

COORDINATION OF PRICING, ADVERTISING, AND PRODUCTION DECISIONS
FOR MULTIPLE PRODUCTS

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ABSTRACT

This research aims to develop and propose mathematical models that can be used to facilitate cross-functional coordination between operations and marketing. We consider a dynamic problem of joint pricing, advertising, and production decisions for a profit maximizing firm that produces multiple products. We assume the firm operates in monopolistic environment, where demand for its products is a function of price and advertising expenditure.

We model the problem as a mixed integer nonlinear program, incorporating capacity constraints, setup costs, and demand seasonality. We first model and solve the pricing problem without advertising. Later, we extend the model to include advertising as decision variables. The demand for each product is assumed to be continuous, differentiable, strictly decreasing in price and concave in advertising. We present a solution approach which can be used for constant pricing, as well as dynamic pricing strategies. Furthermore, the solution approach is more general and applicable for linear as well as nonlinear demand functions.

Using real world data from a manufacturer, we create problem instances, for different demand scenarios at different capacities, and solve for optimal prices for each strategy. We present analytical results that provide managerial insights on how the optimal prices change for different production plans and at different capacities. We compare the firm's profitability for the two pricing strategies, and show that dynamic pricing is valuable at low capacities and when at least one of the products has peak demand in the beginning of the planning horizon.

We show that the optimal allocation of advertising budget across products does not change with budget changes. Moreover, the change is minimal with changes in demand seasonality. It is optimal to increase advertising in periods of higher demand and decrease in periods of lower demand. Hence, firms can use rules of thumb and need not to frequently review the allocation. Numerical results show that the proposed algorithms

have good convergence properties.

Finally, as it is clear from review of academic literature; there are no decision support systems that truly integrate the production/inventory and pricing decisions - specially for multi-product problems. We believe, this work makes valuable contributions in developing solution methodologies that can be incorporated in such decision support systems.

DEDICATION

I dedicate this dissertation to; my mother - who always made my education her first priority, my wife Ayesha - for her unconditional love, support, and care, and my children Raiha, Raza, and Rafay for providing an endless supply of hugs and kisses, and for making every day of my life so delightful.

LIST OF ABBREVIATIONS AND SYMBOLS

α	Parameter for demand function
β	Parameter for demand function
δ_j	Parameter for advertising effectiveness of product j
ϵ	Some small value, used as a stopping criteria for algorithms
η	Price elasticity of demand
Γ_p	Set of feasible prices
γ_{jt}	Demand seasonality factor of product j in period t
κ_0	Parameter for some fixed costs
κ_1	Parameter for some fixed costs
κ_2	Parameter for some fixed costs
μ	Marginal revenue of advertising
Ω	Advertising budget
ω_j	Advertising threshold for product j
ζ_m	Column vector of variables X_{jmn}
A_{jmn}	A parameter defined for ease of notation $A_{jmn} = \frac{\alpha_j}{\beta_j} - c_{jm} - \sum_{t=m}^{n-1} h_{jt}$
a_{jt}	Setup cost of product j in period t
B_{jn}	A parameter defined for ease of notation $B_{jn} = 1/\beta_j\gamma_{jn}$
b_t	Production capacity in period t

c_{jt}	Variable production cost of product j in period t
D	Demand
D_{jt}	Demand of product j in period t
h_{jt}	Inventory carrying cost of product j in period t
J	Number of products
j	Index used for products $j = 1, \dots, J$
k	Index for number of iterations of algorithms
l	An index $l = 1, 2, \dots, L$
m	Index used for time periods $t = 1, \dots, T$
n	Index used for time periods $t = 1, \dots, T$
P	Price
P_{jt}	Price of product j in period t
q_{jt}	Quantity of product j sold in period t
r	Parameter used for advertising effect
S_{jt}	Sales of product j in period t
T	Number of time periods in planning horizon
t	Index used for time periods $t = 1, \dots, T$
v_j	Capacity used per unit of production of product j
W_{jt}	Advertising expenditure of product j in period t
X_{jmn}	Quantity of product j produced in period m and sold in period n
X_{jt}	Production of product j in period t

Y_{jt}	Binary variable for setup of product j in period t
z	Objective function of an optimization problem
z'	Derivative of objective function
z^+	Upper bound on objective function of an optimization problem
z^-	Lower bound on objective function of an optimization problem
$z'_{X_{jmn}}$	First derivative of z with respect to variable X_{jmn}
*	Superscript used to represent optimal value of variables
A, B	Two products produced by firm TIS
MILP	Mixed integer linear program
MINLP	Mixed integer non-linear program
NLP	Non-linear program
OA	Outer approximation
RI	Regeneration interval as defined in Section 4.1
TIS	Abbreviation for name of a manufacturing firm
WA	With advertising
WOA	Without advertising

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Chapter 1

Introduction

The topic of coordination between marketing and operations is of interest to both researchers and practitioners because it addresses key organizational issues that directly affect business performance. Shapiro (1977) highlights key issues and raises more questions than providing answers on how the two functions can collaborate. Decades after this article, we believe most of these questions still need to be answered and issues to be resolved. The modern supply chain business models are essentially based on coordination and alignment of objectives between supply chain partners. The scope of such collaboration spans across companies. Malhotra and Sharma (2002) ask the question that how can companies effectively collaborate with external partners if their own functional areas do not collaborate with each other. They classify the effect of pricing, promotion and advertising decisions on demand and supply management as a key interface between marketing and operations. Issues like capacity decisions, inventory deployment, price-promotion policy, and demand management, are not just marketing or operational issues, rather these lie on the interface between the two. Karmarkar and Lele (2005) highlight that ignoring these interactions can be extremely costly for a firm. They provide case examples to illustrate the problems that may occur when interactions go unrecognized. On the other hand, empirical studies show that a firm can: improve its competitiveness

and profitability (Hausman et al., 2002), align its manufacturing priorities (Hausman and Montgomery, 1997), and improve its delivery performance (Sawhney and Piper, 2002) by coordinating marketing and operational decisions.

This dissertation focuses on researching cross-functional interfaces and developing joint decision models. We solve a real world problem for a manufacturer who produces multiple products on the same equipment with limited capacity. The issue of collaboration between marketing and operations is a multi facet one, and perhaps there is no simple answer as how it can be achieved. Exploratory, empirical and analytical research on interfaces between two functions is required to provide necessary foundation on which new frameworks and theory can be established. Our work aims at providing analytical framework and analysis tools that capture the goals and performance measures of both the functions. Such a framework, when used for joint decision making, will result in inter-functional collaboration and alignment. The presented solution approach and similar other models are useful not only for tactical decisions but also for strategic ones, for example, capacity expansion, and product portfolio decisions. Moreover, a better understanding of interactions between different functions is useful for setting goals, adjusting performance measures, changing behaviors, and even redesigning the organization structure. Thus, the impact is much more than the obvious advantage of solving a tactical problem.

The ability of a company to serve its existing customers and to acquire new customers not only depends upon effectiveness of its marketing programs but also on its manufacturing capabilities. Sometimes these capabilities may not be rigidly defined and they are very much influenced by factors like accuracy of forecasts, frequency of setups and product mix decisions. Interestingly, the very same factors influence the organization's profit and market share - measures that determine performance of marketing function. Marketing programs aim at developing and introducing products (and services) that meet expectations of some target customers and achieve some organizational goals. Customer

expectations are met when the product can fulfill their needs or wants at a price that is no higher than their willingness to pay. Organizational goals are met when the product sales generate target profit or (and) gain target market share. A product's market share is determined by a number of factors including its attributes, quality, delivery, brand image, competitive environment, price etc. Once the product design, attributes, and distribution channels are finalized, the goal of marketing is to decide the selling price and implement an effective communications plan, which creates awareness and persuades customers to buy the product.

Operations management aims at efficient management of direct resources required to produce the goods and services that marketing has decided to include in a firm's product portfolio (Eliashberg and Steinberg, 1993). Management of input resources determines the cost of production, product availability, and in-use performance. In a decentralized decision making process, the two functions make independent decisions for achieving the same goals. Many researchers study the economic benefit of coordination (e.g. Abad (1987) and Freeland (1980)) and identify key interfaces between the two functions. The relative importance of these interfaces depends on the objective, and the product life cycle. For example, when a firm introduces a new product; it may choose either skimming or penetration strategy, focus on improving quality and delivery etc. It will then choose its pricing, advertising, and production (or sourcing) strategies accordingly. On the other hand, a firm may decide to use pricing only for managing demand of a product that is in the decline stage.

Joint pricing and production/inventory planning is a key interface between the two functions as these decisions directly affect the top line as well as the bottom line of a firm. Researchers have studied different models in this context. Most models consider these decisions with infinite capacity and model demand as a linear function of price. Some incorporate setup costs and only few consider multiple products in their analysis. However, in many practical situations, it is common to produce multiple products (or

brands) on the same production line that requires setups and has limited capacity, for example, discrete manufacturing. A related decision, for multi-period problems, is either to maintain a constant price or to change prices dynamically during the planning horizon. Many researchers study dynamic pricing in a retail environment and in context of revenue management. Few papers consider the effect of dynamic pricing on a manufacturing firm's profit. This research investigates how a manufacturer can better manage demand and available capacity by changing prices dynamically. We compare profits from the two pricing strategies. It is easy to see (assuming prices can be changed at zero cost) that dynamic pricing strategy will always be at least as profitable as constant pricing strategy. However, it is worthwhile to investigate situations where dynamic pricing generates significantly higher profit and where the difference is marginal.

In addition to pricing, quality, advertising, promotion, and lead times are the key interfaces that have been studied in academic literature. Similarly, operational decision of finding optimal production schedule to minimize cost is also a key factor in determining profitability. The idea is to manage demand by coordinating one or more of these interfaces such that the firm may maximize its revenue or profit. It has been shown in academic literature that a joint decision making approach leads to higher profits for the organization (e.g. Welam (1977)) .

While many researchers propose different models to capture advertising effect on sales, few consider the combined effect of price and advertising. Marketing literature on optimal advertising policies considers price as given. However, such an assumption ignores the objective of advertising. Is the firm advertising to increase its market share or to maximize its profit? The sales volume, in general, is decreasing in price and increasing in advertising. Let's assume, for the sake of discussion, that the profit is concave both in price and advertising. Then it may not be optimal for a profit maximizing firm to sequentially decide price and advertising. It should in fact prefer a coordinated approach.

Co-op advertising models are well studied in academic literature. These models refer to coordinating advertising and inventory decisions in a manufacturer- retailer supply chain (for example: Xie and Wei (2009), and Huang et al. (2007)) .However, few researchers address the advertising and production interface within a firm. To the best of our knowledge no one has yet addressed the complex interaction between pricing, advertising and production. In this dissertation, we focus on joint pricing, advertising, and production decisions for a manufacturer who produces multiple products and faces capacity constraints. We identify gaps in the literature and present different models to solve problems that have not yet been addressed in academic literature.

The contributions of our work are multi-fold: 1) We extend the works of Gilbert (2000) and Deng and Yano (2006) by solving joint pricing problems for multiple products over multiple periods with setup costs and capacity constraints. 2) We reformulate the problem such that it can be solved as a nonlinear resource allocation problem. 3) We show that the method is applicable to dynamic as well as constant pricing strategies. 4) We develop an exact algorithm that is less restrictive, and works equally well for both linear and nonlinear demand functions. 5) We develop managerial insights in properties of optimal solution and use real world data from a manufacturer to demonstrate applicability of proposed model. 6) In the end we extend the first model to include advertising as a decision variable, thus presenting a single model for joint production, pricing, and advertising decisions.

The remainder of the dissertation is organized as follows. Chapter 2 provides a review of well-researched problems that address coordination of pricing, advertising, and lot-sizing problems in a production setting. Chapter 3 presents a proposed algorithm for solving joint pricing and production problems with capacity constraints and setup costs. Chapter 4 gives numerical examples and provides managerial insights by presenting some properties of optimal solution. Chapter 5 extends the proposed model to incorporate advertising as a decision variable. The extended model iteratively solves joint pricing,

advertising, and production problem. Chapter 6 concludes the contributions of this research and proposes directions for future research.

Chapter 2

Literature Review

In recent years, companies have realized that cross functional integration and joint decision making processes are critical for any business's success. This has led to a new way of thinking and new way of evaluating roles of different business functions. The role of operations management has also evolved from that of cost minimization to that of value maximization. Therefore the strategic role of operations and its impact on overall business value provide compelling reasons to better understand the interaction between operations and pricing decisions (Yano and Gilbert, 2005).

The points of interface between marketing and operations span operational, tactical, and strategic level decision making within an organization (Parente, 1998). Hence the scope of interface includes almost all business critical areas including pricing, promotion, advertising, delivery times, production, inventory, logistics, quality etc. For the purpose of this research we limit our discussions to price, and advertising from the marketing domain and production and inventory decisions from the operations domain. While quite a few researches address these issues in the context of supply chain management or in channels of distribution, we limit our review to the most relevant deterministic models within the manufacturing environment.

Pricing is a key decision that all businesses use to drive revenue and/or profit. While

setting the optimal prices and managing related costs is a challenge for all of them, it is more critical for manufacturing firms. Coordination of these decisions require a system wide approach rather than departmental or functional approaches. Pricing for revenue management has been extensively studied and has revolutionized the way service industries price their products (e.g. airlines, hotel, and car rentals). On the contrary, pricing for profit maximization in manufacturing industries is a relatively less studied domain. Most researchers study relevant problems with simplifying assumptions such as single product, single period, and infinite capacity. In practice, most firms produce multiple products on the same equipment and face capacity constraints. It becomes clear from the literature review that there is a need to develop multi product, multi period models that incorporate manufacturing constraints. Most researchers identify this as an area of future research. Developing strategies that integrate marketing and production decisions for profit maximization may lead to radical improvements in supply chain efficiencies for manufacturing firms (Chan et al., 2004).

We may study the literature on pricing according to one or more of following classifications:

- i. Single or multiple period problems
- ii. Constant or dynamic pricing strategies
- iii. Type of product
- iv. Deterministic or stochastic demand
- v. Functional form of demand
- vi. Fixed and variable production costs
- vii. Capacity and setup constraints

The list is not exhaustive, but it does cover most of the academic literature. In general, almost all such models assume that the demand for different products is mutually

independent, the firm operates in imperfect competition or monopolistic environment, and the customers behave myopically.

In this dissertation, we consider pricing and production decisions for multiple products while also incorporating capacity constraints and setup costs. These decisions are internal to the firm as it involves production and marketing function of the same organization, thus contract-based supply chain coordination models that span across multiple organization are beyond the scope of this review. Here, we limit the review to discrete-time models with deterministic demand functions and to a few that are most relevant to this research. We first discuss some of the most widely used functional forms of deterministic demand. Next, we present the relevant literature under the classification of constant and dynamic pricing strategies for deterministic demand. We list the most relevant ones in Table 2.1. Then, we present the literature on sales response of advertising. Lastly, we review existing literature on joint advertising and production decisions.

2.1 Demand Function

In order to develop a better understanding of consumer demand functions, we need to learn how consumers make their purchase decisions. A typical purchase cycle for a consumer may be described by the following five steps (Lilien et al., 1992) :

- i. Arousal by some internal or external stimuli
- ii. Search for information about products or brands
- iii. Evaluate and compare available options
- iv. Make the purchase decision
- v. Post-purchase experience

During the different stages of purchase cycle, consumer perception of a product is influenced by different operational and marketing decisions of the firm. Consumers would

Table 2.1: Pricing Decisions: List of Papers Most Relevant to this Research

Sr. No.	Paper	No of Products	Time Frame	Capacity Constraint	Setup Cost	Demand Function	Seasonal Demand
1	Whitin (1955)	S	S	-	✓	Linear	-
2	Wagner and Whitin (1958)	S	M	-	✓	NA	NA
3	Arcelus and Srinivasan (1987)	S	S	-	✓	Iso Elastic	-
4	Porteus (1985)	S	S	✓	✓	Linear	-
5	Chen and Min (1994)	S	S	✓	✓	Linear	-
6	Kim and Lee (1998a)	S	S	✓	✓	Iso Elastic	-
7	Thomas (1970)	S	M	-	✓	General	-
8	Kunreuther and Richard (1971)	S	M	-	✓	General	-
9	Kunreuther and Schrage (1973)	S	M	-	✓	Linear	✓
10	Bhattacharjee and Ramesh (2000)	S	M	-	✓	Iso Elastic	-
11	Biller et al. (2005)	S	M	✓	-	General	✓
12	Gilbert (1999)	S	M	-	✓	Linear	✓
13	Deng and Yano (2006)	S	M	✓	✓	General	-
14	Feng (2010)	S	M	✓	-	General	-
15	Morgan et al. (2001)	M	S	-	✓	Linear	-
16	Dobson and Yano (2002)	M	S	✓	✓	Linear	-
17	Gilbert (2000)	M	M	✓	-	Linear	✓
18	This Dissertation	M	M	✓	✓	Linear & Iso Elastic	✓

consider factors like price, quality, features, durability, customer service, lead time etc. In addition, their decisions may also be influenced by word of mouth or experiences of other consumers. While different consumers may give different weights to these factors while making their purchase decisions, the aggregate demand for a product is in fact a function of all these factors. Huang et al. (2012) survey and classify the literature in six major categories, which include: (i) price-, (ii) rebate-, (iii) lead time-, (iv) space-, (v) quality-, and (vi) advertising-dependent demand models. They observe that the price-dependent demand models are the most widely used models. Some researchers also study

Table 2.2: Examples of Deterministic Demand Functions

Model Type	Demand Function	Examples in Literature
Linear	$D(P) = a - bP$, where $a, b > 0$	Eliashberg and Steinberg (1987) Petruzzi and Dada (1999) Gilbert (1999)
	$D(P, r) = a - bP - c(P - r)$, where $a, b, c > 0$ and r is the reservation price	Fibich et al. (2003)
Power	$D(P) = aP^{-b}$, where $a > 0, b > 1$	Bhattacharjee and Ramesh (2000) Kim and Lee (1998a)
	$D(P) = a - bP^c$, where $a, b > 0, c \geq 1$	Chen et al. (2006)
Exponential	$D(P) = \exp(a - bP)$, where $a, b > 0$	Song et al. (2008) Chen et al. (2006)
	$D(P) = a - \exp(cP)$, where $a, c > 0$	Chen and Simchi-Levi (2004)
Logit	$D(P) = 1/(1 + \exp(a + bP))$ where $a, b > 0$	Phillips (2005)

quality-dependent demand. However, little work has been done in developing models for the other four categories.

The two most widely used deterministic demand models are the linear demand model, and the constant elasticity of demand model (also called the iso-elastic demand function) (Lilien et al. (1992), and Lilien and Rangaswamy (2004)). Both models assume that demand for a product is a strictly decreasing function of its price only. A number of researchers use linear demand function in their pricing models, for example Porteus (1985), Chen and Min (1994), Gilbert (1999) etc. The linear function assumes that the price elasticity of demand decreases as price increases. Other researchers use constant elasticity of price to model demand, for example Kim and Lee (1998a), and Arcelus and Srinivasan (1987). The iso-elastic demand function assumes the demand nonlinearly decreases in price, and the price elasticity is constant at all selling prices. Other less commonly used function are the exponential and logit demand functions. Some researchers

use the concept of reservation price, instead of selling price, to model consumer demand (also called as willingness to pay model). Kalish (1985) presents one such model where the customer purchases the product if the price is less than or equal to his reservation price. The author also incorporates uncertainty about product's performance or quality and show that customer's reservation price increases as the uncertainty decreases. A firm can thus use advertising to communicate its product quality and attributes, which decreases uncertainty, and hence allows the firm to select a higher price for the advertised product.

Abad (1988) shows, under some conditions, a class of demand functions lead to a S-shaped profit function. Rosenberg (1991) uses another less commonly used demand function that is concave in price. Table 2.2 gives a classification of demand functions based on their mathematical form, and some examples of their use in literature. All these functional forms assume that demand is function of a single variable price (P). More complex functional forms with two or more variables may be developed first by studying the impact of individual variables and then using an additive or multiplicative interaction between these variables. In Section 2.5 we present a review of deterministic demand models as function of advertising expenditure; then in Section 5.1.1 we propose an additive demand model that is linear in price and nonlinear in advertising expenditure.

2.2 Constant Price Models

The earliest papers that address joint pricing and lot-sizing decisions consider the single product problem in an economic order quantity (EOQ)-type setting. Seminal papers include that of Whitin (1955) and Wagner and Whitin (1958) who introduce price as a decision variable in lot-sizing decisions and prove the well-known planning horizon results. They also consider variations of the basic model with demand and cost functions changing across time periods. Thomas (1970) builds further upon the planning horizon

results of Wagner and Whitin (1958) and shows that prices in periods between setups can be optimized independently. In his model, setup, variable, and inventory holding costs may differ from period to period. Kunreuther and Richard (1971) study the same model, and show that coordinated decisions result in higher profits for a firm. Damon and Schramm (1977) incorporate financial decisions in an integrated model that solves for production, price, and cash flow. They examine each of the three functions and develop a single product model with deterministic demand. They focus on comparing sequential and simultaneous decision making processes. They conduct numerical experiments that demonstrate their proposed simultaneous decision model may result in 25% higher profits.

Since then, many researchers have contributed to the literature by solving different variations of the problem. For example, some consider a single product while others incorporate multiple products in their models, some allow prices to change dynamically whereas others choose a constant price for the planning horizon. Eliashberg and Steinberg (1993), Chan et al. (2004), Yano and Gilbert (2005), Tang (2010), and Chen and Simchi-Levi (2012) are excellent reviews on joint pricing and production literature. We encourage the interested reader to refer to these papers for an extensive literature review. It is evident from these surveys that only a limited number of papers include capacity constraints and few incorporate multiple products in their models.

Porteus (1985) determines a threshold demand beyond which price reduction coupled with investment in setup time reduction stimulates demand and increases profit. Morgan et al. (2001) consider product portfolio decisions and present a multi product model where one production line produces all products with the same frequency. They solve for a price vector that maximizes profit for a fixed production interval. Dobson and Yano (2002) further extend the work to include decisions on which products should be make-to-stock and which ones should be make-to-order.

Kunreuther and Schrage (1973) develop a solution procedure to solve a joint pricing problem for a single product whose demand is deterministic, seasonal, and decreasing in

price. They allow the fixed ordering costs to vary over time and use a seasonal demand function that has a price-insensitive additive component. While their solution approach provides strong bounds for the problem, it does not guarantee optimality. Gilbert (1999) further extends their work: He uses a purely multiplicative seasonal demand factor, assumes that the fixed ordering costs are time-invariant, and proposes an algorithm that solves the problem to optimality. Both papers include setup costs but neither incorporates production capacity constraints.

In another paper, Gilbert (2000) further extends his previous work to the case of constant priced multiple products that exhibit seasonal demands. He uses a linear demand function that is strictly decreasing in price. In the absence of setup costs, but with capacity constraints, he shows that it is more profitable to price aggressively those products whose demands peak early in the season.

All of the above mentioned papers assume that demand linearly decreases in price, and determine the solution space with linear constraints. Although this is a reasonable assumption for many cases, yet demand for some products may exhibit a nonlinear relationship (for example, constant elasticity). Modeling a nonlinear demand function, changes the problem structure, because the constraint set is not linear anymore. Kim and Lee (1998a) use constant elasticity to model pricing, lot sizing, and capacity investment decisions. Arcelus and Srinivasan (1987) use the same demand model and determine optimal markup for profit maximization and average return on inventory. Bhattacharjee and Ramesh (2000) also use a constant elasticity demand function for a single product that has a perishable life. They formulate the profit maximization problem for a monopolistic retailer over multiple time periods and present two inexact algorithms. They solve numerical examples to demonstrate that their algorithms find quality solutions with an optimality gap of less than ten percent.

Few of marketing and production coordination models consider the case of multiple products, perhaps because of the high level of complexity (Eliashberg and Steinberg,

1993). To the best of our knowledge Gilbert (2000) is the only paper that considers joint production and pricing decisions for multiple products and multiple time periods with seasonal demand.

2.3 Dynamic Price Models

Dynamic pricing refers to a pricing mechanism where a seller may balance supply and demand by setting a different price in each period. For example, a retailer of fashion garments may set a higher price when price elasticity of his products is low, and a lower price when the elasticity is high. Dynamic pricing strategies have widely been used in revenue management, specially in situations when the product life is perishable and marginal cost to serve is significantly low (for example, airlines, hotels, etc.). Most of the literature considers case of a single product with limited product availability during the selling period. Some incorporate effect of selling multiple products that may substitute each other. Though the product availability is limited, it is not dependent on any other resource. Few researchers incorporate a limited resource that may be consumed by multiple products. Allocation of manufacturing capacity to different products along with dynamic pricing is one such challenging example. Unlike revenue management, the product mix decisions are not fixed, the product may not be perishable, and the marginal cost to serve a customer is not negligible. Clearly, in such situations, it is in the best interest of a firm to maximize profit instead of revenue. Traditionally, manufacturers have followed a constant list pricing strategy - perhaps because of a lack of knowledge and ability to implement dynamic pricing. Interestingly, while the list prices are constant, firms often offer consumer promotions for limited time periods. Indeed the net effect of such promotions is a temporary price reduction, we tend to agree with Talluri and Van Ryzin (2005) who classify such promotions as a form of dynamic pricing. Hence, many firms implicitly use dynamic pricing to balance supply and demand. Recently, firms

have realized importance of dynamic pricing in managing supply chains. Phillips (2005) observe that dynamic pricing is the most efficient way of maximizing profits.

Elmaghraby and Keskinocak (2003) provides a review of literature and practices in dynamic pricing in inventory considerations. They classify the literature on joint pricing and inventory decisions according to following criteria:

- Stochastic or deterministic demand
- Unlimited or finite capacity
- Single or multiple products
- Fixed ordering costs

They highlight that almost all the research in this area considers pricing decisions for a single product, and identify the consideration of multiple products as a promising direction for future research.

Rajan et al. (1992) study dynamic pricing problem for a seller of single perishable product. They model demand as a function of price and age of the product. The seller places an order after some T periods that is filled instantaneously. The seller's decision is to decide optimal prices, time between orders, and the quantity to order, so as to maximize his average profit. They compare the profitability between constant and dynamic pricing strategies under condition when prices are either monotonically increasing or decreasing over time. They show that for this special case dynamic pricing strategy performs significantly better when the demand is high. Biller et al. (2005) also consider a dynamic pricing and production problem in an industrial environment for the automotive industry. They consider a single facility that must determine prices and production scheduling for a single product over a finite horizon. With the exclusion of setup costs and assumption of concave revenue curve, they develop a heuristic to solve the problem. Deng and Yano (2006) consider the joint pricing and production decisions for a single

product with capacity constraints and setup costs. They make two intriguing observations: i) The optimal prices do not necessarily increase as capacity decreases. ii) Capacity increase does not always have diminishing returns.

Feng (2010) compares dynamic and constant pricing strategies for a single product under uncertain capacity but without setup costs. Netessine (2006) considers dynamic pricing problem for a single product where capacity/inventory is restricted, and only a limited number of price changes is permitted during the planning horizon. Ahn et al. (2007) model another version of the single product problem, they assume that the demand is a function of not only the current price but also the prices in other periods. They develop closed-form solutions and effective heuristics for solving various cases of the problem. Charnsirisakskul et al. (2006) study pricing and manufacturing decisions for a manufacturer who sets prices to manage demand for a single product. The manufacturer may reject some orders and may offer different lead times to different customers. As a consequence, he may set different prices for different customers. They argue, regardless of the pricing mechanism, the manufacturer should coordinate pricing and production decisions. They propose heuristic solutions and solve several numerical examples to show benefit of price differentiation policies. Webster (2002) study a model where a make to order firm can manage demand by changing prices and lead times. The author first considers a model with fixed capacity and then compares its profit with the case when lead time is fixed and price and capacity can be varied over time. He shows that shorter lead time with flexible capacity and dynamic pricing results in higher profits. Aviv et al. (2012) survey a broad range of dynamic pricing models and highlight the need to develop multi-product and multi-resource models. Furthermore, they also discuss extensions of pricing models; including effect of learning and strategic customer behaviors.

2.4 Advertising Decisions

Advertising, in general comprises of set of activities that are executed in an organized manner to communicate attributes or benefits of a product to its potential customers. However, in the context of model building a narrower definition might be more useful. We present two such definitions: i) Advertising is a fixed expenditure which influences the shape or position of a firm's demand curve (Dorfman and Steiner, 1954). ii) Advertising consists of messages delivered to individuals by exposures in media paid for by dollars (Little, 1975).

What advertising does in the market place? is a question that still remains unanswered in the marketing literature (Lilien et al., 1992). There are a number of conflicting opinions and models on how advertising effects sales. The complex interactions between marketing mix, media selection, copy effectiveness, timing, and advertising expenditure make it difficult to arrive at a unified approach. Furthermore, the effect of advertising would vary significantly with product attributes and the stage of product's life cycle. Elasticities are higher in the introduction and early growth stages of a product, and are lower in the maturity stage. This is because, in the later part of product life cycle, customers have accumulated experience and are more knowledgeable about product quality and performance. Sales volumes are usually low in the early stage of product life cycle and hence sales increase due to advertising would represent a large percentage gain in contrast to gain in later periods when sales are repeat purchases (Parsons, 1975). Perhaps that is why; Aaker and Carman (1982) refers the study of relationship between advertising and sales as worse than looking for a needle in a haystack, and Bass (1969a) declares the problem of determining sales response of advertising as the most complex and controversial problem in marketing.

Advertising effects may also vary according to the type of product being advertised. Nelson (1970) categorizes products into two categories: i) 'Experience goods' - are the

products that are used frequently and consumers normally learn about these products from their experiences. They prefer not to spend time and money in collecting information about these products. Example of such products are most consumer packaged goods. ii) ‘Search goods’ - are products that are purchased once off. Examples of such products are technologically innovative products and consumer durables. Clearly, the objectives and effects of advertising may vary according to the product type. Sales response models will also be different. When a customer of search goods makes a purchase she exits the market after making the purchase, whereas a customer of experience goods is likely to make a repeat purchase. One may argue that advertising of search goods is more effective in beginning of planning horizon and is less effective toward the end. Advertising for experience goods, on the other hand, may have the same effect over time. Hence, a model that works best for one product may not perform well for another.

If we don’t really know how advertising influences demand, then what basis do the firms use to decide advertising expenditures ? Interestingly, many empirical studies show that most firms make such decisions without doing any sound cost-benefit analysis. Lilien et al. (1992) quote a number of such examples; some firms use historical spending as reference to decide future plans, some simply spend what they think the firm can afford, many would pick a fix percentage of revenue as their advertising budget, while others may want to match their spending with competition.

Both researchers and practitioners are interested in determining the shape of sales response function. The shape of sales response to advertising refers to rate of change in sales as we increase advertising expenditure. Three typical shapes discussed in literature are: linear, concave, and S-shaped (Figure 2.1). The linear shape assumes every dollar spend on advertising has an equal effect on sales and that the sales keep increasing indefinitely with advertising. The concave shape implies decreasing returns to advertising and a saturation level of sales. The S-shapes implies that both very low levels and very high levels of advertising are least effective. As we increase advertising, the sales

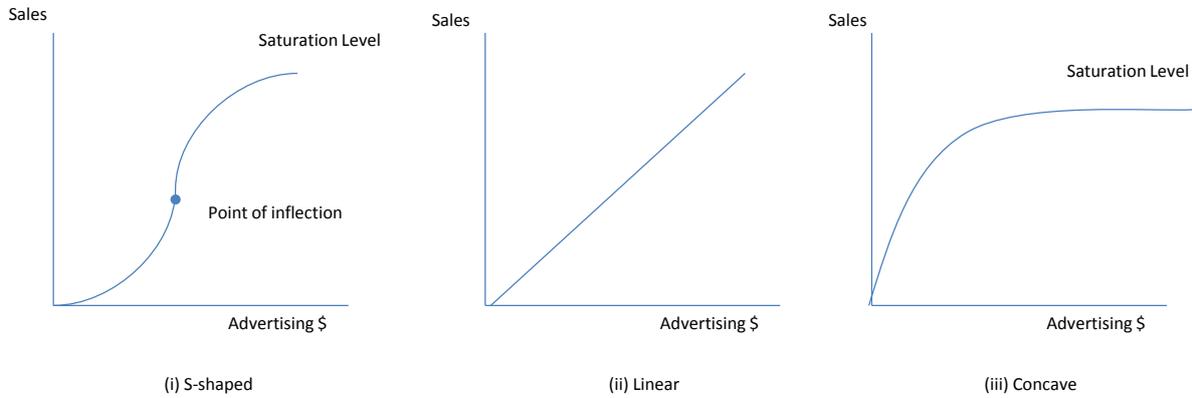


Figure 2.1: Sales Response to Advertising

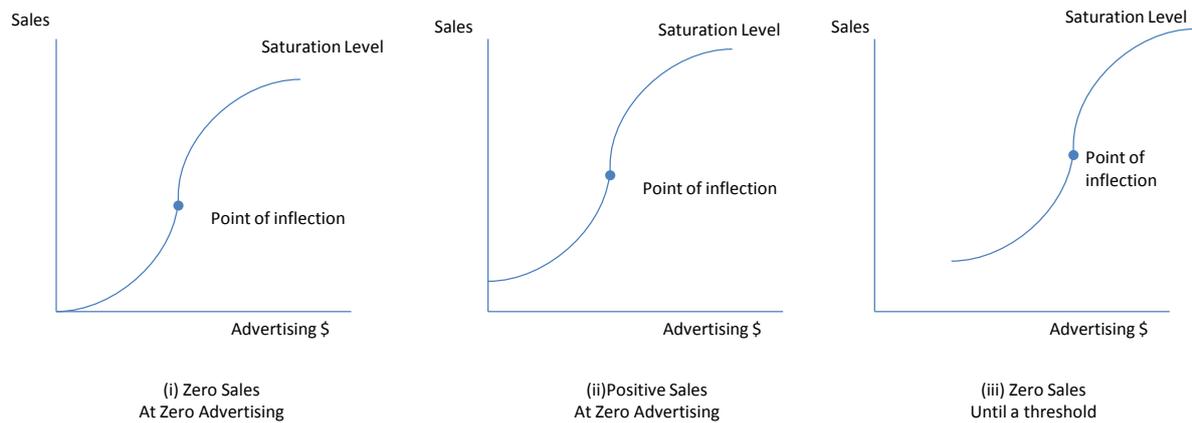


Figure 2.2: Sales at zero advertising

first increase at an increasing rate and then increase at a decreasing rate till it reaches a saturation level. In addition to the shape of response function, another important consideration is how to model sales at zero advertising. Figure 2.2 shows three possible scenarios corresponding to zero advertising: i) zero sale, ii) some positive level of sale, and iii) no sales until a threshold of advertising is reached. Although Figure 2.2 shows a S-shaped curve, the concept is applicable to all response functions irrespective of their shapes. Different researchers evaluate pros and cons of different forms of sales response function (for example Little (2004)) and perhaps agree to disagree with each other. Keeping in view the complex nature of marketing decisions, it is not out of place to conclude that there is no one functional form that may fit all situations.

We present, hereunder, a brief review of existing models that attempt to explain how advertising influences demand. We limit the review to aggregate response models - models that are best suited for understanding the sales response of well established and existing brands. Other models that exist in literature are the flow models and the micro-simulation models, these are best suited for new products. We then compare and contrast these models and discuss conditions in which one may perform better than the others. The discussion is not conclusive and only aims to provide a broad overview of the subject.

2.5 Sales Response to Advertising

2.5.1 A Priori Models

Dorfman and Steiner (1954) are perhaps the first to consider optimal pricing and advertising decisions. In fact they further extend it to optimal quality as well. They show that a monopolistic firm would choose price and advertising budget such that marginal revenue of advertising is equal to price elasticity of demand for the firm's product. They assume the demand is a continuous differentiable function of price and advertising budget, and develop a necessary condition for profit maximization at any level of output.

Vidale and Wolfe (1957) present their model based on results of controlled experiments performed over a number of products and several media. They present a model that can be used to understand the sales response to advertising and subsequently can be used for making advertising budget decisions. The proposed model is based on three concepts: i) In the absence of advertising sales will decay at a constant rate, ii) Sales increase with advertising with a diminishing rate of return and eventually reach a saturation level, and iii) Increase in sales sales per dollar spent on advertising can be estimated by a response constant times the proportion of market that has not yet made purchase decision. The model thus assumes that a customer does not make a repeat purchase

and exits market after making first purchase. Furthermore, the model concludes that a protracted advertising campaign will be more effective than pulsing - a generalization that is not supported by empirical evidence. Another problem with the model is that sales continue to decay without advertising and eventually become zero if not supported by advertising. This is a contradiction to commonly observed fact that many products continue to maintain sales level without any advertising support. The Vidale Wolfe model provides a base upon which other researchers have developed new models, for example; Mahajan and Muller (1986), and Sethi (1973);

Nerlove and Arrow (1962) extend the work of Dorfman and Steiner (1954) by assuming that current advertising spend influences present as well as future sales. They treat advertising expenditure as investment in goodwill of the brand. The goodwill is treated as an asset that depreciates over time and increases with present and future investments. They argue that the net effect of advertising is to shift or change the shape of demand curve. Irrespective of the shape of demand curve, the demand at any point in time can be represented as a function of goodwill. Interestingly, the goodwill variable may be given different meanings, and interpreted in more than one way. For example, Colombo and Lambertini (2003) consider this goodwill as the sales intensity, Lambertini (2005) uses reservation price, and Zielske and Henry (1980) interpret it as product awareness.

Although this model makes intuitive sense, little empirical evidence is available to support it. Furthermore, the model also fails to explain a number of observed phenomenon as described by Little (1979). In a recent paper Buratto et al. (2005) discuss the Nerlove-Arrow's model in detail and present different problems for which the model works well.

Little (1975) proposes his a priori model (BRANDAID) that is a generalization of the above mentioned models, and encompasses a wide variety of products and situations. His model is based on information collected from marketing managers and empirical studies conducted by other researchers. BRANDAID has a modular structure and includes advertising as a sub model. The model is based on the assumption that certain level of

advertising is required to maintain current sales level. This level of advertising is referred to as reference level. Advertising spend less than the reference level would result in lower sales and vice versa. Furthermore, current advertising is assumed to have some carry-over effect on future sales as well. He shows that by appropriate selection of model parameters his model may represent a wide variety of empirically observed phenomenon (e.g. Lodish (1971) and Ray and Sawyer (1971)). Lilien (1979) and Little (2004) provide excellent reviews of different advertising models, and discuss their features and characteristics. From these discussions we may conclude that different researchers have agreed to disagree on how sales can be modeled as a function of advertising.

Basu and Batra (1988) develop the ADSPLIT model for allocating advertising budget among competing brands (the brands have independent demands but compete for allocation of the advertising budget). They present three heuristic procedures to find optimal prices, advertising budget, and its allocation to different brands. The model is versatile and can be used for concave as well as S-shaped response functions. To estimate the model parameters; they propose to use a large number of data points in a regression function. They claim that such a selection makes the model robust. However, they leave the burden of parameter estimation to marketing managers. They use a planning horizon of one year length and assume that there is no carry over effect of advertising beyond the planning horizon. Furthermore, they ignore any interdependencies in the joint utilization of production capacities.

Perhaps we are more interested in their approach of modeling sales response as a function of price and advertising. The function; allows for positive sales at zero advertising, incorporates a saturation level for sales, and permits the user to choose the shape of the curve between these levels. They use constant elasticity of price function to model the effect of price on demand. Furthermore, the price and advertising variables are multiplicatively separable. The general structure of this response function is appealing and it explains quite a few empirically observed phenomenon.

2.5.2 Econometric Models

Rao and Miller (1975) present their study of sales and advertising model that uses sales and advertising/promotional data of fifteen brands of Unilever. They derive advertising response coefficients for each brand and sales districts. They use the results of the study to develop a S-shaped general model of sales response to advertising.

Bass (1969a) highlight the need to develop sales response models that pass the test of theoretical premises and are supported by empirical evidence. They develop a multiple-equation regression model on the premise that not only sales is a function of advertising but advertising is also effected by sales (for example: managers set advertising budget as percentage of sales).

Simon and Arndt (1980) provides a brief review of literature on how advertising effects sales. They raise an interesting observation. Once an advertising plan is finalized the firm ‘buys’ media time or space. Usually there are sizable quantity discounts available in media buying. Hence, sales response to advertising quantity (e.g. gross rating points or size of a newspaper advertisement) may not be the same as sales response to advertising spend in dollars. They conclude that neither advertising quantity nor the advertising spend exhibit an increasing rate of return.

D’Souza and Allaway (1995) and D’Souza and Allaway (1997) argue that although a priori models are simple yet not of practical interest as these are not supported by empirical data. In fact they recommend ‘complexity’ over ‘simplicity’ and encourage use of bigger data sets and more computational power for model development. They propose use of statistical methods for estimation of model parameters that are of practical use for the decision maker. They present a three step data-driven approach that includes model building as well as its usage. To demonstrate the applicability of their approach, they apply their modeling process to a multi-product retailer in a promotion-oriented environment. They develop managerial insights and show how the retailer may increase his profits by better allocation of advertising spend.

While there are conflicting opinions on how to model sales response function, most researchers tend to agree that the curve should either be S-shaped or concave. The S-shape of response function is justified by a priori assumption that initial burst of advertising has increasing returns until it reaches an inflection point; after which the returns follow a diminishing rate of return on advertising. The concave function, on the other hand, is a result of monotonically decreasing returns. Most of the empirical studies support the argument in favor of a concave return function (Simon and Arndt, 1980) . Few studies find evidence in support of the S-shape curve (e.g. Rao and Miller (1975) and Little (1979)). However, lack of empirical evidence does not necessarily mean there is never an increasing return on advertising. It is argued that it does not make economic sense to operate in the convex region of the S-curve. Most firms would practically choose levels in the concave region of the curve - hence no data is available for the convex region.

Mariel and Sandonis (2004) develop a dynamic model of advertising and price competition in duopolistic competition. They use a stock goodwill of advertising to model demand as function of current and past advertising expenditures. They support their model with empirical evidence from German automobile industry. They conclude that the price or advertising action of each player tends to be reinforced by the action of its rival.

Aaker and Carman (1982) show, in many cases, the practitioners tend to over spend on advertising. They give two explanations for this overspending: i) reward structure within the marketing organization encourages managers to spend more on advertising, and ii) difficulty of modeling sales response of advertising. They provide a review of field experiments and econometric studies on understanding the sales effect of advertising. They summarize studies over many markets, product categories, and brands; providing empirical evidence of over-advertising.

Assmus et al. (1984) conduct a meta-analysis of 128 econometric models involving impact of advertising on sales for well established products. They present an analysis of

short-term and long-term response functions as well as goodness of fit for these models. They observe that advertising elasticities differ among product categories and markets. Another interesting observation, based on their study of European and U.S. market for food products, is that either European firms under-advertise or firms in the U.S. over-advertise.

2.5.3 Diffusion Models

While marketing managers may have historical data to predict and model the sales response functions for well established products, it is a daunting task to model sales response of advertising for new product introductions. Diffusion models are used to forecast sales of new products. Bass (1969b) was the first one to develop such a model. He uses theory of adoption and diffusion to model demand of new products and develops the famous “Bass Model”. The model assumes a S-shaped growth pattern for new durables and technologies. The model, initially thought to be limited to consumer durables, also finds extensive applications in a much wider class of products and services (Bass, 2004).

Dodson and Muller (1978) also develop a new product diffusion model that incorporates effect of advertising and word of mouth on sales of a new product. They show the sales response function is concave when the advertising efforts dominate the market conversion; and is S-shaped when the word of mouth is dominant. They assume the product to be a consumer durable, and customers exit the market after making a purchase. Later they also extend their model to incorporate repeat purchases.

Mahajan et al. (1990) is a good review on development and classification of diffusion models in marketing. The authors address different theoretical and practical issues related to these models and suggest directions of further research to make these models more effective and realistic.

2.6 Production and Advertising Models

One of the earliest papers on joint advertising and production decisions is that of Thomas (1971). The author addresses the problem of simultaneously smoothing production and inventory and setting advertising levels. A series of linear models are presented, assuming a deterministic demand-advertising-price relationship. Leitch (1974) also addresses the issue of coordinating production and marketing activities in an uncapacitated environment, where advertising effort can be used to divert the demand such that the firm's production costs are minimized. They develop a linear model and show existence of an optimal solution.

The problem of coordinated pricing and advertising within the context of supply chain is well researched in academic literature. Many researchers study different aspects of the problem and arrive at optimal control policies. Most of the studies assume an unlimited supply and do not incorporate manufacturing capabilities in their models. Krishnamoorthy et al. (2010) examine pricing and advertising decisions of a monopolistic firm in a durable goods market. They develop close form solutions for the problem and conclude that the optimal price is stationary and the optimal advertising effort should decrease over time. Chutani and Sethi (2012) also study the problem for a durable goods supply chain. The manufacturer, who is the stackelberg game leader, announces the wholesale price policy and subsidy rate for each of the retailer. The retailers, acting as followers, then decide their respective prices and advertising effort that maximizes their profits. They conclude that the optimal subsidy rates are independent of the model parameters and are always equal.

Abad (1982) and Abad and Sweeney (1982) develop optimal control policies for a single product, where price and advertising are the control variables and sales and inventory are state variables. They compare and contrast centralized and decentralized decision making processes and use numerical examples to illustrate situations when coordinated

decision making generates higher profits and when the two approaches result in almost same profits.

Thompson and Proctor (1969) develop optimal control policy while considering production, investment in capacity, advertising and price as control variables. Their model is based on the theoretical framework provided by Vidale and Wolfe (1957). Huang and Li (2001) develop a game theory approach to consider co-op advertising policy in manufacturer - retailer setting. They model a case where the manufacturer advertises his product to consumers and at the same time promotes the product at the retail level. The manufacturer's advertising effort is focused on brand building and creating awareness, whereas the retailer focuses on creating desire and stimuli for action such as low price, high quality, and availability. The two advertising efforts, therefore, complement each other. In their model, selling price is assumed to be given and demand is a multiplicative function of the two advertising efforts only.

Erickson (2011) consider the pricing, advertising, and production decisions for a single product assuming infinite production capacity. They model demand as a linear function of price and goodwill of advertising. They model the problem as non cooperative differential game and derive a feedback Nash equilibrium.

Kim and Lee (1998b) develop optimal strategies for coordinating marketing and operations decisions for a uncapacitated single period problem. They model a decentralized, coordinated decision process where the two functions exchange information and iteratively optimize their individual plans. They present two approaches; a marketing dominated coordination approach, and a production dominated coordination approach. Both approaches are similar and are based on finding optimal marginal cost and marginal revenue. However, they show different convergence properties depending upon the shape of marginal cost and revenue functions.

Ulusoy and Yazgac (1995) address the problem of coordinating marketing and production decisions for multiple products. They include pricing, advertising, and lot sizing

as decision variables in their model but do not incorporate manufacturing capacity constraints. They use a dynamic programming-based heuristic procedure to solve the problem. They model demand as function of price and advertising expenditure as proposed by Kotler (1971). However, one problem with this demand function is that it implied zero demand if there is no advertising.

Feichtinger et al. (1994) present a review of dynamic optimal control models in advertising. The review complements an earlier review of Sethi (1977). In addition to advertising, they also include other variables of interest (e.g. quality) in the survey. These reviews cover a wide range of related problems including, sales response to advertising, accumulated goodwill function, pricing, and quality. They include models; with and without competition, and for durable as well as consumer goods. They emphasize the need for building empirical and econometric models that support theoretical sales response functions. It is evident from these reviews that most of the researchers suggest it is optimal to evenly distribute advertising expenditure over the planning horizon. These conclusions are quite opposite to empirically observed phenomenon of pulsing i.e. increased advertising intensity for short time periods. The intuition behind pulsing strategy is that it helps a brand to stand out and being heard in noise. Feinberg (2001) proposes a discrete time S-shape response function that may explain and support the pulsing strategy.

A related stream of research is of joint production and promotions decisions. The promotional decisions of a firm are closely related to pricing decisions as the former is indeed a temporary price reduction. It is worth mentioning that most of the research has focused on optimal production, pricing and promotion decisions and relatively less work has been done to study the effect of advertising decisions operational plans. Perhaps, one of the reasons is that advertising has long been considered as a long term demand driver while price and promotion can impact the demand immediately (Ailawadi and Neslin, 1998).

Neslin et al. (1995) also study the effect of a manufacturer's promotion and advertising decisions on consumer demand. Their modeling framework also includes a portion of demand that is driven by the advertising intensity in the current period. Similar examples can be seen for example in Assuncao and Meyer (1993) and Jørgensen and Zaccour (1999). Since we wish to study the pure impact of advertising on operational decisions, we do not cover the literature that is related to production and promotion decisions. However, the interested reader is referred to Chan et al. (2004), that gives an excellent classification of related research.

To the best of our knowledge there is no existing paper that considers the optimal production decisions in a multi-product environment where the manufacturer has limited production capacity but can manage demand by changing prices and (or) advertising programs. This research makes an attempt to fill this gap by presenting an integrated approach for pricing, advertising, and production decisions.

Chapter 3

Joint Pricing and Production

Decisions

3.1 Problem Statement and Formulation

We develop a multi product, discrete-time model with capacity constraints and setup costs. Specifically, we consider the case of a manufacturer, who produces different products using the same equipment. We assume the demand for each product is a function of its current selling price only. In Chapter 5 we extend the case by including advertising, in addition to price, as a driver of demand. We assume that the demand curves are continuous, differentiable and strictly decreasing in price. The demand functions may vary from period to period. The manufacturer's goal is to determine prices such that he maximizes his profit over a finite planning horizon. While selecting the prices, the manufacturer may follow one of the following two strategies: i) Choose a constant price, for each of the products, which then remains the same through out the selling season or ii) Dynamically change prices throughout the selling season.

In each period, the manufacturer incurs a setup cost for each product if produced in that period, a variable production cost per unit, and a per-unit inventory holding cost

on ending inventory. These costs may vary from period to period. Furthermore, we also assume that the inventories at the beginning and the end of the planning horizon are zero. We assume a fixed production capacity in each period; however, the capacity may vary from period to period.

Since the production capacity is limited, allocation of this capacity to each product is a critical decision. When the prices are decided in isolation, operations may allocate capacity in a fashion that minimizes cost but not necessarily the profit. Such a situation may also result in generating demand that can not be met by the available capacity. A coordinated pricing and production decision process, on the other hand, will not only maximize profit but also result in meeting customer demand in full. We model this problem as a mixed integer nonlinear program (MINLP) and solve it to optimality.

3.2 Demand Function

We make the following assumptions about the aggregate demand function: i) the demand function is continuous, differentiable and convex, ii) the demand is finite, and it tends to zero for sufficiently high prices, iii) the cross price elasticities are zero, which implies that the inverse demand function exists, and iv) the revenue function is concave.

These are not restrictive assumptions and hold true for the most commonly used demand functions (Talluri and Van Ryzin, 2005). For example, the linear demand function $D(P) = \alpha - \beta P$, where $D(P)$ is the demand induced by price P and $\alpha, \beta \geq 0$ are scalar parameters, satisfies these assumptions except that it produces negative demand when $P > \alpha/\beta$. When using this function, we must limit the feasible price set $\Gamma_p = [0, \alpha/\beta]$. Another commonly used demand function is the constant elasticity demand function $D(P) = \alpha P^{-\beta}$, where $\alpha > 0$ and $\beta \geq 0$ are constants. This function is defined for all nonnegative P , so $\Gamma_p = [0, +\infty)$, and it satisfies all of the necessary assumptions.

In the following subsection, we present the problem formulation using a linear demand

function. However, the solution method presented in this paper can be used in general for any demand function that fulfills conditions 1 to 4.

3.3 Model

We now present a mathematical formulation of the problem. Since constant prices can be treated as a special case dynamic prices, we use the later pricing strategy while defining and solving the problem. Table 3.1 presents the notation used for the model. Using this notation, we state the problem as under:

$$(P) \quad \max z_P = \sum_{j=1}^J \sum_{t=1}^T S_{jt}P_{jt} - c_{jt}X_{jt} - h_{jt}I_{jt} - a_{jt}Y_{jt}$$

Subject to:

$$S_{jt} - \gamma_{jt}(\alpha_j - \beta_j P_{jt}) \leq 0 \quad \forall j, t \quad (3.1)$$

$$\sum_{j=1}^J v_j X_{jt} - b_t \leq 0 \quad \forall t \quad (3.2)$$

$$S_{jt} + I_{j,t} - X_{jt} = 0 \quad \forall j, t = 1 \quad (3.3)$$

$$S_{jt} - I_{j,t-1} - X_{jt} = 0 \quad \forall j, t = T \quad (3.4)$$

$$S_{jt} + I_{j,t} - I_{j,t-1} - X_{jt} = 0 \quad \forall j, t = \{2, \dots, T-1\} \quad (3.5)$$

$$v_j X_{jt} - b_t Y_{jt} \leq 0 \quad \forall j, t \quad (3.6)$$

$$S_{jt}, P_{jt}, I_{jt}, X_{jt} \geq 0 \quad \forall j, t \quad (3.7)$$

$$Y_{jt} \in \{0, 1\} \quad \forall j, t \quad (3.8)$$

Constraint set (3.1) ensures that the sales quantity for each product in each period must not exceed the demand for that product in that period. Constraint set (3.2) limits the total quantity produced in a period by the production capacity. Constraint sets (3.3 - 3.5) are the inventory balance constraints, and Constraint set (3.6) ensures that a setup

Table 3.1: Notation Used for The Model

Symbol	Description
j	index for products, $j = \{1, \dots, J\}$
t	index for time periods in the planning horizon, $t = \{1, \dots, T\}$
c_{jt}	unit variable cost for producing product j in period $t \geq 0$
b_t	production capacity available in period $t > 0$
γ_{jt}	seasonal demand factor for product j in period $t > 0$
h_{jt}	unit inventory carrying cost for product j in period $t \geq 0$
β_j	price effect of demand for product $j > 0$
α_j	maximum demand for product $j > 0$
v_j	capacity used per unit of product $j > 0$
a_{jt}	setup cost for product j in period t
P_{jt}	price of product j in period $t \geq 0$
S_{jt}	units of product j sold in period $t \geq 0$
I_{jt}	inventory of product j at end of period $t \geq 0$
X_{jt}	quantity of product j produced in period $t \geq 0$
Y_{jt}	binary variable of setup for product j in period t , $Y_{jt} \in \{0, 1\}$

cost is incurred whenever a product is produced. Lastly, constraints (3.7) and (3.8) are the non-negativity and binary variable constraints respectively. Observe that we use the quantity sold (S_{jt}) as a decision variable and allow for lost sales in the model. Later, we show the optimal solution has the property that there is no lost sale. We exploit this formulation and gain computational advantage while executing the solution method. Furthermore, it ensures there are no nonlinear equality constraints in the model. Thus the proposed algorithm can easily be used for nonlinear demand functions as well.

This problem is a mixed integer nonlinear program, which is a class of problems that are difficult to solve, in general. In the following section we present a solution approach for solving this problem and show that it finds an optimal solution.

3.4 Solution Approach

Grossmann (2002) and Bonami et al. (2008) are two excellent reviews on algorithmic developments for solving MINLP problems. Before we proceed with the solution, it is worthwhile to mention some key characteristics of the problem: i) the binary and the continuous variables are separable in the objective function and constraints, ii) the bi-

nary variables affect the objective and constraints linearly, and iii) there are no nonlinear equality constraints. Duran and Grossmann (1986) study this class of problems and compare the performance of different algorithms. They show that the outer approximation (OA) algorithm provides a tighter formulation and exhibits better convergence properties than the Generalized Benders Decomposition algorithm. Notice that the problem P is feasible for all possible combinations of the binary variables $Y_{j,t}$. Therefore, while applying the OA method, we do not need to add feasibility or integer cuts to the master problem. This reduces the computational effort as the master problem is formulated with fewer constraints, and we do not need to solve feasibility problems.

We develop an algorithm based on the outer approximation methodology. We present the general solution approach in Figure 3.1. We formulate two different problems whose solutions provide upper and lower bounds on the original problem. The first problem, called the primal problem, is a nonlinear program (NLP). We formulate it by fixing the binary variables Y_{jt} to some feasible set to form a restriction of P. Its solution is also feasible for the original problem, and its objective value is a lower bound. A second problem called the reduced master problem is a mixed integer linear program (MILP). We formulate it by introducing outer approximations of the objective function and that of nonlinear constraints to form a relaxation of P. This problem has a structure such that it overestimates the objective function and the feasible solution space of the original problem. Therefore, its solution value is an upper bound for P.

We solve these two problems iteratively where the solution from one problem provides information for updating the other problem for the next iteration. At each iteration, we use the primal solution to update the lower bound and add linearization cuts to the reduced master problem. Next, we solve the master problem, and its solution provides an upper bound for the problem P. We update the primal problem by fixing the binary variables at the current master solution and then repeat the procedure. Observe that at each iteration k we include additional constraints in the reduced master problem.

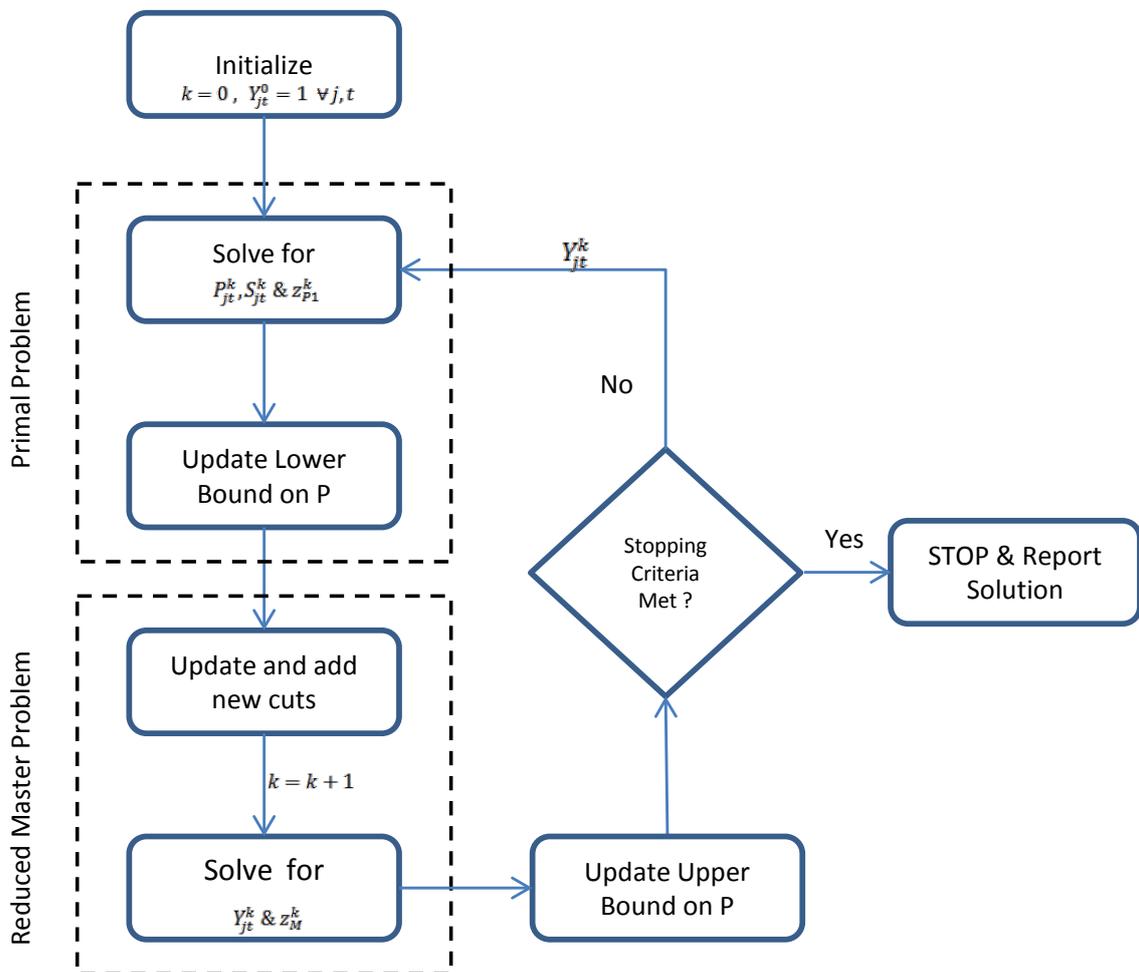


Figure 3.1: General Solution Structure for OA

Hence, its solution generates a sequence of non-increasing upper bounds. The two bounds converge within a finite number of iterations to a solution that is optimal for problem P. In the following subsections, we formally state the two subproblems and present the solution approach.

3.5 Primal Problem

At each iteration k of the algorithm, we solve a restriction of P obtained by fixing all the binary variables Y_{jt}^k at some feasible values. We call it primal problem and state hereunder:

$$(P1^k) \quad \max z_{P1}^k = \sum_{j=1}^J \sum_{t=1}^T S_{jt}^k P_{jt}^k - c_{jt} X_{jt}^k - h_{jt} I_{jt}^k - a_{jt} Y_{jt}^k$$

Subject to:

$$v_j X_{jt}^k - b_t Y_{jt}^k \leq 0 \quad \forall j, t \quad (3.9)$$

and constraint sets (3.1), (3.2), (3.3),(3.4),(3.5) and (3.7).

The primal problem is a nonlinear program that can be solved for example, by using a suitable method of feasible directions. However, such an approach considerably increases computational burden specially for large problems, since an instance of the primal problem is to be solved at each iteration. To improve the efficiency of proposed algorithm we reformulate the primal problem as a resource allocation problem. We exploit the special structure of the new formulation and develop a straightforward recursive algorithm that solves the problem to optimality. First we present following theorem and lemma:

Theorem 1. *A profit maximizing firm, selling multiple products in different periods, will determine selling prices for each product such that the induced demands are equal to the quantity sold for each product in each period (i.e, $S_{jt}^* = D_j(P_{j,t}^*)$ for all j, t at the optimal solution).*

Lemma 1. *For a given sequence of setups, the profit function of P is strictly concave in price.*

Now we present an alternative formulation for $P1^k$. We define set of indexes $N^k = \{(j, m, n), j = \{1, \dots, J\}, m = \{1, \dots, T\}, n = \{m, \dots, T\} : Y_{jm}^k = 1\}$. Let $X_{jmn}^k \geq 0$, be the quantity of product j produced in period m and sold in period n for all $(j, m, n) \in N^k$. For ease of notation, we also define parameters $A_{jmn} = \frac{\alpha_j}{\beta_j} - c_{jm} - \sum_{t=m}^{n-1} h_{jt} \forall (j, m, n) \in N^k$ and $B_{jn} = 1/\beta_j \gamma_{jn}$, $\forall j = \{1, \dots, J\}, \& n = \{1, \dots, T\}$. Furthermore, we also define $\sum_{t=m}^{n-1} h_{jt} = 0$ for $m = n$. We write the objective function at iteration k for a given sequence of setups (i.e, fixed values of Y_{jm}^k), as

$$\max z_{P1}^k = \sum_{(j,m,n) \in N^k} X_{jmn}^k \left(P_{jn}^k - c_{jm} - \sum_{t=m}^{n-1} h_{jt} \right) - \sum_{j=1}^J \sum_{m=1}^T a_{jt} Y_{jm}^k \quad (3.10)$$

It follows from lemma-1 that at the optimal solution $\sum_{m=1}^n X_{jmn}^k = D(P_{jn}^k)$ for all $j \in \{1..J\}$ and $n \in \{1, \dots, T\}$. For a linear demand function $D(P_{jn}) = \gamma_{jn}(\alpha_j - \beta_j P_{jn})$, thus we can write P_{jn}^k as

$$P_{jn}^k = \frac{\alpha_j}{\beta_j} - \frac{1}{\beta_j \gamma_{jn}} \sum_{m=1}^n X_{jmn}^k \quad (3.11)$$

We substitute (3.11) in (3.10) and write the primal problem as

$$(P1^k) \quad \max z_{P1}^k = \sum_{(j,m,n) \in N^k} X_{jmn}^k (A_{jmn} - B_{jn} \sum_{l=1, l \neq m}^n X_{jln}) - B_{jn} X_{jmn}^{k^2} - \sum_{j=1}^J \sum_{m=1}^T a_{jt} Y_{jm}^k \quad (3.12)$$

Subject to:

$$\sum_{j=1}^J \sum_{n=m}^T v_j X_{jmn}^k \leq b_m \quad \forall m \in \{1, \dots, T\} \quad (3.13)$$

$$X_{jmn}^k \geq 0 \quad \forall (j, m, n) \in N^k \quad (3.14)$$

We solve $P1^k$ as a non linear resource allocation problem with multiple resources. We consider the production capacity in each period m as a resource and solve for optimal allocation of capacity to production variables X_{jmn}^k subject to constraints (3.13) and (3.14).

Lemma 2. (*Gibb's Lemma*): Let X_{jmn}^{k*} be the optimal solution to problem $P1^k$, then for each $m = \{1, \dots, T\}$, there exists a $\lambda_m^k \in R$ such that

$$z_{P1_{X_{jmn}}}^{k'}(X_{jmn}^{k*}) = v_j \lambda_m^k \quad \text{if } X_{jmn}^{k*} > 0 \quad (3.15)$$

$$\leq v_j \lambda_m^k \quad \text{if } X_{jmn}^{k*} = 0 \quad (3.16)$$

Where $z_{P1_{X_{jmn}}}^{k'}(X_{jmn}^{k*})$ is the first derivative of z_{P1}^k with respect to X_{jmn} evaluated at the optimal solution of X_{jmn}^{k*} .

Proof. Proof follows directly from Patriksson (2008). □

Gibb's Lemma provides interesting insights in some properties of an optimal solution. It gives a necessary condition that can be used to develop efficient algorithms. Equations (3.15) and (3.16) state a condition that divides the decisions variables into two subsets: one that has all variables pegged to zero at the optimal solution, and other with all

variables strictly positive at the optimal solution. We exploit this property in two ways: first by formulating a problem whose optimal solution is a sparse matrix (specially for large problems), and then by using a simple procedure to find the optimal value of λ_m^k .

We now present some results for solving a single period problem. We then use these results to develop an algorithm for multi period problem. Consider the production decision for some period m and assume that the production quantities for periods $l \neq m$ are fixed at some feasible values. Let X_{jmn}^{k*} be the optimal production plan, then one of the following two cases must be true for each period $m = \{1, \dots, T\}$.

Case-1: Capacity constraint is not binding at optimal solution

In this case constraint (3.13) is a strict inequality for period m i.e., $\sum_{j=1}^J \sum_{n=m}^T v_j X_{jmn}^{k*} < b_m$. This problem can be easily solved by setting the first derivatives $z_{P1X_{jmn}}^{k'} = 0$ and solving for X_{jmn}^{k*} . The optimal solution is

$$X_{jmn}^{k*} = \frac{1}{2} \left(\frac{A_{jmn}}{B_{jn}} - \sum_{l=1, l \neq m}^n X_{jln}^k \right) \quad (3.17)$$

Case-2: Capacity constraint is binding at optimal solution

Let L_m be the number of decision variables for period m . Then it follows from lemma 2 that there exists l_m^* , $1 \leq l_m^* \leq L_m$ such that

$$X_{jmn}^{k*} > 0, \text{ and } z_{P1X_{jmn}}^{k'} = v_j \lambda_m^k(l_m^*) \quad \text{for } l_m = \{1, \dots, l_m^*\}, \text{ and} \quad (3.18)$$

$$X_{jmn}^{k*} = 0, \text{ and } z_{P1X_{jmn}}^{k'} \leq v_j \lambda_m^k(l_m^*) \quad \text{for } l_m = \{l_m^* + 1, \dots, L_m\} \quad (3.19)$$

for each period m , let $N_m^{k+} = \{X_{jmn}^k : X_{jmn}^{k*} > 0, (j, m, n) \in N^k\}$ be the set of l_m^* variables in (3.18) and $N_m^{k0} = \{X_{jmn}^k : X_{jmn}^{k*} = 0, (j, m, n) \in N^k\}$. We write X_{jmn}^{k*} as function of $\lambda_m^k(l_m^*)$

$$X_{jmn}^{k*} = \begin{cases} \left(z_{P1X_{jmn}}^{k'}(0) - v_j \lambda_m^k(l_m^*) \right) / 2B_{jn} & \text{if } X_{jmn}^k \in N_m^{k+} \\ 0 & \text{if } X_{jmn}^k \in N_m^{k0} \end{cases} \quad (3.20)$$

The capacity constraint for period m is

$$\sum_{X_{jmn}^k \in N_m^{k+}} v_j X_{jmn}^{k*} = b_m \quad (3.21)$$

We substitute (3.20) in (3.21) and solve for $\lambda_m^k(l_m^*)$

$$\lambda_m^k(l_m^*) = \frac{\sum_{X_{jmn}^k \in N_m^{k+}} (v_j z_{P1X_{jmn}}^{k'}(0) / B_{jn}) - 2b_m}{\sum_{X_{jmn}^k \in N_m^{k+}} v_j^2 / B_{jn}} \quad (3.22)$$

Clearly, the problem reduces to that of finding the index l_m^* . The objective z_{P1}^k is strictly concave; its derivative $z_{P1}^{k'}$ is decreasing in X_{jmn}^k , and there is only one equality constraint. Efficient algorithms exist for solving this single resource allocation problem (Zipkin, 1980). We use the ranking method of Luss and Gupta (1975) to find the index l_m^* and $\lambda_m^k(l_m^*)$.

Hence, we conclude that, in either case, we can solve the single period problem without much computational effort. We now propose a recursive algorithm (MRA) to solve the multi-period problem and show that it finds an optimal solution for $P1^k$.

3.5.1 Algorithm MRA.

MRA is a recursive procedure that starts with an initial feasible solution, generates an increasing sequence of objective values, and terminates at the optimal solution. During each iteration of the procedure, we solve T single period problems to allocate capacity to the production variables. Each allocation decision is either case-1 or case-2, for either case we can get the optimal capacity allocation as described earlier. The procedure maintains primal feasibility, while solving single period problems, and changes the existing solution only if it improves the primal objective. At the end of each iteration, we compare current objective value with objective value of the previous iteration. We start the next iteration if the current value is higher than the previous value, else we stop and report the current solution.

We present the pseudocode of MRA in Algorithm - 1. We initialize the procedure with all decision variables and objective value equal to zero (lines 1 -2). We then initialize the iteration counter $s = 1$ (line 3). For each period m we first check if the unconstrained solution violates the capacity constraint or not. If the solution is feasible, we accept it and move to period $m + 1$ (lines 7- 8). Else, case-2 holds true, and the capacity constraint is binding for period m at the optimal solution. We rank the derivatives $z'_{P1^{ks}}(0)$, if it exists, in descending order (line 10) and index them as $1, \dots, L_m$. Next, for each $l_m \in \{1, \dots, L_m\}$ we compute $\lambda_m^k(l_m)$ from (3.22). Then we find the smallest $l_m = l_m^*$ that satisfies condition (3.16) of lemma- 2 (lines 11-21). We compute λ_m^k , update current solution (lines 22 - 25), and increment $m = m + 1 \leq T$. Each iteration s comprises of solving T single period problems. At the end of each iteration, we compare the current objective value with the objective value of the last iteration. If the current value is higher than the previous value, we increment $s = s + 1$ and start the next iteration (lines 28 - 30). If the objective value is unchanged, current solution is the optimal solution. We update the solution and terminate the procedure (lines 31 - 35).

Algorithm 1 *MRA*

```
1:  $X_{jmn}^{k0} \leftarrow 0 \forall j = \{1, \dots, J\}, m = \{1, \dots, T\}, \&n = \{m, \dots, T\}$ 
2:  $z_{P1}^{k0} \leftarrow 0$ 
3:  $s \leftarrow 1$ 
4: for  $m = 1, \dots, T$  do
5:   if unconstrained solution is feasible for  $P1^k$  then
6:     compute  $X_{jmn}^{k*}$  from (3.17)
7:      $X_{jmn}^{ks} \leftarrow X_{jmn}^{k*} \forall j = \{1, \dots, J\}, \&n = \{m, \dots, T\}$ 
8:      $X_{jln}^{ks} \leftarrow X_{jln}^{ks-1} \forall j = \{1, \dots, J\}, l \neq m, \&n = \{l, \dots, T\}$ 
9:   else
10:    Rank derivatives  $z_{P1}^{\prime ks}(0)$  w.r.t.  $X_{jmn}^k$  in descending order  $\forall j = \{1, \dots, J\}, \&n = \{m, \dots, T\}$ 
11:    Let  $l_m = 1$  and go to 15
12:    if  $l_m = L_m$  then
13:      stop,  $l_m^* = L_m$ 
14:    end if
15:    compute  $\lambda_m^k(l_m)$  from (3.22)
16:    if  $z_{l_{m+1}}^{\prime ks} > \lambda_m^k(l_m)$  then
17:       $l_m \leftarrow l_m + 1$ 
18:      goto 12
19:    else
20:       $l_m^* \leftarrow l_m$ 
21:    end if
22:     $\lambda_m^k(l_m^*) \leftarrow \lambda_m^k(l_m)$ 
23:    compute  $X_{jmn}^{k*}$  from (3.20)
24:     $X_{jmn}^{ks} \leftarrow X_{jmn}^{k*} \forall j = \{1, \dots, J\}, \&n = \{m, \dots, T\}$ 
25:     $X_{jln}^{ks} \leftarrow X_{jln}^{ks-1} \forall j = \{1, \dots, J\}, l \neq m, \&n = \{l, \dots, T\}$ 
26:  end if
27: end for
28: if  $z_{P1}^{ks} > z_{P1}^{ks-1}$  then
29:    $s = s + 1$ 
30:   go to 4
31: else
32:    $X_{jmn}^{k*} \leftarrow X_{jmn}^{ks} \forall (jmn) \in N$ 
33:   Compute  $P_{jn}^k$  from (3.11)
34: end if
35: return  $z_{P1}^k, P_{jn}^k,$  and  $S_{jn}^k$ 
```

Lemma 3. *For any problem instance of case-2, $\lambda_m^k(l_m^*) \geq 0$ and it is bounded from above by a finite maximum.*

Theorem 2. *The solution from Algorithm MRA is optimal for problem $P1^k$. Furthermore, $\lambda_m(l_m^*)$ are in fact the optimal Lagrange multipliers for the capacity constraints (3.13).*

Lemma 4. *The sequence of nondecreasing objectives in Algorithm MRA converges asymptotically to optimal solution.*

3.6 Reduced Master Problem

The master problem is an alternate formulation whose optimal solution value is also optimal for P. We formulate the master problem by projecting the continuous variables to the discrete space and then identifying suitable continuous points for outer approximation. In other words, for every feasible set of binary variables Y_{jt}^k we solve the problem $P1^k$ and use all these solutions for outer approximation. Duran and Grossmann (1986) show that the resulting master problem and the problem P have the same optimal solution. Notice that the master problem requires predetermination of finite but a large number of outer approximations. Obviously, solving this problem is inefficient. Therefore, instead of solving the master problem, we solve its relaxation - also called the reduced master problem. We get this relaxation by progressively introducing additional cuts at each iteration. Each iteration corresponds to a sequence of production setups. For each given sequence; we solve P, and use its solution to generate cuts for the next iteration of reduced master problem. The iterative process must not cycle i.e. once a sequence of setups has been used for cut generation it must not repeat in subsequent iterations. Hereunder, we formulate the reduced master problem at iteration k :

$$(M^k) \quad \max z_M^k = \pi - \sum_{j=1}^J \sum_{t=1}^T a_{jt} Y_{jt}$$

Subject to:

$$\sum_{j=1}^J \sum_{t=1}^T (P_{jt}^i S_{jt} + S_{jt}^i P_{jt} - P_{jt}^i S_{jt}^i - c_{jt} X_{jt} - h_{jt} I_{jt}) \geq \pi \quad \forall i \in V^k \quad (3.23)$$

$$S_{jt} - \gamma_{jt} (\alpha_j - \beta_j P_{jt}) \leq 0 \quad \forall j, t \quad (3.24)$$

$$\sum_{j=1}^J v_j X_{jt} - b_t \leq 0 \quad \forall t \quad (3.25)$$

$$S_{jt} + I_{j,t} - I_{j,t-1} - X_{jt} = 0 \quad \forall j, t \quad (3.26)$$

$$v_j X_{jt} - b_t Y_{jt} \leq 0 \quad \forall j, t \quad (3.27)$$

$$P_{jt}, S_{jt}, X_{jt}, I_{jt} \geq 0, Y_{jt} \in \{0, 1\} \quad \forall j, t$$

Where $V^k = \{i : P_{jt}^i, S_{jt}^i \text{ are optimal to } P1(Y_{jt}^i), i = 0, 1, 2, \dots, k-1\}$.

The problem M^k is a mixed integer linear program that is solvable, for example, by standard branch-and-bound algorithm. Figure 3.2 gives a graphical depiction of outer approximation. The linearization cuts of the objective function are overestimators of actual objective value. An infinite number of such cuts will result in actual objective function. However, in the OA procedure, we add these cuts one by one. These cuts are added in the vicinity of the optimal solution only and thus keep the problem size manageable. Similarly, the feasible solution space is overestimated by adding linearization cuts of any nonlinear constraints. Therefore, the procedure can solve problems with nonlinear demand function as well. Each instance of problem M^k is a relaxation of problem P . Hence, solution value of the reduced master problem gives an upper bound for problem P . At each iteration k of the algorithm, we add new points to the set V^k and update problem M^k by including linearization cuts at each of the point in V^k . Therefore, solutions of M^k generate a non-increasing sequence of upper bounds.

We present the pseudocode of the solution procedure as Algorithm 2. The procedure

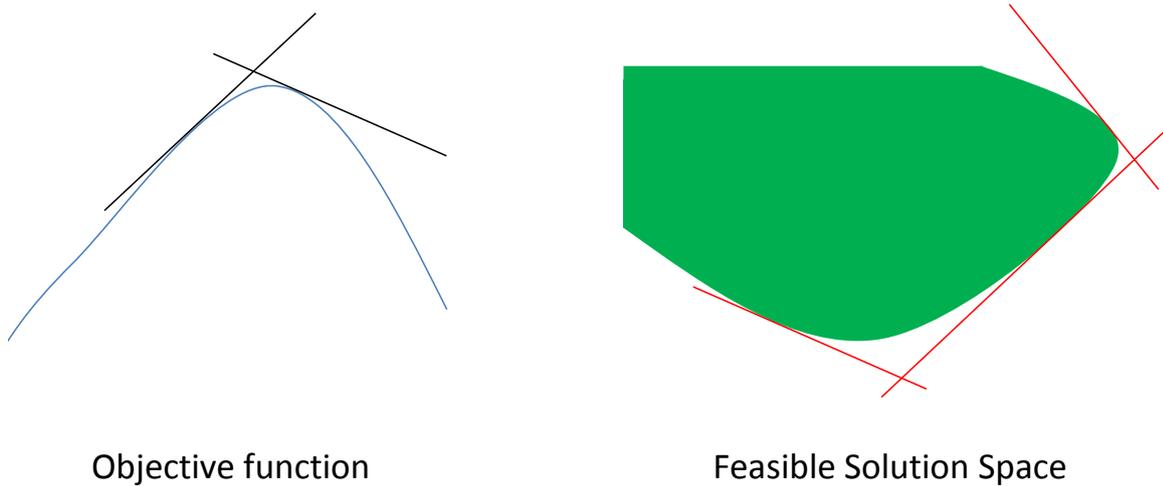


Figure 3.2: Graphical Depiction of Outer Approximation

Algorithm 2 *Outer Approximation*

1: Initialize

$z^+ \leftarrow \infty$. $k \leftarrow 0$. define ϵ .
 $Y_{jt}^0 \leftarrow 1$. $\forall j, t$
 Solve $P1^0$.
 $z^- \leftarrow z_{P1}^0$

2: Solve the reduced master problem

$k \leftarrow k + 1$
 Solve M^k .
 $z^+ \leftarrow \min(z^+, z_M^k)$

3: Terminate ?

If $z^+ - z^- < \epsilon$ Stop. current solution of M^k is optimal for P

4: Solve the primal problem

Solve $P1^k$. Let (P_{jt}^k, S_{jt}^k) be the optimal solution.
 $z^- \leftarrow \max(z^-, z_{P1}^k)$

5: Go to 2.

starts at $k = 0$ by fixing all $Y_{jt}^0 = 1$. This means that we incur a setup cost for each product in each period. Next, we set the upper bound equal to infinity, define a convergence tolerance (ϵ) and solve the primal problem. Let z_{P1}^0 be the objective value for this initial solution, then it is also a lower bound for problem P.

Next, we increment the iteration counter $k = k + 1$ and update the master problem by adding outer approximations at the last solution of $P1^k$. Let z_M^k be the solution value of the master problem at iteration k . At each iteration we update current upper bound $z^+ = \min(z^+, z_M^k)$, check for the termination criteria, and stop if $z^+ - z^- < \epsilon$. Else, we use the current value of Y_{jt}^k to update and then solve problem $P1^k$. We compare its solution with the current lower bound and update the lower bound. The algorithm proceeds iteratively until the lower and upper bounds are within the tolerance ϵ . Duran and Grossmann (1986) and Fletcher and Leyffer (1994) have shown that the OA algorithm can not cycle and terminates in a finite number of steps.

Observe that for dynamic pricing, at each iteration, we add JT number of cuts to the reduced master problem. Each cut corresponds to $P_{jt}^k \forall j, t$. The solution method is same for constant price strategy except that we need to add J number of cuts at each iteration of M^k , where each cut corresponds to P_j^k . Similarly, for a nonlinear demand function, at each iteration, we also add one linearization cut corresponding to each nonlinear constraint. Rest of the procedure is the same, and the algorithm generates two sequences of upper and lower bound that converge at the optimal solution.

Chapter 4

Computational Results and Managerial Insights

4.1 Properties of Optimal Solution

In this section, we first present some analytical results that provide useful insights on pricing decisions for any given production schedule. We then solve a pricing problem using real world data from a manufacturer of industrial gloves. We present computational results for different demand patterns and compare the firm's profitability for the two pricing strategies. We define the term 'Regeneration Interval' (RI), for a product j as a set of consecutive time periods such that the beginning inventory of j in the first period of RI, and the ending inventory of j in the last period of RI are zero. Therefore, all periods except the last period of an RI carry positive inventories. Notice that there may be one or more setups for product j within an RI.

Theorem 3. *The optimal dynamic prices increase strictly (monotonically) during a regeneration interval (RI) if the inventory carrying costs are strictly positive (non-negative), i.e.: $P_{jm}^* < P_{jm+1}^* < \dots < P_{js}^*$ where m is the first period of an RI and s is the last period of the RI. Furthermore, $P_{jn+1}^* - P_{jn}^* = h_{jn}/2$ for all $j = \{1, ..J\}$ and $n =$*

$\{m, m + 1, \dots, s - 1\}$.

Therefore the optimal prices within an RI depend upon the price in the first period and the inventory carrying cost of the product regardless of the seasonality factor. Notice that in case of multiple products an RI may have one or more than one production periods. Within an RI, the price in the first period is the lowest and the price in the last period is the highest. Furthermore, since the prices are strictly increasing in an RI, production plans with longer RIs would result in higher optimal prices and vice versa. Next, we present lemmas 5 and 6 that characterize the price patterns between two consecutive RIs and at different capacities for the same RIs.

Lemma 5. *Let m be the first period of production of product j in an RI and $r = m + s$ be the first period of production of j in the next RI. If there is no strategic incentive of holding inventory (i.e. $c_{jr} \leq c_{jm} + \sum t = m^{r-1}h_{jt}$), then the following must be true for P_{jr}^**

- i). $P_{jr}^* \leq P_{jn}^*$ for $n = m$
- ii). $P_{jr}^* < P_{jn}^*$ for $n = \{m + 1, \dots, r - 1\}$
- iii). $P_{jr-1}^* - P_{jr}^* \geq \frac{1}{2} \sum_{t=m}^{r-2} h_{jt}$

Lemma 6. *The optimal prices decrease monotonically in capacity for a given sequence of setups.*

Assuming that there is no strategic incentive for holding inventory, the optimal dynamic prices follow a saw tooth pattern where the prices are lowest at the beginning of each RI and highest at the end of the RI. Since the maximum price within an RI depends upon the length of the RI, the optimal prices will have a wide (narrow) range if the RIs are long (short). Lemma 5 and 6 together provide valuable insights in understanding how changes in capacity impact the optimal prices. According to lemma 6 we should

expect the optimal price curve to move down (up) if an increase (decrease) in capacity does not change the optimal sequence of setups. In this case the shape of the curve does not change with an increase or decrease in capacity. However, if the optimal solution at higher capacity has longer RIs, the optimal prices do not decrease monotonically in capacity. In fact the optimal prices at higher capacity may actually be higher than the prices at lower capacity.

4.2 Numerical Example

To demonstrate the application of the proposed algorithm, we use real world data from a company (hereinafter referred as TIS) that manufactures and sells gloves to institutional buyers. We collect sales and operational data for two types of gloves (hereinafter referred to as A and B) that are manufactured on the same production equipment. A is a high cost, high margin product whereas B is a low cost low margin product, and its demand is more sensitive to price changes than the demand for A. Note that the products here refer to a group of gloves, where each group may comprise of gloves of different colors and sizes. TIS manufactures these gloves for a variety of applications and may sell similar products in different markets at different prices. For example, two buyers, one from US and one from EU, may purchase similar products at prices that are independent of each other. Each buyer makes her decisions independently, and no two buyers share the pricing information with each other. Since most of the products have specific applications and are sold in different markets, the assumption of zero cross price elasticities is a fair assumption for this company's product portfolio. Currently, TIS management follows a constant pricing strategy, where; each product manager sets her product prices independently, and then communicates the product demand to operations. They realize that such sequential decision making often results in sub-optimal profits, but do not have a method for optimal allocation of production capacity to different products. Moreover,

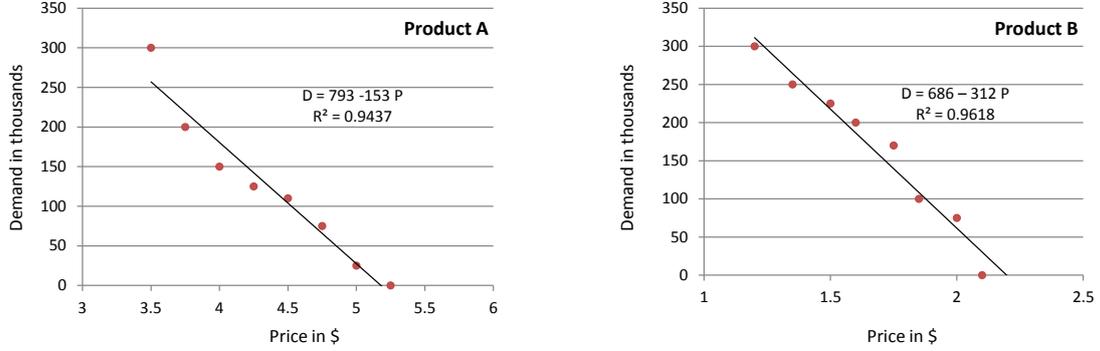


Figure 4.1: Demand Curves for the Two Products.

Table 4.1: Parameters for the Problem.

Products(j)	α_j	β_j	a_{jt}	v_j	h_{jt}	c_{jt}
A	793	153	7500	0.86	0.043	2.85
B	686	312	2000	0.60	0.017	1.10

the management wonders if it will be worthwhile for them to use dynamic pricing?

We collect the recent sale and pricing data of both the products and plot it in Figure 4.1. For each product, we use a linear relationship to approximate demand as function of its price. We summarize different problem parameters in Table 4.1, note that the variable, inventory, and setup costs do not change over the planning horizon for this particular data set.

We examine four scenarios for demand variations: i) demand for both the products is stationary, ii) demand for both the products is increasing and peaks at the end of the season, iii) demand for both the products is decreasing and peaks at the beginning of the season, and iv) demand for one of the product peaks at the beginning of the season while the demand for the other product peaks at the end of the season. Table 4.2 presents the values of seasonal factors γ_{jt} for the four scenarios, and Figure 4.2 presents a graphical description of these scenarios. We keep capacity constant over the planning horizon, although at different levels (*i.e.* 30, 40, 50, 60, & 70). We create problem instances for each

Table 4.2: Seasonal Demand Factors for Scenarios 1-4

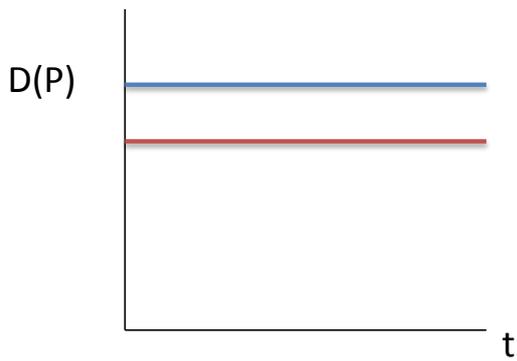
Scenario	j	t					
		1	2	3	4	5	6
1	A	0.167	0.167	0.167	0.167	0.167	0.167
	B	0.167	0.167	0.167	0.167	0.167	0.167
2	A	0.1	0.1	0.1	0.2	0.2	0.3
	B	0.1	0.1	0.1	0.2	0.2	0.3
3	A	0.3	0.2	0.2	0.1	0.1	0.1
	B	0.3	0.2	0.2	0.1	0.1	0.1
4	A	0.1	0.1	0.1	0.2	0.2	0.3
	B	0.3	0.2	0.2	0.1	0.1	0.1

demand scenario at different capacities and solve for optimal dynamic as well as constant prices.

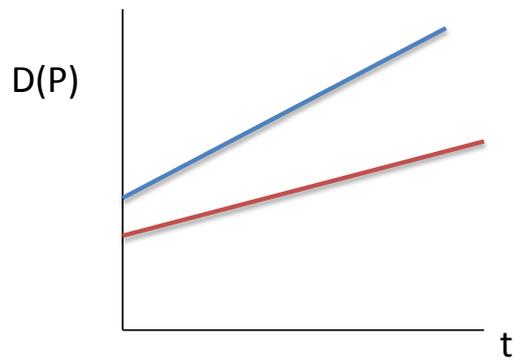
We present the optimal dynamic prices for the four scenarios in Figures 4.3 to 4.6. At first, the prices seem to follow unpredictable patterns. However, a close examination shows the price changes are explained well by theorem 3 and lemmas 5 & 6. For each of the product, the prices rise with constant slope during each RI followed by a price decrease in the first period of the next RI.

In Scenario 2 (Figure 4.4) the demand peaks at the end of the season and the optimal solutions at different capacities are characterized by few RIs spanning over multiple periods. Therefore, the optimal prices move in a narrow range. Whereas in Scenario 3 (Figure 4.5), the optimal solutions are characterized by RIs of shorter lengths. Hence, the prices move in a wider range.

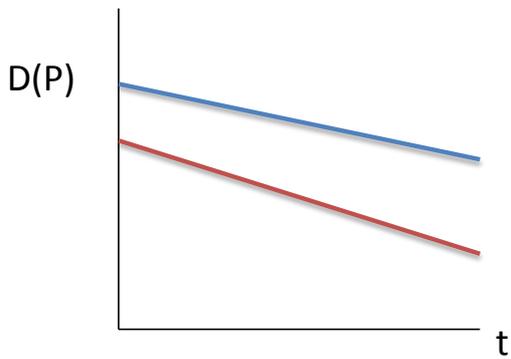
The two parameters that influence the optimal lengths of RIs are the capacity and seasonal demand factors. In general, one should expect longer RIs at higher capacities and hence lesser variability in prices. Similarly, one should expect shorter RIs and higher price variability when the demand peak is in the beginning of the planning horizon. We



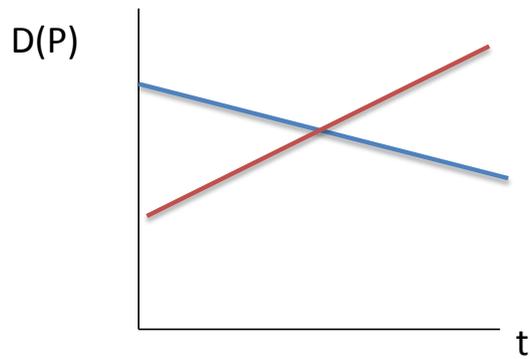
(1) Uniform Demands



(2) Demands peak in the end of planning horizon



(3) Demands peak in the beginning of planning horizon



(4) One product has peak demand in the beginning and the other in the end

Figure 4.2: Four Demand Scenarios

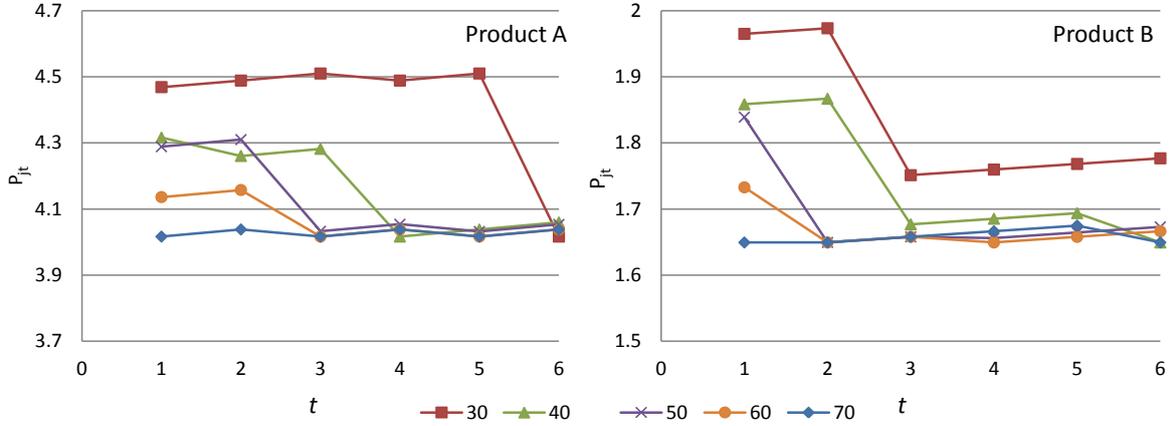


Figure 4.3: Optimal Dynamic Prices for Scenario 1

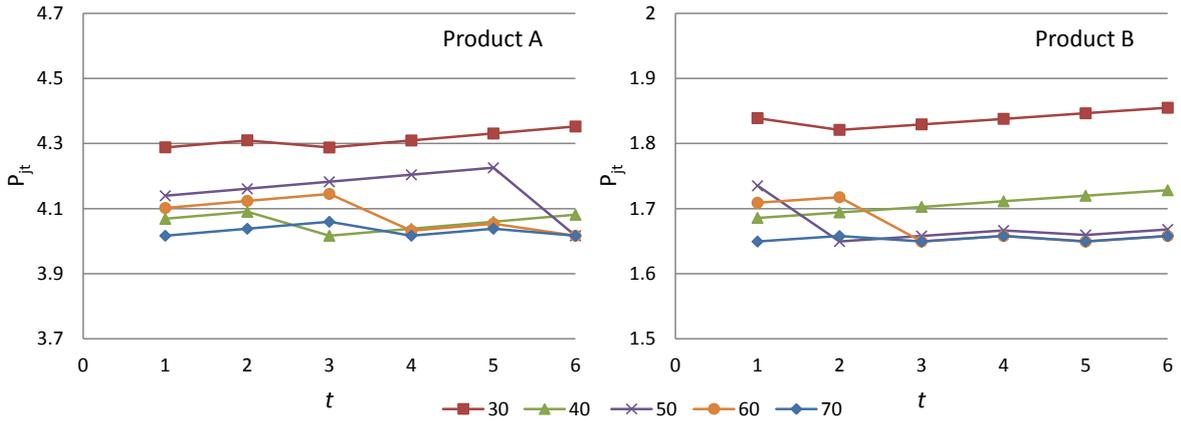


Figure 4.4: Optimal Dynamic Prices for Scenario 2

illustrate this pattern in Figure 4.6. The demand for product B is highest in the beginning of the planning horizon. We observe a high variability in prices of product B at lower capacities. Whereas, the prices tend to move in a narrow range at higher capacities.

We solve the same problem instances for constant pricing strategy and summarize the optimal prices in Figure 4.7. Since A is a high margin product and its demand is less sensitive to price changes, we observe a high variation in its price across different demand scenarios. Whereas B has lower margins and its demand is more sensitive to price changes, hence its optimal price does not vary much with changes in seasonality. Observe that the optimal constant prices do not follow a monotonic relationship with

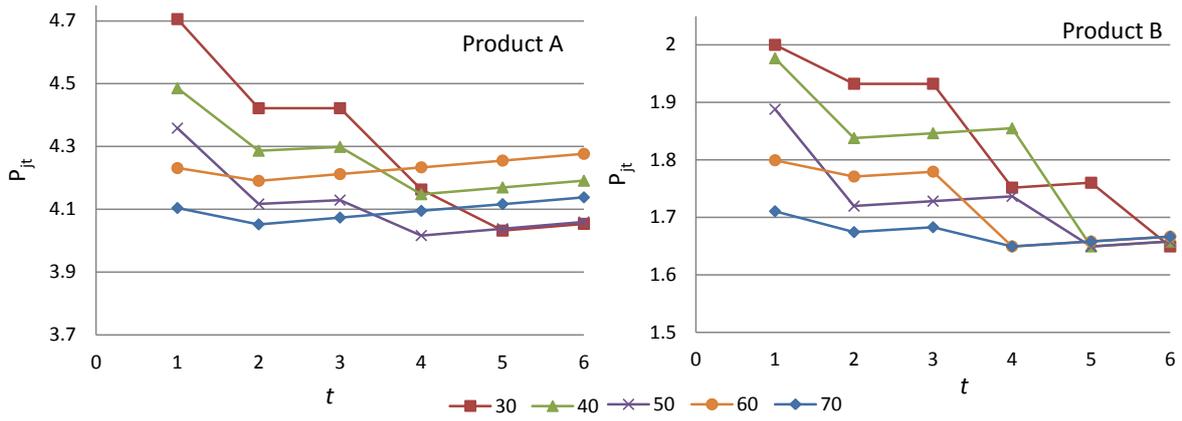


Figure 4.5: Optimal Dynamic Prices for Scenario 3

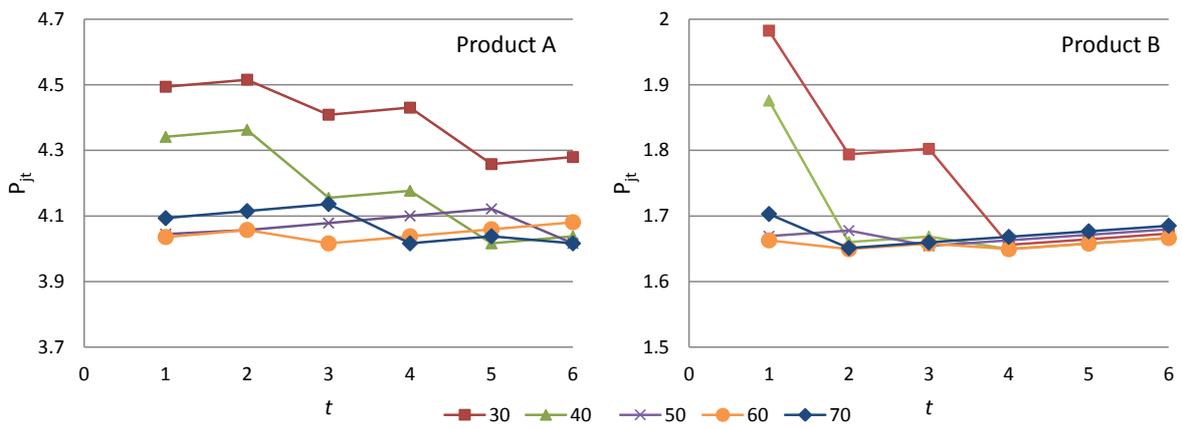


Figure 4.6: Optimal Dynamic Prices for Scenario 4

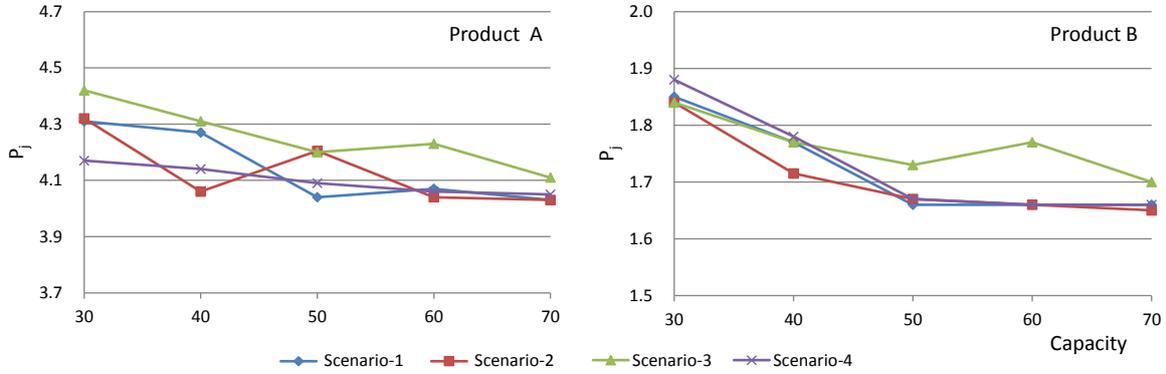


Figure 4.7: Optimal Constant Prices

capacity. These results complement the conclusions of Gilbert (2000) who shows, in the absence of setup costs, that the optimal prices for multiple products are monotonically decreasing in capacity.

In this section, we compare the optimal profits for the dynamic and constant pricing strategies. We assume, we can change the price of a product from period to period at negligible or no additional cost. Although this assumption may not be true in some situations yet it is predominantly used in dynamic pricing literature (Chen and Hu, 2012). There can be situations where frequent price changes are not feasible due to the associated costs (Netessine, 2006). Some of these costs may be indirect or implicit costs that are difficult to quantify. We contrast the profitability of the two strategies without explicitly incorporating the cost of price changes. The results provide a general framework that may be used by TIS to identify situations for which constant pricing is better and others for which dynamic pricing is more suitable.

We summarize the results in Table 4.3, where we show the optimal profits for each demand scenario at varying capacities. We also record the percentage improvement in profit that can be achieved with dynamic pricing. The results show that dynamically changing prices is valuable for the firm only at lower capacities and/or when the demands follow either scenario-3 or scenario-4. It is just marginally better for scenarios 1 and 2. These results are also explainable by the properties of the optimal solution. The dynamic

Table 4.3: Optimal Profits for the Two Pricing Strategies

Capacity	Scenario-1			Scenario-2			Scenario-3			Scenario-4		
	CP	DP	% Imp									
30	219.5	223.3	1.7%	235.5	235.7	0.08%	187.8	199.5	6.2%	219.0	230.4	5.2%
40	248.1	251.0	1.2%	258.3	258.5	0.08%	221.6	230.0	3.8%	245.8	253.4	3.1%
50	261.5	262.5	0.4%	262.6	264.5	0.72%	239.1	249.1	4.2%	258.7	261.0	0.9%
60	265.7	267.0	0.5%	267.7	267.7	0.00%	255.4	257.9	1.0%	265.8	266.4	0.2%
70	268.1	268.1	0.0%	268.3	268.3	0.00%	265.9	266.8	0.3%	268.3	268.4	0.0%

Note: DP = Dynamic Price, CP = Constant Price, % Imp = percentage improvement

prices vary in a narrow range at higher capacities and when demand follows scenario 1 or 2. Hence, these will be quite close to optimal constant price and therefore, the profits for the two strategies will not be greatly different. However, in scenarios 3 or 4 and especially at lower capacities the optimal solution comprises of shorter RIs. Hence the dynamic prices vary over a wider range and generate substantially higher profits.

4.3 Comparison with Uncoordinated Approach

In this section, for the sake of completeness, we compare the profitability between coordinated and uncoordinated decision approaches. We use the dynamic pricing strategy for the former approach and a sequential decision making process for the later. Figures 4.8 and 4.9 present the results, as absolute profit and as percent improvement in profit, computed at different capacities for each of the scenarios. Clearly, coordinated decisions generate higher profits for all the given scenarios; difference in profitability is significantly higher at low capacities (average 17.7% for 30) and is relatively higher for higher capacities (average 5% for 70). Similarly, the benefit is much higher when demand peaks in the beginning of the season.

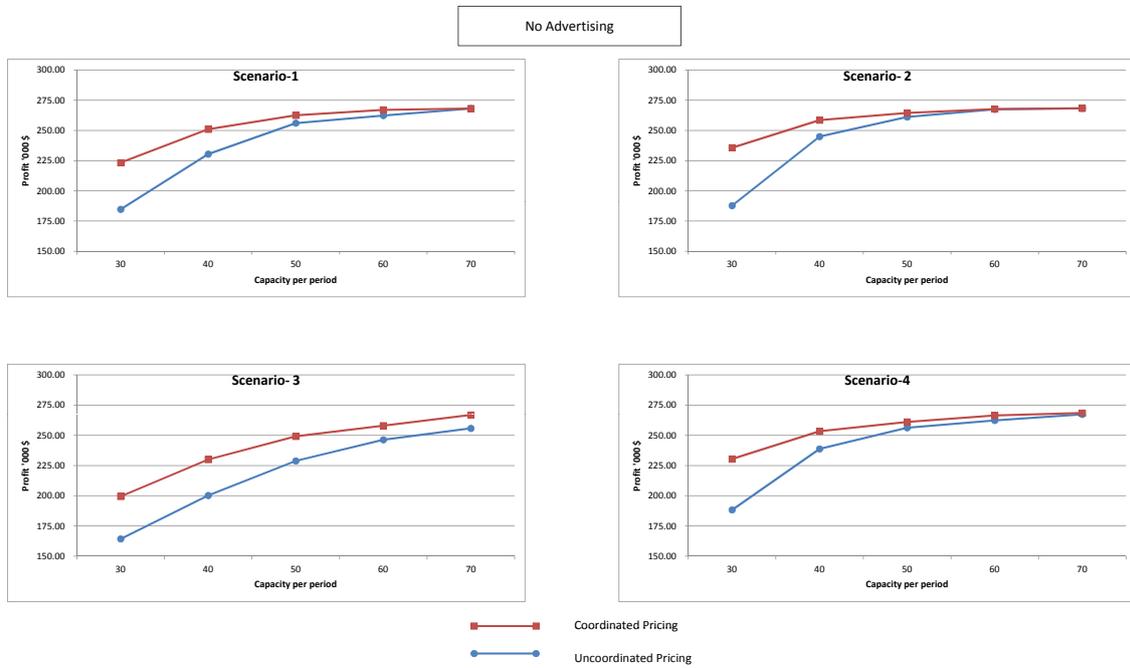


Figure 4.8: Profit with and without Price Coordination

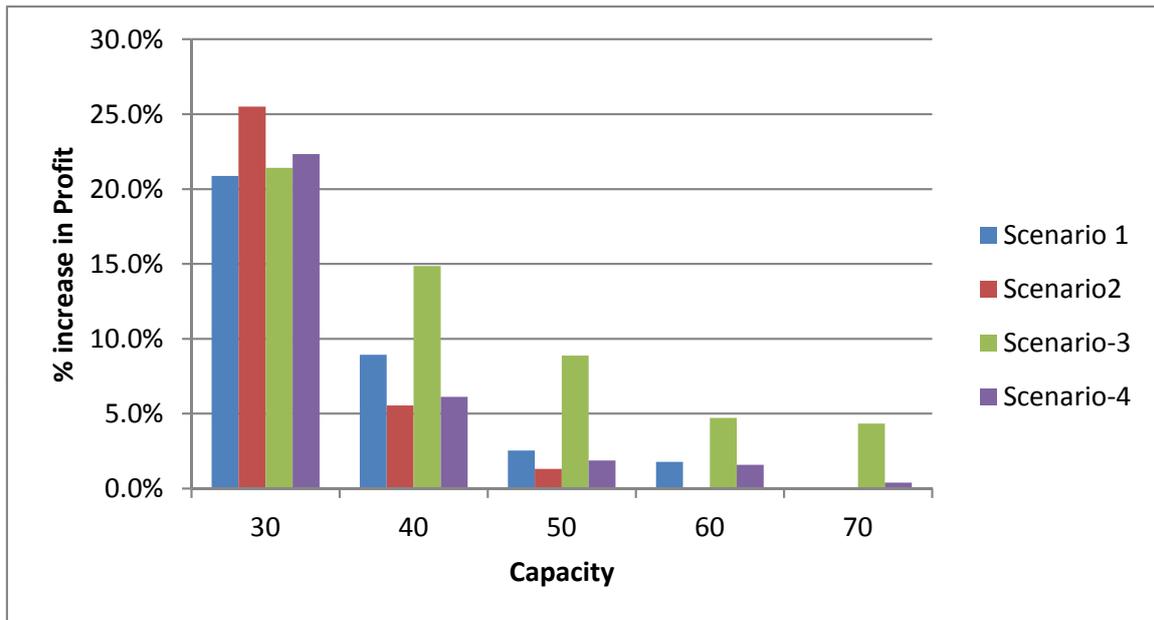


Figure 4.9: Percent Increase in Profit due to Price Coordination

Chapter 5

Joint Advertising, Pricing, and Production Decisions

In this chapter, we consider the situation where, in addition to price, marketing may influence demand by advertising a product. We assume demand to be a function of price and dollars spent on advertising. All other problem characteristics, as described in Chapter-3, remain the same. Since we are interested in knowing the aggregate demand, we do not consider the individual marketing activities or plans that are by the marketing manager. We assume that the manager can estimate aggregate effect of all the activities on demand. This assumption is quite realistic, and all the models discussed in Section 2.5 make similar assumptions. In practice, marketing managers receive a share of advertising budget for each brand or product, which they can use to manage different marketing activities over a given period. Thus, the decision variable is the allocation of a given budget to individual products or brands. This decision problem is of practical significance as it integrates two main marketing decisions with operational plans. Almost all manufacturing firms that advertise their own brands need to make these decisions. The impact of advertising is two fold as it influences revenue as well as the total cost. The logic for coordination of advertising is similar to that of coordinating pricing. Either variable

influences market demand, which in turn, drives production decisions. A marketing plan may induce demand that either exceeds current production capacity or may be fulfilled at a higher cost, for example, by overtime. Similarly, uncoordinated advertising plans for two products that use same production capacity may result in higher production costs and lower profits. Over-advertising is another related issue. Aaker and Carman (1982) points out that marketing managers may tend to over-spend on advertising and firms may improve profitability by reducing advertising expenses. Hence, optimal advertising decisions may be essential for firms profitability. Most of the academic literature on optimal advertising consider advertising as a stand alone decision, some incorporate pricing as a decision variable in their models, but they ignore capacity constraints. To the best of our knowledge no one has presented a decision model that incorporates pricing and advertising decisions along with capacity constraints and setup costs.

5.1 Defining Advertising

We define, for this dissertation, advertising expenditure as the amount of money spent on all marketing activities that communicate product attributes and (or) performance characteristics. Such activities may include media spending, market activation, sales force efforts etc. Advertising, here, refers to the combined marketing expenditure on all such activities measured in money terms. The net effect of such spendings is an increase in demand for a given price. We call this change in demand as ‘advertising effect’. Similarly, we call change in demand due to a change in price at given advertising expenditure as ‘price effect’. Note that this definition of advertising effect does not include promotions or temporary price cuts of any form. We assume that such activities result in a net price reduction and are, in fact, included in the price effect.

5.1.1 Demand as Function of Price and Advertising

As discussed in Section 2.5 many researchers have proposed a number of sales response models and there is no clear agreement on how advertising effects sales. We believe the sales response to advertising should: i) depict diminishing returns of advertising, ii) must have positive sales at zero advertising for some feasible price, and iii) should attain a finite sales volume at all possible advertising levels. Specifically, we model demand as:

$$D_{jt} = \alpha_j - \beta_j P_{jt} + \delta_j W_{jt}^r \quad (5.1)$$

Where $D_{jt} \geq 0$, $P_{jt} \geq 0$, and $W_{jt} \geq 0$ are the demand, price, and advertising spend for product j in period t , and $\alpha_j > 0$, $\beta_j > 0$, $\delta_j > 0$, and $0 < r < 1$ are parameters. Demand in (5.1) is separable in advertising and price effects, hence ensuring for each (j, t) there is positive demand for $0 \leq P_{jt} < \alpha_j/\beta_j$ and $W_{jt} = 0$. Parameter $0 < r < 1$ implies diminishing returns to advertising and that the marginal return on advertising approaches to zero for very large values of W_{jt} . The total demand is an additive function of price, and advertising, we prefer not to use a multiplicative form as such a form implies zero sales unless the product is advertised.

This functional form of demand assumes the total sale is the sum of price effect and advertising effect and that there is no interaction between the two effects (Figure 5.1). There are two underlying assumptions behind this demand model: i) Customers are price sensitive, and they make their purchase decisions based on current price only, ii) Advertising results in an increased awareness of product availability and its attributes. The potential market size increases as more consumers become aware of product availability. Communication about product quality and other attributes generates trial and may increase customer willingness to pay. The net result is the sum of the two effects. Note that there is no explicit carryover effect of advertising from period to period. This may seem as a simplifying assumption, and in some sense it is. However, the carryover effect

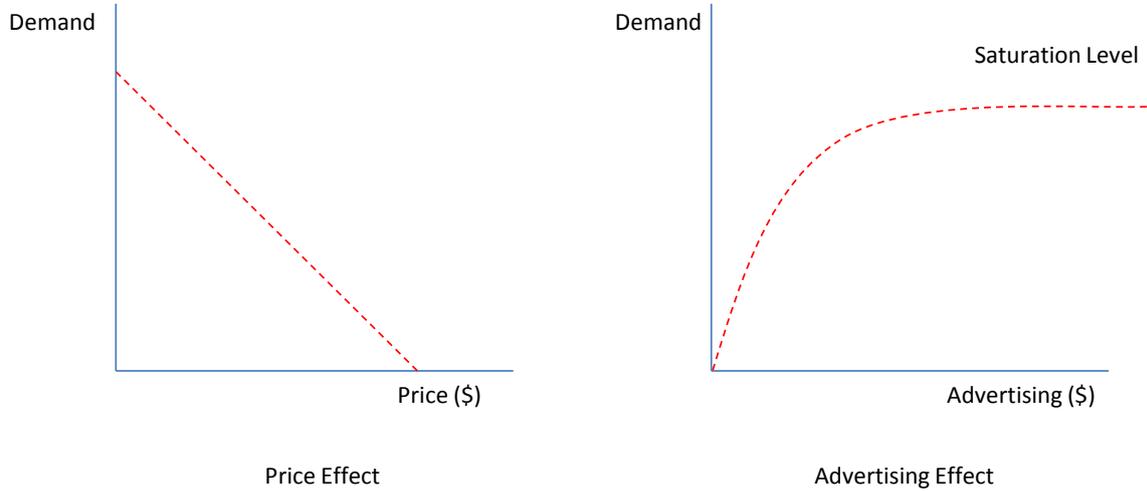


Figure 5.1: Price and Advertising effects

of advertising may implicitly be included by updating the parameters between planning horizons.

5.2 Mathematical Model for Price, Advertising, and Production Decisions

Consider a multi-product (index j) and multi-period (index t) problem of a manufacturer with limited production capacity. He has to make following decisions:

- Production Decisions: Production quantities (X_{jt}) and setups (Y_{jt})
- Marketing Decisions: Selling prices (P_{jt}) and advertising expenditure (W_{jt})

Traditionally, these decisions are made in isolation. While there is a growing trend of coordinating pricing decisions, allocation of advertising budget and execution of advertising are seldom coordinated with operational decisions. We propose a coordinated decision approach for price, advertising, and production. Hereunder, we present a mathematical model for the problem (hereinafter referred to as Q), and then propose a solution procedure. Problem Q is a mixed integer nonlinear program, and its structure is similar

to problem P as described in sections 3.1 and 3.3. Additional decision variables W_{jt} are introduced in the objective function of Q. Each product's demand is a function of price as well as advertising, and we have an additional constraint so that total advertising expenditure may not exceed the advertising budget (Ω). We assume that the firm has already decided the maximum advertising budget, and the problem at hand is to find optimal allocation to different products and periods. However, the proposed model and solution procedure may also be used for finding optimal advertising budget after dropping the budget constraint from Q. Now we formally state problem Q hereunder:

$$(Q) \quad \max z_Q = \sum_{j=1}^J \sum_{t=1}^T S_{jt}P_{jt} - c_{jt}X_{jt} - h_{jt}I_{jt} - a_{jt}Y_{jt} - W_{jt}$$

Subject to:

$$S_{jt} - \gamma_{jt}(\alpha_j - \beta_j P_{jt} + \delta_j W_{jt}^r) \leq 0 \quad \forall j, t \quad (5.2)$$

$$\sum_{j=1}^J v_j X_{jt} - b_t \leq 0 \quad \forall t \quad (5.3)$$

$$S_{jt} + I_{j,t} - X_{jt} = 0 \quad \forall j, t = 1 \quad (5.4)$$

$$S_{jt} - I_{j,t-1} - X_{jt} = 0 \quad \forall j, t = T \quad (5.5)$$

$$S_{jt} + I_{j,t} - I_{j,t-1} - X_{jt} = 0 \quad \forall j, t = \{2, \dots, T-1\} \quad (5.6)$$

$$v_j X_{jt} - b_t Y_{jt} \leq 0 \quad \forall j, t \quad (5.7)$$

$$\sum_{t=1}^T \sum_{j=1}^J W_{jt} - \Omega \leq 0 \quad (5.8)$$

$$S_{jt}, P_{jt}, I_{jt}, X_{jt}, W_{jt} \geq 0 \quad \forall j, t \quad (5.9)$$

$$Y_{jt} \in 0, 1 \quad \forall j, t \quad (5.10)$$

Constraint set (5.2) ensures that the sales quantity for each product in each period may not exceed the demand for that product in that period. Constraint set (5.3) limits the total quantity produced in a period by the production capacity. Constraint sets (5.4 - 5.6) are the inventory balance constraints, constraint set (5.7) ensures that a setup cost is incurred whenever a product is produced, and constraint (5.8) is the advertising budget constraint. Lastly, constraints (5.9) and (5.10) are the non-negativity and binary variable constraints respectively.

5.3 Solution Approach

In this section we extend the solution approach of Section 3.4 and present an algorithm that solves problem Q to optimality. Observe that algorithm OA solves Q if variables W_{jt} are fixed to some feasible values. We exploit this structure of the problem by defining and solving three sub-problems: QP, QI, and QM. Each sub-problem is obtained by fixing some of the decision variables and then solving for the others. These sub-problems are solved iteratively until a convergence criteria is met. We show that the solution thus obtained is also optimal for Q.

5.3.1 Problem QP

Problem QP is a restriction of Q, obtained by fixing variables Y_{jt} and W_{jt} at some feasible values. We define sets of indexes $R_t = (j, t) : Y_{jt} = 1$ for $t = 1..T$. Now, let $\kappa_0 = \sum_{j=1}^J \sum_{t=1}^T (a_{jt}Y_{jt} + W_{jt})$ such that $\sum_{j=1}^J \sum_{t=1}^T W_{jt} \leq \Omega$. In other words κ_0 is the total fixed cost of setups and advertising for a given problem instance. Also, let $\bar{\alpha}_j = \alpha_j + \delta_j W_{jt}^r$, then we may write QP as:

$$(QP) \quad \max z_{QP} = \sum_{j=1}^J \sum_{t=1}^T (S_{jt}P_{jt} - c_{jt}X_{jt} - h_{jt}I_{jt}) - \kappa_0$$

Subject to:

$$S_{jt} - \gamma_{jt}(\bar{\alpha}_j - \beta_j P_{jt}) \leq 0 \quad \forall j, t \quad (5.11)$$

$$\sum_{(j,t) \in R_t} X_{jt} - b_t \leq 0 \quad \forall t \quad (5.12)$$

$$S_{jt} + I_{j,t} - X_{jt} = 0 \quad \forall j, t = 1 \quad (5.13)$$

$$S_{jt} - I_{j,t-1} - X_{jt} = 0 \quad \forall j, t = T \quad (5.14)$$

$$S_{jt} + I_{j,t} - I_{j,t-1} - X_{jt} = 0 \quad \forall j, t = \{2, \dots, T-1\} \quad (5.15)$$

$$S_{jt}, P_{jt}, I_{jt}, X_{jt} \geq 0 \quad \forall j, t \quad (5.16)$$

$$X_{jt} = 0 \quad \forall j, t \notin R_t, t = 1, \dots, T \quad (5.17)$$

5.3.2 Problem QI

We formulate the Intermediate problem (QI) by fixing the variables X_{jt} , I_{jt} , and Y_{jt} for problem Q. One advantage of fixing these variables is that all operational costs are also fixed and the only cost that enters the objective function is the advertising cost. Let κ_1 be the total cost of production, inventory, and setups, then we can write the intermediate problem as:

$$(QI) \quad \max z_{QI} = \sum_{j=1}^J \sum_{t=1}^T (S_{jt} P_{jt} - W_{jt}) - \kappa_1$$

Subject to:

$$\sum_{j=1}^J \sum_{t=1}^T W_{jt} - \Omega \leq 0 \quad (5.18)$$

$$S_{jt} - \gamma_{jt}(\alpha_j - \beta_j P_{jt} + \delta_j W_{jt}^r) \leq 0 \quad \forall j, t \quad (5.19)$$

$$W_{jt}, P_{jt} \geq 0 \quad \forall j, t$$

Lemma 7. *For an optimal solution of Problem QI, there exist some P_{jt}^* and W_{jt}^* for all j, t such that there is no lost sales at the optimal solution.*

According to Lemma 7 if at some feasible solution the demand generated is more than the quantity available for sale, it is possible to find a better solution by increasing price and (or) reducing advertising expenditure. Let q_{jt} be the quantity sold then we may write $P_{jt} = \frac{\alpha_j}{\beta_j} - \frac{q_{jt}}{\gamma_{jt}\beta_j} + \frac{\delta_j W_{jt}^r}{\beta_j}$ at the optimal solution. Now we rewrite QI as:

$$(QI) \quad \max z_{QI} = \sum_{j=1}^J \sum_{t=1}^T \left(\frac{q_{jt}\delta_j W_{jt}^r}{\beta_j} - W_{jt} \right) - \kappa_2 \quad (5.20)$$

Subject to:

$$\sum_{j=1}^J \sum_{t=1}^T W_{jt} - \Omega \leq 0 \quad (5.21)$$

$$W_{jt}, P_{jt} \geq 0 \quad \forall j, t \quad (5.22)$$

$$\text{Where } \kappa_2 = \kappa_1 + \sum_{j=1}^J \sum_{t=1}^T \left(\frac{\alpha_j q_{jt}}{\beta_j} - \frac{q_{jt}^2}{\gamma_{jt}\beta_j} \right)$$

Lemma 8. *The objective function z_{QI} is concave in advertising expenditure W_{jt} .*

Lemma 9. *Let W_{jt}^* be the optimal advertising expenditure for product j in period t , then there exists a unique solution of QI:*

$$W_{jt}^* = \begin{cases} \frac{K_{jt}}{\sum_{j=1}^J \sum_{t=1}^T K_{jt}} \Omega & \text{if constraint(5.21) is binding} \\ K_{jt} & \text{if constraint(5.21) is not binding} \end{cases} \quad (5.23)$$

$$\text{Where } K_{jt} = \left(\frac{q_{jt}\delta_j^r}{\beta_j} \right)^{\frac{1}{1-r}}$$

Lemma 10. Let $\eta = -\frac{P}{D} \frac{\partial D(P)}{\partial P}$ be the price elasticity of demand, and $\mu = \frac{\partial D(W)}{\partial W} P$ be the marginal revenue of advertising, then optimal solution of QI must satisfy following necessary conditions for all j, t :

$$(i.) \quad \mu_{jt} \geq \eta_{jt}, \quad (5.24)$$

$$(ii.) \quad \frac{\mu_{11}}{\eta_{11}} = \frac{\mu_{12}}{\eta_{12}} = \dots = \frac{\mu_{jt}}{\eta_{jt}} = \dots = \frac{\mu_{JT}}{\eta_{JT}}. \quad (5.25)$$

Condition (5.24) implies that, at a given solution, if the marginal revenue of advertising is less than the ordinary price elasticity, it is profitable for the firm to reduce advertising expenditure. Condition (5.25) implies that the marginal benefit of advertising dollars is equal for all W_{jt} at the optimal allocation of advertising budget.

5.3.3 Problem QM

Problem QM is a relaxation of Q. The objective function of QM is a linearization of objective function of Q. Since the objective function of Q is concave, its linearization is always an over estimation of the objective value for all feasible solutions. Observe that all feasible solutions of Q are also feasible for QM, hence its solution is an upper bound for Q. We formally define the problem QM hereunder.

$$(QM) \quad \max z_{QM} = \Delta - \sum_{j=1}^J \sum_{t=1}^T a_{jt} Y_{jt}$$

Subject to:

$$\sum_{j=1}^J \sum_{t=1}^T (P_{jt}^i S_{jt} + S_{jt}^i P_{jt} - P_{jt}^i S_{jt}^i - c_{jt} X_{jt} - h_{jt} I_{jt} - W_{jt}) \geq \Delta \quad \forall i \in G \quad (5.26)$$

$$S_{jt} - \gamma_{jt} (\alpha_j - \beta_j P_{jt} + \delta_{jt} W_{jt}^r) \leq 0 \quad \forall j, t \quad (5.27)$$

$$\sum_{j=1}^J v_j X_{jt} - b_t \leq 0 \quad \forall t \quad (5.28)$$

$$v_j X_{jt} - b_t Y_{jt} \leq 0 \quad \forall j, t \quad (5.29)$$

$$S_{jt} + I_{j,t} - I_{j,t-1} - X_{jt} = 0 \quad \forall j, t \quad (5.30)$$

$$P_{jt}, S_{jt}, X_{jt}, I_{jt} \geq 0, Y_{jt} \in \{0, 1\} \quad \forall j, t$$

$$\text{Where } G = \{i : P_{jt}^i, S_{jt}^i \text{ are optimal to } QI(Y_{jt}^i), i = 0, 1, 2, \dots\}. \quad (5.31)$$

The L.H.S of constraint set (5.26) is obtained by linearization of objective function of Problem Q. Observe that it is an upper bound on Δ such that $z_Q \leq z_{QM}$. Constraint set (5.27) ensures sales may not exceed demand, constraint set (5.28) ensures production does not exceed capacity, constraint set (5.29) imposes a fixed setup cost for each product and period with positive production, and lastly (5.30) are inventory balance constraints. The set G defined in (5.31) is the set of points that are used to define cuts in (5.26). Notice that the structure of problem QM is same as problem M^k . As discussed in Section 3.4, outer approximation is a suitable method for solving such problems.

Figure 5.2 gives an outline of the solution procedure and its pseudo code is presented as Algorithm 3. Let k be an index for iterations, z_Q^+ be an upper bound for solution of Q, and z_Q^- be a lower bound for solution of Q. We use the superscript k to denote the value of variables and objective functions at each iteration. At the initialization step, set $k = 0$, $z_Q^+ = \text{inf}$, $z_Q^- = 0$, and define a convergence tolerance (ϵ). Furthermore, fix the variables $Y_{jt}^0 = 1$, and $W_{jt}^0 = 0$ for all j, t . Thus, at the initialization step we assume a setup for each product in each period and that no product is advertised.

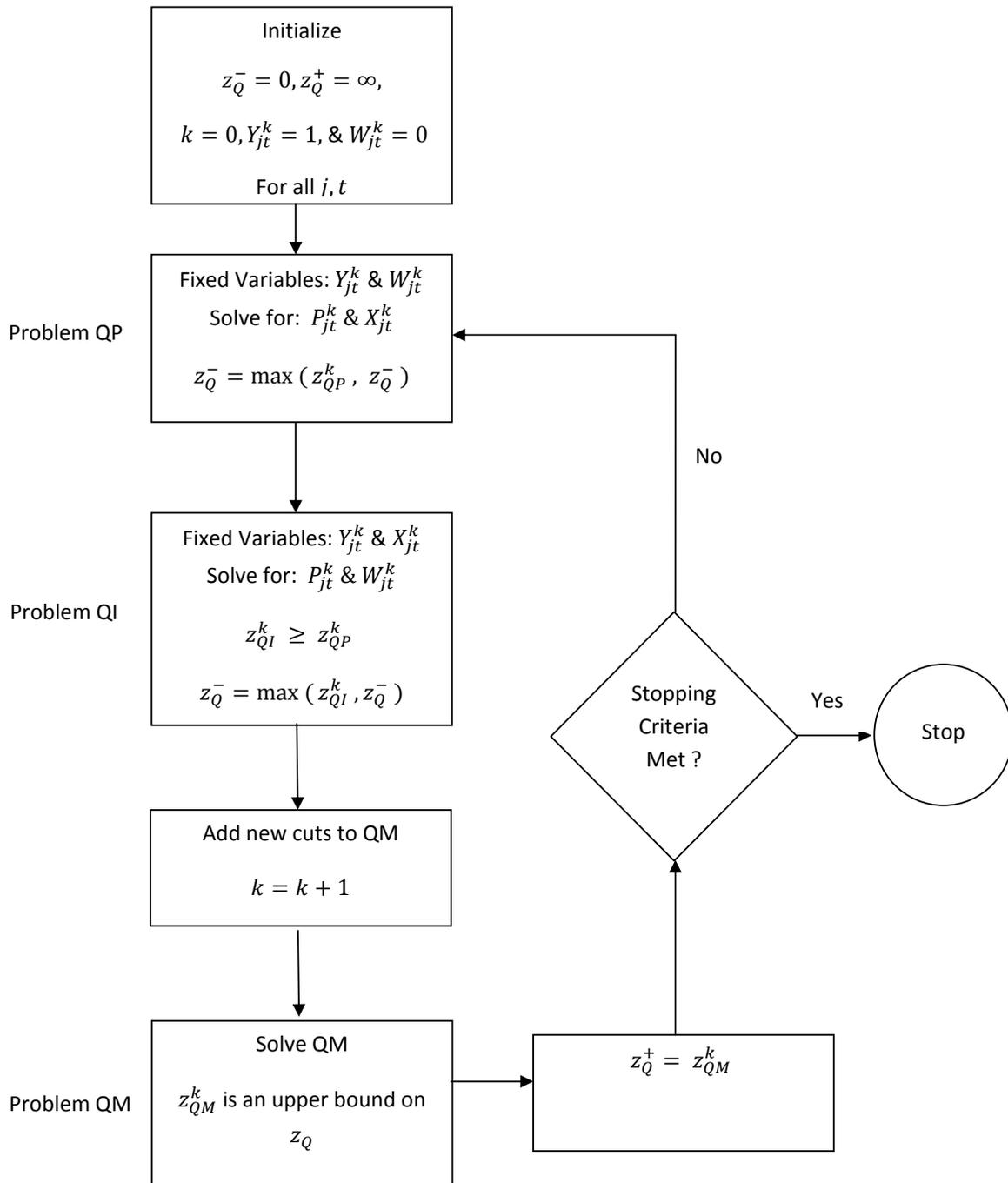


Figure 5.2: Solution approach for Problem Q

Algorithm 3 *Outer Approximation 2*

1: Initialize

$z_Q^+ \leftarrow \infty$. $z_Q^- = 0$. $k \leftarrow 0$. define ϵ .
 $Y_{jt}^0 \leftarrow 1$. $W_{jt}^0 = 0$, $\forall j, t$

2: Problem QP

Generate an instance of QP for fixed Y_{jt}^k & W_{jt}^k
Solve QP by using Algorithm 1.

3: Problem QI

Generate an instance of QI for fixed Y_{jt}^k & X_{jt}^k
Solve QI from (5.23).

$k \leftarrow k + 1$

$z_Q^- \leftarrow \max(z_Q^-, z_{QI}^k)$

4: Problem QM

Update set G in (5.31) for $i = \{0, 1, \dots, k - 1\}$.

Update and solve QM.

$z_Q^+ \leftarrow z_{QM}^k$

5: Check for Stopping Criteria

IF $z_Q^+ - z_Q^- \leq \epsilon$ Stop. current solution of QI is also optimal for Q

6: ELSE go to 2

We create a problem instance of QP for the fixed values of setup and advertising variables. We solve the problem instance by using Algorithm 1. Now fix the variables X_{jt}^k at current solution and solve for optimal advertising expenditure from (5.23) and find z_{QI}^k . This objective value is a new lower bound for problem Q. Set $z_Q^- = \max(z_Q^-, z_{QI}^k)$. Next, increment the iteration counter ($k = k + 1$) and update set G for $i = \{0, \dots, k - 1\}$. At each iteration k we generate one new member of set G . Hence, we sequentially add new constraints at each iteration. These new constraints cut off the previous solution and give a new solution such that $z_{QM}^k \leq z_{QM}^{k-1}$. Solution from each iteration tightens the upper bound for Q and the sequence of solutions is always non-increasing. Solutions of QI tighten the lower bounds and the process iterates until a stopping criteria is met.

Lemma 11. *Algorithm 3 generates a sequence of solutions such that at any iteration k of the algorithm:*

$$\max z_{QP}^{k-1} \leq \max z_{QI}^k \leq \max z_Q^k \leq \max z_{QM}^k$$

As discussed in Section 3.4, the optimal solution of QM is also optimal for Q. Furthermore, the sequence of improving bounds in Lemma 11 converges within a finite number of iterations, and the method will not loop back to a previous solution (Duran and Grossmann (1986) and Fletcher and Leyffer (1994)).

5.3.4 Model with Advertising Threshold

Once an advertising budget and its allocation is finalized, the firm purchases advertising space or time in different media. In problem Q we assume that the variables W_{jt} may assume any non-negative value, however small it may be. However, in most practical situation it would require some minimum amount of budget to implement advertising, for example a minimum size of ad in print or a minimum length of TV commercial. Hence, there may be situations when an optimal solution of Q is not a practical one. To handle these situations we introduce a minimum advertising spend (ω_j) for each product j such that either $W_{jt} \geq \omega_j$ or $W_{jt} = 0$ for all j, t . This means that if the optimal allocation of advertising budget for product j in period t is less than the minimum amount of money required to execute advertising, then the firm should either increase the allocation to ω_j or do not advertise product j in t . In either case a new allocation of budget is desired after fixing one or more variables either to zero or ω_j . To model this problem, we add the new constraints to QI and call it problem QIA. Hence, QI is a relaxation of QIA, and if its optimal solution is feasible for QIA it is also also optimal solution of QIA. Notice that addition of these constraints only changes the feasible solution space and does not

change the sales response function itself. Problem QIA is stated hereunder:

$$(QIA) \quad \max z_{QIA} = \sum_{j=1}^J \sum_{t=1}^T \left(\frac{q_{jt} \delta_j W_{jt}^r}{\beta_j} - W_{jt} \right) - \kappa_2 \quad (5.32)$$

Subject to:

$$\sum_{j=1}^J \sum_{t=1}^T W_{jt} - \Omega \leq 0 \quad (5.33)$$

$$W_{jt} \geq \omega_j \vee W_{jt} = 0 \quad \forall j, t \quad (5.34)$$

We propose an iterative solution procedure that: solves a relaxation of QIA, checks for feasibility, pegs one variable to zero or to ω_j value at each iteration, and terminates when the optimal solution of relaxed problem is also feasible for QIA. Figure 3.1 shows flow chart of the proposed procedure and the pseudo code is presented as Algorithm 4. Let z_{QIA}^+ and z_{QIA}^- be upper and lower bounds respectively for z_{QIA} . Let k be iteration counter that is initialized as $k = 1$. Notice that QI is a relaxation of QIA, and we already have a closed form solution of QI. Solve QI and record its objective value as z_{QIA}^+ . Now let set U be a set of indexes (j,t) , where $U = \{(j, t) : 0 < W_{jt} < \omega_j\}$. In other words, it is the set of indexed of variables that do not satisfy constraint set 5.34. Clearly, if U is empty, then solution of QI is also feasible and hence also optimal for QIA. Else, fix all variables $W_{jt} \in U$ equal to ω_j . Solve QI with fixed values for set U . The new solution is a lower bound for z_{QIA} . Update lower bound and unfix the variables.

Now for all $(j, t) \in U$, compute $\Delta_{jt} = \{\frac{\delta_j}{\beta_j} q_{jt} W_{jt}^r - W_{jt}\}$. Let (l, n) be the product and period such that $\Delta_{ln} = \min \Delta_{jt} \forall (j, t) \in U$. Next, increment the counter k , fix $W_{ln} = 0$ and solve QI once again. Stop if the current objective value is less than the current lower bound z_{QIA}^- , else record this as a new lower bound and repeat the procedure until stopping criteria is met.

Lemma 12. *Algorithm 4 terminates in a finite number of iterations ($\leq TJ$) to an optimal solution of QIA.*

Algorithm 4 requires little computational effort as its relaxation QI has close form solution. Moreover, at each iteration at least one variable is pegged to zero value. Since there is a finite number of variables, the algorithm terminates in no more than TJ number of iterations.

Algorithm 4 *Algorithm QIA*

1: Initialize

$k \leftarrow 1$
 $z_Q^+ \leftarrow \infty$
 $z_Q^- = 0$
 Let set $O = \{\}$

2: Update Upper Bound QI

Solve QI $\forall (j, t) : (j, t) \notin O$
 Let $z_{QIA}^+ \leftarrow z_{QI}^k$
 Let set $U = \{(j, t) : 0 < W_{jt} < \omega_j\}$

3: Check for Stopping Criteria

IF $U = \{\}$ then stop. current solution of QI is also optimal for QIA
 ELSE

4: Update Lower Bound

For all $(j, t) \in U$, Fix $W_{jt} = \omega_j$
 Solve QI $\forall (j, t) : (j, t) \notin O$
 Let $z_{QIA}^- \leftarrow z_{QI}^k$

5: Increment Iteration

$k \leftarrow k + 1$
 Unfix $\forall W_{jt} \in U$

6: Peg one variable to 0

Compute Δ_{jt} , $\forall (j, t) \in U$
 Find (l, n) such that $\Delta_{ln} = \min \Delta_{jt} \forall (j, t) \in U$
 Fix $W_{ln} = 0$
 Let $O = O \cup (l, n)$
 Solve QI $\forall (j, t) : (j, t) \notin O$

7: Check for Stopping Criteria

IF $z_{QIA}^- > z_{QI}^k$ stop. z_{QI}^{k-1} is also optimal for QI
 ELSE

8: Update Lower Bound

Let $z_{QIA}^- \leftarrow z_{QI}^k$
 Goto 2

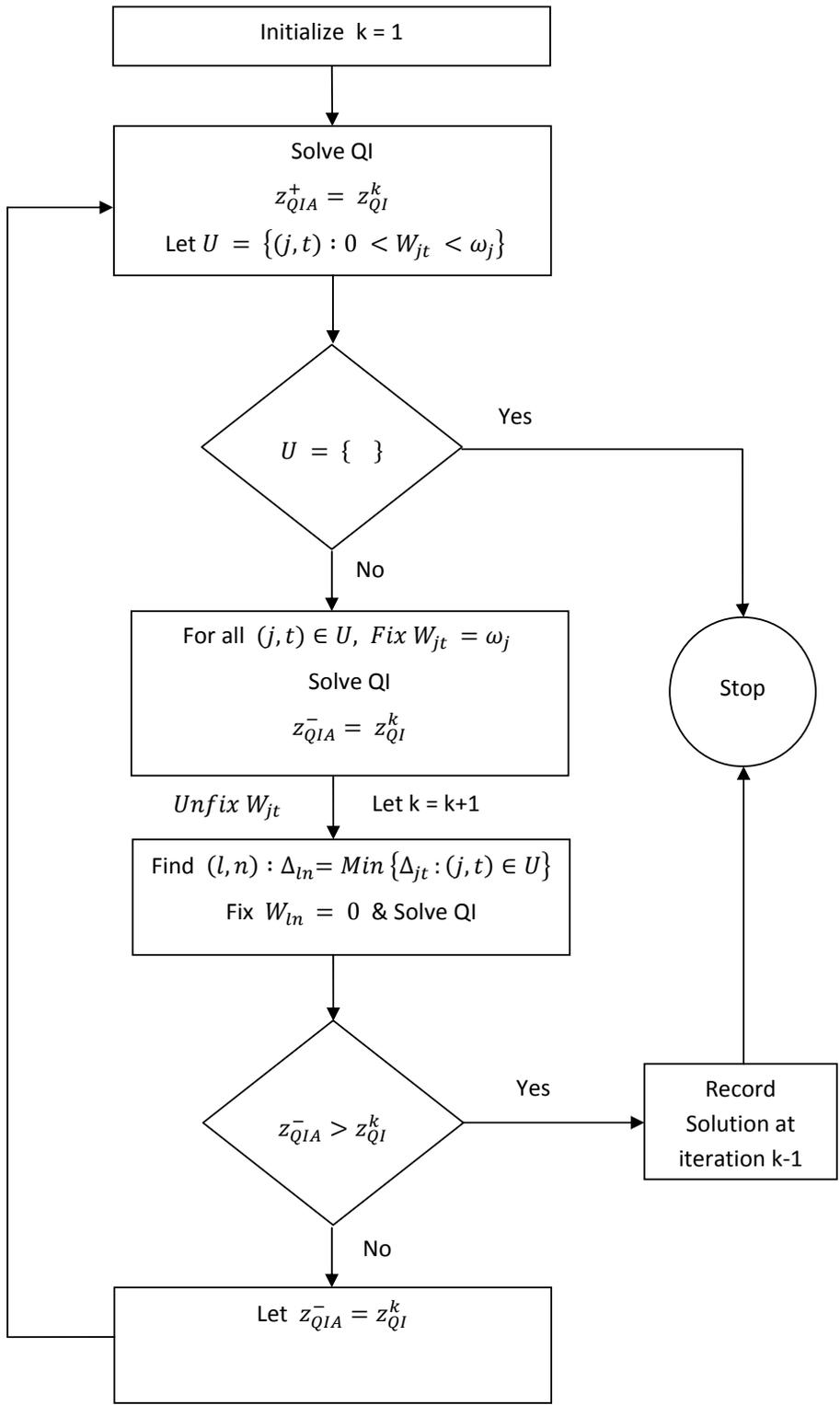


Figure 5.3: Solution Procedure for Problem QIA

Table 5.1: Optimal Profits with and without Advertising

Capacity	Scenario-1			Scenario-2			Scenario-3			Scenario-4		
	WA	WOA	% Imp									
30	229.0	223.3	2.5%	242.4	235.7	2.8%	205.8	199.5	3.2%	237.1	230.4	2.9%
40	257.2	251.0	2.5%	268.1	258.5	3.7%	236.5	230.0	2.8%	262.7	253.4	3.7%
50	270.5	262.5	3.0%	274.0	264.5	3.6%	257.2	249.1	3.3%	270.8	261.0	3.8%
60	275.8	267.0	3.3%	278.0	267.7	3.8%	266.8	257.9	3.5%	276.3	266.4	3.7%
70	277.3	268.1	3.4%	278.6	268.3	3.8%	276.2	266.8	3.5%	278.5	268.4	3.8%

Note: WOA = Without Advertising, WA = With Advertising, % Imp = percentage improvement

5.4 Numerical Examples and Managerial Insights

We solve some numerical examples to demonstrate working of Algorithm 3. We use the same set of data as presented in Tables 4.1 and 4.2 and compare the results with advertising (WA) and without advertising (WOA). As a base case we assume an advertising budget $\Omega = 2000$, advertising effectiveness coefficients $\delta_j = 15$, and $r = 0.5$. We solve problem instances for each of the four scenarios (as given in Table 4.2 and Figure 4.2) at different capacities and summarize computational results in Table 5.1. As expected, for each of the scenario the firm's profit improves with advertising. Figures 5.4 and 5.5 give a graphical comparison of these results. The two parameters that influence profit most, for a given advertising budget and effectiveness, are capacity and seasonality of demand. When the demand is uniform profit increases with additional capacity. However, the largest percentage improvement is observed when the demand for at least one product peaks in the beginning of the season. This observation is similar to the one observed in Section 4.2, where we compared profitability of dynamic pricing with that of constant pricing.

Figure 5.5 shows that the maximum percentage increase in profit for a \$2,000 budget is 3.8% ; which happens when at least one product has peak demand in the early part of season (Scenarios 2 & 4). This observation is an interesting insight that may be useful for decision making. It tells us that a profit maximizing firm may generate higher profits if it

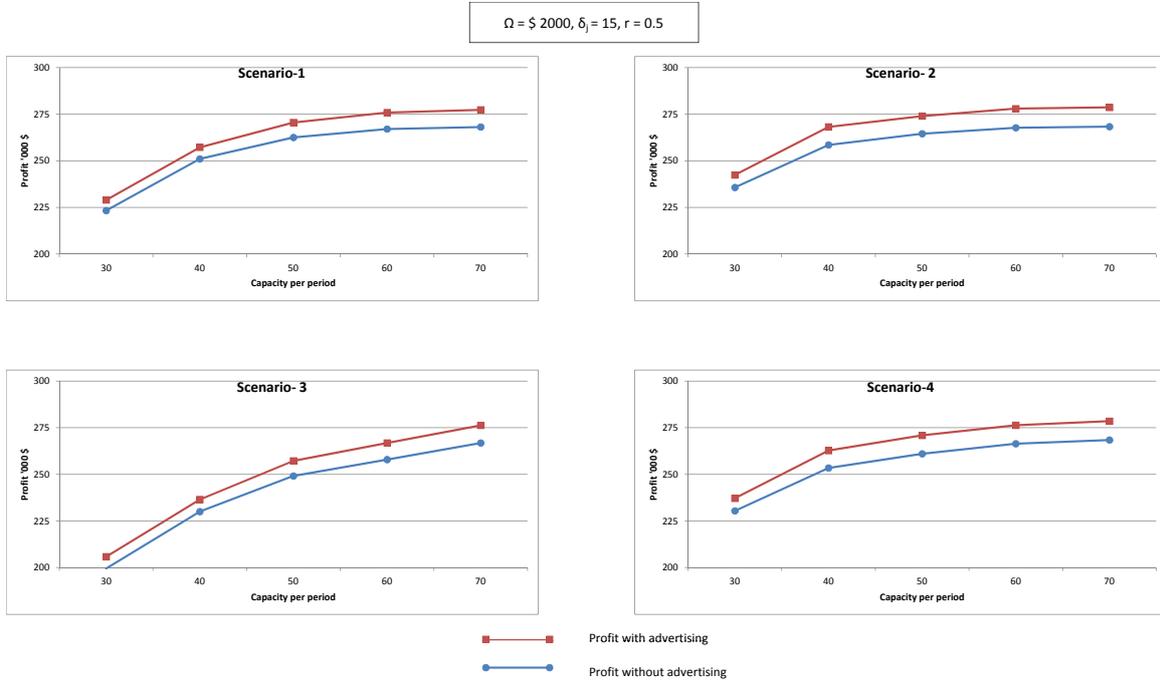


Figure 5.4: Optimal Profits with and without Advertising

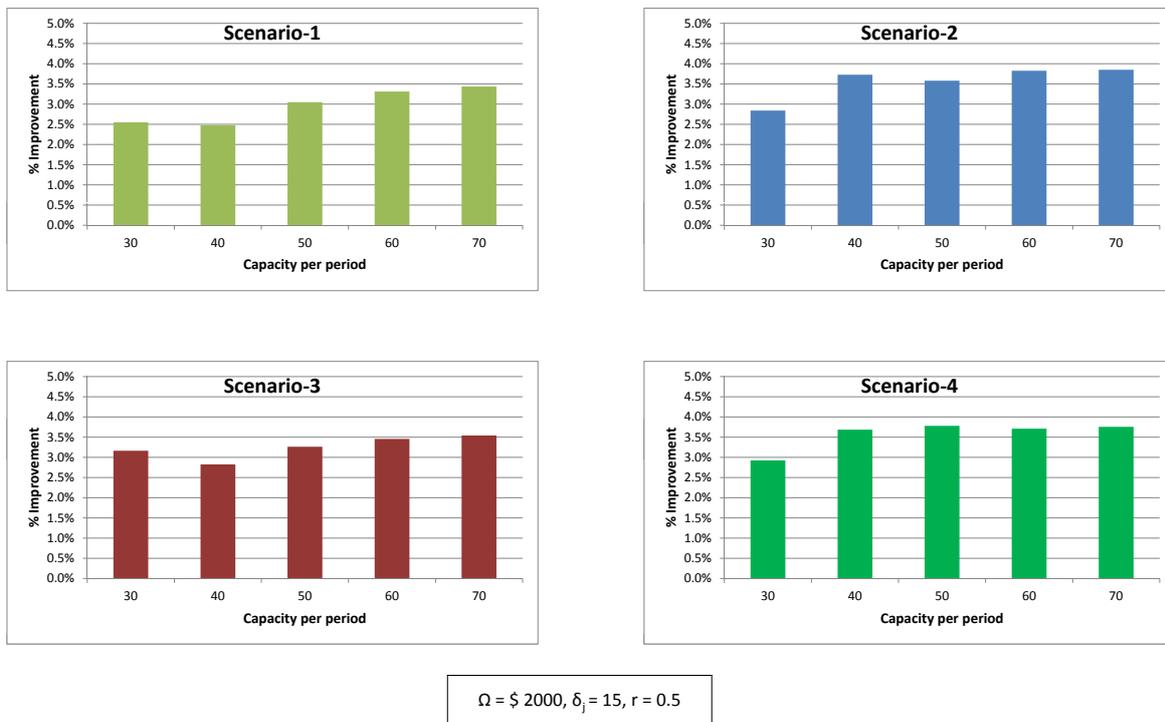


Figure 5.5: Percent Improvement in Profit with Advertising

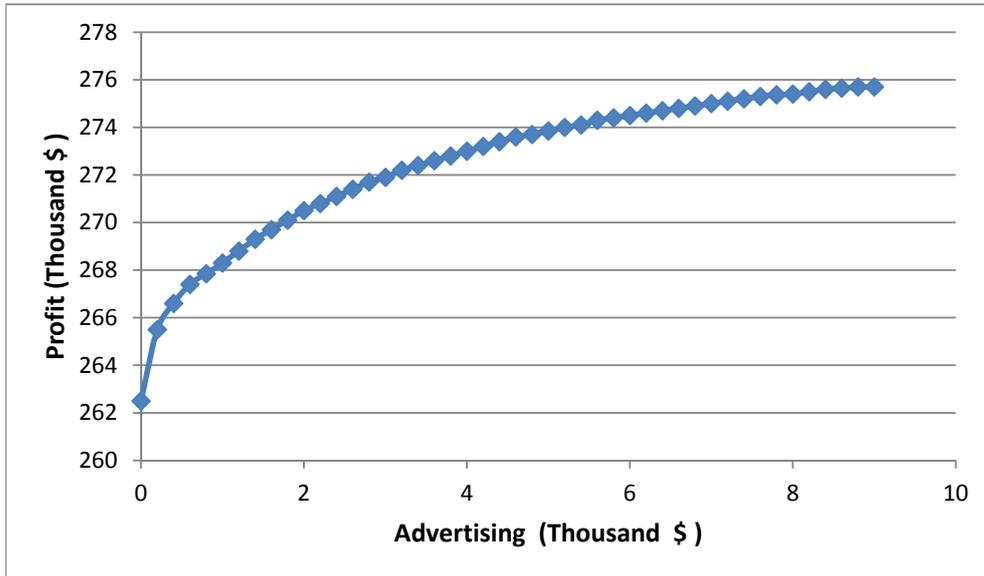


Figure 5.6: Profit as Function of Advertising (Scenario-1, Capacity = 50)

can align its planning horizon with demand seasonality. Typically, we would like to believe that operations would prefer to have peak demand at the end of the planning horizon. This is because it gives enough time to build inventory and be able to meet the demand. However, our results show that if the firm coordinates marketing and operations decisions, it is beneficial to have the peak demand at the beginning of the planning horizon. This way the firm may charge higher prices at the beginning and lower prices at the end of the planning period, and maximize its profits.

Next, we solve a series of problems for different values of the advertising budget. Figure 5.6 shows total profit as a function of advertising budget. The data used here is for Scenario-1 at a capacity of 50 per period, however, the results are similar for other scenarios, as well. Profit increases at a diminishing rate and reaches a saturation level at around 8,000.

The results, presented so far, show the overall effect of advertising, capacity, and seasonality. In addition, one might be interested in looking at the allocation of budget across products and time periods. Figure 5.7 shows the allocation of budget between the

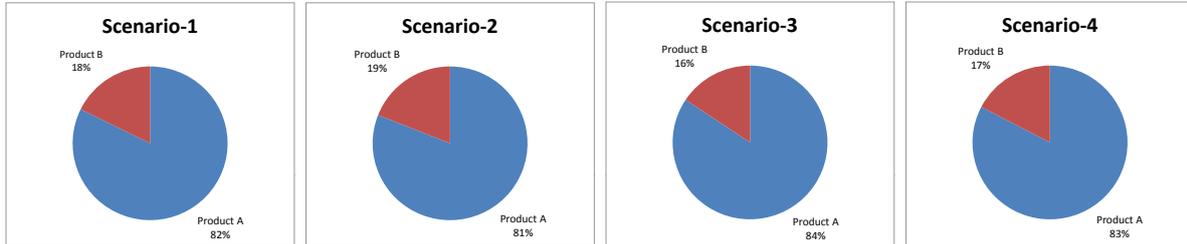


Figure 5.7: Optimal Allocation of Budget for two Products

two products for each demand scenario. We solve a large number of problem instances for different budgets, and make following two observations:

- i. The optimal allocation as a percent of the total budget, for each product, does not change with an increase or decrease in the budget.
- ii. The allocation does not change significantly with the changes in seasonality factors.

This means the budget allocation is quite robust with respect to changes in advertising budget and seasonality. Hence, firms can use rules of thumb and need not to frequently review the allocation. For example, decisions related to budget cuts can be made quickly without the effort of re optimization.

Figure 5.8 presents, yet another interesting observation. It shows a typical allocation (for the given problem data) of advertising budget for different time periods. The four scenarios correspond to the season demand factors (Figure 4.2). Observe that the optimal allocation of advertising budget over time follows the demand seasonality pattern. It is optimal to decrease advertising in periods of low demand and increase during periods of high demand. Usually, firms have excess capacity in periods of low demand. For some, it might make intuitive sense to advertise heavily in periods of low demand and make use of excess production capacity. However, the computational results show it is not in the best interest of the firm.

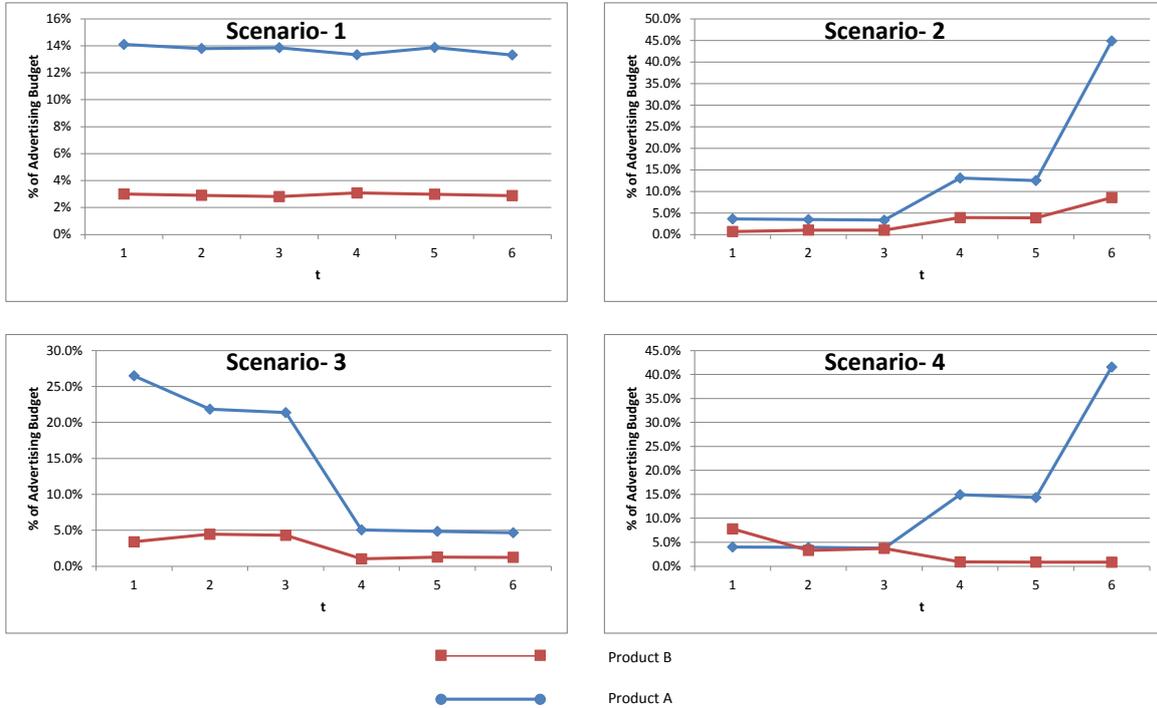


Figure 5.8: Advertising Spend per Period

5.4.1 Comparison with Uncoordinated Approach

In this section, we compare the profits of uncoordinated decisions to that of coordinated decisions. For the sake of this comparison, we assume that in an uncoordinated decision making approach; marketing sets prices P_{jt} to values that maximize total revenue, and then allocates the advertising budget W_{jt} in proportion of sales revenue of each product. We use, as an example, the sales data of TIS for the two products. Product A is a low volume high margin product that generates 40% of the sales revenue. Product B is a high volume, low margin product which generates 60% of sales revenue. TIS allocates its advertising budget in the same proportion to the two products. We compute and compare the profits at different capacities for each demand scenario. Figure 5.9 presents the results in a graphical form. We observe that coordinated decisions are much valuable at lower capacities and when the demand peak occurs in the beginning of planning horizon. Coordination of pricing, advertising, and production decisions, gives substantially higher

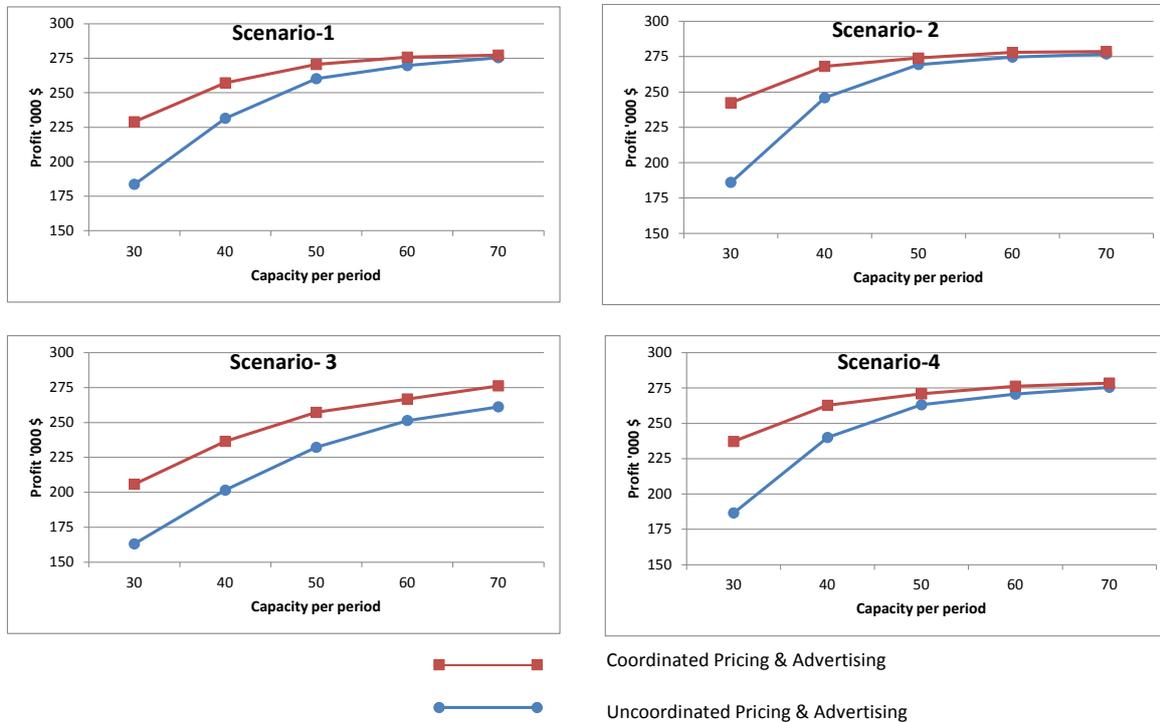


Figure 5.9: Profitability with and without Coordination

profits at low capacities. This difference becomes marginal when the firm adds more capacity. In Figure 5.9, we observe, the profits for coordinated approach at capacity 40 are equal to the uncoordinated profits at capacity 50. It is clear from this analysis that coordination of these decisions may allow the firm to achieve same profit, and reduce its investment in capacity. This can be very valuable when capacity expansion is capital intensive.

It is of interest to see how the profits vary for different advertising expenditures. We consider a base case; with a capacity of 50 per period, and stationary demand. Figure 5.10 gives a comparison of profitability for different advertising budgets. The profits for coordinated approach are consistently higher at all advertising levels. More importantly, for uncoordinated approach, the profit reaches a maximum and then starts decreasing at higher levels of advertising because such allocation is suboptimal. We also look at the advertising for each product in each period. Figure 5.11 presents this comparison

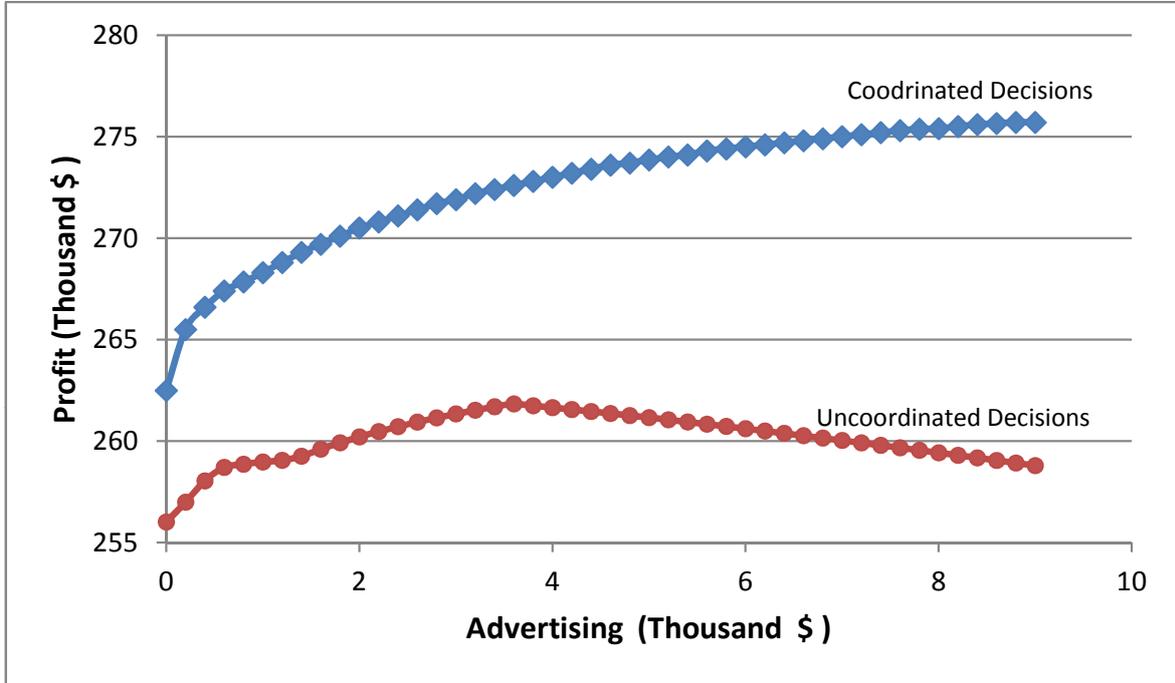


Figure 5.10: Profit as Function of Advertising (Scenario-1, Capacity = 50)

for both the products and for each demand scenario. We observe, in all these cases, product A is under-advertised during periods of peak demand, whereas product B is over-advertised in all periods. Over-advertising results in a higher demand that may result in lost sales or dictate the need to invest in more capacity. Typically, in such situations, operations would meet the increased demand by using excessive over-time or outsourcing production at a higher cost. Ironically, such decisions may result in higher sales revenues, but have a negative impact on profitability. This happens because excessive advertising has lower marginal returns. Under-advertising, on the other hand, not only results in less than optimal demand but also limits the firm's ability to increase price. This example illustrates that coordination of operations and marketing decisions may contribute in more than one ways to increase profitability:

- i. Optimal prices result in better demand management
- ii. Advertising effectiveness may be maximized by coordinating demand and supply

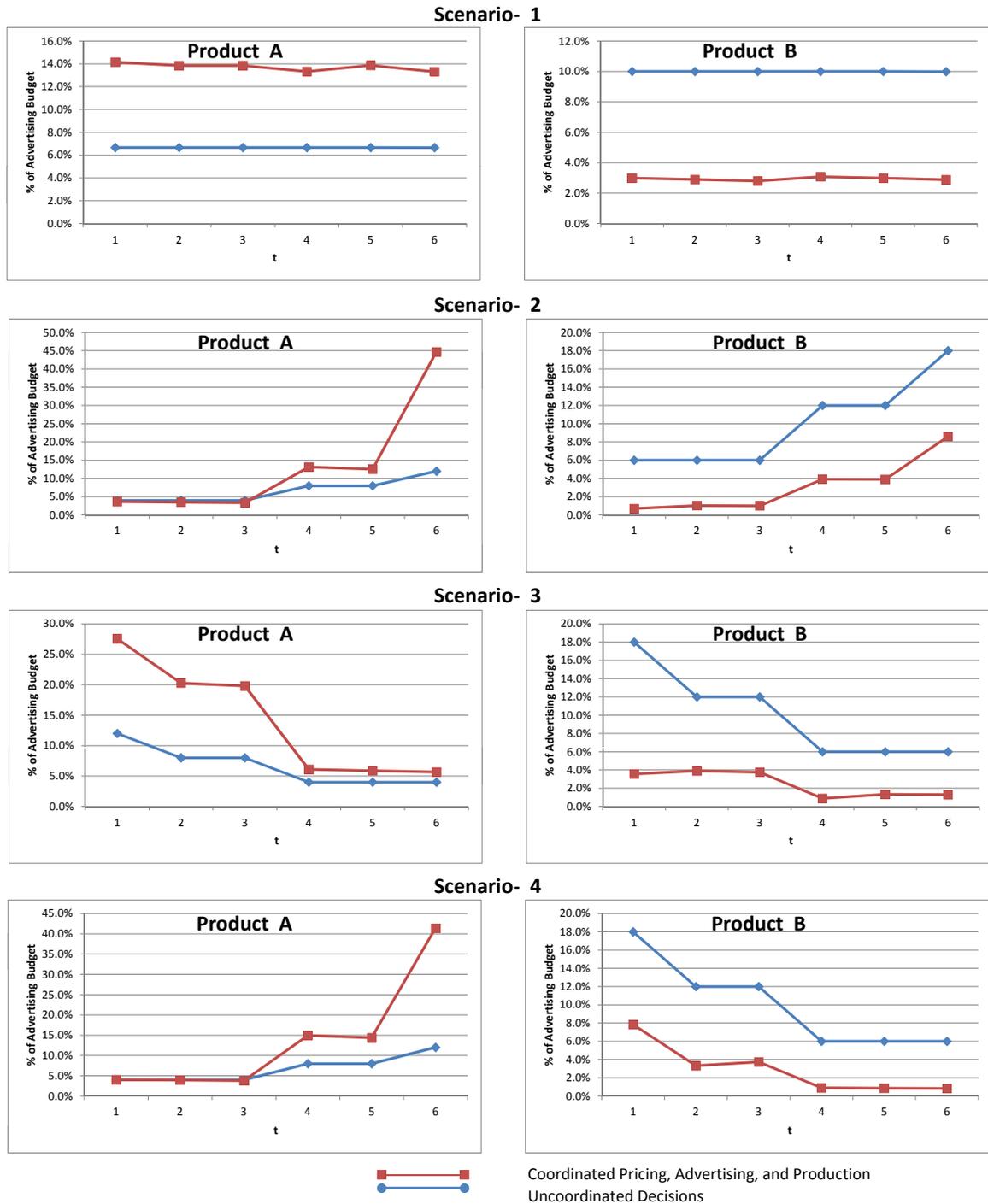


Figure 5.11: Advertising Budget Allocation for Coordinated Vs Uncoordinated Approach (Scenario-1, Capacity = 50)

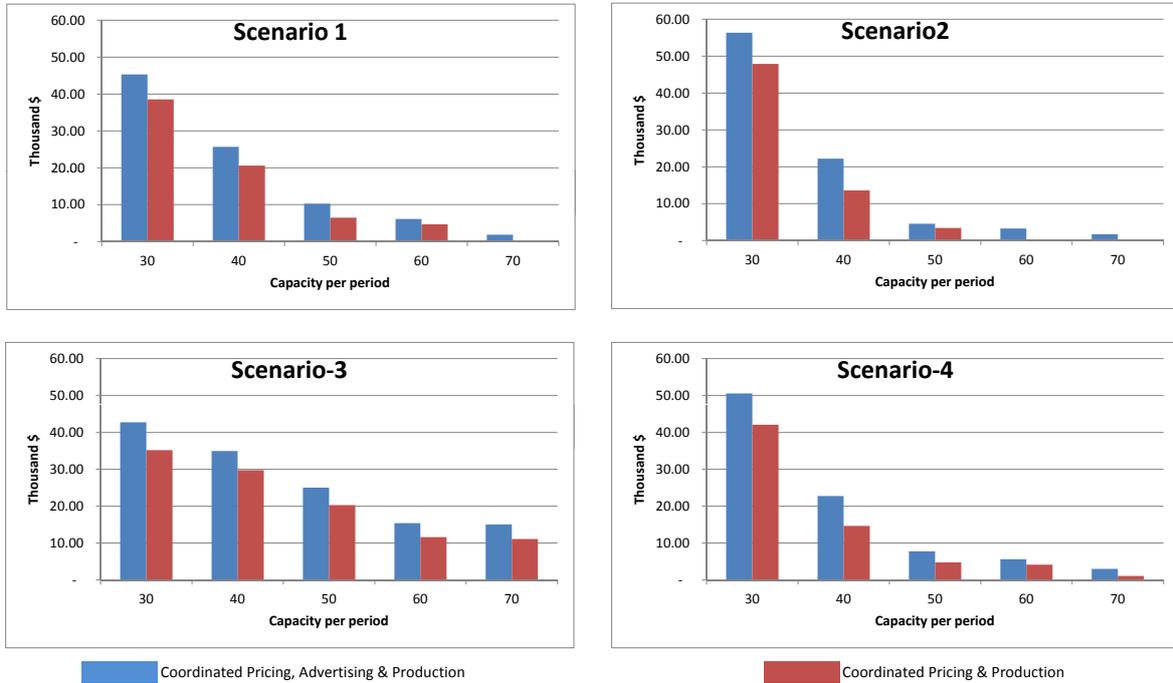


Figure 5.12: Increase in Profit due to Coordinated Decisions

- iii. Eliminate over-advertising, thus reducing waste
- iv. Reduce investment in capacity, or defer capacity expansion

In Figure 5.12 we summarize the effect of coordination on profit. For each demand scenario and at different capacities, we show the increase in profit due to coordination of pricing, advertising, and lot sizing decisions. Although, in this example, price coordination has greater impact on profitability, the overall effect depends on parameters of demand function. For example, the parameters for demand function may differ significantly during the product life cycle. It can be argued that highly effective advertising may decrease price sensitivity, and as a result the advertising effect can be greater than price effect. In either case, the results demonstrate that by including advertising as a decision variable the firm may achieve significant increase in profit in comparison to coordinating price and production only.

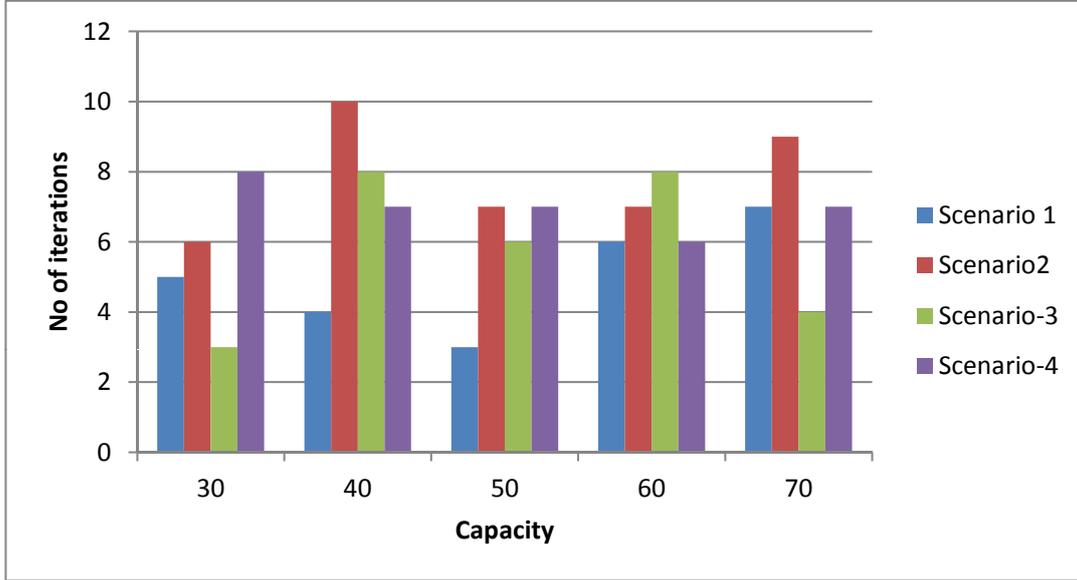


Figure 5.13: Required Number of Iterations for Convergence of OA2 ($\epsilon = 0.1\%$)

5.5 Convergence Properties of Proposed Algorithms

Lastly, we discuss convergence properties of Algorithms OA1 and OA2. As stated in Chapter 4, the proposed algorithms, in general, converge asymptotically. However, we use results from the computational experiments (base case as used earlier in this section) and report observed number of iterations on an average. We use a stopping criteria of $\epsilon = 0.1\%$ i.e. stop when the difference between upper and lower bounds is less than or equal to 0.1% of the upper bound. Figure 5.13 shows the number of required iterations for different capacities and seasonal factors. The algorithm converged within a few iterations for all the test problems. In most cases, the desired tolerance was achieved in 6 to 8 iterations while 2 being the best case and 10 being the worst case for convergence. To test the performance for larger problem sizes we generate additional problem instances. We define the problem size as product of the number of periods and the number of products (TJ). We solve ten problem instances for each of the following:

- i. Two products and six periods i.e. $TJ = 12$.

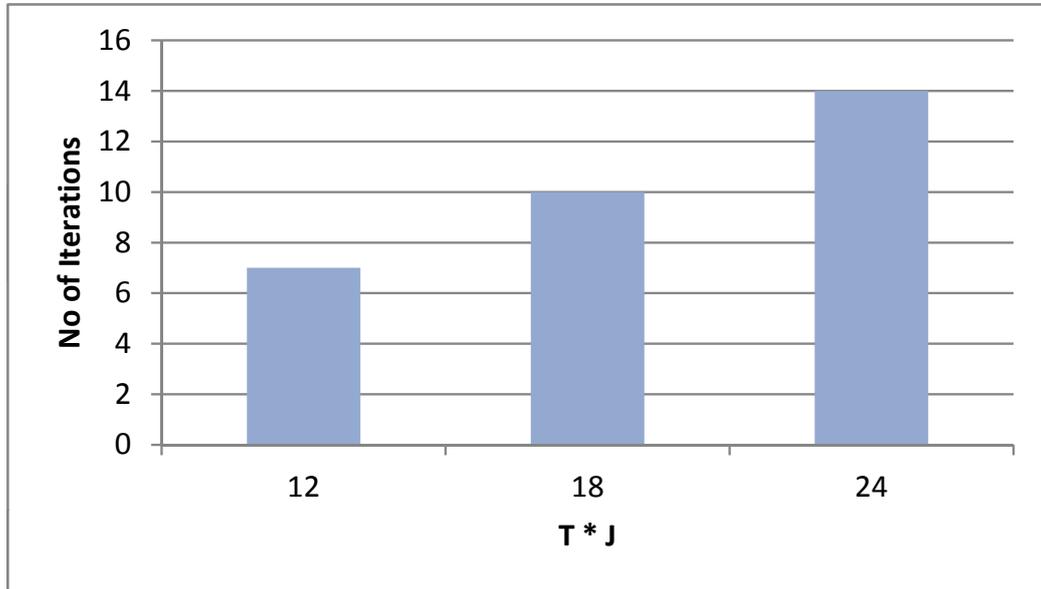


Figure 5.14: Convergence of OA2 for different problem sizes($\epsilon = 0.1\%$)

ii. Three products and six periods i.e. $TJ = 18$.

iii. Three products and eight periods i.e. $TJ = 24$.

Figure 5.14 presents the computational results. On an average problems with $TJ = 6$ took 7 iterations to converge, $TJ = 18$ required 10, and $TJ = 24$ were solved in 14 iterations.

Chapter 6

Conclusion and Future Research

Coordination and alignment of cross functional goals is a well researched topic in academic literature. Many researchers address this issue and demonstrate the value of coordinated decision making. Most of the related research is in the context of supply chain management, and proves that coordination strategies do result in higher profits for all the stakeholders.

What needs to be done is quite clear, but how to do it? is a question that still needs to be answered - specially when the stake holders are part of the same organization. Few researchers present analytical methods that model capacity and setup constraints of a manufacturing firm. Even fewer address the solution approach for multiple products.

This research shows how mathematical models and analytical tools can be used for coordinated decision making within an organization. We address the problem of a manufacturer who produces multiple products on the same equipment. We formulate and solve his problem in discrete-time setting with capacity constraints and setup costs. Assuming a monopolistic market, and that the demand is a strictly decreasing function of the price, we present a solution approach which can be used for constant pricing, as well as dynamic pricing strategies.

We present two models; the first model is for coordinating pricing and production

decisions, whereas, the second model incorporates advertising decisions, in addition to price and production. We solve both the problems to optimality, and present computational results to demonstrate excellent convergence properties of proposed algorithms. Furthermore, the solution approach is more general and applicable for linear as well as nonlinear demand functions.

We present properties of the optimal solution and show the relationship between optimal prices and production plan. We define the term Regeneration Interval (RI) as a set of consecutive periods such that: beginning inventory of the first period and ending inventory of the last period are zero. The optimal prices increase monotonically within an RI, and drop sharply at the first period of next RI. Hence the prices follow a see-saw pattern where a rising trend corresponds to the length of an RI and a fall indicates the start of a new RI.

Using real world data from a manufacturer, we create problem instances, for different demand scenarios at different capacities, and solve for optimal prices for each strategy. We show that optimal prices do not always decrease monotonically in capacity. We compare the firm's profitability for the two pricing strategies, and show that dynamic pricing is valuable only at low capacities and when at least one of the product has peak demand in the beginning of the season. There is only a marginal difference between the two strategies when the capacity is high or the demands of both products peak at the end of the season.

In the second model, we incorporate advertising, in addition to price, as a demand driver. One version of this model allows for all non negative values of advertising, whereas, a second version incorporates a minimum threshold of advertising. Solution approach in both cases is to define three subproblems and then solve them iteratively until a convergence criteria is met. We present some analytical as well as computational results, and managerial insights. Two valuable insights are: i) The optimal allocation of advertising budget between different products is quite robust and does not change with an increase

or decrease in the budget, ii) The optimal allocation over different time periods follows a pattern that is similar to demand seasonality. It is optimal to increase advertising in periods of higher demand and decrease in periods of lower demand. Such insights may allow firms to use rules of thumb and need not frequently review the allocation. For example, decisions related to budget cuts can be made quickly without the effort of re-optimization.

Although some of the results are specific to the functional form of demand, it is possible to extend the results to other functional forms. The structure of the solution approach is quite versatile, and it can be used for solving similar problems. For example, instead of advertising, we may include quality or lead time in the demand function. Algorithm OA2 can easily be modified to solve these new problems.

We would like to extend the current work to incorporate more complex inter-product interactions, for example, when cross price elasticities are not zero, and when the products are substitutes or complements. Another interesting extension is to incorporate the customer behavior models like the reference price model. A similar solution approach can be developed to incorporate promotional decisions along with the pricing and lot sizing decisions. Our model assumes demand is a function of the current price only, thus implying that the customers do not behave strategically. Such customer behavior, though out of scope of the present analysis, can be included in future extensions of current work. Finally, as Tang (2010) observe, there are no decision support systems that truly integrate the production/inventory and pricing decisions. They attribute this gap to lack of efficient algorithms for general, multi-period, multi-product models. We believe, this work makes valuable contributions in developing solution methodologies that can be incorporated in such decision support systems.

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APPENDIX

Theorem 1. A profit maximizing firm, selling multiple products in different periods, will determine selling prices for each product such that the induced demands are equal to the quantity sold for each product in each period (i.e, $S_{jt}^* = D_j(P_{jt}^*)$ for all j, t at the optimal solution).

Proof. a)- The firm may choose different prices in different periods for the same product. Let us suppose that the firm chooses prices P_{jt}^* , such that for some j and t the resulting demand $D(P_{jt}^*) > S_{jt}^*$. Where S_{jt}^* is the maximum quantity of j available for sale in t . Let $C(S_{jt}^*)$ be the total cost, and $\pi^* = \sum_{j=1}^J \sum_{t=1}^T P_{jt}^* S_{jt}^* - C(S_{jt}^*)$ be the profit for implementing this plan. Since demand is strictly decreasing in price, there exists another price $P'_{jt} > P_{jt}^*$ such that $D(P'_{jt}) = S_{jt}^*$. Therefore, the firm may increase the price to P'_{jt} without any change in the cost. The new profit will be $\pi' = \sum_{j=1}^J \sum_{t=1}^T P'_{jt} S_{jt}^* - C(S_{jt}^*)$. Since $P'_{jt} > P_{jt}^*$ the profit $\pi' > \pi^*$ which implies P_{jt}^* are not optimal.

b)- The firm determines constant prices for each product during the planning horizon. Let P_j^* be the optimal price and $S_{j,t}^*$ be the quantity of j sold in period t . Let us suppose the quantity sold is less than the induced demand for some product in at least one of the periods i.e., $S_{j,t}^* < \gamma_{j,t}(\alpha_j - \beta_j P_j^*)$ for some j and t . Then the total quantity sold in the planning horizon must also be less than the total demand for product j during the same period.

$$\sum_{t=1}^T S_{j,t}^* < \alpha_j - \beta_j P_j^*$$

Let $C(S_{jt}^*)$ be the total cost for this optimal solution, then the optimal profit π^* is:

$$\pi^* = \sum_{j=1}^J \sum_{t=1}^T P_j^* S_{j,t}^* - C(S_{jt}^*) \quad (6.1)$$

$$< \sum_{j=1}^J P_j^* (\alpha_j - \beta_j P_j^*) - C(S_{jt}^*) \quad (6.2)$$

Since the demand is decreasing in price, there exist some $\hat{P}_j > P_j^*$ such that $\sum_{t=1}^T S_{j,t}^* = \alpha_j - \beta_j \hat{P}_j$ at the same cost $C(S_{jt}^*)$. The profit $\hat{\pi}$ at price \hat{P}_j will be :

$$\hat{\pi} = \sum_{j=1}^J \hat{P}_j (\alpha_j - \beta_j \hat{P}_j) - C(S_{jt}^*) \quad (6.3)$$

Since $\hat{P}_j > P_j^*$ it follows that $\hat{\pi} > \pi^*$. Therefore, P_j^* are not optimal. Hence, at the optimal solution the induced demand is equal to the quantity sold and there are no lost sales.

□

Lemma 1. For a given sequence of setups, the profit function of P1^k is strictly concave in price.

Proof. Let z_{jt} be the profit for product j in period t , then $z_{jt}(P_{jt}) = \gamma_{jt}(\alpha_j - \beta_j P_{jt})(P_{jt} - c_{jt}) - h_{jt}I_{jt} - a_{jt}Y_{jt}$. Differentiating twice with respect to P_{jt} we get,

$$\frac{\partial^2 z_{jt}}{\partial P_{jt}^2} = -2\beta_j \gamma_{jt} < 0 \quad (6.4)$$

Since the objective function in P1^k is a sum of z_{jt} 's that are strictly concave in price, it is also strictly concave.

□

Lemma 3. For any problem instance of case-2, $\lambda_m(l_m^*) \geq 0$ and it is bounded from above by a finite maximum.

Proof. We first show that $\lambda_m(l_m^*)$ is bounded from above by a finite positive number. Observe that $\max z_{P1X_{jmn}}^{k'}(0) = A_{jmn} \forall (j, m, n) \in N$. Since each A_{jmn} is a finite positive number $\sum_{(jmn) \in N} z_{P1X_{jmn}}^{k'}(0)$ is bounded from above by a finite positive number. It follows from (3.22) that $\lambda_m(l_m^*)$ must also have a finite upper bound.

Now, let us suppose $\lambda_m(l_m^*) < 0$ for some l^* . Then from (3.22) :

$$\begin{aligned} & \sum_{X_{jmn} \in N_m^+} \frac{v_j z_{P1X_{jmn}}^{k'}(0)}{2B_{jn}} < b_m. \\ \Rightarrow & \sum_{X_{jmn} \in N_m^+} v_j \left(\frac{A_{jmn}}{2B_{jn}} - \sum_{l=1, l \neq m}^n X_{jln} \right) < b_m \end{aligned} \quad (6.5)$$

But the term in parenthesis in (6.5) is the unconstrained optimal solution X_{jmn}^* , therefore

$$\sum_{X_{jmn} \in N_m^+} v_j X_{jmn}^* < b_m \quad (6.6)$$

But this is a contradiction as by definition of case-2 the capacity constraint must be binding. Hence our initial assumption is wrong and we conclude $\lambda_m(l_m^*) \geq 0$. \square

Theorem 2. The solution from Algorithm MRA is optimal for problem $P1^k$. Furthermore, $\lambda_m(l_m^*)$ are in fact the optimal Lagrange multipliers for the capacity constraints (3.13).

Proof. Let θ_m^* and μ_{jmn}^* be the optimal Lagrange multipliers for capacity and non-negativity constraints respectively. The KKT conditions for the optimal solution are :

$$A_{jmn} - 2B_{jn}X_{jmn}^* - 2B_{jn} \sum_{l=1, l \neq m}^n X_{jln}^* - v_j \theta_m^* + \mu_{jmn}^* = 0 \quad \forall (j, m, n) \in N \quad (6.7)$$

$$\sum_{j=1}^J \sum_{n=m}^T v_j X_{jmn}^* \leq b_m \quad \forall m \in 1..T \quad (6.8)$$

$$\theta_m^* \left(\sum_{j=1}^J \sum_{n=m}^T v_j X_{jmn}^* - b_m \right) = 0 \quad \forall m \in 1..T \quad (6.9)$$

$$\mu_{jmn}^* X_{jmn}^* = 0 \quad \forall (j, m, n) \in N \quad (6.10)$$

$$\theta_m^* \geq 0 \quad \forall m \in 1..T \quad (6.11)$$

$$X_{jmn}^* \geq 0, \mu_{jmn}^* \geq 0, \quad \forall (j, m, n) \in N \quad (6.12)$$

By construction, Algorithm (1) starts with a feasible solution and maintains feasibility at each iteration. Hence $X_{jmn} \geq 0$ and constraint (6.8) are satisfied during all iterations. We define $W = \{X_{jmn}^* : X_{jmn}^* > 0\}$, it follows from (6.10) that for all $X_{jmn}^* \in W$, $\mu_{jmn}^* = 0$ and (6.7) implies:

$$v_j \theta_m^* = A_{jmn} - 2B_{jn}X_{jmn}^* - 2B_{jn} \sum_{l=1, l \neq m}^n X_{jln}^* \quad (6.13)$$

But the right hand side of above equation is the first derivative of $z_{P_1}^k$ evaluated at X_{jmn}^* . Therefore, it follows from Lemma 2 and Lemma 3 that $\theta_m^* = \lambda_m(l_m^*) \geq 0$. Now for $X_{jmn} \notin W$ we write (6.7):

$$\mu_{jmn}^* = v_j \lambda_m(l_m^*) - A_{jmn} - 2B_{jn} \sum_{l=1, l \neq m}^n X_{jln}^* \quad (6.14)$$

From (3.16) we have:

$$A_{jmn} - 2B_{jn} \sum_{l=1, l \neq m}^n X_{jln}^* \leq v_j \lambda_m(l_m^*) \quad \forall X_{jmn}^* \notin W \quad (6.15)$$

Hence $\mu_{jmn}^* \geq 0$ and KKT conditions are satisfied. \square

Lemma 4 . The sequence of nondecreasing objectives in Algorithm MRA converges to optimal solution.

Proof. The proof follows from Mjelde (1983). Let us define ζ_m as the column vector of variables X_{jmn} that are produced in period m . Then any feasible solution of $P1^k$, at end of iteration s of MRA, may be written as $\zeta^s = \{\zeta_1^s, \dots, \zeta_m^s, \dots, \zeta_T^s\}$. We introduce following notation:

$$\begin{aligned} \zeta_{m-}^s &= (\zeta_1^s, \dots, \zeta_{m-1}^s) && \text{for } m \geq 2 \\ \zeta_{m+}^s &= (\zeta_{m+1}^s, \dots, \zeta_T^s) && \text{for } m \leq T - 1 \\ \zeta^s &= (\zeta_1^s, \zeta_{1+}^s) = (\zeta_{T-}^s, \zeta_T^s) \\ \zeta^s &= (\zeta_{m-}^s, \zeta_m^s, \zeta_{m+}^s) && \text{for } 2 \leq m \leq T - 1 \end{aligned}$$

Let $z_{P1}^k(\zeta^s)$ be the value of objective function at ζ^s , then by construction of MRA we have:

$$\begin{aligned} z_{P1}^k(\zeta^s) &\leq z_{P1}^k(\zeta_{m-}^{s+1}, \zeta_m^s, \zeta_{m+1}^s) \\ &\leq z_{P1}^k(\zeta_{m-}^{s+1}, \zeta_m^{s+1}, \zeta_{m+1}^s) \\ &\leq z_{P1}^k(\zeta^{s+1}) \end{aligned}$$

Since the feasible region of $P1^k$ is compact, it contains a sequence that converges to

$\bar{\zeta}$. Therefore, monotonicity of $z_{P_1}^{ks}$ with respect to s implies, that

$$z_{P_1}^k(\bar{\zeta}) = \lim_{s \rightarrow \infty} z_{P_1}^k(\zeta^s)$$

□

Theorem 3. The optimal dynamic prices increase strictly (monotonically) during a regeneration interval (RI), if the inventory carrying costs are strictly positive (non-negative), i.e: $P_{jm}^* < P_{jm+1}^* < \dots < P_{js}^*$ where m is the first period of an RI and s is the last period of the RI. Furthermore, $P_{jn+1}^* - P_{jn}^* = h_{jn}/2$ for all $j = \{1, \dots, J\}$ and $n = \{m, m+1, \dots, s-1\}$.

Proof. We first consider a regeneration interval with a single period of production. Let m be the first period of production for product j in the RI. From (3.11) we write the optimal prices:

$$\begin{aligned} P_{jn}^* &= \frac{\alpha_j}{\beta_j} - B_{jn} X_{jmn}^* & \forall j &= \{1, \dots, J\} \ \& \ n = \{m, m+1, \dots, s-1\} \\ &= \frac{\alpha_j}{\beta_j} - \frac{1}{2} \left(z_{P_1 X_{jmn}}^{k'}(0) - v_j \lambda_m^* \right) \\ &= \frac{1}{2} \left(\frac{\alpha_j}{\beta_j} + c_{jm} + v_j \lambda_m^* + \sum_{t=m}^{n-1} h_{jt} \right) \end{aligned} \quad (6.16)$$

Therefore, $P_{jn+1}^* - P_{jn}^* = h_{jn}/2$ for all $j = \{1, \dots, J\}$ and $n = \{m, m+1, \dots, s-1\}$.

Now we consider a RI with two periods of production. Let o be the second period of production such that $m < o \leq s$. Observe that the optimal prices for $n = \{m, \dots, o-1\}$ are given by (6.16). For periods $n = \{o, o+1, \dots, s\}$ the optimal prices are given by:

$$P_{jn}^* = \frac{\alpha_j}{\beta_j} - B_{jn} (X_{jmn}^* + X_{jon}^*) \quad \forall j = \{1, \dots, J\} \ \& \ n = \{o, o+1, \dots, s\} \quad (6.17)$$

It follows from Lemma-2 :

$$\begin{aligned}
z_{P_{1X_{jmn}}}^{k'}(X_{jmn}^*) &= z_{P_{1X_{jmn-1}}}^{k'}(X_{jmn-1}^*) \\
A_{jmn} - 2B_{jn}(X_{jmn}^* + X_{jon}^*) &= A_{jmn-1} - 2B_{jn-1}(X_{jmn-1}^* + X_{jon-1}^*) \\
B_{jn}(X_{jmn}^* + X_{jon}^*) &= B_{jn-1}(X_{jmn-1}^* + X_{jon-1}^*) - \frac{1}{2}h_{jn-1}
\end{aligned} \tag{6.18}$$

Now we substitute (6.18) in (6.17)

$$\begin{aligned}
P_{jn}^* &= \frac{\alpha_j}{\beta_j} - B_{jn-1}(X_{jmn-1}^* + X_{jon-1}^*) + \frac{1}{2}h_{jn-1} \\
&= P_{jn-1}^* + \frac{1}{2}h_{jn-1} \\
\Rightarrow P_{jn}^* - P_{jn-1}^* &= \frac{1}{2}h_{jn-1} \quad \forall j = \{1, \dots, J\} \ \& \ n = \{o, o+1, \dots, s\}
\end{aligned}$$

The same procedure can be extended for more than two periods of production within an RI. □

Lemma 5. Let m be the first period of production of j in an RI and $r = m + s$ be the first period of production of j in the next RI. If there is no strategic incentive of holding inventory (i.e. $c_{jr} \leq c_{jm} + \sum t = m^{r-1}h_{jt}$), then the following must be true for P_{jr}^*

$$\begin{aligned}
i). \quad & P_{jr}^* \leq P_{jn}^* \quad \text{for } n = m \\
ii). \quad & P_{jr}^* < P_{jn}^* \quad \text{for } n = \{m+1, \dots, r-1\} \\
iii). \quad & P_{jr-1}^* - P_{jr}^* \geq \frac{1}{2} \sum_{t=m}^{r-2} h_{jt}
\end{aligned}$$

Proof. We write the optimal prices for product j in periods m and r as:

$$P_{jr}^* = \frac{1}{2} \left(\frac{\alpha_j}{\beta_j} + c_{jr} + v_j \lambda_r^* \right) \tag{6.19}$$

$$P_{jm}^* = \frac{1}{2} \left(\frac{\alpha_j}{\beta_j} + c_{jm} + v_j \lambda_m^* \right) \tag{6.20}$$

Now Let us suppose $P_{jr}^* > P_{jm}^*$. It follows from (6.19) and (6.20)

$$\begin{aligned} c_{jr} + v_j \lambda_r^* &> c_{jm} + v_j \lambda_m^* \\ \Rightarrow \lambda_r^* &> \lambda_m^* \end{aligned} \quad (6.21)$$

Since $X_{jmr}^* = 0$:

$$\Rightarrow v_j \lambda_m^* \geq \frac{\alpha_j}{\beta_j} - c_{jm} - \sum_{t=m}^{r-1} h_{jt} \quad (6.22)$$

Also $X_{jrr}^* > 0$:

$$\Rightarrow v_j \lambda_r^* < \frac{\alpha_j}{\beta_j} - c_{jr} \quad (6.23)$$

From (6.22) and (6.23) we conclude:

$$\lambda_m^* \geq \lambda_r^* \quad (6.24)$$

But (6.21) and (6.24) are contradictory and can not be true at the same time. Hence our initial assumption is incorrect and $P_{jr}^* \not> P_{jm}^*$, $\Rightarrow P_{jr}^* \leq P_{jm}^*$.

This completes the proof for part (i) of the theorem. Proof of parts (ii) and (iii) follows from Theorem 3.

□

Lemma 6. The optimal prices decrease monotonically in capacity for a given sequence of setups.

Proof. let 1 be the index used to denote optimal solution for some period m with capacity b_m^1 , and 2 be the index used to denote optimal solution for m at capacity b_m^2 such that $b_m^2 > b_m^1$.

Observe that if the capacity constraint (3.13) is not binding at optimal solution for capacity b_m^1 , then it must be non-binding for solution at capacity b_m^2 as well. Therefore the optimal solution will be the same at both capacities. If the capacity constraint is binding at b_m^1 but not binding at b_m^2 , the optimal prices at higher capacity are equal to revenue maximizing prices. Therefore, $P_{jn}^2 \leq P_{jn}^1$.

Now consider a scenario when constraint (3.13) is binding for optimal solutions at either capacities. One of the following must be true for the two solutions:

i). The production variables X_{jmn}^{1*} that were equal to zero in the first optimal solution at b_m^1 are also zero at the new optimal solution at b_m^2 . Since $b_m^2 > b_m^1$, it follows from (3.22) and (3.20) that $\lambda_m^{2*} < \lambda_m^{1*}$, and $X_{jmn}^{2*} \geq X_{jmn}^{1*} \forall (j, m, n)$. Therefore $\sum_{t=1}^n X_{jtn}^{2*} \geq \sum_{t=1}^n X_{jtn}^{1*}$. It follows from (3.11) that $P_{jn}^2 \leq P_{jn}^1$.

ii). At least one of the variables X_{jmn}^{1*} that were equal to zero in the first optimal solution, is strictly positive in the new solution. Now this means $l_m^{2*} > l_m^{1*}$, and since the variables in index l_m are arranged in decreasing order of partial derivatives $z_{P1}(0)$, it follows from algorithm- 1 and lemma -2 that $\lambda_m^{2*} < \lambda_m^{1*}$. Rest of the argument is the same as described in (i) above; hence $P_{jn}^2 \leq P_{jn}^1$. \square

Lemma 7. For any optimal solution of Problem QI, there exist some P_{jt}^* and W_{jt}^* for all j, t such that there is no lost sales.

Proof. Let us suppose that at some optimal solution P_{jt}^* and W_{jt}^* , the demand is greater than sales for some j, t . Now we consider following three cases:

a) Let us fix the advertising variables to current solution W_{jt}^* . Let $\bar{\alpha}_j = \alpha_j + \delta_{jt} W_{jt}^{*r}$, then $D_{jt}^* = \bar{\alpha}_j - \beta_j P_{jt}^*$ for all j, t . It follows from Theorem 1 that there exists another price P'_{jt} such that the profit at P'_{jt} is strictly greater than profit at P_{jt}^* . Furthermore, there is no lost sales at the new solution. Hence, P_{jt}^* is not the optimal price.

b) Let us fix the price to current solution P_{jt}^* . Since demand is increasing in advertising we can find some $W'_{jt} < W_{jt}^*$ such that the demand at (P_{jt}^*, W'_{jt}) is equal to sales. The

two solutions have same revenue, however the new solution has a lower advertising cost and hence higher profit. Moreover, since $W'_{jt} < W^*_{jt}$ the advertising budget constraint 5.18 is also satisfied. Therefore, W^*_{jt} could not have been an optimal solution.

c) We allow both P^*_{jt} and W^*_{jt} to vary and look for an alternate solution. It follows from cases (a) and (b) above that there exists another solution P'_{jt} and W'_{jt} such that the profit at the new solution is strictly greater than the original profit. Moreover there is no lost sales at the new solution. This completes the proof. □

Lemma 8. The objective function z_{QI} is concave in advertising expenditure W_{jt}

Proof.

$$\frac{\partial^2 z_{QI}}{\partial W_{jt}^2} = \frac{q_{jt}\delta_j}{\beta_j} r(r-1)W_{jt}^{(r-2)} \quad (6.25)$$

Since $0 < r < 1$, the second derivative of the objective function w.r.t. W_{jt} is strictly negative and hence z_{QI} is a concave function of W_{jt} □

Lemma 9.

Let W^*_{jt} be the optimal advertising expenditure for product j in period t , then there exists a unique solution of QI:

$$W^*_{jt} = \begin{cases} \frac{K_{jt}}{\sum_{j=1}^J \sum_{t=1}^T K_{jt}} \Omega & \text{if } \text{constraint}(5.21) \text{ is binding} \\ K_{jt} & \text{if } \text{constraint}(5.21) \text{ is not binding} \end{cases} \quad (6.26)$$

Where $K_{jt} = \left(\frac{q_{jt}\delta_j r}{\beta_j} \right)^{\frac{1}{r-1}}$

Proof. The objective function of Problem QI is concave and it has one linear constraint. Let θ^* be the optimal Lagrange multiplier for constraint 5.21, then it follows from KKT

conditions that at the optimal solution:

$$\frac{q_{jt}\delta_j r}{\beta_j} W_{jt}^{*r-1} - 1 - \theta^* = 0 \quad (6.27)$$

$$\theta^* \left(\sum_{j=1}^J \sum_{t=1}^T W_{jt}^* - \Omega \right) = 0 \quad (6.28)$$

$$\sum_{j=1}^J \sum_{t=1}^T W_{jt}^* - \Omega \leq 0 \quad (6.29)$$

$$\theta^*, W_{jt}^* \geq 0 \quad (6.30)$$

Now from 6.27

$$\begin{aligned} W_{jt}^{*r-1} &= \frac{\beta_j}{q_{jt}\delta_j r} (1 + \theta^*) \\ W_{jt}^* &= \left(\frac{\beta_j}{q_{jt}\delta_j r} (1 + \theta^*) \right)^{\frac{1}{r-1}} \\ &= K_{jt} (1 + \theta^*)^{\frac{1}{r-1}} \end{aligned} \quad (6.31)$$

$$\text{Where: } K_{jt} = \left(\frac{q_{jt}\delta_j r}{\beta_j} \right)^{\frac{1}{1-r}}$$

If the budget constraint 5.21 is not binding at the optimal solution i.e. $\sum_{j=1}^J \sum_{t=1}^T W_{jt}^* - \Omega < 0$, then $\theta^* = 0$ follows from 6.28, and from 6.31 $W_{jt}^* = K_{jt}$.

Else, $\theta^* > 0$, and

$$\begin{aligned} \sum_{j=1}^J \sum_{t=1}^T W_{jt}^* - \Omega &= 0 \\ \sum_{j=1}^J \sum_{t=1}^T K_{jt} (1 + \theta^*)^{\frac{1}{r-1}} - \Omega &= 0 \\ (1 + \theta^*)^{\frac{1}{r-1}} &= \frac{\Omega}{\sum_{j=1}^J \sum_{t=1}^T K_{jt}} \end{aligned} \quad (6.32)$$

Now, from 6.32 and 6.31 we have

$$W_{jt}^* = \frac{K_{jt}}{\sum_{j=1}^J \sum_{t=1}^T K_{jt}} \Omega$$

□

Lemma 10. Let $\eta = \frac{P}{D} \frac{\partial D(P)}{\partial P}$ be the price elasticity of demand, and $\mu = \frac{\partial D(W)}{\partial W} P$ be the marginal revenue of advertising, then optimal solution of QI must satisfy following necessary conditions for all j, t :

$$(i.) \quad \mu_{jt} \geq \eta_{jt}, \quad (6.33)$$

$$(ii.) \quad \frac{\mu_{11}}{\eta_{11}} = \frac{\mu_{12}}{\eta_{12}} = \dots = \frac{\mu_{jt}}{\eta_{jt}} = \dots = \frac{\mu_{JT}}{\eta_{JT}}. \quad (6.34)$$

Proof. From (6.27) we can write:

$$\begin{aligned} \frac{q_{jt} \delta_j^r W_{jt}^{r-1}}{\beta_j} &= 1 + \theta^* \\ \frac{P_{jt} \delta_j^r W_{jt}^{r-1}}{P_{jt} \beta_j / q_{jt}} &= 1 + \theta^* \\ \frac{\mu_{jt}}{\eta_{jt}} &= 1 + \theta^* \\ \Rightarrow \theta^* &= \frac{\mu_{jt}}{\eta_{jt}} - 1 \end{aligned} \quad (6.35)$$

But $\theta^* \geq 0$, hence it follows from (6.35) that relationships (i) and (ii) must hold at the optimal solution. □

Lemma 11. Algorithm OA2 generates a sequence of solutions such that at any iteration k of the algorithm $\max z_{QP}^{k-1} \leq \max z_{QI}^k \leq \max z_Q^k \leq \max z_{QM}^k$.

Proof.

- i. Observe that every solution of Problem QP at some iteration k is also feasible for Problem QI at iteration $k + 1$. Hence solution of QP is a lower bound for QI i.e.

$$\max z_{QP}^{k-1} \leq \max z_{QI}^k.$$

ii. At any iteration k Problem QI is a restriction of problem Q. Hence $\max z_{QI}^k \leq \max z_Q^k$.

iii. At any iteration k Problem QM is a relaxation of problem Q. Hence $\max z_Q^k \leq \max z_{QM}^k$.

Therefore, $\max z_{QP}^{k-1} \leq \max z_{QI}^k \leq \max z_Q^k \leq \max z_{QM}^k$ must be true for all k . \square

Lemma 12. Algorithm 4 terminates in a finite number of iterations ($\leq TJ$) to an optimal solution of QIA.

Proof. For any iteration k of Algorithm 4, consider the following cases:

i. Set U is empty:

If $U = \{\}$, then constraint set (5.34) is satisfied and the solution must be optimal to QIA.

ii. Set U has exactly one member:

First consider the simplest case when the set U has only one element (l, n) at the current optimal solution of QI. By construction, Algorithm 4 computes the objective value for $W_{ln} = 0$ and $W_{ln} = \omega_j$, it updates current bounds, selects the best solution, and terminates. The solution thus found is optimal.

iii. Set U has more than one member:

Now consider the case when the set U has more than one member at some iteration k . Observe that Algorithm 4 computes lower bound in step 4 and then attempts to improve this bound by fixing one variable to zero. Suppose we randomly select some member of U and set corresponding $W_{jt} = 0$. Since q_{jt} is fixed for each (j, t) ; a decrease in advertising effect must be compensated by an equal increase in price effect. Let superscripts 1 and 2 denote the prices for $W_{jt} > 0$ and $W_{jt} = 0$ respectively, then:

$$\begin{aligned}
\beta_j(P_{jt}^1 - P_{jt}^2) &= \delta_j W_{jt}^r \\
(P_{jt}^1 - P_{jt}^2) &= \frac{\delta_j W_{jt}^r}{\beta_j}
\end{aligned} \tag{6.36}$$

Multiplying both sides of (6.36) by q_{jt} gives an expression for decrease in revenue:

$$q_{jt}(P_{jt}^1 - P_{jt}^2) = \frac{\delta_j}{\beta_j} q_{jt} W_{jt}^r$$

There is a decrease in cost equal to W_{jt} , therefore :

$$\begin{aligned}
\text{Decrease in profit} &= q_{jt}(P_{jt}^1 - P_{jt}^2) - W_{jt} \\
&= \frac{\delta_j}{\beta_j} q_{jt} W_{jt}^r - W_{jt} \\
&= \Delta_{jt}
\end{aligned} \tag{6.37}$$

Therefore, selecting some $W_{l,n} = 0$ such that $\Delta_{ln} = \min \Delta_{jt} \forall (j, t) \in U$ will provide a solution whose objective value is greater than or equal to the objective value obtained by fixing any other member of U to zero. Hence it must be optimal to fix $W_{ln} = 0$.

Once we fix some variable to zero value, it follows from (5.25) that the optimal value of remaining members of U must increase in the next iteration. Thus one or more than one members of U become feasible during each iteration. Since the total number of decision variables is JT , the algorithm terminates in no more than JT iterations.

□