DEVELOPMENT OF THE EQUATIONS FOR A NUMERICAL MODEL OF A STEAMFLOOD TO BE APPLIED TO A WATERFLOODED RESERVOIR

by

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A THESIS

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NOMENCLATURE

\[ A = \text{area} \]
\[ A_f = \text{formation area} \]
\[ A_F = \text{extent of steam front} \]
\[ a_i = \text{thermal diffusivity of species } i \]
\[ b = \text{width of the reservoir} \]
\[ B = \text{ratio of the latent heat to the total volumetric heat capacities of the steam zone} \]
\[ B_i = \text{parameter related to characteristic time of steam zone growth} \]
\[ C_p = \text{specific heat capacity at constant pressure} \]
\[ D = 1 - S_{or} - S_{wc} \]
\[ f = T M_j, j=1,2; \text{heat capacity term} \]
\[ f_{1l} = \text{value of water cut at economic limit} \]
\[ f_s = \frac{df_w}{dS_w} \]
\[ f_{sl} = \text{value of } f_s \text{ at economic limit} \]
\[ f_{sc} = \text{value of } f_s \text{ at breakthrough} \]
\[ f_{sq} = \text{steam quality} \]
\[ G = \text{oil-water viscosity ratio} \]
\[ h = \text{thickness of reservoir} \]
\[ h_i = \text{specific enthalpy of phase } i \]
\[ k = \text{absolute permeability} \]
\[ k_h = \text{thermal conductivity} \]
\[ k_{ri} = \text{relative permeability of phase } i \]
L = length

$L_c$ = characteristic length of the steam zone

$L_s$ = steam zone length

$L_v$ = latent heat of vaporization of water

$L_L$ = linear operator

$L_N$ = nonlinear operator

$M$ = volumetric heat capacity

$M_g$ = rate of steam condensation per unit reservoir volume and time

$N_{st}$ = Stefan number

$N_p$ = total oil production

$p_c$ = capillary pressure

$r(t)$ = radius at time t

$S$ = saturation

$S_c$ = critical water saturation (constant)

$S_{or}$ = immobile oil saturation (constant)

$S_{wb}$ = breakthrough water saturation

$S_{wc}$ = immobile water saturation (constant)

$S_{wl}$ = water saturation value at economic limit

$S_w(x)$ = water saturation function in x-direction

$W_i$ = cumulative gross production of the total field

$X$ = dimensionless distance

$X_1$ = coordinate where $S_w = 1 - S_{or}$

$X_c$ = highest value of $X$ where water occurs

$X_1$ = coordinate where economic limit is reached

$u$ = volumetric velocity per unit reservoir cross-section

$v$ = steam front velocity
\( V \) = volume
\( w \) = water mass injection rate per unit injection area
\( x \) = space coordinate
\( y \) = space coordinate
\( z \) = space coordinate
\( Z \) = function defined by: \( L_{SF}^{D}(Z_{D}, t_{D}) = T_{D}(t_{D})Z_{D}(Z_{D}) \)
\( \lambda \) = time in Marx-Langenheim equation: \((x,y)<t\)
\( \mu \) = viscosity
\( P \) = perimeter of the steam front
\( \rho_{i} \) = density
\( \tau \) = time
\( \phi \) = porosity
\( \Phi \) = heat injection rate per unit injection area
\( \psi_{ncd} \) = conductive heat flux to the hot liquid zone
\( \psi_{ncv} \) = net convective heat flux to the hot liquid zone

Subscripts

\( D \) = dimensionless quantity
\( f \) = rock formation
\( F \) = steam front
\( i \) = initial
\( inj \) = injection point
\( n \) = normal to surface
\( o \) = oil
\( ob \) = overburden
\( R \) = reservoir
\( s \) = steam
ub = underburden
w = water
1 = refers to energy balance
2 = refers to steam balance

Superscripts
I = inner side of surface of steam front
II = outer side of surface of steam front
- = spatial average
+ = upper bounds
+ = vector
NOMENCLATURE FOR HISTORICAL DEVELOPMENT

Buckley–Leverett:

\( A \) = cross-sectional area
\( f_D \) = fraction of flowing steam comprising displacing fluid
\( Q_T \) = total amount of displacing fluid
\( q_T \) = total rate of flow through section
\( S_D \) = saturation of displacing fluid
\( \theta \) = time
\( u \) = distance along path of flow
\( \Delta u \) = incremental distance along path of flow
\( \phi \) = porosity

Welge:

\( f \) = fractional flow
\( f' \) = derivative of fractional flow with respect to \( S_w \)
\( l_{oil} \) = fractional flow of oil
\( L \) = length of section
\( Q_I \) = initial flow rate
\( S_{av} \) = average water saturation
\( s \) = saturation of phase
\( t \) = time
\( v \) = velocity
\( x \) = distance, \( 0 \leq x \leq L \)
Lauwerier:

\[ b = \text{thickness of layer} \]
\[ \eta = \text{dimensionless distance in y-direction} \]
\[ \theta = \text{ratio of specific heat of water to specific heat of oil} \]
\[ T = \text{temperature} \]
\[ \tau = \text{dimensionless time} \]
\[ \xi = \text{dimensionless distance in x-direction} \]
\[ V = \text{unit function} \]

Marx-Langenheim:

\[ D = \text{overburden thermal diffusivity} \]
\[ H_0 = \text{constant heat injection rate} \]
\[ h = \text{pay thickness} \]
\[ k = \text{overburden thermal conductivity} \]
\[ M = \text{formation heat content} \]
\[ \Delta T = T_i - T_0 \]
\[ T_i = \text{injection temperature} \]
\[ T_0 = \text{initial formation temperature} \]
\[ t = \text{time since start of heat injection} \]
\[ x = \text{dimensionless variable} \]

Mandl-Volek:

\[ C = \text{specific heat per unit mass} \]
\[ f_v = \frac{u_w}{u_w + u_0} \]
\[ F = \text{ratio between instantaneous and initial heat injection} \]
\[ k_{he} = \text{effective thermal conductivity in liquid zone} \]
\[ k_{hf} = \text{thermal conductivity in cap and base rock} \]
\( L_v \) = latent heat per unit mass of steam

\( S \) = saturation; without subscript: water saturation in liquid zone

\( \bar{S} \) = average water saturation in steam zone

\( S^+ \) = boundary value of water saturation

\( t \) = time

\( t_c \) = critical time (onset of convective heat transport into liquid zone)

\( t_D \) = \( \frac{t_c}{v_c} \), dimensionless time

\( v_c \) = critical velocity of the CF (onset of convective heat transport into liquid zone)

\( W_{w}^{W_{st}} \) = mass rates of hot-water and steam injection through unit cross-section

\( \delta Q_{st} \) = rate of local heat loss from steam zone to cap and base rock

\( \xi \) = length of steam zone

\( \bar{\xi} \) = lower bound length of steam zone

\( \overline{\xi} \) = upper bound length of steam zone

\( \rho \) = density

\( \rho_{sc} \) = average of specific heat per unit volume of steam zone

\( \sigma \) = characteristic dimensionless parameter

\( \phi \) = porosity

Subscripts:

1 = steam zone

2 = liquid zone

st = steam

w = water

o = oil

f = base and cap rock
Myhill and Stegemeier:

\[ A \quad = \quad \text{area/injector} \]
\[ A_s \quad = \quad \text{area of steam zone} \]
\[ E_{hs} \quad = \quad \text{thermal efficiency of steam zone} \]
\[ E_{hs} \quad = \quad \text{average thermal efficiency of steam zone} \]
\[ F_{os} \quad = \quad \text{ratio of oil displaced from steam zone to water as steam injected} \]
\[ f_{sd} \quad = \quad \text{steam quality, injector bottom-hole} \]
\[ h_D \quad = \quad \text{ratio of enthalpy of vaporization to liquid enthalpy} \]
\[ M_1 \quad = \quad \text{average heat capacity of steam zone} \]
\[ Q \quad = \quad \text{rate of heat injection} \]
\[ t \quad = \quad \text{time} \]
\[ V_{b, st} \quad = \quad \text{bulk volume of steam zone} \]

van Lookeren:

\[ A_{LD} \quad = \quad \text{dimensionless group for scaling a linear steam zone} \]
\[ A_{RD} \quad = \quad \text{dimensionless group for scaling a radial steam zone} \]
\[ C \quad = \quad \text{heat capacity} \]
\[ F_1(t) \quad = \quad \text{ratio of heat in steam zone to sum of heat in steam zone and heat in cap rock and underlying rock} \]
\[ h \quad = \quad \text{reservoir thickness} \]
\[ \bar{h}_{st} \quad = \quad \text{areally averaged steam zone thickness} \]
\[ H \quad = \quad \text{enthalpy of steam at } T_{st} \text{ relative to } T_f \]
\[ M^* \quad = \quad \text{mobility ratio of temperature } T_f \]
\[ r \quad = \quad \text{radius} \]
\[ t \quad = \quad \text{time} \]
\[ T \quad = \quad \text{temperature} \]
\[ V_{b, st} \quad = \quad \text{bulk volume of steam zone} \]
*w* = mass flow rate  
*x* = linear distance  
*ρ* = density  

**Subscripts:**  
*e* = top endpoint of steam zone at *x* sub *e* or *r* sub *e*  
*f* = formation  
*i* = initial  
*st* = steam  
*1* = at Point 1  

**Neuman**  
*A* = area  
*C* sub *w* = heat capacity of water  
*h* = steam zone thickness  
*L* sub *v* = heat of vaporization of water  
*M* sub *s* = volumetric heat capacity of steam  
*Q* = heat injection rate  
*t* = time  
*t* sub *+* = time until steam injection slowed or stopped  
*t* sub *- τ* = time after first heating to steam temperature  
*T* = temperature  
*V* = volume  

**Vogel**  
*A* = projected area  
*h* = thickness of steam zone  
*K* = thermal conductivity
Q = cumulative heat consumption
\dot{Q} = rate of heat consumption
t = time
T = temperature

Subscripts:
l = heat loss zones overlying or underlying the steam zone
s = steam zone

Yortsos and Gavalas
See nomenclature of main text.
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ABSTRACT

The purpose of this study is to develop a mathematical formulation and the equations for use in a numerical model that will simulate the performance of a steamflood applied to a waterflooded reservoir. Through the application of such a process the incremental oil recovery can hopefully be determined as accurately as possible.

An accurate prediction is critical in determining the economic feasibility of such an operation. Furthermore, such a model will present the opportunity to evaluate the effects of parametric changes on the recovery efficiency. Also critical to the study is the consideration of the high water saturations due to the precession of the waterflood. The effects on the development and movement of the steam zone are examined as well as the effects on the formation of steam-override.

For the study, the van Meurs and van der Poel theory is adopted which is capable of defining water saturations at any stage of a waterflood project. The water saturation profile thus obtained is then defined as the initial water saturation profile in the reservoir at the onset of the steamflood.

A general analytical solution is presented which follows closely the Yortsos and Gavalas upper bounds theory. The solution yields two upper bounds for the volume of the steam zone for a three-dimensional geometry. The results for steam zone volume growth are then used in calculating the
incremental oil recovery based on the Myhill and Stegemeier oil recovery equations.

A tentative procedure for a numerical model solution is also presented only to be detailed in another study.
CHAPTER I
INTRODUCTION

With the ever increasing worldwide demand for petroleum and the continuously diminishing supplies (Fig. 1), it has become necessary to increase our exploration efforts to discover new fields that will provide the world with the oil it needs to meet future requirements.

New reserves are and will continue to be harder to find and more expensive and difficult to produce, thus underscoring the reality that the world's oil is a finite resource. For this reason it is becoming increasingly necessary to focus attention not only on finding new fields, but also on improving the recoveries from actively producing ones, with the hope that this will add significantly to potential and proven reserves.

Primary methods of production usually recover only 20 to 30 percent of the oil in the reservoir, thus leaving behind a considerable unexploited amount of the original oil in place. New and existing enhanced recovery methods can be applied to producing reservoirs to improve declining production, or can be applied to restart production from shut-in wells. In contrast, the exploration game is a game of chance and can take a long time before it can discover and produce a new field.

Secondary recovery techniques have been in application for decades now and the most successful and predominant method has been the waterflood. However, in many fields waterfloods have reached or surpassed their maturity leaving behind a significant unproduced portion of the oil in place.
Figure 1. World Oil Discoveries and Production
(Exxon Background Series)
The use of a steamflood as a tertiary recovery mechanism, applied to a waterflooded reservoir, for the express purpose of obtaining incremental production needs to be examined as an alternative process to more complex and expensive methods.

An attempt is made in this study to describe the process of augmenting a waterflood project in its maturity with a steamflood. A numerical model will be developed which will, as accurately as possible, determine the incremental recovery from such an application in order to judge its economic feasibility. To achieve this, first a mathematical model needs to be formulated that describes a waterflooded reservoir under a steamflood with initial conditions chosen to coincide with the most recent or final conditions of the waterflood project.

The model is developed as a normal steamflood following the theory for upper and lower bounds developed by Yortsos and Gavalas with the addition of a known water saturation distribution profile based on van Meurs and van der Poel's theory of viscous fingering. The profile replaces what otherwise would have been an inferred average water saturation value. This is a very critical consideration because of the very high water saturations particularly close to the injection well, where $S_w \approx 1 - S_{or}$, due to the precession of the waterflood. The problem is further compounded due to the tendency of steam to override the formation. The effect of the excess water on the steam-front formation and movement is central to this study and differentiates this model from other existing steamflood models.

This study considers a viscous reservoir with an adverse mobility ratio. The van Meurs and van der Poel theory was selected particularly for its
treatment of viscous conditions since the Buckley-Leverett \(^3\) and Welge \(^4\) theories are basically for more favorable mobility ratios. \(^5\)

For the solution to the problem the Yortsos and Gavalas \(^1\) analytical modeling solution is followed which yields two upper bounds for the extent of the steam-zone volume. The volume obtained is then used with Myhill and Stegemeir's \(^6\) equation for the calculation of incremental recovery. A numerical solution method is tentatively suggested as a follow-up study.
Numerical models that simulate reservoir operations have been used extensively in petroleum engineering. Their technology is constantly refined and with the advances of modern computer and software technology can now describe very complex processes that take place in the reservoir. They have not only assisted in providing us with solutions to problems unsolvable before but have also improved our understanding of the mechanisms of recovery.

A reservoir simulator, if used properly, is a very useful tool. It does not, however, replace sound engineering judgment based on past experiences. In the sections that follow, a review with historical sequence briefly examines the focal points of the most important mathematical works regarding waterflood and steamflood theories and experiments. The treatment is considered necessary because of the continuity that exists in the development of the theories.

2.1 Waterflooding Theories

Waterflooding is one of the oldest and most widely used secondary recovery methods and the two following works are considered pioneering developments of the waterflooding theories.
2.1.1 Buckley–Leverett\textsuperscript{2}

The most important original work for waterflooding was developed by Buckley and Leverett, whose description of the fluid displacement mechanisms is still used today even in the more advanced waterdrive studies.

Using a material balance equation and neglecting gravity and capillary pressures, Buckley and Leverett expressed the linear flow of two incompressible fluids and their saturation advancement through a small element of sand as

\[
\left( \frac{\partial S_D}{\partial \theta} \right)_u = - \frac{q_T}{\phi A} \left( \frac{\partial f_D}{\partial u} \right) \theta \tag{2.1}
\]

or

\[
\left( \frac{\partial u}{\partial \theta} \right)_{S_D} = \frac{q_T}{\phi A} \left( \frac{\partial f_D}{\partial S_D} \right) \phi \tag{2.2}
\]

From the above equation the total amount of displacing fluid entering the system is

\[
\Delta u = \frac{q_T}{\phi A} \cdot \frac{df_D}{dS_D} \tag{2.3}
\]

The graph of the above solution is water saturation versus distance yields triple values of water saturation for part of its length. This physical impossibility was attributed by Buckley–Leverett to the formation of a discontinuity or "shock front" due to the omission of the capillary pressures in the formation of their theory. They evaluated the position and strength of the shock front from material balance considerations. Numerous other methods have been proposed to deal with the discontinuity but they are beyond the scope of this study.
2.1.2 Welge

Welge derived an analytical method for computing average water saturation values to determine oil recovery. Starting from the Buckley-Leverett equation, he arrived at the following expression for the constant saturation plane velocity:

\[
\left( \frac{\Delta x}{\Delta t} \right)_S = \left( \frac{dx}{dt} \right)_S = - \frac{\partial S}{\partial t} \frac{\partial S}{\partial x} = \nu \frac{df}{dS} = \nu f'.
\] (2.4)

From the above equation Welge further derived a relationship for the velocity near the outlet where \(x = L\), the reservoir length

\[
\frac{1}{f'} = \frac{1}{f'(S)} = \frac{v \Delta t}{L} = Q_1.
\] (2.5)

where \(Q_1\) is the cumulative injection when that particular saturation reaches the outlet. Hence an average water saturation can be determined from the above to be

\[
S_{av} = S_2 - \int_1^{f_2} \frac{df}{f'} = S_2 - \frac{f_2 - 1}{f'_2}
\] (2.6)

or

\[
S_{av} - S_2 = (f_{oil})_2 Q_1
\] (2.7)

where 1 denotes the inlet point and 2 denotes the outlet point.

2.2 Steamflooding Theories

The following are works based on a historical sequence and present the evolution of the steamflood theories. Each work is referred to by its
author(s) name(s) and concentrates on the focal points which are supplemented by their mathematical expressions. The nomenclature for these theories is given in a separate nomenclature section for ease of use.

2.2.1 Lauwerier

The pioneer work for describing the transportation of heat in an oil layer by the injection of hot liquid belongs to Lauwerier who developed equations for predicting temperature distributions as a function of time and position.

The following equations were formulated based on the assumption that the thermal conductivity is zero in the direction of flow and infinite perpendicular to it, thus maintaining a uniform temperature in the water layer. In addition, the thermal equilibrium at the fluid/sand grain interface is assumed to be established instantaneously.

\[
\text{For } |\eta| > 1 \quad \theta \frac{\partial^2 T_2}{\partial n} = \frac{\partial T_2}{\partial \tau} \quad (2.8)
\]

\[
\text{For } |\eta| = 1 \quad \frac{\partial^2 T_2}{\partial \tau^2} + \frac{\partial T_2}{\partial \xi} - \frac{\partial T_2}{\partial n} = 0 \quad (2.9)
\]

\[
\text{For } \tau = 0 \quad T_1 = T_2 = \begin{cases} T_0 & \text{if } \xi < 0 \\ 0 & \text{if } \xi > 0 \end{cases} \quad (2.10)
\]

The solution of the above system yields the temperature distribution in the water layer as:

\[
T_1 = T_0 \text{ erfc} \left( \frac{\xi}{2\sqrt{\theta (\tau - \xi)}} \right) V (\tau - \xi) \quad (2.11)
\]

If the injection temperature changes to \( T'_0 \) at time \( \tau' \) then the
solution can be found by superposition of the two time intervals, i.e.,

\[
T_1 = T_0 \text{ erfc} \left( \frac{\xi}{2 \sqrt{\theta(t - \xi)}} \right) - (T'_0 - T_0) \text{ erfc} \left( \frac{\xi}{2 \sqrt{\theta(t - t' - \xi)}} \right) \tag{2.12}
\]

for \( t > t' + \xi \).

2.2.2 Marx and Langenheim \(^8\)

Lauwerier's results were not used in the subsequent classic work by Marx and Langenheim which described a method for estimating thermal invasion rates

\[
H_0 = 2 \int_0^t \frac{k \Delta T}{\sqrt{\pi \Delta T (t - \lambda)}} \left( \frac{dA}{d\lambda} \right) d\lambda + Mh \Delta T \frac{dA}{dt} \tag{2.13}
\]

and cumulative heated areas

\[
A(t) = H_0 \frac{MhD}{4 \ k^2 \Delta T} \left( e^{x^2} \text{ erfc} \ x + \frac{2x}{\sqrt{\pi}} - 1 \right) \tag{2.14}
\]

or

\[
\frac{dA}{dt} = \frac{H_0}{Mh \Delta T} \left( e^{x^2} \text{ erfc} \ x \right) \tag{2.15}
\]

for a linear steam drive by assuming zero or negligible heat transfer ahead of the steam zone.

2.2.3 Willman \(^9\)

Willman et al. reported on the results of a laboratory study of oil recovery by steam injection and found by combining the Buckley-Leverett solution to the Marx-Langenheim equation. The steam zone residual oil saturation was shown to be essentially independent of initial saturation.
In addition, the heat required to exploit a reservoir appeared to be virtually independent of the amount of oil in place.

Their calculations assumed that the fractional flow of water across the steam zone boundary was equivalent to the fractional flow of water across an isothermal hot water boundary at steam temperature.

2.2.4 Mandl and Volek

Mandl and Volek, while studying the effect of heat convection from the steam zone into the oil/water zone, developed a theory suggesting that the Marx-Langenheim solution for the growth of a steam zone was not valid beyond a critical time, $t_c$. That time corresponded to when the heat transfer occurring across the condensation front into the hot liquid zone changed from mostly conductive to mostly convective flow, i.e., the zone downstream of the advancing condensation front is heated by hot water moving through the condensation front. Prior to the critical time, all injected heat is within the steam zone and the results of Mandl and Volek are the same as those of Marx and Langenheim.

The general heat balance for condensation front by Mandl and Volek is given by

$$C_w \left[ W_{st}(t) + W_w(t) \right] (T_1 - T_2) + W_{st}(t) L_v - \dot{Q}_{st}(t)$$

$$+ k_{he} \frac{\partial T_2}{\partial x} (x^+, t) = v(t) \left[ \phi \rho_{st} L_v \bar{s}_{st} + \frac{\rho_{1 c_1}}{T_1 - T_2} \right]$$

$$+ \phi \xi(t) \left[ \rho_{st} L_v - C_w \left[ \rho_w(t_1) - \rho_{st}(T_1) \right] (T_1 - T_2) \right] \frac{d}{dt} \bar{s}_{st} \quad (2.16)$$
where

- $W_{st}$ = mass rate of steam injection
- $W_w$ = mass rate of hot water
- $k_{he}$ = effective thermal conductivity in liquid zone
- $\xi$ = length of steam zone
- $\phi$ = porosity

with the additional condition

$$W_{st}(t)L_v - Q_{st}(t) \geq \phi \rho_{st} L_v \left[ v(t) \bar{S}_{st} + \xi(t) \frac{d \bar{S}_{st}}{dt} \right]. \quad (2.17)$$

When the heat flow across the condensation front is neglected, the above heat balance equation reduces to

$$W_{st}(t)L_v - Q_{st}(t) - \phi \rho_{st} L_v \left[ v(t) \bar{S}_{st} + \xi(t) \frac{d \bar{S}_{st}}{dt} \right] = \rho_{w} (T_1) \ddot{C}_{st} \left[ \rho_{w} (T_1) - \rho_{st} \right] \frac{d}{dt} \bar{S}_{st}$$

$$- \left[ W_{st}(t) + W_w(t) \right] C_w \left\{ T_1 \right\} \quad (2.18)$$

since

$$T_2 = 0 \quad \text{at} \quad x = \xi \quad \text{and} \quad T(\xi^{-}, t) = T_1, \quad T(\xi^{+}, t) = T_2 = 0, \quad \frac{\partial T_2}{\partial x} = 0$$

and whose solution yields for variable injection rates

$$\bar{S}(t_D) = \int_{0}^{t_D} F(\tau_0) e^{\sigma(t_D - \tau_D)} \text{erfc} \sqrt{\sigma(t_D - \tau_D)} d \tau_D \quad (2.19)$$
\[
\frac{d \xi_D}{dt_D} = F(t_D) - \sigma \int_0^{t_D} F(\tau_D) \left[ \sqrt{\frac{1}{\pi \sigma(t_D - \tau_D)}} \right] d\tau_D \\
- e^{-\sigma(t_D - \tau_D)} \text{erfc} \left( \sqrt{\frac{\sigma(t_D - \tau_D)}}{} \right) d\tau_D
\] (2.20)

or for constant injection rates (\(F = 1\))

\[
\xi_D(t_D) + \frac{1}{\sigma} \left[ e^{\sigma t_D} \text{erfc} \left( \frac{\sigma t_D}{\sqrt{\pi}} \right) \right] - 1
\] (2.21)

and

\[
\frac{d \xi}{dt_D} = e^{\sigma t_D} \text{erfc} \left( \frac{\sigma t_D}{\sqrt{\pi}} \right) . \] (2.22)

For this solution to be logically consistent with the physical model it must satisfy the condition

\[
W_{st}(t) - \dot{Q}_{st}(t) > 0
\] (2.23)

and since \(d\overline{S}_{st}/dt\) has been neglected the condition becomes

\[
v \geqslant v_c = \left[ \frac{W_{st}(t) + W_w(T)}{C_w} \right] \frac{1}{\rho_1 c_1}
\] (2.24)

which leads to the critical time equation

\[
t_c = \frac{b^2 \left( \rho_1 c_1 T_1 + \phi \rho_{st} L \overline{S}_{st}^2 \right)}{4 T_1^2 k_{hf} \rho_1 c_f} \sigma t_D \] (2.25)

where

\[b^2 = \text{width of the reservoir, squared.}\]
Beyond this critical time the above heat balance equations fail to describe the process.

The equation defining $\bar{\xi}(t_D)$ is considered as an upper bound for the steam zone growth rate. A lower bound $\check{\xi}(t_D)$ is developed from the following general equation that describes the heat balance beyond $t_c$.

$$W_{st}(t)L_v - \dot{Q}_{st}(t) - \phi_{st}L_v \left[ v(t)\bar{\xi}_{st} + \xi(t) \frac{d}{dt} \bar{\xi}_{st} \right] = 0 \quad (2.26)$$

the solution of which yields the difference between bounds

$$\sigma \Delta \xi_D(t_D) \equiv \sigma(\bar{\xi}_D - \xi_D) = \frac{2\sqrt{\sigma(t_D-t_{CD})}}{\sqrt{\pi}}$$

$$\left[ \frac{v_{CD}(t_D) - \check{V}_D(t_D)}{3} \right] - \frac{\sigma(t_D-t_{CD})}{3\sigma} \frac{d}{dt} \left[ V_{CD}(t_D) - \check{V}_D(t_D) \right] \right] \right) . \quad (2.27)$$

For a constant injection rate, the equation becomes

$$\sigma \Delta \xi_D(t_D) = \sigma(\bar{\xi}_D - \xi_D) = \frac{2\sqrt{\sigma(t_D-t_{CD})}}{\sqrt{\pi}}$$

$$\left\{ V_{CD} + \left[ \frac{\sigma(t_D-t_{CD})}{3} - 1 \right] V_D(t_D) - \frac{\sigma(t_D-t_{CD})}{3\sqrt{\pi}} \frac{1}{\sqrt{\sigma}} \right\} \quad (2.28)$$

yielding an approximation of the steam zone drive for $t_D > t_{CD}$ given by

$$\xi_D(t_D) = \frac{\bar{\xi}_D + \check{\xi}_D}{2} = \bar{\xi}_D(t_D) - \frac{\Delta \xi_D(t_D)}{2} . \quad (2.29)$$
The critical time concept is a significant contribution for a system wherein there is no gravity override of the steam zone and all points on the condensation front advance at the same rate.

The Mandl and Volek model can also determine the water saturation in the liquid zone

\[ 1 - f_v (S^+, T_1) = \left[ 1 - S^+ - S_{o1} \right] \alpha(t) \]  \hspace{1cm} (2.30)

given the fractional flow function \( f_v(S,T) \) for different temperatures.

2.2.5 Myhill and Stegemeier\(^6\)

Myhill and Stegemeier ignored the contribution of the condensate zone to the produced oil assuming that it was negligible. They defined thermal efficiency as the ratio of heat remaining in the steam zone to the total heat injected,

\[ E_{hs} = \frac{V_1 M_1 \Delta T}{Q_t} \] \hspace{1cm} (2.31)

By using Mandl and Volek's equations, Myhill and Stegemeier obtained the upper and lower bounds of the steam zone by relating them to the fraction of the injected heat present and the heat efficiency:

\[ \overline{E}_{hs} = E_{upper \ bound} - \left( \frac{1}{1 + h_D} \right) \Delta E \] \hspace{1cm} (2.32)

where \( \overline{E}_{hs} \) approaches zero as the steam quality decreases.
2.2.6 van Lookeren\textsuperscript{11}

The previous theories describe conditions in one or two dimensional geometries, without considering gravity segregation. A method was developed by van Lookeren for linear and radial systems to calculate the approximate shape of the growing steam zone.

**Case 1:** Even steam condensation between cap and base rock

\[ h_{st} = \left[ A_{LD} \left( \frac{1 - M^X}{\cos t_g} \right) - t_g \right] (X_e - X). \tag{2.33} \]

The shape of this interface is linear.

**Case 2:** Steam condensation at only the leading edge of steam zone

\[ h_{st} = \sqrt{2A_{LD} h (X_e - X)}. \tag{2.34} \]

The shape of this interface is parabolic between cap and base rock and the average tilt of this steam/liquid front is twice that of Case 1.

For radial systems the above equations become

**Case 1:**

\[ \frac{h_{st}}{A_{RD} h} = \left( \ln \frac{r_e}{r} - \frac{1}{2} + \frac{1}{2} \frac{r^2}{r_e^2} \right)^{\frac{1}{2}}. \tag{2.35} \]

**Case 2:**

\[ \frac{h_{st}}{A_{RD} h} = \ln \left( \frac{r_e}{r} \right)^{\frac{1}{2}}. \tag{2.36} \]

The growth of the steam zone with the tilted condensation front can now be determined by integrating the solutions according to Marx-Langenhein's or Mandl-Volek's approaches, yielding for the bulk volume of the steam zone.
\[ V_{b, st} = \frac{W_{st,i} H_{st}}{\rho_1 C_1 (T_{st} - T_f)} t (1 - f_p) F_1(\tau). \] (2.37)

The external radius of steam zone is then

\[ r_e(t) = \left( \frac{V_{b, st}}{\pi h_{st}} \right)^{\frac{1}{2}}. \] (2.38)

2.2.7 Neuman\textsuperscript{12}

Another approach to segregated flow was studied and modelled by Neuman and considers the presence of a hot water zone below a steam zone affected by gravity override. He assumed that the time it takes the steam to rise to the top of a permeable reservoir is negligible compared to the time required to heat the total reservoir and assumed simple values for oil and water saturations throughout the steam zone. He calculated the volume of the steam zone to be proportional to the net heat injected into the reservoir, given by

\[ V = \int_0^t \frac{dA}{dt} h (t - \tau) d\tau \] (2.39)

or

\[ V = \frac{Q C_w t}{M_s (L_v + C_w \Delta T_s)} \] (2.40)

and independent of the heat losses from the steam zone. He also developed a system of equations that specify a schedule of reduced heat injection rate, \( Q_t \), such that the area of the steam zone remains constant.
\[ Q(t) = \frac{Q_0}{\pi} \int_0^{t^*} \frac{1}{\tau \sqrt{t - \tau}} \, d\tau \]  

(2.41)

or

\[ Q(t) = \frac{2}{\pi} Q_0 \arctan \frac{t^*}{t-t^*} \]  

(2.42)

where \( t^* \) is the time at which areal growth stops. Total heat injected is

\[ Q_T = \dot{Q}_0 \left\{ 1 + \frac{2}{\pi} \left( \frac{t}{t^*} - 1 \right) \arctan \left( \frac{t}{t^*} - 1 \right) + \frac{1}{\pi} \ln \left[ 1 + \left( \frac{t}{t^*} - 1 \right)^2 \right] \right\} \]  

(2.43)

with steam reduction volume

\[ V = V^* \frac{2}{\pi} \left( \frac{t}{t^*} - 1 + \frac{t}{t^*} \arccot \sqrt{\frac{t}{t^*} - 1} \right) \]  

(2.44)

where \( V^* \) is steam zone volume at the time, \( t^* \), and the rate of increase of steam zone volume after \( t^* \) is

\[ \frac{dV}{dt} = \frac{2V^*}{\sqrt{\pi t^*}} \arctan \left( \frac{t^*}{t-t^*} \right)^{\frac{1}{2}} . \]  

(2.45)

The model predicts that the oil produced from the heated zone is directly proportional to the net heat injected as steam, an idea which is now central to calculating oil recoveries from steam floods.

2.2.8 Vogel

Vogel introduced some simplified expressions for heat calculations by considering Neuman's equations to calculate ultimate heat flow to the region underlying the steam chest.
\[ Q_1 = \int_{A(t)=0}^{A(t)} 2 \, k_1 \Delta T \, \frac{t-\lambda}{\pi a_1} \, dA(t). \] (2.46)

Assuming that the steam overlay is instantaneous (\( \lambda = 0 \)) over the entire project area (\( A(t) \)), the above equation becomes

\[ Q_1 = 2k_1 A \Delta T \sqrt{\frac{t}{\pi a_1}} \] (2.47)

and since heat retained in the steam chest is

\[ Q_s = Ah (\rho C)_s \Delta T \] (2.48)

the equation for total underground heat requirement is

\[ Q_{\text{total}} = Ah (\rho C)_s \Delta T + 2k_{11} \, A \Delta T \sqrt{\frac{t}{\pi a_1}} \]

\[ + 2k_{12} \, A \Delta T \sqrt{\frac{t}{\pi a_{12}}} \] (2.49)

where the overlying and underlying zones are 1 and 2, respectively.

---

2.2.9 Yortsos and Gavalas

The assumption that negligible heat flows into the hot liquid zone until time, \( t_c \), and the development by Mandl and Volek of upper and lower bounds for \( t > t_c \) are made for convection dominated reservoirs. This is true for high injection rates, but such assumptions lead to considerable deviations with low injection rates which do not favor convection dominated processes.

Yortsos and Gavalas, in a two-part study, attempted to clarify this concern by constructing analytical models that provide a detailed account
of heat transfer into the hot liquid zone. Beginning with basic partial
differential equations coupled with interfacial conditions, they developed the
following integral forms.

For thermal energy, the equation is given as

\[
\Delta T \frac{d}{dt} \int_{v(t)} M_1 dV + \Delta T A_F(t) \left[ \psi_{hcv}(t) + \psi_{hcd}(t) \right] \\
+ \int_{A_f(t)} \left( -k_{hf} \frac{\partial T_f}{\partial n} \right) dA = \left[ W_s(t) + W_s(t) \right] C_{pw} \Delta T \\
+ W_s(t)L_v
\]

(2.50)

where

\[ M_1 = \text{volumetric heat capacity} \]
\[ \psi_{hcd} = \text{conductive heat flux to the hot liquid zone} \]
\[ \psi_{hcv} = \text{convective heat flux to the hot liquid zone} \]
\[ k_{hf} = \text{thermal conductivity} \]

For the latent heat, the equation is given as

\[
\frac{d}{dt} \int_{v(t)} M_2 dV + \Delta T A_F(t) \psi_{hcd}(t) \\
+ \int_{A_f(t)} \left( -k_{hf} \frac{\partial T_f}{\partial n} \right) dA = W_s(t)L_v
\]

(2.51)

where \( M_2 \) is the volumetric heat capacity.
They established bounds on the integral heat losses for continuously advancing fronts as

\[
\frac{k_h T_s - T_i}{\sqrt{\pi a_f}} \frac{1}{\sqrt{t}} \int_{A_f(t)} dA < \int_{A_f(t)} \left( -k_h \frac{\partial T_f}{\partial n} \right) \ dA
\]

\[
< \frac{k_h (T_s - T_i)}{\sqrt{\pi a_f}} \int_{A_f(t)} \frac{dA}{\sqrt{t - \lambda(x,y)}}
\]  

as well as bounds on the heat fluxes to the hot liquid zone.

Combining the integral balance with the heat loss inequalities and observing that the resulting equations are comprised of linear and nonlinear segments, they arrived at the following bound

\[
\frac{d}{dt} \int_0^{L_{sf}(t)} f_j [x,t,L_{sf}(t)] \ dx + \frac{c}{\sqrt{t}} L_{sF}(t) \ < \Phi_j (t), \ t>0, \ j=1,2
\]

for a one-dimensional geometry, the solution of which yields

\[
L_{si}^+(t) = \frac{\exp \left( \frac{2c \sqrt{t}}{M_j \Delta T} \right)}{M_j \Delta T} \int_0^t \Phi_j(\tau) \exp \left( \frac{2c \sqrt{\tau}}{M_j \Delta T} \right) d\tau.
\]

This delivers two upper bounds for the steam zone, \(L_{s1}^+\) and \(L_{s2}^+\), depending on the dominance of either the conductive or convective forces. Similar expressions for multiple geometries are also considered.

In the second part of the study, Yortsos and Gavalas\(^{14}\) concentrated on the development of assymptotic solutions for small and large times of steam zone displacement.
CHAPTER III
MATHEMATICAL FORMULATION

Before a computer simulation of a reservoir can be developed, a mathematical model is required whose terms are suitable for describing the physical and mechanical conditions of the reservoir. From the mathematical formulation, a set of partial differential equations results, with appropriate initial and boundary conditions. These equations are usually too complex to be solved exactly by analytical methods and, hence, assumptions and approximations must be made in such a way that a valid mathematical model describing the process exists and is also amenable to a numerical simulator.

3.1 Waterflooding Mechanics

Central to the performance of any drive mechanism is the familiar idea of effective permeability. Ideal conditions for a piston-like displacement are only possible when the mobility ratio is $M < 1$. This exists when oil flow is ahead of an oil/water interface, in the presence of a residual oil saturation.

In this study we are concerned with viscous reservoirs where mobility ratios range from ten to several hundreds, a situation which creates serious problems in the recovery effort.

In the development of the model the abandonment conditions of the waterflood mechanism are considered. This means the flood has surpassed breakthrough. Because of the gradual increase in the water saturation behind the flood front, the quantity $\frac{dP_c}{dS_w}$ is considered small and is neglected.
3.2 Water Saturation Distribution

A water saturation profile is needed to describe the distribution of water saturations throughout the reservoir at the end of the waterflood. The final profile is needed because it will be established as the initial profile for the steamflood model. The importance of establishing such a profile is evident when we consider the increase in heat capacity due to the increased water saturation which will affect the growth and propagation of the steam front. Unfortunately, conventional water saturation relations obtained from the fractional flow theory, or the capillary-saturation interdependence, consider mainly the displacement process rather than the final saturation distribution.

3.2.1 Horizontal Distribution

Water saturation values, as was stated before, can range from an almost \((1-S_{or})\) near the injector well to \(S_{wc}\) near an all oil producing well. Therefore a water saturation profile is needed to describe conditions between the injector well and the producer well.

In the Buckley-Leverett frontal advance theory it is assumed that the initial displacement of oil by water is a smooth interface. The Welge approach to immiscible displacement also considers favorable mobility ratios. However, these conditions, although not at all unusual, are not compatible with our study which assumes high mobility ratios.

Studies by van Meurs and van der Poel showed that for high mobility ratios the displacement of oil by water exhibits viscous fingering. This phenomenon was evident even in laboratory systems where the porous media constructed was as uniform as possible.
They also presented a schematic representation of the water saturations in a formation with three distinct regions as shown in Fig. 2. The following assumptions were made based on an idealized oil field produced by linearly encroaching edge water:

1. the field is rectangular and has a constant thickness
2. from one side of the rectangle edge water is encroaching
3. the production is obtained from an infinite number of regularly distributed wells covering the entire field
4. the formation is homogeneous
5. when a predetermined water-cut value is reached the wells are closed
6. all the producing wells have the same gross production rate
7. gravity and capillary effects are negligible
8. initial oil saturation is 100 per cent.

Three regions are thus considered where in Region 1 the wells have been abandoned, in Region 2 the wells are producing oil and water, and in Region 3 the wells produce only clean oil. The variable $X$ is a dimensionless length.

Considering the regions separately, van Meurs and van der Poel determined the water saturation distributions as a function of the cumulative gross production as follows:²

**Region 1**

In this region water is injected at the point where $X = 0$ and oil is produced at the point where $X = X_1$. Thus, the amount of oil displaced in this case is equal to the amount of water injected. Hence, for
Figure 2. Illustration of how formation can be divided into three regions (top), accompanied by water saturation distribution diagram (bottom). (After van Meurs and van der Poel 2)
the water saturation distribution is given by the following equation:

\[ S_w = S_{w1} - \frac{D}{G-1} + \frac{\sqrt{G}D}{G-1} \sqrt{\frac{W_i}{x}}. \] (3.1)

The point \( x_1 \) is defined by

\[ x_1 = \frac{W_i}{MD} \] (3.2)

and the point \( x_1 \) is defined as

\[ x_1 = W_i f_{sl} \] (3.3)

where

\[ f_{sl} = \frac{1}{GD} \left[ f_1 + G (1 - f_1) \right]^2. \] (3.4)

For the case where \( X < x_1 \) the water saturation is easily expressed as

\[ S_w = 1 - S_{or} \] as shown by Fig. 2.

Region 2

In this region both water and oil flow simultaneously and the total production rate depends on the position \( X \) of the well because of the withdrawal of fluid from the wells located between the injector well and the well at point \( X \). Hence, for

\[ x_1 < X < x_c \]

the water saturation distribution is given by the following equation:
\[ S_w = S_{wc} - \frac{D}{G-1} + \frac{1}{G-1} \sqrt{\frac{GD}{f_{sl}}} \sqrt{\frac{1}{1-W_i f_{sl}}} \left( \ln \frac{1}{1-W_i f_{sl}} \right) \left( \ln \frac{1}{1-X} \right). \] (3.5)

The point \( X_c \) is defined by

\[ X_c = 1 - (1 - W_i f_{sl})^{f_{sl}/f_{sc}} \] (3.6)

where

\[ f_{sc} = \frac{GD}{D + \sqrt{D(G-1) S_{wc}}} \] (3.7)

The equation (3.5) becomes identical to equation (3.1) when \( X \) is

\[ X = X_1 = W_i f_{sl} \] (3.8)

which can be shown easily by direct substitution of equation (3.8) into equation (3.5). For the case where \( X = X_c \) the saturation is given by

\[ S_c = S_{wc} + \sqrt{\frac{DS_{wc}}{G-1}} \] (3.9)

Region 3

In this region, according to the assumptions, the wells produce only oil and thus

\[ S_w = 0. \] (3.10)

At any stage in the waterflood the appropriate region can be selected so as to obtain the corresponding water saturation distribution equation. The water saturation profile thus obtained will become the initial prevailing condition for the steamflood.
The van Meurs and van der Poel development assumes a one-dimensional linear displacement process and ignores the effects of gravity as it was stated in the assumptions. However, from physical considerations in a very viscous reservoir, gravity segregation cannot be ignored and a vertical water saturation profile will be needed in addition to the horizontal profile.

3.3 Steamflood Model

The engineering evaluation of a steamflood is often based on a simplified mathematical description of reservoir heating by hot fluid injection. This description, presented by Marx and Langenheim, was modified by Mandl and Volek and later expanded by Yortsos and Gavalas.

The theory, when combined with basic fluid flow considerations, formulates a steamflood model that determines the incremental oil recovery rates. In this study the theory is extended to consider the precession of the waterflood by incorporating the a priori known water saturation profile into the integral balance equations.

It is the effects of the precession of the waterflood on the steamflood application that the three-dimensional model attempts to simulate. However, such a theoretical treatment, which includes also the effects of preheating in the hot liquid zone, faces some difficulties. The heat capacities are saturation-dependent which is a critical point in this study. The steam front shape for a three-dimensional model is still a complicated problem for which satisfactory solutions have yet to be obtained. These difficulties are dealt with through the adoption of integral balance equations which allow for an analytical approach to the problem in lieu of a numerical one.
As with all complex mathematical modeling problems, to develop the governing equations certain assumptions must be made. It is assumed that the reservoir is homogeneous, isotropic and has a constant thickness. The steam quality and steam injection rate can either be constant or vary with time. The geometry examined is three-dimensional linear or radial and the equations are cast in uniform notation for both cases. A one degree symmetry is assumed to exist, however, in the geometry of the reservoir.

3.3.1 Mathematical Equations

The equations describing a three phase fluid flow are familiar. Using Darcy's Law for the velocities (Appendix A), the equations for each phase, when combined with the mass balances, give the partial differential equations governing the flow in the reservoir sand.

Looking at Fig. 3, which represents a reservoir undergoing a steam-flood process, it is seen that two distinct regions are formed. One is the region which the steam has overtaken and is called the steam zone. The other is the region where the steam has not arrived yet and where the oil and water flow together towards the producing wells and is called the hot liquid zone.

The equations describing mass and energy transfer inside the two regions at the reservoir in the absence of steam distillation of oil are as follows.

**Steam Zone**

**Water Mass Balance:**

\[
\phi \frac{\partial}{\partial t} \left( \rho_w S_w \right) + \nabla \cdot \left( \rho_w u_w \right) = -M_g
\]  

(3.11)
Figure 3. Reservoir notation for steamflood in a plan and cross section view. (After Yortsos and Gavalas)
Steam Mass Balance:

\[ \phi \frac{\partial}{\partial t} (\rho_s S_s) + \nabla \cdot (\rho_s \mathbf{u}_s) = M_g \]  

(3.12)

where \( M_g \) is a condensation term given by equation (A-4).

Oil Mass Balance:

\[ \phi \frac{\partial}{\partial t} (\rho_o S_o) + \nabla \cdot (\rho_o \mathbf{u}_o) = 0 \]  

(3.13)

Volumetric Balance:

\[ S_s + S_w + S_o = 1 \]  

(3.14)

Combining equations (3.11) and (3.18) yields

Total Water Mass Balance:

\[ \frac{\partial}{\partial t} (\rho_w S_w) + \nabla \cdot (\rho_w \mathbf{u}_w + \rho_s \mathbf{u}_s) = 0 \]  

(3.15)

Thermal Energy Balance:

\[
\phi \rho_o S_o \frac{\partial h_o}{\partial t} + \phi \rho_w S_w \frac{\partial h_w}{\partial t} + \phi \rho_s S_s \frac{\partial h_s}{\partial t} + (1 - \phi) \rho_R \frac{\partial h_R}{\partial t} \\
= \nabla \cdot k_{hr} \mathbf{V} T_R - (\rho_o \mathbf{u}_o \cdot \nabla h_o + \rho_w \mathbf{u}_w \cdot \nabla h_w + \rho_s \mathbf{u}_s \cdot \nabla h_s) \\
- M_g L_v
\]  

(3.16)

where \( L_v = h_s - h_w \) and is the Latent Heat of Vaporization for water.

Hot Liquid Zone

Oil Mass Balance:

\[ \frac{\partial}{\partial t} (\rho_o S_o) + \nabla \cdot (\rho_o \mathbf{u}_o) = 0 \]  

(3.17)
Water Mass Balance:

\[ \frac{\partial}{\partial t} (\rho_w S_w) + \nabla \cdot (\rho_w \mathbf{u}_w) = 0 \]  

(3.18)

Thermal Energy Balance:

\[ \phi \frac{\partial}{\partial t} \left[ \rho_w S_w h_w + \rho_o S_o h_o \right] + (1 - \phi) \rho_R \frac{\partial h_R}{\partial t} = \nabla \cdot k_{hr} \nabla T_R - \nabla \cdot \left[ \rho_w u_w h_w + \rho_o u_o h_o \right] \]  

(3.19)

Surrounding Formations

Thermal Energy Balance:

\[ (1 - \phi) \rho_f \frac{\partial h_f}{\partial t} = \nabla \cdot k_{hf} \nabla T_f \]  

(3.20)

Relations Across Steam Front

From Fig. 4 the following relations hold across the steam front to allow for possible discontinuities in saturations, temperature and volumetric velocities.

Total Oil Mass Balance:

\[ \rho_o^I (u_{on}^I - \phi S_o^I v_n) = \rho_o^H (u_{on}^H - \phi S_o^H v_n) \]  

(3.21)

Total Water Mass Balance:

\[ \rho_s^I (u_{sn}^I - \phi S_s^I v_n) + \rho_w^I (u_{wn}^I - \phi S_w^I v_n) = \rho_w^H (u_{wn}^H - \phi S_w^H(x) v_n) \]  

(3.22)

where \( S_w^H(x) \) is the water saturation distribution function in the hot liquid zone and for this study is given by equation (3.5), for a region where oil
Figure 4. Diagram of interface between the steam zone and the hot liquid zone. (After Yortsos and Gavalas)
and water flow together as

\[
S_w^I(x) = S_{wc} - \frac{D}{G-1} + \frac{1}{G-1} \sqrt{\frac{GD}{f_{sl}}} \ln \frac{1}{1-W_{f_{sl}}} .
\]  

(3.5)

Thermal Energy Balance:

\[
\rho_w \frac{I}{h_w} \frac{I}{u_{wn}} + \rho_s \frac{I}{h_s} \frac{I}{u_{sn}} + \rho_o \frac{I}{h_o} \frac{I}{u_{on}} - (k_{hR} \frac{\partial T}{\partial n})^I = \left[ \phi (\rho_w \frac{I}{h_w} \frac{I}{S_w^I} + \rho_s \frac{I}{h_s} \frac{I}{S_s^I} + \rho_o \frac{I}{h_o} \frac{I}{S_o^I}) + (1 - \phi) \rho_{R^I_{hR}} \right] n
\]

\[
= \rho_w \frac{I}{h_w} \frac{I}{u_{wn}} + \rho_o \frac{I}{h_o} \frac{I}{u_{on}} - (k_{hR} \frac{\partial T}{\partial n})^I
\]

\[
- \left[ \phi (\rho_w \frac{I}{h_w} \frac{I}{S_w^II} + \rho_o \frac{I}{h_o} \frac{I}{S_o^II}) + (1 - \phi) \rho_{R^II_{hR}} \right] n .
\]  

(3.23)

The equations from (3.11) to (3.23) describe the steam injection process applied to a waterflooded viscous oil reservoir. They include preheating into the liquid zone and an a priori known length dependent water saturation distribution. They also consider heat and mass transfer across two different regions coupled by interphase transport through a moving front and accompanied by phase change. Considering the last two features, the equations can be classified as a Stefan\textsuperscript{17} type formula.

Despite the complexity of the problem, for the study of the rate of growth of the steam zone in the light of integral aspects, an analytical approach is possible.
CHAPTER IV
SOLUTIONS

4.1 Solution Procedures

The equations (3.11) through (3.23) define the mathematical model. Although complex, this set of equations is amenable to two forms of solution: 1) an analytical and approximate solution, and 2) a numerical solution.

In the context of determining integral aspects of the process, the use of a reliable analytical model could be more suitable than highly sophisticated and expensive numerical simulators. However, numerical models are needed to obtain information regarding finer characteristics.

4.1.1 Analytical Procedure

In order to develop an analytical solution the Yortsos and Gavalas procedure is followed very closely. The approach rests on adoption of integral balances and the decoupling of the heat and mass transfer equations whenever possible.

From the conservation equations and over the steam zone the following hold.

For the Oil Mass Balance

$$\phi \frac{\partial}{\partial t} (\rho_o s_o) + \nabla \cdot (\rho_o u_o) = 0.$$  (3.13)
For the Total Water Mass

\[
\frac{\partial}{\partial t} \left( \rho_w S_w + \rho_s S_s \right) + \nabla \cdot (\rho_w \mathbf{u}_w + \rho_s \mathbf{u}_s) = 0.
\]  

(3.15)

For the Thermal Energy Balance

\[
\phi \rho_w S_w \frac{\partial h_w}{\partial t} + \phi \rho_s S_s \frac{\partial h_s}{\partial t} + \phi \rho_o S_o \frac{\partial h_o}{\partial t} + (1 - \phi) \rho_R \frac{\partial h_R}{\partial t}
= \nabla \cdot k_{HR} \nabla T_R - (\rho_w \mathbf{u}_w \cdot \nabla h_w + \rho_s \mathbf{u}_s \cdot \nabla h_s + \rho_o \mathbf{u}_o \cdot \nabla h_o)
- M_g L_v.
\]  

(3.16)

Combining the above equations and making use of the following interface conditions:

1. Total Water Mass Balance

\[
\rho_s (u_{sn}^I - \phi S_s^I v_n) + \rho_w^I (u_{wn}^I - \phi S_w^I v_n) = \rho_w^I (u_{wn}^I - \phi S_w^I v_n)
\]  

(3.22)

2. Total Oil Mass Balance

\[
\rho_o^I (u_{on}^I - \phi S_o^I v_n) = \rho_o^I (u_{on}^I - \phi S_o^I v_n)
\]  

(3.21)

3. Thermal Energy Balance

\[
\begin{align*}
\rho_w^I h_w^I u_{wn}^I + \rho_s^I h_s^I S_s^I + \rho_o^I h_o^I u_{on}^I - & (k_{HR} \frac{\partial T_R}{\partial n})^I \\
- \left[ \phi (\rho_w^I h_w^I S_w^I + \rho_s^I h_s^I S_s^I + \rho_o^I h_o^I S_o^I) + (1 - \phi) \rho_R^I h_R^I \right] v_n &= \rho_w^II u_{wn}^II + \rho_o^II h_o^II u_{on}^II - (k_{HR} \frac{\partial T_R}{\partial n})^II \\
- \left[ \phi (\rho_w^II h_w^II S_w^II + \rho_o^II h_o^II S_o^II) + (1 - \phi) \rho_R^II h_R^II \right] v_n,
\end{align*}
\]  

(3.23)
we can proceed to develop the analytical solution based on the following assumptions:

1. the average volumetric heat capacity of the steam zone is not expected to vary appreciably with time
2. the steam zone temperature is considered spatially uniform and equal to the temperature of the injected steam.

Integrating the conservation equation and making use of the interface conditions, the integral balances yield (Appendix B) the following inequality for a three-dimensional reservoir with one degree of symmetry (Fig. 5).

\[
\frac{d}{dt} \int_{V_F(t)} f_j (x,z,t) \, dx \, dz + \frac{C}{\sqrt{t}} \left\{ \frac{h \left[ A_{so}b(t) + A_{su}b(t) \right]}{2} \right\} < \Phi_j(t), \quad t>0, \ j=1,2 \quad (4.1)
\]

Defining a functional form of the steam zone volume, \( V_F(t) \), as

\[
f[V_F(t)] = \frac{h \left[ A_{so}b(t) + A_{su}b(t) \right]}{2} \quad (4.2)
\]

and introducing the uniform propagation assumption that \( f(V_F) \) is an increasing function [i.e., \( f'(V_F)>0 \)] allows to define \( V_j^+(t) \) by the equation

\[
\frac{d}{dt} \int_{V_j^+(t)} f_j (x,z,t) \, dx \, dz + \frac{C}{\sqrt{t}} \cdot f\left[ V_j^+(t) \right] = \Phi_j(t) \quad (4.3)
\]

where

\[
V_j^+(t) = 0 \quad , \quad j = 1,2
\]
Figure 5. Three-dimensional reservoir with one degree of symmetry. (After Yortsos and Gavalas)
or in dimensionless form

\[ 2 \frac{B_j}{B_1} V_j^+ D(t_D) + \frac{1}{\sqrt{t_D}} f_D[V_j^+ D(t_D)] = \Phi_j D(t_D) \]  (4.4)

where

\[ V_j^+(o) = 0 \quad , \quad j = 1, 2 \] .

Since \( f'(V_F) > 0 \) it can be shown that \( A_j^+ D(t_D), (j = 1, 2) \) are two independent upper bounds of \( V_{FD}(t_D) \) for any injection rates.

To solve equation (4.4) we must find the functional form of \( f_D(V_{FD}) \).

To achieve this assume

\[ f_D(V_{FD}) = mV_{FD} \]  (4.5)

where \( m \) is a positive constant. The solution now depends on this functional form. For example, for vertical fronts \( f_D(V_{FD}) = V_{FD} \) and equation (4.4) reduces to its one-dimensional analog. The assumption \( f_D(V_{FD}) = mV_{FD} \) is more general than the assumption of vertical fronts. Such fronts have the property that the area \( (A_{sFD}^+) \) of any cross section of the steam zone parallel to the bedding plane \((x, y)\) is a separable function of time and the vertical coordinate \( z \). In dimensionless notation

\[ A_{sFD}^+(z_D, t_D) = T_D(t_D)Z_D(z_D), \quad 0 < t_D, \quad 0 < z_D < 1 \]  (4.6)

where \( T_D, Z_D \) are arbitrary functions. Fronts with this property are called "separable fronts." Thus, the ratio of areas of any two sections is constant with time; particularly the ratio of heading to trailing edges and vertical sweep efficiency remains constant. The parameter \( m \) assumes the form
\[ m = \frac{Z_D(0) + Z_D(1)}{2 \int_0^1 Z_D(z_D) \, dz_D} \]  

(4.7)

which is the ratio of the volume of a frustrum of a cone with sides \( Z_D(0) \) and \( Z_D(1) \) (Fig. 6) to the volume of the steam zone.

Hence, we can define upper bounds \( v_j^+(t_D) \):

\[ 2 \frac{B_j}{B_1} v_j^+(t_D) + \frac{m v_j^+ D(t_D)}{\sqrt{t_D}} = \Phi_j D(t_D) \]  

(4.8)

where

\[ v_j^+(0) = 0 \quad , \quad j = 1, 2 \ . \]

The equation (4.8) for constant injection rates admits the solution

\[ v_{1D}^+(t_D) = \frac{1}{m^2} \left[ m \sqrt{t_D} - 1 + \exp \left( -m \sqrt{t_D} \right) \right] \]  

(4.9)

and

\[ v_{2D}^+(t_D) + \frac{F}{m^2} \left[ m \sqrt{t_D} - B + B \exp \left( -\frac{m \sqrt{t_D}}{B} \right) \right] . \]  

(4.10)

In equation (4.10) the variable \( F \) expresses the ratio of the latent heat to the total heat injected as

\[ F = \frac{f_{sq}}{f_{sq} + N_{st}} \]  

(4.11)

where

\[ N_{st} = \frac{C_{pw} \Delta T}{L_v} \]  

(4.12)

is the Stefan number and \( B \) expresses the ratio of the latent heat to the
Figure 6. Frustum of a cone for a three-dimensional reservoir with one degree of symmetry. (After Yortsos and Gavalas 1)
total volumetric heat capacities of the steam zone as

\[ B = \frac{1}{1 + N_{st} \left( \frac{1}{t_{sq}} + \frac{\rho_{o} S_{o} C_{po}}{\rho_{s} S_{s} C_{pw}} + \frac{(1-\phi) \rho_{R} C_{pR}}{\rho_{s} S_{s} C_{pw}} \right)} \]  \hspace{1cm} (4.13)

Equations (4.9) and (4.10) are two upper bounds for steam volume growth, each one corresponding to the mode that dominates heat transfer in the hot liquid zone at different times, i.e., conductive mode or convective mode, respectively. They complement each other in providing an overall lowest upper bound for the steam volume growth. Of course, if an exact representation of the heat transfer in the hot liquid zone were possible, the two bounds would have been identical. The solution also accounts for the extreme variations in water saturation as an integrated quantity through the convective heat flux term as is shown in Appendix B.

Having now obtained the two upper bounds the incremental oil recovery from the application of the steamflood can be determined. To determine the recovery the Myhill and Stegemeier\(^6\) equation for volume of oil displaced can be used as

\[ N_{p} = V_{j} \Delta S \hspace{0.5cm} j = 1,2 \]  \hspace{1cm} (4.14)

where \( V_{j} \) is one of the two upper bounds. Depending on the value of \( m \) and the dimensionless time, \( t_D \), equation, \( x \) will yield the appropriate recovery.

From the above solution it can be seen that the growth of the steam zone volume is a function of the dimensionless time (\( t_D \)) and the parameter \( m \). The upper bounds are smaller when \( m = 1 \), i.e., when the three-dimensional equations reduce to their one-dimensional analogs. The effect is more
pronounced as \( m \) assumes larger values. When \( m > 1 \) the steam zone propagates rapidly along the top of the reservoir thus creating a tongue that grows faster than the remaining zone. As \( m \) further increases the upper bound \( V^{+}_{2D} \) assumes control of the growth of the steam zone at an earlier time beyond which it yields a more accurate estimate of the steam zone volume than the one given by \( V^{+}_{1D} \).

In physical terms \( V^{+}_{1D}(t_{D}) \) is the dimensionless volume of the steam zone when every point of the steam zone loses heat to the surroundings at the same rate but where no heat is transported through the front to the hot liquid zone. In contrast \( V^{+}_{2D}(t_{D}) \) is the steam zone volume when every point of the steam zone loses heat to the surroundings at the same rate but there is no conductive heat flux to the hot liquid zone. Thus, the high water saturation values in the hot liquid zone mean a reduced net convective heat flux through the front. This, in turn, will increase (as expected on physical grounds) the latent heat capacity of the steam zone which will finally control the growth of the steam zone.

4.1.2 Numerical Procedure

In order to develop a numerical solution an existing steamflood numerical model can be adopted and tailored to the conditions of this study or a simpler one can be devised. In either case the main framework of the model may consist of the following equations representing conservation of energy and conservation of mass for water, steam, and oil, respectively.

Conservation of Energy:

\[
\phi \rho_{w} \frac{\partial h_{w}}{\partial t} + \phi \rho_{s} \frac{\partial h_{s}}{\partial t} + \phi \rho_{o} \frac{\partial h_{o}}{\partial t} + (1 - \phi) \rho_{R} \frac{\partial h_{R}}{\partial t} = \nabla \cdot k_{h} R \nabla h - (\rho_{w} u_{w} \cdot \nabla h_{w} + \rho_{s} u_{s} \cdot \nabla h_{s} + \rho_{o} u_{o} \cdot \nabla h_{o}) - M_{L} V \cdot (3.16)
\]
Conservation of Mass:

For water

\[ \phi \frac{\partial}{\partial t} (\rho_w S_w) + \nabla \cdot (\rho_w \dot{u}_w) + M_g = 0 \]  \hfill (3.15)

For steam

\[ \phi \frac{\partial}{\partial t} (\rho_s S_s) + \nabla \cdot (\rho_s \dot{u}_s) = M_g \]  \hfill (3.12)

For oil

\[ \phi \frac{\partial}{\partial t} (\rho_o S_o) + \nabla \cdot (\rho_o \dot{u}_o) = 0 \]  \hfill (3.13)

The above equations are accompanied by the following conditions:

1. \( S_w + S_s + S_o = 1 \)  \hfill (3.14)

2. \( S_w = S_w(x) \) as an initial condition

where

\[ S_w(x) = S_{wc} - \frac{D}{G-1} + \frac{1}{G-1} \sqrt{\frac{GD}{f_{sl}}} \sqrt{\ln \frac{1}{1-W_f f_{sl}}} \left( \frac{1}{1-X} \right). \]  \hfill (3.5)

To proceed with the solution of the above mathematical model a particular finite difference representation scheme will have to be adopted for each equation. This will allow the equations to be written in a discrete form to permit a numerical solution. As it can be seen the heat balance equation is of a parabolic type since the second partial derivative with respect to time is absent while the mass balance equation is of an elliptic type. For the heat balance, since it is a parabolic type equation, the
choice is among the various methods available for solution of implicit equations. For the mass balance, which is an elliptic type equation, the choice for a finite difference representation is between an implicit or an explicit representation. However, combinations of explicit and implicit techniques are also feasible depending on the method of solution employed.

For all reservoir modeling equations an iterative method may be better than a direct one in order to avoid problems with round-off errors and ill-conditioned matrices. For instance, an optimum overrelaxation factor can be estimated by a set of trials in a particular problem and then used unaltered during the runs. If the solution method is iterative the stability of the solution process and convergence must be of great concern also. A stability analysis can be attained by either a matrix method, which requires the eigenvector of the matrix, or a method that relies upon the Fourier analysis, i.e., the Neumann method.

Having chosen the finite difference techniques to be employed, the partial differential equations are then discretized for the purposes of computation. This process ultimately leads to the consideration of truncation errors, which can be determined only if the computational procedure is capable of producing the exact solution of the finite difference equation.

Let $D[u]_i^n$ represent the partial differential equation, with the derivatives evaluated at $x = i \Delta x$, $t = n \Delta t$, and let $D_{fd}[u]$ be its finite difference form. The truncation error can then be written as

$$E = D_{fd}[u] - D[u]_i^n$$

and $E \to 0$ as $\Delta x$ and $\Delta t$ tend to zero.
Depending upon the degree of the polynomial employed, the maximum discretization error can be calculated over a time domain \( 0 < t < t\), and for a particular finite difference method yielding the required sufficient conditions for convergence.

Another major consideration for the solution of the equations is the choice of grid type. This choice is between a block and a lattice-centered grid and since most reservoir simulators employ block-centered grids, the former seems an appropriate choice for this study.

In a block-centered grid and for a three-dimensional geometry, a point \( P \) is associated with the block center having indices \((i,j,k)\) as seen in Fig. 7. In the scheme the interfaces carry \((i \pm \frac{1}{2}, j \pm \frac{1}{2}, k \pm \frac{1}{2})\) as indices for a collection of blocks \(i = 1,2, \ldots, N_x\) in the x-direction, \(j = 1,2, \ldots, N_y\) in the y-direction and \(k = 1,2, \ldots, N_z\) in the z-direction.

The finite difference techniques employed to represent the partial differential equations are independent of the type of grid system employed. The difference arises only in the representation of the boundary conditions. For a block-centered grid the Dirichlet and Neumann type boundary conditions are considered. The Dirichlet condition requires that the dependent variable be specified on the boundary while the Neumann condition requires that the gradient of the dependent variable be specified on the boundary.

Finally, a numerical model algorithm will have to be developed that will solve the equations over the entire grid block through time. In the algorithm, at each time step the unknowns of the problem will be calculated for each phase. As the algorithm arrives at each new block that is a distance \( x \) from the injector well the new unknowns will be calculated.
Figure 7. Schematic representation of a three-dimensional block centered grid.
based on the values at the previous step. In this manner, as the steam front propagates through each block of the formation, it will "pick-up" the appropriate water saturation value for use in solving the discretized equations. For the purposes of running the model, the problem of vertical variation in $S_w$ can be handled with a second or third degree polynomial curve that will represent the relation between $S_w$ and depth of reservoir based on physical grounds, i.e., small $S_w$ at the top and increasing gradually with depth.

The data obtained from the solution of the unknowns of the problem can be used within the algorithm to calculate incremental oil recoveries, thus yielding a complete picture of the dependence of the recoveries on the variation of the parameters of the problem.
5.1 Conclusions

The following conclusions are drawn on the basis of the theory and general solution developed:

1. A logical procedure has been presented and incorporated into the analytical model by Yortsos and Gavalas to formulate a steamflood model applied to a reservoir that has been waterflooded just prior to steam injection.

2. Equations have been developed that describe the steamflood process applied to a waterflooded reservoir with water saturation functionally dependent on the distance from the injector well.

3. The parameter \( m \) of the analytical solution is evaluated in terms of the physical variables of the process. This helps to yield a reliable evaluation of the gravity override effects.

4. It is difficult to evaluate the success of the mathematical model since data from an actual case cannot be obtained.

5.2 Recommendations

It is recommended that:

1. A numerical model and algorithm be developed to produce a numerical solution to the problem.
2. A bench-scale model be constructed to study the behavior of the process and the data obtained used to calibrate the numerical model.

3. Alternative water saturation profiles be developed from other considerations than the one presented in this study. They might include assumptions based on various permeability/water saturation relationships.

4. A study be conducted to examine for relatively low mobility ratios, the use of Welge's method for developing water saturation profiles for comparison with this study.

5. For viscous waterflooded reservoirs a vertical water saturation distribution be developed to account for the gravity segregation effects. Average values obtained from gravity segregation studies can be used or an independent development can provide a distribution profile derived from the frontal advance theories.
REFERENCES


APPENDIX A

Multiphase Fluid Flow Equations

Applying the continuity equation to a control volume of each region yields the mass balance equations as follows:

For water phase

\[ \phi \frac{\partial}{\partial t} (\rho_w S_w) = - \nabla \cdot (\rho_w \vec{u}_w) - M_g \]  \hspace{1cm} (A-1)

For steam phase

\[ \phi \frac{\partial}{\partial t} (\rho_s S_s) = - \nabla \cdot (\rho_s \vec{u}_s) + M_g \]  \hspace{1cm} (A-2)

For oil phase

\[ \phi \frac{\partial}{\partial t} (\rho_o S_o) = - \nabla \cdot (\rho_o \vec{u}_o) \]  \hspace{1cm} (A-3)

where \( M_g \) is an interphase mass transfer term allowing for steam condensation when the equations are used to represent a steamflood operation and given as

\[ M_g = - \phi \left[ \frac{\partial (\rho_S v_s)}{\partial t} \right] \text{ constant element mass} \]  \hspace{1cm} (A-4)

The volumetric velocity vectors are expressed in terms of the potential for water, steam and oil as:
\[ u_w = - k \frac{k_{rw}}{\mu_w} (\nabla \rho_w + \rho_w \delta \nabla z) \quad (A-5) \]

\[ u_s = - k \frac{k_{rs}}{\mu_s} (\nabla \rho_s + \rho_s \delta \nabla z) \quad (A-6) \]

\[ u_o = - k \frac{k_{ro}}{\mu_o} (\nabla \rho_o + \rho_o \delta \nabla z). \quad (A-7) \]
APPENDIX B

From the conservation equations (3.16, 3.19, and 3.23), by integration over the steam zone and by making use of the interface conditions we obtain for the thermal energy

\[ \Delta T \frac{d}{dt} \int_{V(t)} M_1 dV + \Delta T A_F(t) \left[ \psi_{hc_F}(t) + \psi_{hc_F}(t) \right] \]

\[ + \int_{A_F(t)} \left( - k_h \frac{\partial T_f}{\partial n} \right) \text{II} dA = \left[ w_s(t) + w_w(t) \right] C_{pw} \Delta T \]

\[ + w_s(t)L_v \]

(B-1)

where \( T_i \) is the reference temperature, \( T = T_s - T_i \) and \( M_1 \) is the volumetric heat capacity of the steam zone given as

\[ M_1 = \sum_{i=w,s,o} \phi C_{pi} \rho_i S_i + (1 - \phi) \rho_R C_{pR} + \phi \frac{L_v \rho_s S_s}{\Delta T} \]  

(B-2)

The term \( \psi_{hcF}(t) \) is the area averaged net convective heat flux through the front per unit \( \Delta T \) and given as

\[ \psi_{hcF}(t) + \left[ \sum_{i=w,o} C_{pi} \int_{A_F(t)} \rho_i \text{II} (u_i \text{II} - \phi S_i \text{II} v_i) dA \right] - (1 - \phi)C_{pR} \int_{A_F(t)} \rho_R v_n dA \]

(B-3)

where

\( \text{II} \)  

54
where $S_w^I = S_w^I(x)$.

The term $\psi_{\text{hcd}F}(t)$ is the conductive heat flux through the front per unit $\Delta T$ and $w_w$ and $w_s$ are water and steam injection rates at the injection well and is given as

$$\psi_{\text{hcd}F}(t) = \left[ \int_{A_F(t)} \left( - k_{hf} \frac{\partial T}{\partial n} \right)^I dA \right] / \Delta T \ A_F(T). \quad (B-4)$$

Similarly the latent heat (or steam mass) integral balance is given as

$$\frac{d}{dt} \int_{V(t)} M_2 dV + \Delta T A_F(t) \ \psi_{\text{hcd}F}(t) + \int_{A_F(t)} \left( - k_{hf} \frac{\partial T}{\partial n} \right)^I dA$$

$$= w_s(t) L_v \quad (B-5)$$

where

$$M_2 = \phi \frac{L_v \rho_s S_s}{T}. \quad (B-6)$$

is the volumetric latent heat capacity of the steam zone.

Equations (B-1) and (B-6) constitute the basic integral conservation equations.

In steam injection the temperature function satisfies the constraint

$$0 \leq \phi(x,y,t) \leq T_s - T_i \quad , \quad (B-7)$$

thus, for any $(x,y)$ inside the steam zone the following inequalities are satisfied

$$\frac{k_{hf}}{\sqrt{\pi a_f}} \frac{(T_s - T_i)}{\sqrt{t}} < - k_{hf} \left( \frac{\partial T}{\partial n} \right)^I < \frac{k_{hf}}{\sqrt{\pi a_f}} \frac{(T_s - T_i)}{\sqrt{t - \lambda (x,y)}}. \quad (B-8)$$
Integrating equation (B-8) yields the integral heat losses for continuously advancing fronts as

\[ \frac{k_h f(T_s - T_i)}{\sqrt{\pi} a_f} \frac{1}{\sqrt{t}} \int A_f(t) dA < \int \frac{\Delta T_i}{A_i(t)} dA. \]

(B-9)

Since the hot liquid zone is at a temperature lower than \( T_s \) then over the steam front,

\[ \Delta T \mathcal{A}_F(t) \psi_{hcd F}^\prime(t) = \int \left( -k_h R \frac{\partial T_R}{\partial n} \right) II A_F(t) dA > 0. \]

(B-10)

Similarly, the net heat flux through the front for a spontaneous process must be positive; hence

\[ \Delta T \mathcal{A}_F \left[ \psi_{hcv_f}^\prime(t) + \psi_{hcd_f}^\prime(t) \right] = \left\{ \Delta T \sum_{i=w,o} C_{pi} \right\}
\]

\[ \cdot \int A_F(t) \rho \phi^\prime \left( u_{in}^\prime - \phi S_i^\prime v_n^\prime \right) dA - (1 - \phi) \Delta T \mathcal{C}_{pR}
\]

\[ \cdot \int A_F(t) \rho R v_n dA + \int A_F(t) \left( -k_h R \frac{\partial T_R}{\partial n} \right) II dA > 0. \]

(B-11)

Combining the integral balances with the above inequalities on heat losses we can proceed to establish upper bounds for the steam zone volume.

For a one dimensional geometry and in uniform notation the integral valances become: 1
\[ \Delta T \frac{d}{dt} \int_{0}^{L_{SF}(t)} M_{1}dx + \pi_{F}(t) \Delta T \left[ \psi_{hcv_{F}}(t) + \psi_{hcd_{F}}(t) \right] \]

\[ + \frac{2}{h} \int_{0}^{L_{SF}(t)} \left( - k_{hf} \frac{\partial T_{F}}{\partial n} \right) II dx \]

\[ = \left[ w_{S}(t) + w_{w}(t) \right] C_{pw} T + w_{S}(t)L_{v} \]  \hspace{1cm} (B-12)

for the thermal energy and

\[ \Delta T \frac{d}{dt} \int_{0}^{L_{SF}(t)} M_{2}dx + \pi_{F}(t) \Delta T \psi_{hcd_{F}}(t) \]

\[ + \frac{2}{h} \int_{0}^{L_{SF}(t)} \left( - k_{hf} \frac{\partial T_{F}}{\partial n} \right) II dx = w_{S}(t)L_{v} \]  \hspace{1cm} (B-13)

for the latent heat.

Equations (B-12) and (B-13) are nonlinear with respect to \( L_{SF}(t) \) and can be written as a combination of a linear and a nonlinear part:

\[ L_{L_{i}} \left[ L_{SF}(t) \right] + L_{N_{j}} \left[ L_{SF}(t) \right] = \Phi_{j}(t), \ j=1,2 \]  \hspace{1cm} (B-14)

where

\[ \Phi_{1}(t) = w_{s}L_{v} + (w_{w} + w_{s})C_{pw} \Delta T \]  \hspace{1cm} (B-15)

and

\[ \Phi_{2}(t) = w_{s}L_{v} \]  \hspace{1cm} (B-16)
The linear operator $L_{L_i}^j$ has a common structure in both equations and for continuously advancing fronts is given as

$$L_{L_i}^j \left[ L_{S_F}^j(t) \right] = \frac{d}{dt} \int_0^{L_{S_F}^j(t)} f_j[x,t; L_{S_F}^j(t)] dx + c \int_0^{L_{S_F}^j(t)} \frac{dx}{\sqrt{t-\lambda(x)}},$$

$\quad j=1,2$ \hspace{1cm} (B-17)

where

$$C + \frac{2k_r \Delta T}{h \sqrt{\pi a_f}}$$ \hspace{1cm} (B-18)

and

$$f_j[x,t; L_{S_F}^j(t)] = \Delta T M_j, \quad j=1,2.$$ \hspace{1cm} (B-19)

Hence using the inequalities (B-9), (B-10), and (B-11) to bound the nonlinear part of (B-14) we arrive at

$$\frac{d}{dt} \int_0^{L_{S_F}^j(t)} f_j[x,t; L_{S_F}^j(t)] dx + \frac{c}{\sqrt{t}} L_{S_F}^j(t) < \Phi_j(t), \quad t>0, \quad j=1,2$$ \hspace{1cm} (B-20)

and define $L_{S_i}^{+}(t)$ by

$$\frac{d}{dt} \int_0^{L_{S_i}^{+}(t)} f_j[x,t; L_{S_F}^j(t)] dx + \frac{c}{\sqrt{t}} L_{S_i}^{+}(t) = \Phi_j(t), \quad t>0, \quad L_{S_i}^{+}(0) = 0, \quad j=1,2.$$ \hspace{1cm} (B-21)

Parts $L_{S_1}^{+}(t)$ and $L_{S_2}^{+}(t)$ would be identical if an exact representation of the heat transfer in the hot liquid zone was available.
Inequality (B-20) becomes the following for a three-dimensional geometry with one-degree of symmetry

\[
\frac{d}{dt} \int_{V(t)} f_j(x,z,t) \, dx \, dz + \frac{c}{\sqrt{t}} \left[ h \cdot \frac{A_s^{ob}(t) + A_s^{ub}(t)}{2} \right] < \Phi_j(t),
\]

\[t > 0, \ j = 1, 2. \]  \hspace{1cm} (B-22)