defined understanding of the geometric functions. Her procedures demonstrated she did not know the rules governing reflection or rotation. She attempted to perform them using a limited visual notion. Bev's and Fran's results are addressed in the summary for this section.

In the performance of the transformations, Andy, Cynthia, Danni, and Fran relied on a numeric method of counting to place vertices and then produce images. But their understanding of the functions could be somewhat deciphered from their talk while producing the images. In their use of rotations, both Fran and Cynthia demonstrated a solid understanding of the function. They gave a pivot point and the number of degrees rotated, for example, Fran would say, "… you have a rotation at C, around C of 180°." Cynthia remarked, "Okay, with rotation it doesn't change [shape] …." With regard to reflection, both Andy and Danni used the concept and Andy was able to give a rule for the function relating the figure, image, and line of reflection; but, Danni's description was less precise saying, "It flipped." Fran, Cynthia, and Danni all used the concept of a translation. Again, it was done by counting and replicating the previous shape in the new position. These four participants exhibited sufficient knowledge of geometric transformations to create a path from ABCD to A'B'C'D'.

One element of their solutions should be addressed. The participants' images of the rigid motions suggest that they had a limited sense of appropriate notation. With the exception of Andy, there was a minimum of precision in labeling to indicate corresponding parts of the images and the transformation being used. The purpose of the labeling would be to indicate the order in which the images were produced and which function was being used.

Using dilation to enlarge an image. To describe the function dilation, it is necessary to locate the projection point and determine a ratio of similarity (or proportionality). Of the participants, only Andy recalled the term dilation. He was also able to determine the ratio of similarity by comparing three pair of corresponding sides. Danni and Fran also determined the ratio. Danni found the ratio in a manner similar to Andy's method. Fran thought it necessary to find the length of all the sides and then described the change as the sides "grew". Cynthia simply doubled the length of line segments by counting. All of them projected the enlargement through a vertex of a quadrilateral but did not identify it as a projection point.

Summary. Of the four that had sufficient knowledge of the topic to create a path from ABCD onto A'B'C'D' and present it through the production of images, there appeared a need to develop detail in the rules for the functions and precision in notation. With the notable exception of dilation, they all recalled the terms for the transformations, naming them as they produced the images. (It should be noted that the terms reflection, slide, and rotation were given in the task description; dilation was not.) But, there appeared to be an incomplete understanding of reflection and dilation. All appeared on the video recordings to demonstrate procedural knowledge from a visual-numeric orientation. When pressed for rules, they often reduced the functions to "flipping" or some other imprecise way of expressing the function.

It is problematic to address the complications that arose from Bev's and Elizabeth's work. Their problems were different. Bev's struggle may have resulted from not having recent experiences with transformations. Her high school experience may have been limited and her college course did not contain this topic. Elizabeth's situation was different in that her college course did contain transformations. Also, she was able to produce a verbal similarity transformation but she was not able to produce the necessary images to present her findings. This may imply that she did not have the necessary visual-numeric connection to the rules governing the functions to draw the images.

Data Analysis

The analysis of this task focused on (a) the implications of two sets of data having the same mean; (b) the participants' use of mathematics to describe the variability of the two sets of data; and (c) recommendations presented as a result of examining the data. It was found that the task failed to engage the participants mathematically. The findings are below.

Interpreting the common mean. Of the six participants, three addressed the implications that the mean was the same for both sets of data. Andy said, "so, generally they are getting about the same … amount of people." Elizabeth and Fran made similar remarks. Bev and Danni did not make any remarks regarding the mean. Danni explained this during her interview that, "I looked at the mean when I was doing the problem, but …. I didn't really take that into account." At the very end of her session, Cynthia reflected on the common mean but did not pursue the fact.

Describing the variability of the data. All of the participants in some form noted the difference between the highest and lowest pieces of data in each set. But, none found the range of the two sets. This would have provided some elementary method by which to compare the data about the two theaters.

In examining the data within the sets, there were some vague references to the relationship of the data with the mean of Theater B. Cynthia said, "Theater B attendance was average through the week." Fran commented that, "attendance is more around the same for B than A.” The participants noted the differences in the attendance per day between the two theaters through a visual examination but failed to mathematically describe the difference. In contrast to the other participants, Fran did try to statistically contrast the two sets. In an effort to find relevant information, she ordered the data and found the median. Unfortunately, she mislabeled it the mode. Elizabeth also tried to describe the data, reviewing mean, median, and mode, but became frustrated. All the participants concentrated on measures of central tendency. They did not address variability either in the form of finding the range or variance.

Recommendations regarding the attendance data. The recommendations presented by the participants varied in quality and number. Andy produced a series of seven questions, two contrasting the personnel needs of the theaters and the others focused on some element that would affect attendance. Fran also produced several recommendations all focused on factors that would affect the attendance at the theaters. Cynthia’s recommendations also remained focused in the parameters of the task.

Danni and Elizabeth, in their effort to make sense of the task, moved outside of the parameters of the task. Danni began to connect to her personal experience with movie theaters, which have multiple showing at one time. This diversion exhibited her frustration. She voiced this during the interview with, “I don’t feel like I knew a lot about … like what I needed to do.” Elizabeth introduced the variable of ticket prices. She also became frustrated and quickly terminated the problem solving session.

Summary. From Danni’s comment above and others, it became apparent that the participants were not cognizant that the underlying concept to be explored was the variability of the data sets. Mathematically, this could be explored by computing the range for each set and comparing them. In more detail, the variance could have been computed using the technology that they were provided. A discussion contrasting the two sets of data could have been further developed with the production of visual representations in the form of graphs. Either circle or bar graphs would be appropriate with this discrete data sets.

There were two factors that may have contributed to the limited nature of the participants' responses. One was the format of the task. It was a data analysis task worded in an open-ended form based in an application. This may have been interpreted by the participants that the response should be similar. The other factor may be confusion as to what mathematics were required, as Andy expressed in the interview, "I was trying to … get some numbers. But, I can't really do this with numbers too much." Bev revealed her conflict in, "I have never seen one of those problems in math before." Evidently, the participants would have felt more confident if the task had been presented with two sets of data and specific questions guiding them into what descriptive data were required. It was apparent that none of the participants exhibited awareness of how to contrast the variability of the data.

Chapter Summary

The analysis presented in this chapter supports the idea that each problem solver had unique methods and strengths in approaching the different tasks. One give a more complete answer to one task but fail to do so for another. This could be observed in a limited use of representations and expressed through inadequate use of terminology. Of the tasks, the one that elicited the least metacognitive activity was data analysis. The participants struggled in their efforts to express the mathematical implications of the data sets. The findings in this chapter were examined more deeply for themes in response to the research questions framed at the initiation of this study. These themes are presented in the next chapter, Discussion.

CHAPTER VI:

DISCUSSION

The purpose of this study was to delve into the problem solving practices of preservice secondary mathematics teachers. It is essential that mathematics teacher educators and others involved in PSMTs' education construct a perception of PSMTs' practices with respect to recent movements in mathematics education, in particular, the mathematical standards and Standards for Mathematical Practice outlined in the CCSS in which the practices have always been the sign of excellent classroom practice. This study is one attempt to provide some information by structuring tasks around concepts outlined in the CCSS and investigating practices through metacognition: regulation of cognition, representations/strategies, and declarative ability.

To frame the discussion of the themes found in the study, I returned to the topics reviewed in the literature and framed in the research questions. First, themes surrounding the PSMTs' mathematical knowledge in relationship to the chosen tasks were developed. Second, the topic of the impact of multiple representations use was explored. Third, a common theme regarding the utilization of the stages of cognitive and metacognitive categories was addressed. And, finally, the participants' declarative ability and terminology use was appraised.

PSMT Performance

PSMTs' knowledge base has been described as "thin and rule bound" (Ball, 1990). The findings from this study do support the idea that some concepts in the secondary curriculum have not been developed to the depth that PSMTs can express them effectively. For example, it was not apparent that the six participants connected the algebra task to the expression of a function defining a relationship between two variables. A factor affecting performance was when the participants most recently experienced a concept. The geometry task asked for the recall of transformations. The participants from the two institutions had experienced these concepts at different points and levels in their education. The topic was contained in the mathematics coursework for the larger institution but not the smaller regional one. But, whether the participant experienced the topic in their college mathematics coursework was not a definitive predictor of success. Also, the key concept underlying the data analysis task, measuring the variability of the data sets, was not apparent to any of the six participants.

The participants solved the algebra task to different levels. Three of the six produced an algebraic model of the relationship. The purpose of forming the model for application was only implied by two, Danni and Elizabeth. The third, Bev, stated her rationale as simply, "Every math problem has a formula." It was interesting that each of the three participants derived their symbolism differently: one by relating to her numeric statement, one by synthesizing numeric and geometric information, and one by decomposing the figures into geometric areas. Of the remaining three participants, it was clear that two, Fran and Cynthia, were seeking an "algebraic formula" but failed to find one. Fran did not succeed in that she appeared stuck in a recursive definition of the sequence and Cynthia did not have the numeric background to translate her findings into a rule. In the work of the sixth participant, Andy, it was not evident whether the difficulty lay in the perception of the need for a function or the appropriate knowledge of how to translate his verbal expression of the relationship into algebraic symbols. For Fran and Andy, it appeared that the difficulty lay in the underlying principle of assigning a variable. This was verified in the follow-up interviews when they were able to create the relationship once that element was supplied. Although both Andy and Fran noted that the first factor of their numeric statements related to the number of the figure in the sequence, they did not recognize the significance and use that knowledge. This synthesis of the participants' work implies that some PSMTs may benefit from attention to the process of algebraic modeling of functions.

Another finding within the algebra task solutions was the relatively unsophisticated method employed to discover the figure containing at least 10,000 tiles by most participants. Only Elizabeth recognized the value of using an inequality and she failed to do so correctly. The other two participants that developed an algebraic expression for the relationship, Bev and Danni, set up equations and then decided to drop that technique for the numeric strategy of guess and test. Two other participants who persisted in the task, Andy and Fran, also used guess and test. Although four used the same method, only Bev and Fran found the least figure to have 10,000 tiles. One explanation for this may be their efficient use of alternative methods of representations, particularly sketching, to make sense of their answers. This finding illustrates that although these participants have had multiple experiences with function and solving quadratic equations/inequalities, they may not have been in the form of application. Both of these findings in the algebra task imply that these participants may have difficulty scaffolding how to model real world situations which is part of the CCSS.

The analysis of the geometry task exposed two elements that may affect the ability to perform transformations: prior experience with the geometric functions and the capacity to produce transformed images. Andy, Bev, and Cynthia attended the regional university at which the college geometry course did not involve a close study of transformations. They had to draw on their high school experience and any other that may have been part of their coursework. Danni, Elizabeth, and Fran participated in a capstone course that did include this topic. The different experiences did affect performance in this task. Of the six, Bev failed to complete the task. Apparently, her recall from her high school experience was not strong enough to support a solution. Cynthia also had to rely on recall from her high school career but she was able to complete the task producing images sometimes without reference to the properties of the functions. Danni and Fran performed the task solidly, perhaps not in the most expedient manner but well. The two that stood out were Andy and Elizabeth's solutions. Andy executed the task with a minimum of rigid motions and used precise notation. This was a reflection of a recent experience in creating a *discovery* lesson on this topic. Andy had to begin from the concept he wanted the student to learn and present it in a form in which the student discovers the concept without it being directly delivered through scaffolding. The other case, Elizabeth, demonstrated a lack of connection between the functions and the physical production of images. She had a grasp of the terms but did not have the ability to draw. The implications from this condensing of the data is the importance of the capstone courses being based in the mathematics that the PSMT will teach and the value of requiring PSMTs to reflect on the mathematics to the point of understanding what is necessary to construct the concepts, such as in Andy's case.

Finally, the data analysis task revealed that the PSMTs focused on measures of central tendency, which are a valid part of descriptive statistics. But, they did so to the exclusion of measures of dispersion. This task attempted to prompt them into considering the concept of variability, a key concept as proposed by the GAISE Report (2004). All of the participants mentioned the highest and lowest pieces of data or how the data varied, but none approached the expression of the idea from a mathematical perspective. This may have been a result of the phrasing of the task in that it was in a very open-ended and narrative form. Although, Fran said this type of task was a part of her program, the other participants appeared unfamiliar with a task like this. Bev commented, "I've never seen one of those problems in math before. I mean like … Okay, they would have the table and they would ask you questions about it but you would have to find the average and all that." Similarly, Andy said, "I was trying to, you know … get some numbers." Also, Danni mentioned, "I felt like, I don't know … I don't feel like I knew a lot about … like what I needed to do." This may have been, as I wrote above, that the wording of the task did not engage the participants mathematically or that they were not appropriately familiar with the concept of variability to explore the data which could have been accomplished by finding the range of each set in order to develop a contrast between the data sets or by computing the variance to discuss the variability with respect to the common mean. The implication from their work is that the PSMTs had a limited perception of how to describe data. They primarily focused on measures of central tendency and failed to investigate the data in the form recommended by the American Statistical Association as expressed in, "… statistical problem solving and decision making depend on understanding, explaining, and quantifying the variability in the data (*GAISE*, p. 6)."

The Use of Multiple Representations

There were incidents in which it could be observed that the use of more than one representation supported the participants' results. There were also moments in which it could be seen that failing to use an alternative representation led to precipitous conclusions. This was most clearly detected in the solutions produced for the algebra task.

Using more than one representation for the task was the basis for the mathematics by Bev, Elizabeth, and Fran. Bev and Elizabeth used the structure of the figures to support their construction of an algebraic expression. Though Bev's procedure was somewhat clouded by her reluctance to express her method of obtaining her expression, from the drawing, it was clear that the figures' structure formed the foundation of her algebraic symbolization. Elizabeth's process was more transparent. She worked in a cyclic manner, referring to the figures and producing new ones until she was confident about the pattern. After writing the relationship, she returned to identify the components by producing a correlation to both numeric and geometric elements of her solution. For both of these participants there was a strong geometric and algebraic connection. Fran employed the geometric connection as a tool of evaluation. After producing a solution, she would return to the structure of the figures to evaluate her conclusions. This supported her ability to find the least figure that would have 10,000 tiles.

A failure to use alternative representations, which may have assisted their work, can be seen in Andy, Cynthia, and Danni's solutions. In one case, Cynthia, it was apparent that there was a complete breakdown of connection between numeric and visual representations of the figures. This was perhaps based in a limited experience with sequences and other patterns within the study of function. Both Andy and Danni appeared to rely on mental-visual images of the figures. Andy did not pursue a visual or an algebraic representation to the relationship. This was questioned during his interview, but he was not able to express why he did not. But, he did recognize that the lack of doing so hindered his ability to produce or evaluate his final answer. His self-description may give some explanation of this: "I'm a numbers person." Danni's failure to produce figures may have been an impediment to her evaluation of finding the least figure in that she appeared fixated on the dimensions of a square rather than the construct she used to produce her function, a rectangle. The sketching of the 100th figure may have triggered a re-evaluation of her answer.

Progression through the Cognitive and Metacognitive Categories

One of the most significant findings of this study was the limited use of evaluation and internalization by the participants in their problem solving. Both Pólya's (1945) framework for problem solving and Yimer and Ellerton's (2010) cognitive/metacognitive categories consider this an essential part of problem solving. Internalization, similar to Pólya's stage of looking back, involves reflecting on the entire process for depth of mathematical rigor and elegance, also adaptability to other situations (such as, solving similar problems and perhaps scaffolding in the classroom).

The dominant cycle through the cognitive and metacognitive categories when the participant experienced uncertainty was to immediately stop work once a possible answer was found. This could be seen in Andy's finding of the least figure that contains 10,000 tiles. Once he found an answer that appeared probable, he stopped without using his method of guess and check to methodically evaluate values less or more than 100. It could also be seen in Elizabeth's answer to the geometry task. She found a path to place ABCD onto A'B'C'D' and concluded, "Okay, that is pretty much all I can tell you about it." Bev's work by her own admission in her final interview was based in this cycle. In reviewing the participants' work, it is reasonable to remark that all the participants operated in this cycle on the data analysis task. The lack of producing a specific mathematical plan to explore the data may have prevented any further progress through the stages (Figure 41).

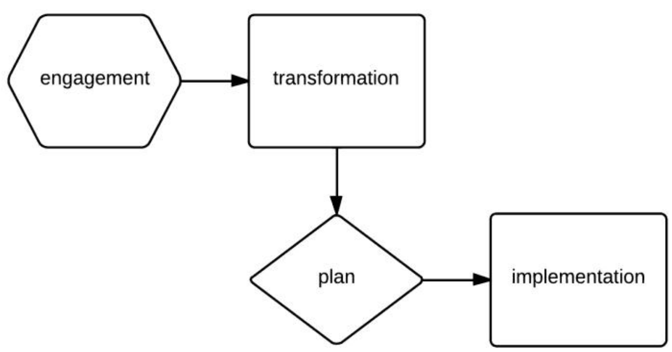


Figure 41. Cycle 1 includes engagement, transformation, and implementation.

Much of the problem solving included the category evaluation but to a limited degree. The participants would stop, look over the immediate work, and then continue. There were notable exceptions, such as when Cynthia while working on the algebra task realized that her original pattern did not fit the figures she produced and later in the geometry task that there was an alternative method to produce the similarity transformation. There were also missed opportunities to expend more time in evaluation. If she had evaluated the inequality produced, Elizabeth may have realized the mistake in the structure and solution method. Similarly, if she had expended time in evaluation, Danni may have found the most correct answer to the second part of the algebra task. The cycle Cynthia and others used is modeled in Figure 42.

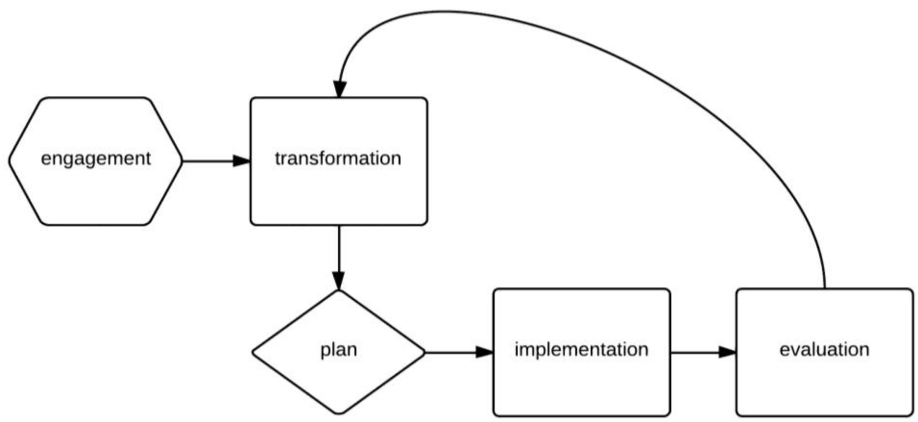


Figure 42. Cycle 2 includes engagement, transformation, implementation and evaluation.

The use of internalization was not evident in the work or recordings with any of the participants. Elizabeth was the closest to exampling this stage when in the algebra task she returned to the description to check to see if she answered all the components of the first part of the task. Her evaluation of her progress was more checking off of a list rather that a reflection over the entire process. Two structures within the study were included in an attempt to prompt this type of reflection: the participants writing at the end of the task and the reviewing of the video recording. It was found that the writing, instead of inspiring reflection, served as a recounting of the procedures used and stimulated little insight. In fact, by the second task, the participants were writing their method as they progressed instead of at the end of the task as a reflection. In viewing the recordings, some of the participants, particularly Bev, Elizabeth and Fran, began volunteering what they were thinking at a particular point. This may indicate some elements of reflection but not internalization. There were missed opportunities in which this process could have supported the participants' work. With both Fran and Andy, if they had returned to the beginning of the algebra task and reviewed their thoughts and notes, perhaps they would have reconnected with their realization that the number of the figure was the first factor in their numeric statements and then had the basis from which to write an algebraic function. A suggested cycle including internalization may be found in Figure 43.

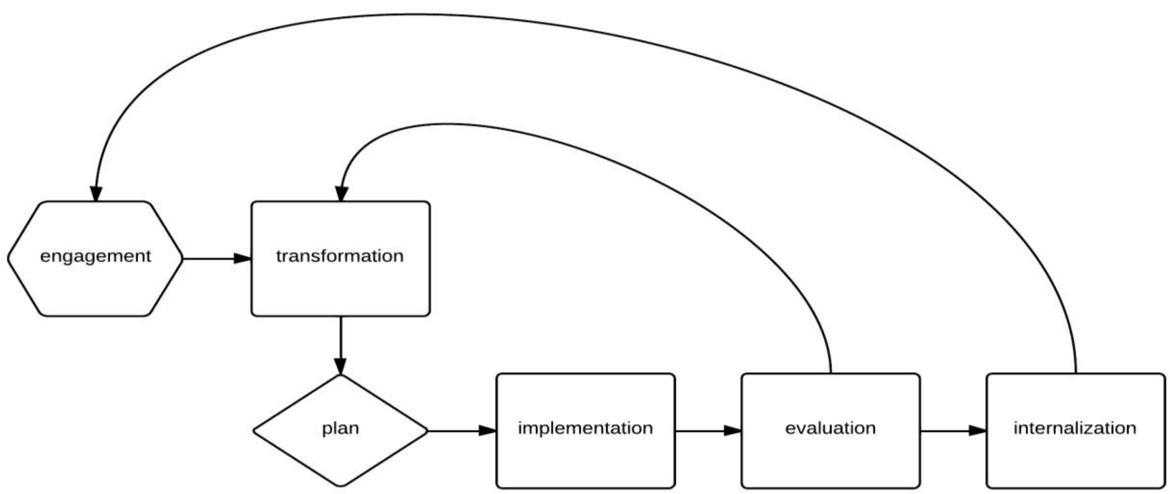


Figure 43. Cycle 3 includes the previous categories with internalization.

Declarative Ability and Terminology

The lack of reflection was also exposed in their responses to interview questions. When asked why she felt it necessary to produce an algebraic expression, Bev answered, "I don't like questions like that. I don't know." Another example was found when Danni was pressed about the properties of a reflection, she replied, "Umm … it is the same shape but flipped over the line. … It is the same shape but instead of going up, you go down. Umm … you do the … not exactly the opposite …." When questioned about how or why a procedure was used, the participants' explanations would sometimes diminish to a visual-mental description without mathematical reference. This lack of specificity is not unusual in mathematics. Clark et al (2008) found this was not uncommon in their work with inservice teachers.

The geometry and data analysis tasks indicated that the participants' recall of terminology specific to the two topics was limited. For the geometry task, Andy alone recalled the term dilation. The majority of the other participants recognized the need to enlarge an image of ABCD to the size of A'B'C'D' in a ratio of 1:2 but did not use the function's term. It was not evident that some used the properties of the function. In the data analysis task, one elementary method to compare the two sets of data was to compute the range. During the interview, Bev and Danni were able to recall the term and describe/use it but the remaining participants could not.

In this discussion of precision in language it would be remiss not to mention how the participants expressed their answers in the solving of the algebra task. One goal of the task was to express the relationship between any figure and its total number of tiles. Three participants were able to do so algebraically. None used function notation. And, only one expressed the relationship completely: Danni's "[n(n+2)] +2 = total number of tiles for nth figure**.**" This revealed either this task did not trigger a connection to function or the participants did not consider using the notation a fundamental part of the task. Also, Danni and Elizabeth were the only participants to state the second solution as an inequality. Danni said, "So, the figure that would require at least ten thousand tiles would be any figure that had … at least a hundred columns." (The correct answer should be at least 99 columns.) And, Elizabeth let her inequality represent her answer: "$n \leq 70.70$." Again, the other participants that found an answer to this part of the task stated the 99^{th} or 100^{th} figure without implying that the answer was a boundary on an interval.

This is not to imply that the participants were not capable of expressing themselves. During the stages of transformation and implementation, many used complete thoughts to articulate their exploration of the tasks and their methods of procedure. There are numerous examples of this given in the previous chapters. Also, I will repeat that the participants were asked to work the tasks through a think-aloud not explain the task as if they were scaffolding it for a class.

Limitations

The limitations of this study were centered in the tasks, setting, and interviews. The tasks were chosen across topics in the CCSS. They were based in the secondary curriculum employing concepts which some of the participants had not experienced in their college career. If they had studied the topic in college, it did appear to affect the level with which the task was completed and the level of math talk, for example, Bev's lack of experience in transformations was very apparent in her work on the geometry task. But, it was not a complete determiner of success or failure, for example, Elizabeth, who had a geometry course with transformations, failed to produce images on the same task.

There were also complications in the wording of two tasks. The geometry task presentation included the terms reflection, translation, and rotation, but not dilation. Only Andy recalled and used the term dilation. That three terms for transformations were given and not the fourth may have affected the other participants' responses, either prompting the terms use or implying that these three were the only ones to be used. Also, the data analysis task was worded to prompt a reply in first person. This may have affected the participants' responses since their previous experiences may have been phrased in a more objective manner.

With regard to the setting, unfortunately, for the participants at the larger institution, the room assignment had to change several times during the project. There was a lack of finding a consistent place for the think-alouds, availability changed. It did not appear to concern the participants. They had no difficulty finding the locations and were supplied with the same tools each time.

Finally, the interview questions were not framed exactly the same between the participants. Due to the nature of the individuality of the problem solvers' procedures, a question may be asked of one participant but not another. For example, in the geometry task, Danni employed the transformation reflection, but not rotation. Fran used rotation, not reflection. It was appropriate to ask Danni how a reflection was produced and why she used that particular one which may or may not draw a contrast to rotation; whereas, Fran was asked how she produced a rotation and why. Again, this might produce a contrast to reflection or not. Both responses gave information about their procedural ability and flexibility.

Implications

The analysis and summarization of findings suggest the existence of a relationship between fully developed elements of metacognition and successful problem solving. The decomposition of the participants' problem solving through the lens of metacognition led to recognition of skills that support solution attainment and detection of incidents in which incomplete understanding existed. The findings have implications for practice, research, and theory development.

The findings indicate that there is a role for mathematics teacher educators and mathematics professors in supporting both PSMTs' skills that may not have been studied deeply since their secondary experience and developing concept clarity for the purpose of explanation. The research revealed that some PSMTs struggled with composing algebraic relationships. Even though they may consider themselves very capable of applying mathematical operations, as evidenced in their self-rating, it was not apparent from their task solutions that all could effectively apply the principles behind using a variable to express a relationship. The data also revealed a necessity for instructors to reintroduce concepts and vocabulary from secondary curricula, such as range, dilation, and variability, giving experiences that require their correct verbal use. The need to support these seemingly elementary skills and terminology at the college level may be seen in the descriptions of Bev's and Danni's geometric problem solving. Bev was not able to recall sufficient knowledge of transformations to find a satisfactory solution or to use the terminology easily; whereas, Danni did not struggle with the task. Danni's knowledge was supported by her experiences with transformations at the college level. The need to reinforce skills is also supported by the evidence that all the participants responded to the data analysis task with limited reference to the variability of the data, failing to compare the dispersion of the two sets of data in the most basic manner, range.

The implications for research may be seen in: (1) the value of employing a structure such as Yimer and Ellerton's (2010) framework to examine problem solving; and, (2) the importance of creating tasks that focus on the underlying concepts embedded in the secondary curriculum. These two mediums allowed an inspection of the participants' fundamental understanding of the ideas upon which the three tasks were based and granted a view of the participants' articulation of procedure and concept definition. Similar techniques may be used to evaluate in future studies the introduction of methods supporting metacognitive components of problem solving, such as an emphasis on multiple representations, articulation of procedure and concept, and reflection over problem solving experiences.

Schoenfeld (1981) found proficient problem solvers spent more time in the early stages of problem solving, engagement and transformation. This study supported his theory, but the PSMT had to make a necessary connection to prior knowledge before proceeding. It appeared at times that the connections were not strong. One example was the difficulty the participants had relating pattern, sequence, and function. This supports the relevance of re-visiting basic concepts with the purpose of developing connections and the ability to articulate them.

Recommendations for Future Research

In pursuit of methods to support PSMTs development of metacognitive strategies and reflection, there may be several avenues of research to follow. Brown (1987) contended that metacognitive abilities continue to develop throughout one's life. The value of incorporating more tasks of this form may be explored. The tasks may be considered elementary but the algebra task uncovered two problematic areas: forming an algebraic model and recognizing patterns. The geometry task revealed the ability to produce images numerically using a grid but would the participants be able to produce the images using the properties of the functions. Also, the statistics task revealed the focus on measures of central tendency with a lack of attention to measures of dispersion or spread.

The research project contained a component of writing. It was found that writing immediately after the conclusion of a problem solving task was not productive for reflection. It would be interesting to determine if the participants returned to their work after an expanse of time and wrote their method or an alternative method of solving the problem would there be more reflection. Another alternative would be to ask the PSMTs to reflect on examples of high school students' work.

Conclusion

This project has presented a perspective of PSMTs' knowledge and problem solving practices through an examination of six participants' solutions to three tasks selected from topics embedded in the CCSS. Multiple components of the Standards for Mathematical Practice can be found within this study, such as, make sense of problems and persevere in solving them; model with mathematics; and attend to precision. Regulation of cognition structured through the framework by Yimer and Ellerton (2010) provided the structure of Standard 1 in describing how a student progresses through problem solving, including incidents of debugging. An analysis of representations and the strategies they support provided the means to study how the participants modeled during their responses to the tasks. Standard 4 refers to modeling real world situations which were not presented in the algebra and geometry tasks but provided opportunities in which, "… a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another (p. 7)." Standard 6, "They try to use clear definitions in discussion with others and in their own reasoning (p. 7)" was extended to include the participants' declarative abilities during problem solving and expressing conclusions.

The findings support the idea that a relationship between problem solving and the development of metacognitive skills exists. Since a person's metacognition continually evolves, it is relevant to pursue techniques that support it so that it in turn may support problem solving. Such techniques as pressing for clarity in explanation and reflection of mathematical procedures by those involved in mathematics may do so.

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Appendix A: Questionnaire

The purpose behind this questionnaire is to determine if you are willing to lend your experience and expertise to a doctoral dissertation project. If you are, please indicate so and answer the few questions below. If not, thank you for your time.

I. Are you willing to give approximately four hours to complete this project? This time would be planned around your schedule.

____Yes

____No

II. On a scale of 1 to 10, 10 being very confident, how confident are you about solving problems that are based in the high school curriculum? Why?

III. Demographics:

1. What is your gender?	___ female	___ male
2. How would you describe yourself?	___ Asian American	___ African American
	___ Hispanic	___ White
	___ Other	___ Prefer not to say

3. Do you plan to enroll in your internship next semester? ___ Yes __No

4. How may I contact you? ____

Thank you!

Appendix B: Think-aloud Protocol

Instructions:

Thank you for agreeing to be part of this study entitled: "A Study of Preservice Secondary Mathematics Teachers' Metacognition and Language".

There are two phases to this activity. The first one is to work a problem, (algebra, geometry, data analysis) through a method called a *think-aloud* and the second is to write your method of finding the solution:

Step 1: Work this problem like you usually would but while you are working please let your thoughts be known by speaking them. A goal of this study is to understand your problem solving process. As you know, the only way I am able determine that is to listen to what you are thinking. So, please verbalize any thoughts. There will be no judgments about right or wrong thoughts. Simply say them.

Your activity will be video recordings. But, there will be every effort to protect your anonymity. For organization purposes, a pseudonym will be used and the recordings will be destroyed at the end of the study. Here are paper and pencils. If you would like to use your calculator, that is fine. For the purposes of videotaping, please keep your work in this area.

Do you have any questions?

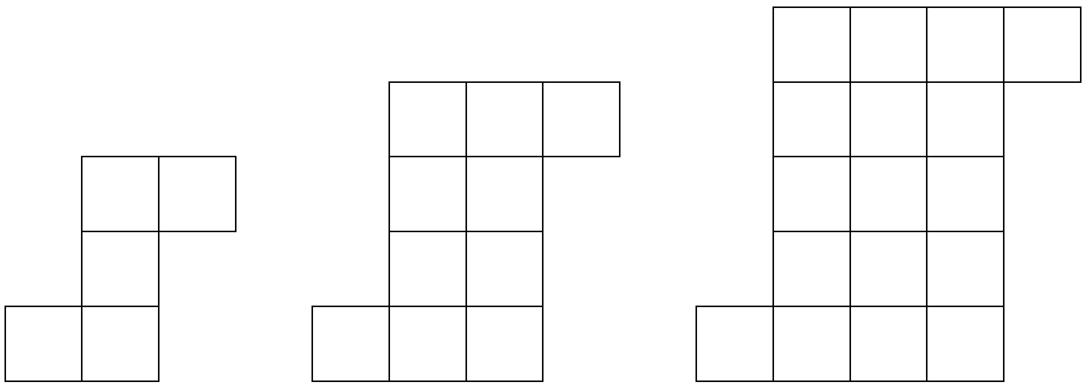
Step 2: In this second phase, I ask you to write how you solved the problem. If you want to write it in steps, you may but please include any thoughts that helped you get to that step.

Thank you for participating,
Hazel Truelove

Appendix C: Mathematical Tasks

Question 1: Algebra:

The first three figures in a pattern of tiles are shown below.

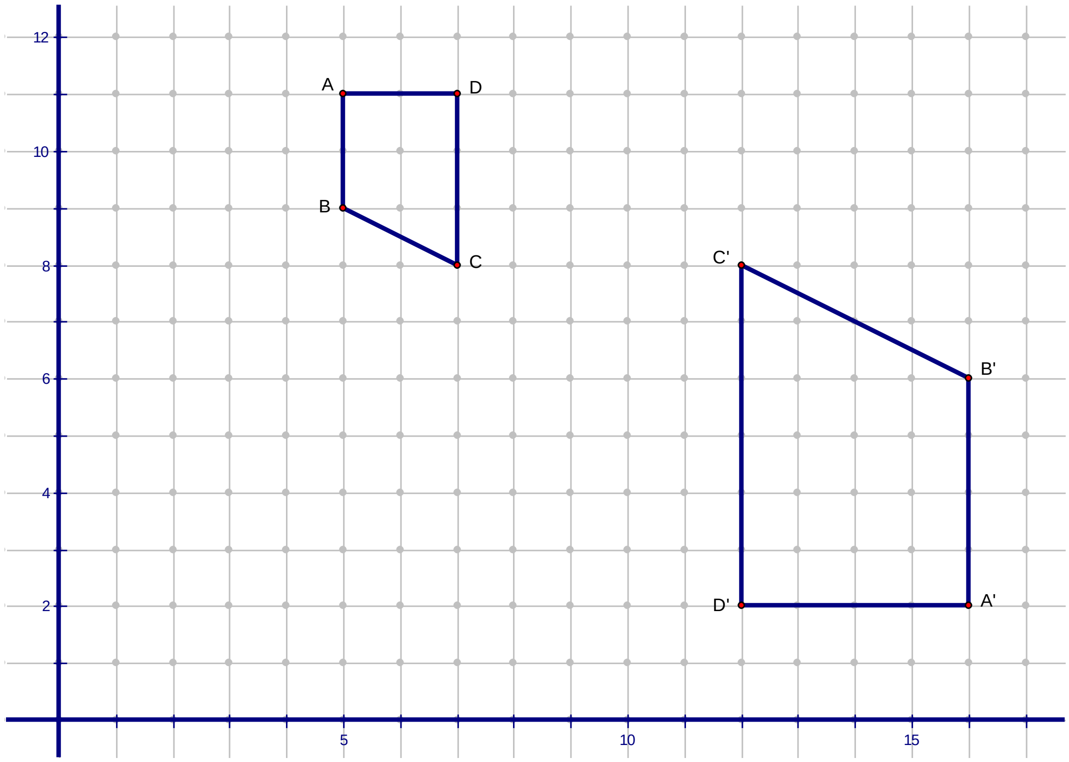


Describe the 20th figure in this pattern, including the total number of tiles it contains and how they are arranged. Then explain the reasoning that you used to determine this information. Write a description, algebraically or in words, which could be used to define any figure in the pattern.

Which figure would require at least 10,000 tiles?

Question 2: Geometry:

Describe a similarity transformation that takes quadrilateral ABCD onto quadrilateral A'B'C'D' as shown. Indicate any lines of reflection, slides, or rotations around a point.



Question 3: Data Analysis:

The table below shows the average daily attendance at two movie theaters over the past year.

	Theater A	Theater B
Sunday	110	92
Monday	97	67
Tuesday	90	65
Wednesday	24	66
Thursday	10	71
Friday	91	98
Saturday	124	87
Mean	78	78

Imagine that you are interviewing to be manager of Theater A or B. Which theater would you prefer and why?

Would you make any recommendations concerning personnel or some other consideration if this data is typical of each theater's attendance?

Appendix D: Cognitive and Metacognitive Categories

Yimer and Ellerton (2010)

1. **Engagement**: initial confrontation and making sense of the problem.
   A. Initial understanding (jotting down the main ideas, making a drawing)
   B. Analysis of information (making sense of the information, identifying key ideas relevant information for solving the problem, relating it to a certain mathematical domain)
   C. Reflecting on the problem (assessing familiarity or recalling similar problems solved before, assessing degree of difficulty, assessing the necessary store of knowledge one has in relation to the problem)
2. **Transformation-Formulation**: Transformation of initial engagements to exploratory and formal plans.
   A. exploring (using specific cases or numbers to visualize the situation in the problem)
   B. conjecturing (based on specific observations and previous experiences)
   C. Reflecting on conjectures or explorations whether they are feasible or not
   D. Formulating a plan (devising a strategy either to test conjectures or devising global or local plans)
   E. Reflecting on the feasibility of the plan vis-à-vis the key feature of the problem
3. **Implementation**: A monitored acting on plans and explorations.
   A. Exploring key feature of plan (breaking known plan into manageable sub plans where necessary)
   B. Assessing the plan with the conditions and requirements set by the problem
   C. Performing the plan (taking action by either computing or analyzing)
   D. Reflecting on the appropriateness of actions.
4. **Evaluation**: Passing judgments on the appropriateness of plans, actions, and solutions to the problem
   A. Rereading the problem whether the result has answered the question on the problem or not
   B. Assessing the plan for consistency with the key features as well as for possible errors in computation or analysis
   C. Assessing for reasonableness of results
   D. Making a decision to accept of reject a solution
5. **Internalization**: Reflecting on the degree of intimacy and other qualities of the solution process.
   A. Reflecting on the entire solution process
   B. Identifying critical features in the process
   C. Evaluating the solution process for adaptability in other situations, different ways of solving it, and elegance
   D. Reflecting on the mathematical rigor involved, one’s confidence in handling the process and degree of satisfaction.

Appendix E: Interview Protocol

Before we begin reviewing the recording of your last problem, between when you worked the problem and now, did any thoughts about the problem occur?

Okay, I am going to play back the video recording of your last session on the (algebra, geometry, data analysis) task. As we watch the video, please tell me what you were thinking. I may stop the recording at certain points to ask you questions. But, feel free to interrupt the recording if you want to clarify any of your thoughts or actions.

First question: *What did you think when you first saw the problem?*

Start the recording.

Interview questions:

What were you thinking at this point?
Why did you do …?
Why did you decide to …?

Let's look at your writing.

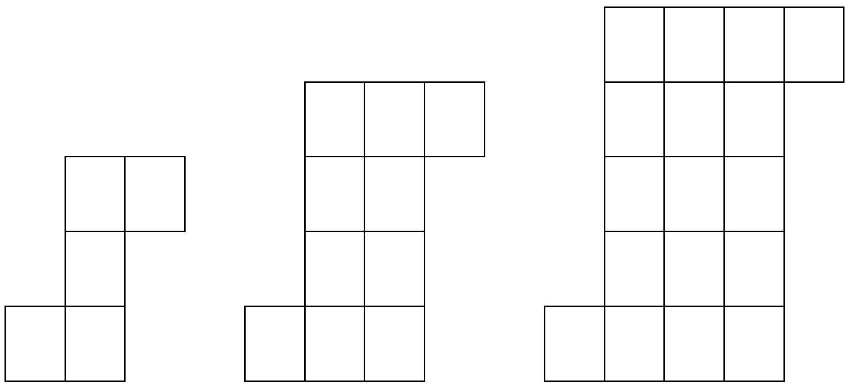
Would you like to revise any of it?
When you wrote this, what were you thinking?
Why did you do …?

Thank you. Do you want to add anything?
Let's move on to the (geometry, data analysis) task.

Appendix F: Mathematics Task Solutions

Question 1: Algebra

The first three figures in a pattern of tiles are shown below.



Understanding the problem:

I am told that these three figures establish a pattern. That means that any figure of the pattern will be similar in shape.

The pattern is presented with figures (visual representation), so that feature must be important in some way.

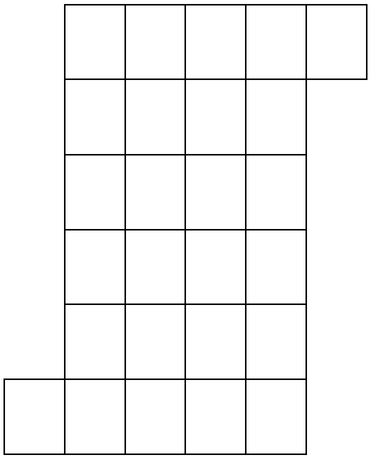
My goal is to find the quantity of tiles in the 20th figure and then find which figure has at least 10,000 tiles. So, first I am looking for the quantity of tiles; second, I am looking for a figure number.

Developing a plan:

Organizing the data:

Figure	Tiles	Difference	Rows	Columns
1	5		3	1
2	7	3	4	2
3	10	5	5	3
4	17?	7?	6?	4?

So, from the figures I notice that each one increases by one row and one column (recursively). The first one is 3 rows, one column + 2 'extra' tiles; the second is 4 rows, 2 columns +2; and, the third is five rows and three columns + 2. Also, the number of columns coincides with the figure number and the number of rows is 2 more. So, the fourth figure would be 6 rows and 4 columns + the two 'extra' tiles. Drawing the fourth figure, you have:



If you let n = the number of the figure, then you know that n will also be the number of columns in the figure and n + 2 will be the number of rows. An algebraic expression for this relationship could be $S(n) = n(n+2) + 2$ or $S(n) = n^2 + 2n + 2$.

This drawing allows the confirmation of the relationship of # of tiles and figure # developed in the table. It also extends the pattern of differences.

[Developing an algebraic relationship allows the finding of the number of tiles in any figure and in turn allows the finding of any figure given the number of tiles.]

Executing the plan:
To find the number of tiles in the 20th figure: $S(20) = 20^2 + 2(20) + 2 = 442$.
Finding the figure with at least 10,000 tiles: $n^2 + 2n + 2 \geq 10{,}000$.
Or, $n^2 + 2n - 9998 \geq 0$. This can be solved using multiple means. Using equation solver on a graphing calculator: $n \geq 98.994 \text{ } or \text{ } n \leq -100.9$.
The second value is eliminated because it would imply a negative figure number and our figure numbers are increasing. Suppose the first value. Again, the set of numbers we are working with are whole numbers. Rounding gives 99.

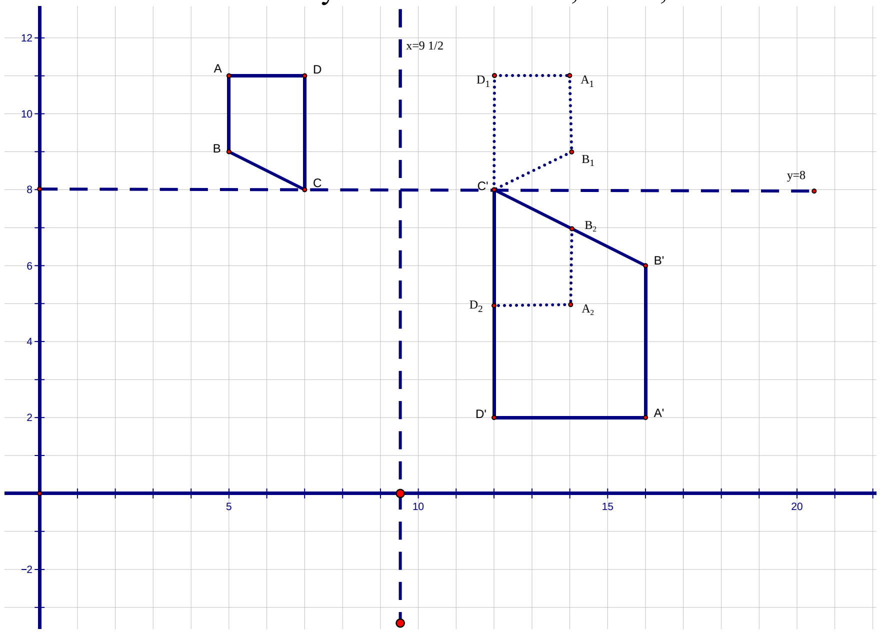
Looking back:
Checking the first solution: the 20th figure would have 20 columns and 22 rows, plus 2; this fits the established pattern.
Checking the second solution: $99^2 + 2(99) + 2 = 10{,}001$
$98^2 + 2(98) + 2 = 9802$ and $100^2 + 2(100) + 2 = 10{,}202$
So, the 99th figure is the 'smallest' figure that has at least 10,000 tiles. In other words, figure 99 or greater will be a solution for this question, but the 99th is the lowest that satisfies the question.

Other means by which to explore this problem:
Graphically: you could plot the points (1, 5), (2, 10), and (3, 17). One who has explored functions and is comfortable with graphing calculators could visually recognize the monotone increasing function and/or generate the function with embedded software.
Recognizing patterns and using finite differences: you could recall that the generation of odd values while comparing a sequence of numbers, implies 'perfect squares'. Using a table:

Figure	# tiles	relationship
1	5	$1^2+2(1)+2$
2	10	$2^2+2(2)+2$
3	17	$3^2+2(3)+2$
n		n^2+2n+2

Question 2: Geometry:

Describe a similarity transformation that takes quadrilateral ABCD onto quadrilateral A'B'C'D as shown. Indicate any lines of reflection, slides, or rotations around a point.



Understanding the problem:

A similarity transformation implies that the smaller figure is to be transformed using rigid motion (reflections, translations, rotations, etc.) to 'move' the first figure 'onto' the second. The labeling of the figures helps support the idea of 'corresponding parts' and their location on the two figures. Also, the smaller figure must be enlarged to be onto the second. This is dilation.

Developing a plan:

This can be accomplished in several ways: two reflections and dilation or a rotation, translation, and dilation. The transformations could be done in different orders or combinations, such as using a glide reflection.

Usually, to fit the idea of elegance and efficiency in mathematics, it is done with a minimum of transformations.

Executing the plan:

One example is above: a reflection over the line x = 9.5, then a reflection over y = 8, and finally a dilation using a projection point of C' and a factor of 2.

Looking back:

There are multiple ways to accomplish the similarity transformation.

The figure ABCD and A'B'C'D' are similar in that the corresponding sides of the quadrilaterals are in proportion, 1:2 (one method of determining similarity).

Question 3: Data Analysis:

The table below shows the average daily attendance at two movie theaters over the past year. Imagine that you are interviewing to be manager of Theater A or B. Which theater would you prefer and why? Would you make any recommendations concerning personnel or some other consideration if this data is typical of each theater's attendance?

	Theater A	Theater B
Sunday	110	92
Monday	97	67
Tuesday	90	65
Wednesday	24	66
Thursday	10	71
Friday	91	98
Saturday	124	87
Mean	78	78

<u>Understanding the problem:</u>

The attendance has been recorded for two theaters over a year. The average attendance for each day of the week for Theater A and B are given.

The mean is the same.

The focus is on attendance.

<u>Developing a plan:</u>

Since the mean is given and data is summarized, it may not be advantageous to examine other measures of central tendency, such as mode and median.

From a visual inspection, it is apparent that the main difference between the two sets is the variability. The data for attendance at Theater A varies from 10 to 124; whereas, the data for Theater B is fairly consistent, from 65 to 98.

So, the range is a basic way in which to compare the two theaters' attendance. At a more advanced level, one may compare the data to the mean by computing the variance or standard deviation.

Once this is done, a choice between the two can be made according to personal criteria. And, recommendations can be made for each theater.

<u>Executing the plan:</u>

The range for Theater A is from 124 to 10 which is 114.

The range for Theater B is from 98 to 65 which is 23. This implies that Theater A has much more variation in attendance that Theater B.

Depending on one's criteria, one may choose A or B or neither, but must have a justification.

For the second question of recommendations, staffing and marketing strategies for each could be discussed.

<u>Looking back</u>:

This question is open-ended. The decision as to theater does not have a set correct answer. But, any statements must be justified in the attendance data given.

Appendix G: Tables for Analysis

Solutions of Tasks:

		Andy	Bev	Cynthia	Danni	Elizabeth	Fran
ALGEBRA	Found the total tiles of the 20th figure -how arranged	**numeric pattern of columns x rows + 2;** (Moved past recursive definition) -general description after 9th figure	**decomposition of figures** into square (n^2) + two rows of n+1; confirmed with calculator	**numeric pattern of adding consecutive odds** (skipped one term)	tables 1. figure to total tiles 2. figure, columns, rows **numeric pattern of columns x rows + 2**	**numeric pattern of columns x rows + 2** with sketches	Drawing; **numeric pattern of columns x rows**; confirmed with drawing
	Created **model** for the relationship between figure and total number of tiles expressed in words or algebraically	Verbal relationship only (after 9th figure; none after the 20th figure)	Sketch and expression $\mathbf{2(n + 1) + n^2}$	none	**[n(n+2)] +2 = total number of tiles for nth figure.**	**(n+2)(n) + 2**	Attempted recursive definition
	Found the figure with at least 10,000 tiles	Guess and check 100 x 102; did not evaluate answer (did not have the means to evaluate)	Guess and check: 99 x 101; evaluated answer using her expression	none	Tried equation; guess and check 100 x 102; tried to rationalize answer	Tried inequality; $n^2 + 2n + 2 \leq 10{,}000$ $n \leq 70.70$; did not evaluate answer	Guess and check; 100 x 102 evaluated using drawing revised to 99 x 101
	Notes	• 3 participants did not express the relationship between the figure and total number of tiles using algebraic notation: one did not appear to attack the development, one did not have the prerequisite recall/knowledge, one appeared focused on the recursive definition of the terms. • 3 were able to write an expression for the relationship; it appeared all used different methods; one used her numeric statement; one used the geometric decomposition; one used a combination of reference to the figures (rows and columns) and her verbal description. • one was able to write a partially numeric, partially written equation for the relationship • The 3 that developed an expression of the relationship had difficulty using it to find the figure with at least 10,000 tiles • Two found the most correct answer for the inequality: Bev and Fran. One used guess and check and an expression; the other used guess and test and a sketch. The common element was the visual representation. • all understood the idea of 'at least' but none used the idea and their expression to write an inequality correctly. One did attempt an inequality but did not write it correctly.					

GEOMETRY	Used rigid motion to move ABCD to A'B'C'D'	2 reflections; lines of reflection clearly marked; figures labeled appropriately;	Two attempts; Notation on corresponding vertices mistaken; no other notation; neither complete	1st: rotation and two slides; incomplete Angles of rotation noted, slides noted 2nd: dilation, rotation, slide (errors in the dilation and slide)	2 reflections, slide One line of reflection given; no other notation of lines or vertices	No written work; Rotation, slide	2 rotations, slide; No notation to indicate angles of rotation; Image after slide has vertices labeled.
	Used dilation as function to enlarge the image of ABCD	Found ratio of sides (did not state, but used); **dilated** thru C'	No dilation Noticed that it should enlarge	2nd: Enlargement; the sides doubled	found ratio of sides; enlarged using "scale factor of two"	No written work: Found ratio of all corresponding sides (mistake w/oblique)	Found ratio of all corresponding sides; The figure 'grew'
	Notes	Only one participant was able to indicate all transformations, labeling appropriately the requirements for each rigid move (as required by the task) Three were able to demonstrate the similarity transformation with the drawing of images; one was able to do so verbally; one was able with one mistake; one was not able to complete the task apparently from a lack of understanding of dilation. Only one participant recalled the term 'dilation' All **dilated** using a **point of projection**.					
DATA ANALYSIS	Interpreted the common mean	"so, generally they are getting about the same … amount of people."	No	no	No	"They have the same amount of people coming in per [week]."	"their mean is the same; so, I would say both theaters are very similar."
	Found the range for the sets of data	Describes Th A as spread out; Th B as evenly distributed.	Noted the highest and lowest attendance for each theater	Described the sets but did not compute the range	Noted the highest and lowest attendance for each theater	Describes Th A as weekends have a higher amount; Th B is very consistent.	No mention of comparing the highest and lowest values
	Found variability with respect to the mean	No	no	Theater B attendance was 'average' through the week	No	no	"attendance is more around the same for B than A."
	Focused on attendance	Yes	Yes	Yes	Introduced personal experience	Introduced the variable of 'money coming in'.	yes
	Notes	None of the participants approached the task focusing on variability.					

Elements of Cognition/metacognition:

		Andy	Bev	Cynthia	Danni	Elizabeth	Fran
ALGEBRA	**Regulation of Cognition (persistence in problem solving)**						
		Minimal use of evaluation; no evidence of internalization. Would pause at the end of a solution to look over material just done.	Minimal use of evaluation; no evidence of internalization. Ended the session immediately after finding a final answer.	Evaluation of the first conjecture and solution led to debugging. Persistence was shown in finding a workable plan; but a satisfactory function for the figures nor solution for the 2nd part was not found	Minimal use of evaluation; no evidence of internalization. Ended the session immediately after finding a final answer.	Use of first four stages. Extensive period of evaluation and possible internalization after the production of the 20th figure. None at the conclusion of the task.	Used four categories used an alternative problem solving strategy (drawing) to evaluate answers found by numeric means.
	Understanding of the requirements for the task	Did not seek an algebraic means of expression the function but did recognize its relevance Though it appeared that he used conservation of area, decomposition of figures, sequence patterning, he did not use the language	"I don't remember this from high school." Understood the need for an algebraic model but could not describe why or how.	-frustrated that could not find equation *lack of connection to the rule of 'adding consecutive odds' prevented the expression of the function	"I don't know …. I guess maybe I was just trying to … put it in more general terms and then to use that for the next part of the problem." <u>Interview</u>: -during engagement, found familiarity with similar experiences in college MH class (inductive proof)	-use of the term sequence from the beginning:, recognized the advantage of finding the nth term suggests a strong connection to sequences "Sequences have a pattern to them. And, I noticed the pattern."	Sought an algebraic formula Pattern of problem solving: Draw, numeric, draw
	Use of Representations (strategies)						
	• Role of numeric expressions	exclusively numeric work	worked backwards numerically to check her visual representation and algebraic expression (This discovered in the interview.)	-numeric patterned listing	Numeric tables -used numeric expression of 20th figure to write function	Informal tables	used numeric pattern to find answer to the 20th figure used guess and check with calculator to find second

	• Role of visual representations	no sketches although visualization supported his plan (columns x rows) and aided recognition of the task	visually decomposed figures into geometric shape	examination of the presented figures led to adding consecutive odds -used drawing for checking	no drawing although visualization supported her tables and plan	Constant referral to the figures Drew 4^{th}, 5^{th}, 20^{th}	Drew first drew to confirm answers supported the correct solution to the 2^{nd} part
	• Role of alg. symbolization • Was the rule between figure and total number of tiles found? • Was it modeled in function notation? • Was the solution to the 2^{nd} component expressed as an interval?	no alg. notation/was able during the interview The rule was expressed in words only. *he noticed at the 8^{th} total that the first factor correlated to the # of the figure* No. No. Once 100x102 was found, stopped; no expression as to the number of the figure or an interval	A geometric model was drawn and an expression was written No. No. The solution to the inequality was not expressed as an interval. "the 99^{th} figure would require 10,001 tiles …."	No algebraic notation.	"[n(n+2)+2=total number of tiles for the nth figure" Yes, the answer was expressed as ""So, the figure that would require at least ten thousand tiles would be any figure that had … at least a hundred columns." (should be at least 99 columns)	(n+2)(n) + 2 Above: row and column Below: area, extra No. $n \leq 70.70$ "at least 70"	*at the 13^{th} total: "then I noticed that this [the first factor] went along with whatever number [figure] it was." No. "So, that would be the ninety-ninth figure which would give you ten thousand and one tiles."
	• Role of writing/ language	"1 column, 1 row" for reflection		"The 20^{th} figure would be in the shape of the original shape but it would contain at least ~~118~~ ~~290~~ 407 tiles." The	writing to communicate her actions	-used a process of note-taking, reference to the figures, informal tables, drawing, etc. The referencing was	

				writing acted as communication and an object of reflection/organization.		cyclic.	
	Expression of Declarative knowledge (precision in language)						
	• DK/exploration of task during transformation	What I am doing is taking, multiplying the row by the column and the two outside ones at the beginning and the end just kind of leaving them off. So, that is going to be one times three which will be three plus two is five.	Verbal expression of attributes of the figures was not given during transformation. <u>Given after a solution</u>: "Because each of these had one more than whatever figure it was [pointing to the top row] and so that is where the plus one [is]. And, there are two bars so that is times two and that was a perfect square in the middle.	Verbal expression was in phrases, not sentences.	"I don't really want to focus on the one … at the bottom left and on the top right …. I kind of just want to focus on the number of the rectangle."	"Right now, I am just trying to see what the sequence … would be." "So, it is always going to be whatever it is in the middle plus two."	"I notice that the pattern is … growing. Now, the two side ones on the top and bottom … they are going to stay the same in each one."
	• DK/procedure during implementation	Procedures expressed clearly during implementation; "So, by the pattern, it is just columns, that would be whatever figure you are on, and the rows … plus two and then you multiply those two together, then add the two tiles on the outside." (moment of insight **at 9th** figure)Ex. Procedure	Procedures were not expressed during implementation; only after Bev had come to a conclusion. "To figure out how many blocks, it is two times (n + 1) plus n squared. So, two times 21 plus 400. Two times 21 is 42 plus 400 equals 442."	Procedures were expressed in writing at beginning of work and completed/ corrected during conclusion of work to communicate During procedures, the work was expressed in phrases	Procedure: "So, to find how many tiles are in that center rectangle I'm going to multiply twenty times twenty-tow because that is how you find area."	Procedure: tendency to confuse terms and patterns; Rows and columns Following pattern and then changes summarized "it is going to have twenty columns and twenty-two rows and it's going to have one extra right here and one extra right here."	Procedure: "So, we need to multiply that [the number of columns and rows] to get over ten thousand"

		for finding the total tiles for the 9th figure and guess and check to find an answer for the 2nd component.					
	• Precision in language/ mechanics	Used the term 'pattern' -did not generalize the relationship -did not identify solution for 2nd part	The solution was only expressed in words with reference to the 20th figure. It was generalizable to any figure in an expression. To solve the inequality, she set up an equation. Did not solve, but used to check her guess in guess and check	Solution for the 20th figure was found - it was not generalized.	The solution to the 20th figure was found and generalized to "[n(n+2)+2=total number of tiles for the nth figure" 20x22=440+2=442 -set up equation for 2nd part; struggled to solve -use of "nth figure" and "area" implies across topics connections	Rows and columns Set up inequality incorrectly and solved it incorrectly.	Diagram of the 20th figure is labeled incorrectly. Difficulty writing an algebraic rule for the figures. "Where (x,y) start at (1, 3) and increase by 1 each time."

		Andy	Bev	Cynthia	Danni	Elizabeth	Fran
GEOMETRY	**Regulation of Cognition (persistence in problem solving)**						
		Pattern of first 3 stages and then short period (30-45 seconds) of evaluation. It did not appear that he experienced internalization.	Pattern of first 3 stages and then Evaluation for 30 seconds. No internalization.	Did Cynthia move into internalization after first attempt?	Pattern of first 3 stages. No evaluation or internalization. Once done, did not think about alternative pathway.	Pattern of first 3 stages. No evaluation or internalization. Once done, did not think about alternative pathway.	Check amount of time each spends on engagement and transformation

	Understanding of the requirements for the task	-his notation indicated that he had a good understanding of the requirements of the task	*Lack of recent/meaningful experience hindered completion of the task.	"I am trying to think of the transformations, all the transformations …. It has been so long since I have thought of them."	Accomplished the similarity transformation but did not follow directions or conventions for solving this type of task.	Accomplished the similarity transformation but did not demonstrate the transformations	Concerned with expressing the transformations; most of her time in cog./metacog. Transformation was spent trying
	Use of Representations (strategies)						
		-**clear notation and sketches** -reflections were done numerically, with precision (counting) -notation for corresponding vertices	-did not perform geometric transformations with precision **-no lines of reflection, angles of rotation, etc.** **-notation for corresponding vertices incorrect**	The use of the 'axes' to support rotation. -writing for organization and reflection **Indicated angles of rotation** Used line segment notation	-no labeling of lines of reflection or corresponding vertices	-no representations used even though she re-read the requirements for the task several times -no notation used	-took notes on coordinates of vertices, area -labeled third image A''B''C''D'' -no angles of rotation
	Expression of Declarative knowledge (precision in language)						
	• DK/exploration of the task during transformation	-he did not verbalize any exploration of the task	"It looks like … the first one, the small one ABCD, is flipped down where A and D are at the bottom and then … let me see… [using hands] then flipped around that way [motioning with hands]. Or, you could say that it is rotated at C and then flipped." (lack of precision)	"It's rotated … clockwise… let's see … 180." "now, it needs to slide." "It can't be 360 because it will be back in the same position." Appropriate terminology but complicated	-no verbal planning	"Let's see [using hand motions]. Now, if you, if you rotated it …. The C would still be here [pointing at C and C']. So, it looks like some type of rotation cause if …. The point [C] would still be here, it would still slide down to the …. But, and A and then D. So, yeah, it is definitely going to be a rotation. "	…okay. [pointing at the line y = 8] I notice that C is on the same line … [as C'], so it is going to be … it is probably going to be a rotation and then a slide. "I just now noticed … that it is a similarity transformation, so it is a similar figure."

						-dropped this plan; picked it up again.	
	• DK/procedure during implementation	<u>Verbalized procedures clearly</u>: -"one line of reflection … would be at … y = 8." "D would reflect over to … (7, 5)…." "Let's just reflect this first. C would go to C' …." -used the term *dilate*: "I would have to dilate the original figure by two." There is a line of reflection … at … x equals nine and a half. We reflect A_1 B_1 C_1 D_1 and dilate it by two." Was not confident about the rule for dilation.	Contradiction between what she said she was doing and what she did due to an incomplete understanding of transformations: "rotate around C"	Uses the basic vocabulary of slide and rotate accurately. Descriptions of procedures are not fluid in that she is moving in and out of transformation and implementation repeatedly. (result of not having recently experienced this type of activity?) It will be 180 clockwise. Dilated the figure but did not use the rules defining dilation. No recall of the term	"I first know that I want C on top, not on the bottom. So … I want to … reflect it … over y = 8." "All I am doing now is reflecting it." "I … now … I see I need C on the left side [of the quadrilateral] and not the right. So, I am going to reflect it again over x = 7." "I want to count how big it is …. And, so … I know that the scale factor is two … because the sides are being doubled." "now I just have to … translate it over … to the right."	Verbal implementation only; so, there were no procedures expressed -she was able to produce two figures during the interview	"Okay … you have a rotation at C, around C of 180 degrees." -she was explicit about the center of rotation and how many degrees she was rotating it. She did not indicate the rotation with any notation. you are going to have a … slide of … five units to the right "Well, the … length and the width of the square … are going to be times two." "Oh, I am doing the Pythagorean Theorem. (oblique sides).
	• Precision in language/ mechanics	<u>Interview:</u> Verbalized some of the rule for reflection <u>Interview</u>: described the way that he found the ratio of	Interview: recalled only reflections, slides, and rotations. Was not able to precisely describe/define the geometric transformations	"okay, with rotation it doesn't change [shape], it just rotates around the point." Uses the basic vocabulary of slide	-no use of 'dilation' but found the scale factor -no use of translate but slide the figure "Yes, it is flipped.	"I forget what it is called whenever something … gets twice as big." -no mention of ratio -there were no figures to discuss	We are going to have a … slide … what is the …geometric word for that? Translation. "(x + 5, y)". "Well, the … length

		similarity but did not use terms such as ratio or proportion.	-described the ratio of the figures ABCD is “2 times smaller” than A’B’C’D’ (not precise)	and rotate accurately as seen in her writing. Imprecise: reflects more appropriately would be corresponds to.	But, what do you mean by flipped?” D: “It is the same shape but instead of going up, you go down. Umm … you do the … not exactly the opposite ….”	except the failed rotation at the bottom of the page. “… reflection, you have to reflect about an axis and I knew I needed to find the middle of something.” -mistake on oblique sides	and the width of the square … are going to be times two.” “Okay, a similar figure … always is a ratio.” The ratio would be … one to two “Oh, I didn’t remember that. That is a lot better than ‘grow’.”

		Andy	Bev	Cynthia	Danni	Elizabeth	Fran
DATA ANALYSIS	**Regulation of Cognition (persistence in problem solving)**						
		Minimal evidence of evaluation. No evidence of internalization.	No evidence of evaluation or internalization	Evaluation at end of session.	No clear path of engagement, transformation, and implementation.	Abbreviated cycle of engagement, transformation, implementation	Clear path of engagement, transformation, and implementation. Minimal evaluation or internalization.
	Understanding of the requirements for the task	“I was trying to … get some numbers. But, I can’t really do this with numbers too much. So, I was just trying to think of questions ….”	“I’ve never seen one those problems in math before.”	Reflected on the implication of a common mean at conclusion.	I don’t feel like I knew a lot about … like what I needed to do.	“I’m mad!”	Had not expected one in this project

Use of Representations (strategies)						
	Writing only.	Writing only.	Writing only.	Writing only.	Writing only.	Some numeric Ordered the data.
Expression of Declarative knowledge (precision in language)						
• DK/exploration of the task during transformation	"so, generally they are getting about the same … amount of people." "Theater A is a little spread out…. Theater B is a little more … evenly distributed."	"Well, B has got steady numbers…. A … one day you have 110 … and one day you have ten." "… the question (2) is about personnel?" <u>Interview</u>: referenced personal experience.	"For Theater A, Wednesday and Thursday are their slowest days … Theater B has a constant flow throughout the entire week …." <u>Interview</u>: "I noticed that they had the same mean…. They still averaged out to have the same …." -"B would have an average crowd" <u>Problem solving:</u> referenced the two theaters.	"I'm just looking at … each day and comparing the numbers." "The lowest is 65 and the highest is 98…. So, every day is in that range. But, in Theater A, you have all the way from 10 to 124." <u>Interview:</u> "I looked at the **mean** when I was doing the problem, but … I didn't really take that into account. <u>Problem-solving</u>: Entered references from her personal experience.	"A has higher weekend attendance, B has more consistency. Both have the same attendance per were."	"their mean is the same; so, I would say both theaters are very similar." "… it looks like … Theater A has a lot more dips, like highs and lows. And, Theater B is pretty … it is more around the mean."
• DK/procedures during implementation	No discussion of computation of range or variance	No discussion of computation of range or variance. Misinterpretation of the task: "Theater A not predictable…."	No discussion of computation of range or variance	No discussion of computation of range or variance	No discussion of computation of range or variance	Ordered the data; found median
• Precision of language	No use of the term **range** -difficulty recalling range in the interview	No use of the term **range** - in the interview; "the range is very broad"	-no use or recall of **range**	"so [for Theater B] every day is in that range" BUT **range** was not computed.	- knowledge of measures of central tendency (mode, median, mean) -no range	-found the medians, called it 'mode' -confused mode, median, and range (interview)