AN ANALYTICAL AND EXPERIMENTAL STUDY OF THE
MINIMUM PressURES IN CIRCULAR PIPE BENDS

by

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A DISSERTATION

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<tr>
<td>A</td>
<td>Cross-sectional area of the pipe bend</td>
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<tr>
<td>a</td>
<td>Pipe radius</td>
</tr>
<tr>
<td>C</td>
<td>Constant defined by Equation 71</td>
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<tr>
<td>( C_p )</td>
<td>Pressure coefficient</td>
</tr>
<tr>
<td>( C_{P_{\text{wall}}} )</td>
<td>Pressure coefficient at the wall</td>
</tr>
<tr>
<td>( C_{P_{\text{min, core}}} )</td>
<td>Coefficient of the minimum pressure in the central core (called the minimum core pressure coefficient)</td>
</tr>
<tr>
<td>( C_{P_{\text{min, wall}}} )</td>
<td>Coefficient of the minimum pressure at the wall (called the minimum wall pressure coefficient)</td>
</tr>
<tr>
<td>( C_{\text{wall}} )</td>
<td>Constant defined by Equation 72</td>
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<tr>
<td>f</td>
<td>Signal frequency from the flow meter or calibrator in hertz</td>
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<td>( I_{a, b} )</td>
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<td>Constants related to the free vortex motion in the central core, defined by Equations 9 and 13</td>
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<td>n*</td>
<td>Coordinate normal to wall (= a-r)</td>
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<td>P</td>
<td>Total pressure</td>
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<td>p</td>
<td>Static pressure</td>
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<td>Q</td>
<td>Flow rate in gallons per minute</td>
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LIST OF SYMBOLS (Continued)

\( \mathbf{q} \) Velocity vector in the central core

\( \dot{q}_c \) Volume flow exchange between the shedding layer and the central core

\( \dot{q}_{SL} \) Volume flow in the shedding layer

\( R \) Radius of curvature of pipe bend

\( \text{Re} \) Pipe Reynolds number \( \left( \frac{2 \rho W a}{\mu} \right) \)

\( r, \psi, \theta \) Toroidal coordinates defined in Figure 6

\( U, V, W \) Central core velocity components defined in Figure 5

\( u, v, w \) Viscous layer velocity components

\( u^*, v^*, w^* \) Velocity components in the toroidal coordinate system defined in Figure 6

\( \mathbf{v} \) Velocity vector in the viscous layer

\( v_m \) Maximum tangential velocity

\( \overline{W} \) Average axial velocity through the pipe

\( W_1 \) Central core velocity immediately upstream of the bend

\( x, y, \theta \) Cylindrical coordinates defined in Figure 5

\( \delta \) Viscous layer thickness

\( \delta_1 \) Shedding layer thickness

\( \delta_2 \) Average boundary layer thickness
LIST OF SYMBOLS (Continued)

\( \delta_{2,0} \) Boundary layer thickness at the inside wall
\( \zeta \) Vorticity vector
\( \eta_m \) Distance from wall at which maximum tangential velocity occurs in the shedding layer.
\( \theta_{\min} \) Deflection angle at which the wall pressure is a minimum
\( \theta_{vv}, \theta_{ww}, \theta_{vw} \) Integrals defined in Equation 37
\( \mu \) Dynamic viscosity
\( \nu_0 \) Variable defined in Equation 40
\( \rho \) Liquid density
\( \tau \) Stress tensor
\( \tau_\theta \) Axial component of the wall shear stress
\( \tau_\psi \) Tangential component of wall shear stress
(\( \cdot \))' Dimensionless quantity
(\( \cdot \)_1) Refers to uniform conditions immediately upstream of the bend
(\( \cdot \)_0) Refers to conditions at the edge of the viscous layer
(\( \cdot \)) Average
CHAPTER I
INTRODUCTION

Introductory Remarks

The flow of fluids in pipe bends has been a topic of practical consideration for a number of years. A recent and largely unexplored region of interest in curved pipe flow has evolved around the ducting of the liquids which are employed as the propellants for liquid rocket engines. With these liquids there may arise the undesirable phenomenon of liquid cavitation near the inside wall of pipe bends that are located in the ducting system on the suction side of the turbopumps.

The presence of cavitation in these components can have a number of detrimental effects on the overall design and performance of the propulsion system. Because of the hysteresis effects associated with the formation and disappearance of cavitation bubbles, the vaporous pockets formed in a cavitating pipe bend may persist in a supercooled state for some distance downstream before collapse. These bubbles form "weak spots" in the liquid and, if they are convected downstream into a pump, they can provide the nuclei for premature inception of cavitation within the pump itself. Once this happens, the formation and collapse of the bubbles markedly degrades pump
efficiency and induces undesirable pressure fluctuations which can be transmitted to the rocket engine combustion chamber.

To suppress cavitation within the pump, it is necessary to bring the liquid to the inlet section under a sufficient "net positive suction head" (NPSH) defined as the total pressure at the pump inlet minus the vapor pressure of the propellant at the pump inlet. The required inlet pressure may be obtained either by auxiliary pumps or by pressurization of the propellant tanks. The tank pressure that would be required is

\[ P_{\text{tank}} = \text{NPSH} + \text{feedline friction loss} + \text{vapor pressure} - \text{propellant head}. \]

The NPSH in the equation above is that at which the pump is known to operate satisfactorily, and its value will increase if the upstream ducting components, in this case the pipe bends, alter the conditions at the pump inlet from the ideal conditions presupposed for design purposes. Therefore, through the above relationship, the cavitation of pipe bends located on the suction side of the turbopumps can directly influence the required pump inlet pressure along with the objectional weight penalties associated with the auxiliary pressurization components. For this reason it is desirable to design the propellant pumps and the ancillary ducting components so that the lowest pressure possible is required at the pump inlet to assure cavitation free operation.

Considerable effort has been devoted to the design of pumps\(^1\)-\(^6\) so that the NPSH required for proper operation is minimized; however, the
influence of the upstream ducting components (pipe bends) on cavitation within the pump has not been adequately determined nor has there been any effort, beyond the exploratory phase, expended in achieving an understanding of the cavitation characteristics of flow in curved pipes.\(^7\)

A number of factors are known to affect the inception of cavitation, however, their relationships to each other and the manner in which they exercise their influence on the cavitation process are not altogether known with any degree of certainty.\(^8\)\(^-\)\(^12\) Therefore the investigation of the cavitation characteristics of any hydraulic device eventually involves the full scale experimental determination of these effects. Fortunately most of these factors exercise only a small influence on the inception of cavitation and then in a manner which usually results in a conservative hydrodynamic design when it is desirable to operate free of cavitation. As one would intuitively reason, the minimum static pressure sustained in a ducting component and the vapor pressure of the liquid are the two most important factors influencing the conditions at which cavitation first appears. Hence if a specified liquid is to be employed, a knowledge of the minimum pressure intensity occurring in the ducting component must at least be determined before any statements can be made regarding the probability of cavitation.

Due to the many variables involved and the complex three-dimensional nature of curved pipe flow, there exists at the moment no
completely satisfactory means of predicting the minimum pressure within a pipe bend. However, various aspects of this type of flow have been experimentally and analytically studied to a limited extent. The object of this investigation is to attempt an approximate analytical technique for predicting the minimum pressure levels in pipe bends and to experimentally examine some of the features of curved pipe flow in 90-degree pipe bends.

**Flow Characteristics in a Pipe Bend**

Fluid particles passing through a pipe bend follow very complex, three-dimensional trajectories. The flow conditions of the fluid at the entrance to the bend are established by the upstream ducting system, and as a result the velocity is generally nonuniform because of obstructions that disarrange the flow or because of viscous retardation of the flow at the walls. Due to the variations of the entering fluid velocity, nonuniform centrifugal forces act on the fluid as the particles traverse individual curved trajectories through the pipe bend. This unbalance of centrifugal force leads to the formation of a secondary flow system which is oscillatory\(^\text{13}\) in nature and is directed outwards at the center of the pipe and inwards near the wall with the fluid elements moving along the pipe in two sets of spirals separated by the central plane of curvature which contains the centerlines of the two elbow tangents. The influence of the elbow on the flow extends into the downstream
tangent until the secondary flow system is damped from viscous forces and fully developed pipe flow is again established. 14

To balance the centrifugal forces acting upon the fluid, a pressure gradient across the bend is generated by the curvilinear flow. The pressure at the outside of the bend becomes larger than the initial static pressure, attaining its maximum value part way through the bend, and the pressure at the inside, nearer the origin of the bend, decreases from its initial value until some minimum value is reached part way through the bend. Thus, as the flow undergoes the transition from rectilinear to curvilinear motion, a positive pressure gradient in the direction of flow is initially imposed on the outer wall of the elbow and then a negative gradient is generated as the static pressure readjusts to a uniform value when the flow leaves the bend. Conversely, on the inner wall, a negative axial pressure gradient is initially present as the pressure decreases to some minimum value at approximately midway through the bend and then a positive gradient is formed as the pressure increases back to a uniform value across the duct downstream of the turn (Figure 1).

When the turning radius is sharp and the flow rates high, the centrifugal forces acting on the flow are large and hence the positive axial pressure gradients may be of sufficient magnitude and extent so that the slow moving fluid particles near the wall lack sufficient momentum to traverse the region of increasing pressure. If this is the case, the particles will reverse their original direction of motion
Figure 1. Schematic of the Pressure Distribution on the Inner and Outer Walls in the Plane of Curvature of an Elbow
and create a local region of eddies and vortexing near the duct boundaries. That is, the main flow fails to adhere to the walls of the duct or it "separates"; a very undesirable phenomenon since energy losses which accentuate the resistance to flow are created by the vortexing action.

In the center of the pipe, the pressure profile from the inside to the outside of the bend is primarily governed by the inviscid momentum balance on the particles moving in curvilinear paths (Figure 2). For fully developed turbulent flow at the bend entrance, this balance is approximately that which occurs for potential flow. Near the wall, however, the profile becomes distorted by the viscous effects present there. These effects are such that the minimum pressure in a bend cross sectional plane occurs at some distance, $\delta_{2,0}$, from the inside wall rather than immediately adjacent to it.

If the fluid is a liquid with a sufficiently high vapor pressure, the static pressure near the inside wall of the duct may depress to a value equal to or less than the vapor pressure and thus cause cavitation within the elbow. This is an entirely different phenomenon from separation although both may occur simultaneously.

Weske\textsuperscript{15} has illustrated three curved flow regions which are more distinctly delineated for turbulent flow than for laminar flow because of the flatter entrance velocity profile of the former. These regions are (Figure 3):
Figure 2. Schematic of the Pressure Distribution Across a Bend at the Station of Minimum Pressure.
Figure 3. Schematic Drawing Showing Flow Phenomena in a Curved Duct (Reference 15)
1) The "core" or central body of the fluid in which the velocity component in the axial direction is large compared to the transverse velocity components. In this region, the flow is largely unaffected by viscosity; thus, pressure and inertia forces predominate over viscous forces. Up to a bend deflection angle of about 45 degrees, the axial velocity distribution approximates free-vortex motion for symmetrical entrance flow, the product of radius and velocity being roughly constant across the section. The portion of the flow contained in the "core" will vary according to the entrance velocity profile and the distance through the bend. As the flow progresses past a deflection angle of about 45 degrees, the developing secondary currents which arise from the retarding action of viscosity near the walls begin to distort the potential flow processes in the central core, and as a result the particles of maximum velocity are displaced towards the outside of the bend. These secondary currents are initially insignificant (especially for turbulent flow with its flatter velocity profile) and confined to a narrow layer near the walls but they grow slowly in intensity and size with distance through the bend to eventually consume most or all of the potential flow. Eventually equilibrium is established between the various viscous and dynamic influences that cause the initial oscillations in transition from rectilinear to curvilinear motion. When this occurs, as in a bend of large deflection angle or a helical coil, flow variations in the axial directions cease to exist and the flow is termed "fully developed".
2) The "shedding layer" near the wall in which the velocity component normal to the wall is nearly zero and the velocity component in the peripheral direction parallel to the wall is of the same order of magnitude as the component of velocity in the axial direction. The fluid in this layer has lost a large part of its kinetic energy on account of viscous stresses and it therefore flows towards the inside of the bend, in the direction of the negative peripheral pressure gradient imposed by the core flow. In many respects this "shedding layer" resembles the three-dimensional boundary layer on the sweptback wings of high-speed aircraft and on bodies of revolution immersed in a moving fluid and yawed with respect to the wind axis.

3) The "region of eddying flow" at the inside of the bend where the opposing shedding layers impinge as they follow the curvature of the wall. In this region, the total energy of the flow is much less than in the core and the fluid is in a state of random turbulence. Distinction is made between eddy flow and separated flow which occurs when the radius of curvature is small. A complete reversal of flow arises in the latter case when the viscous layer at the inner wall lacks sufficient momentum to penetrate the region of positive axial pressure gradient near the bend exit.
CHAPTER II
THEORETICAL ANALYSIS

Analyses of Curved Pipe Flow

Fully developed curved pipe flow is more amenable to analysis than the transition type of flow that occurs in a finite elbow, and at the entrance and exits of a coiled pipe. This is because the velocity profiles are similar at different stations and hence all velocity derivatives in the stream direction disappear entirely. As far as can be determined, all analytical treatments of viscous curved pipe flow which could be applicable to the present problem are restricted to the fully developed case.

Dean\textsuperscript{16, 17} first obtained a solution to the governing Navier Stokes equations for fully developed laminar flow in a coiled pipe. The resulting truncated solution was obtained as a small perturbation from straight pipe laminar flow and therefore is not applicable to high velocities or to small radii of curvature. Dean's method has been extended to include elliptical and square flow cross sections,\textsuperscript{18, 19} but the restrictive assumptions for these analyses remain the same as for Dean's original analysis.
The possibility of obtaining solutions for curved pipe flow at large Reynolds number (in the turbulent flow regime) exists with the shedding layer concept. Using this approach the flow field is divided into a central inviscid central core and a viscous layer at the wall which assumes the characteristics of a three-dimensional boundary layer. The solutions in the two regions are obtained in the same manner as for an external flow field.

This technique was first applied by Adler and later by Barua to laminar flow. Weske applied the method to turbulent flow as did Ito, both essentially using it as an order of magnitude analysis to determine the dimensionless parameters that influence the loss coefficients. Weske's solution contained an undetermined parameter, the pitch of the twisting flow, and hence one must resort to properly obtained experimental data in order to use his analysis.

**Analytical Model**

The shedding layer concept for analyzing the viscous flow in a curved pipe consists of a division of the flow field into two primary components to which different simplifying assumptions may be applied regarding the nature of the flow. All viscous effects are assumed to be confined to a thin layer adjacent to the wall and all pressure gradients to a central core. Hence, the vortices generated by the curvilinear motion consist of flow in the core from the inside to the outside of the pipe and back again to the inside through a shedding layer which is
driven by the pressure gradient across the bend. A mass balance is formed between the core and the shedding layer that must be considered; therefore, the solutions for each region are coupled to one another through this relationship and the viscous layer boundary conditions.

For fully developed turbulent entrance conditions, the core flow has been experimentally observed to approximate potential flow\(^{15}\) at least up to the deflection angle at which the minimum pressure occurs. Therefore, for the present analysis it will be assumed that the core flow is irrotational.

Before a solution can be obtained, assumptions to remove the three-dimensionality of the problem must be introduced. Therefore, it is assumed that, at the deflection angle where the pressure is a minimum, all velocity gradients in the axial direction vanish. This assumption, which is the condition for fully developed flow, is physically inconsistent within the transition region of a pipe bend because the flow is in the process of changing from fully developed rectilinear motion to fully developed curvilinear motion and thus the derivatives of the flow properties in the streamwise direction cannot be expected to vanish, except perhaps for stationary values which are unlikely for all the velocity components. Thus, in essence it has been assumed that in the transition region, the streamwise derivatives of the velocity components are negligible in comparison with the component derivatives in the two
remaining coordinate directions. This is, admittedly, a rather serious assumption of undetermined consequences, but at the present time the problem does not appear tractable without the simplifications it introduces. It may be noted that the assumption uncouples the solution from the initial flow conditions at the entrance to the bend in that no variation from the fully developed turbulent flow can be accommodated as a problem variable.

The specification of a zero axial pressure gradient identifies a cross sectional plane of the bend where the pressure in the bend is a minimum or a maximum. Therefore, the analysis will now be independent of the deflection angle $\theta$ and will be considered applicable only to that cross section where the minimum pressure occurs ($\theta_{\text{min}}$).

One further assumption will be made pertaining to the flow in the central core. This assumption is that the flow in the core will be regarded as being two dimensional with no velocity component normal to the plane of the bend. On the basis of an unpublished reference, Barua $^{21,22}$ justified the same assumption for fully developed coiled pipe flow. The data of Weske indicates that this condition is also approximately valid for bend flow. Figure 4 illustrates the fluid motion as it is now assumed to exist in the cross section to which the present development is applicable.
Figure 4. Shedding Layer Model
Inviscid Core. - In its usual form Euler's equation for the conservation of momentum for the steady motion of an inviscid fluid in the absence of body forces is

\[(\vec{q} \cdot \nabla) \vec{q} = -\frac{1}{\rho} \nabla p\]  

(1)

Employing vector identities, this equation may be written as

\[\vec{q} \times (\nabla \times \vec{q}) = -\frac{1}{\rho} \nabla p - \frac{1}{2} \nabla \vec{q}^2\]  

(2)

or in terms of total pressure and vorticity

\[\vec{q} \times \vec{\zeta} = -\frac{1}{\rho} \nabla P\]  

(3)

where the fluid is assumed to be homogeneous and to have a constant density. \(\vec{\zeta}\) is the vorticity vector and \(P\) is the total pressure defined by Bernoulli's equation for steady, incompressible flow

\[P = p + \frac{\rho \vec{q}^2}{2}\]  

(4)

Since limited experimental data have indicated that in the transition region of pipe bends the total pressure is approximately uniform in the central core for a fully developed turbulent flow at the entrance (Weske\textsuperscript{15} and others), Equation 3 becomes

\[\vec{q} \times \vec{\zeta} = 0\]  

(5)

which is the requirement for Beltrami flow. In general, the vorticity and the velocity vectors will not coincide, therefore the only nontrivial solution for the previous equation is

\[\vec{\zeta} = 0\]  

(6)

That is, the flow is irrotational in the region of uniform total pressure.
Refer now to a cylindrical coordinate system for a description of the core flow only (see Figure 5). Let \( y \) be the coordinate normal to the plane of the bend, intersecting it at the bend origin. Let \( x \) be the radial coordinate normal to the \( y \) axis and \( \theta \) the angular position of \( x \) rotated about \( y \) measured from the bend entrance. Let \( U, V, \) and \( W \) be the components of velocity in the \( x, y, \) and \( \theta \) directions, respectively. In this coordinate system, the components of the vorticity equation, (6), are:

\[
x \frac{\partial W}{\partial y} - \frac{\partial V}{\partial \theta} = 0 \tag{7a}
\]
\[
\frac{\partial U}{\partial \theta} - \frac{\partial xW}{\partial x} = 0 \tag{7b}
\]
\[
\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} = 0 \tag{7c}
\]

The important assumption will now be made that the components of velocity are invariant in the \( \theta \) direction. In addition, it is assumed that any motion normal to the plane of symmetry is confined to the viscous layer. Therefore, in the central core the \( y \) component of velocity is zero. Under these assumptions the components of the vorticity become

\[
\frac{\partial W}{\partial y} = 0 \tag{8a}
\]
\[
\frac{\partial xW}{\partial x} = 0 \tag{8b}
\]
\[
\frac{\partial U}{\partial y} = 0 \tag{8c}
\]
Figure 5. Cylindrical Coordinate System \((x, y, \theta)\)
The first two of the above relationships yields the distribution of the axial velocity $W$ within the inviscid core

$$xW = K_1$$  \hspace{1cm} (9)

To obtain two similar relationships for the $x$ component of the velocity, it is necessary to employ the continuity equation which, for an incompressible fluid, is

$$\nabla \cdot \mathbf{q} = 0$$  \hspace{1cm} (10)

In the cylindrical coordinate system, the equation of continuity takes the form

$$\frac{\partial xU}{\partial x} + x \frac{\partial V}{\partial y} + \frac{\partial W}{\partial \theta} = 0$$  \hspace{1cm} (11)

Now by exercising the previously stated assumptions, this equation becomes

$$\frac{\partial xU}{\partial x} = 0$$  \hspace{1cm} (12)

which with the vorticity equation, (8c), yields the solution

$$xU = K_2$$  \hspace{1cm} (13)

For a specified flow condition, $K_1$ and $K_2$ are constants which are independent of the coordinate position. They must be determined by the requirements of conservation of mass exchange between the inviscid core and the boundary layer, and hence will be functions of the Reynolds number and the bend radius of curvature of the pipe bend.

Since the cylindrical coordinate system is not convenient for representing the geometry of a circular pipe, transformation will now be made to the toroidal coordinate system (Figure 6). In this coordinate
Figure 6. Toroidal Coordinate System \((r, \psi, \theta)\)
system $U$ is the velocity component in the radial direction normal to the wall of the pipe, $V$ is the velocity component in the $\psi$ direction tangential to the wall of the pipe and $W$ is the component in the $\theta$ direction parallel to the axis of the pipe.

In the toroidal coordinate system, the velocity components in the core become

$$U = \frac{K_2 \sin \psi}{R + r \sin \psi} \quad (14a)$$

$$V = \frac{K_2 \cos \psi}{R + r \sin \psi} \quad (14b)$$

$$W = \frac{K_1}{R + r \sin \psi} \quad (14c)$$

where it is emphasized that the upper case letters $U$, $V$, $W$, now and henceforth refer to the core flow in the toroidal coordinate system.

Now Equations 14 may be more conveniently written in a dimensionless form. Thus

$$U' = \frac{K_2' \sin \psi}{R' + r' \sin \psi} \quad (15a)$$

$$V' = \frac{K_2' \cos \psi}{R' + r' \sin \psi} \quad (15b)$$

$$W' = \frac{K_1'}{R' + r' \sin \psi} \quad (15c)$$
where

\[ U', V', W' = \frac{U}{W}, \frac{V}{W}, \frac{W}{W} \]

\[ r' = \frac{r}{a} \]

\[ R' = \frac{R}{a} \]

\[ K' = \frac{K}{aW} \]

and \( W \) is the mass averaged axial velocity through the bend. From Bernoulli's equation, (4), the pressure in the central core becomes

\[ p' = P' - \frac{K_2^{12} + K_1^{12}}{2 (R' + r' \sin \psi)^2} \quad (16) \]

where

\[ p' = \frac{p}{\rho \overline{W^2}}, \quad P' = \frac{P}{\rho \overline{W^2}}. \]

Now since the inviscid core flow at the bend entrance is uniform, Bernoulli's equation may be written there as

\[ P' = p_1' + \frac{1}{2} W_1'^2 \quad (17) \]

where \( p_1' \) is the dimensionless static pressure at the bend entrance.

Since the total pressure \( P' \) is uniform within the region of interest, e.g., the transition portion of the central core, it can be eliminated from Equations 16 and 17. Thus, the pressure coefficient becomes

\[ C_P = \frac{p_1 - P}{\rho \overline{W^2}/2} = \frac{K_2^{12} + K_1^{12}}{(R' + r' \sin \psi)^2} - W_1'^2. \quad (18) \]
The minimum pressure, which is of most concern in cavitation studies, will occur at \( \psi = 270 \) degrees; hence, the minimum pressure coefficient for the inviscid core will be

\[
P_{\text{min, core}} = \frac{p_{1} - p_{\text{min}}}{\rho \frac{W^{2}}{2}}
\]

\[
= \frac{K_{1}^{2} + K_{2}^{2}}{[R' - (1 - \delta_{2,0})]^{2}} - W_{1}^{2}
\]  

(19)

where \( \delta_{2,0} \) is the boundary layer thickness at the inside of the bend and \( W_{1} \) is generally assumed to approximate unity.

The constant \( K_{2} \), as one would deduce from Equation 15 is a measure of the intensity of the secondary circulation and, as such, is necessarily dependent upon the Reynolds number and the radius of curvature for given bend inlet conditions. It is recognized, however, that in any event, \( K_{1} \) will be much greater than \( K_{2} \), because the axial velocity will be much greater than the transverse velocity. Hence, the secondary motion will not significantly affect the pressures (see Equation 19) except by its influence on the boundary layer.

**Viscous Layer: Basic Equations.** - The thickness of the boundary layer at the inside wall and the constants in Equation 14 remain to be determined. Since \( U \) and \( V \) are components of the rotating flow in the core they are related to the mass flow in the shedding layer, hence, an attempt will be made to obtain an approximate solution to the governing viscous flow equations.
The momentum equation for the steady motion of an incompressible viscous fluid is

\[ \rho (\vec{v} \cdot \nabla) \vec{v} = -\nabla p + \nabla \cdot \tau \]  

(20)

where \( \tau \) is the stress tensor. This equation together with the continuity equation

\[ \nabla \cdot \vec{v} = 0 \]  

(21)

completes the set of conservation relations governing the flow in the viscous layer. Since the flow is assumed to be turbulent, \( \vec{v} \) and \( p \) represent the time-averaged quantities of velocity and pressure, respectively. To cast these equations into a tractable form, reference must be made to a specific coordinate system, which for the present application is the toroidal coordinate system. The three components of the momentum equation plus the continuity equation, when written in the toroidal coordinate system, become

\[ \rho \left\{ u \frac{\partial u}{\partial r} + v \frac{\partial u}{\partial \psi} + \frac{w}{R+r \sin \psi} \frac{\partial u}{\partial \theta} - \frac{v^2}{R+r \sin \psi} - \frac{\sin \psi}{R+r \sin \psi} w^2 \right\} = -\frac{\partial p}{\partial r} + \frac{\partial \tau_{rr}}{\partial r} \]

\[ + \frac{1}{r} \frac{\partial \tau_{r\psi}}{\partial \psi} + \frac{1}{R+r \sin \psi} \frac{\partial \tau_{r\psi}}{\partial \theta} + \frac{\tau_{rr}(R+2r \sin \psi)}{r(R+r \sin \psi)} - \frac{\tau_{\psi\psi}}{r} \]

\[ + \frac{\tau_{r\psi} \cos \psi}{R+r \sin \psi} - \frac{\tau_{\theta\theta} \sin \psi}{R+r \sin \psi} \]  

(22a)
\[ \psi \text{ component} \]

\[
\rho \left\{ u \frac{\partial v}{\partial r} + v \frac{\partial v}{\partial \psi} + \frac{w}{r \sin \psi} \frac{\partial v}{\partial \theta} + \frac{uv - \cos \psi}{R+r \sin \psi} w^2 \right\} = 1 - \frac{1}{r} \frac{\partial p}{\partial \psi} + r \frac{\partial \tau_{r \psi}}{\partial r} \\
+ r \frac{\partial \tau_{\psi \psi}}{\partial \psi} + \frac{1}{R+r \sin \psi} \frac{\partial \tau_{\psi \theta}}{\partial \theta} + \frac{\tau_{r \psi} (2R+3r \sin \psi)}{r(r+r \sin \psi)} \\
+ \frac{\tau_{\psi \psi} \cos \psi}{R+r \sin \psi} - \frac{\tau_{\theta \theta} \cos \psi}{R+r \sin \psi} \\
\]

(22b)

\[ \theta \text{ component} \]

\[
\rho \left\{ u \frac{\partial w}{\partial r} + v \frac{\partial w}{\partial \psi} + \frac{w}{r \sin \psi} \frac{\partial w}{\partial \theta} + \frac{\cos \psi}{R+r \sin \psi} w v + \frac{\sin \psi}{R+r \sin \psi} u w \right\} \\
= - \frac{1}{R+r \sin \psi} \frac{\partial p}{\partial \psi} + \frac{\partial \tau_{r \theta}}{\partial r} + \frac{r}{r \sin \psi} \frac{\partial \tau_{\psi \theta}}{\partial \theta} + \frac{1}{R+r \sin \psi} \frac{\partial \tau_{\theta \theta}}{\partial \theta} \\
+ \frac{\tau_{r \theta} (R+3r \sin \psi)}{r(R+r \sin \psi)} + \frac{2 \tau_{\psi \theta} \cos \psi}{r+r \sin \psi} \\
\]

(22c)

Continuity Equation

\[
\frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \psi} + \frac{1}{R+r \sin \psi} \frac{\partial w}{\partial \theta} + \frac{R+2r \sin \psi}{r(r+r \sin \psi)} u + \frac{\cos \psi}{R+r \sin \psi} v = 0 \\
\]

(23)

Consider now the viscous layer thickness \( \delta \) to be small in comparison with the inside radius \( a \) of the pipe. Consider the order of magnitude of the following variables as 27

\[
\begin{align*}
u & \approx \delta \\
v & \approx 1 \\
w & \approx 1 \\
\delta & \ll 1 \\
\tau_{ij} & \approx \mu \frac{\partial v_i}{\partial x_j}
\end{align*}
\]
Substituting the order of magnitude approximations into the Navier-Stokes equations and the continuity equation and neglecting all terms of the order \( \delta \) or higher yields

\[
\frac{v^2}{r} + \frac{\sin \psi}{R + r \sin \psi} w^2 = -\frac{1}{\rho} \frac{\partial p}{\partial r} \tag{24}
\]

\[
u \frac{\partial v}{\partial r} + \frac{v}{r} \frac{\partial v}{\partial \psi} - \frac{\cos \psi}{R + r \sin \psi} w^2 = \frac{1}{\rho} \left( -\frac{1}{r} \frac{\partial p}{\partial \psi} + \frac{\partial \tau_{r \psi}}{\partial r} \right) \tag{25}
\]

\[
u \frac{\partial w}{\partial r} + \frac{v}{r} \frac{\partial w}{\partial \psi} + \frac{\cos \psi}{R + r \sin \psi} vw = \frac{1}{\rho} \frac{\partial \tau_{r \theta}}{\partial r} \tag{26}
\]

\[
u \frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \psi} + \frac{R + 2r \sin \psi}{r(R + r \sin \psi)} u + \frac{\cos \psi}{R + r \sin \psi} v = 0 \tag{27}
\]

where the derivatives with respect to \( \theta \) have been assumed negligible in accordance with the previous discussion. The \( r \) momentum equation, (24), reflects the balance between the centrifugal forces and the pressure gradient across the viscous layer. If it is assumed that the pipe bend radius of curvature is sufficiently large, then the pressure gradient across the viscous layer will be negligible in conformance with the usual boundary layer assumptions. Thus Equations 24 and 25 become

\[
0 = \frac{\partial p}{\partial r} \tag{28}
\]

and

\[
u \frac{\partial v}{\partial r} + \frac{v}{r} \frac{\partial v}{\partial \psi} - \frac{\cos \psi}{R + r \sin \psi} w^2 = \frac{1}{\rho} \left( -\frac{1}{r} \frac{\partial p}{\partial r} + \frac{\partial \tau_{r \psi}}{\partial r} \right) \tag{29}
\]
Now Equations 26, 27 and 29 form the set of equations that must be solved to obtain the flow properties within the viscous layer. The boundary conditions are:

1) At \( r = a \) and \( \psi = \psi \)
   
   \[
   u = v = 0 \\
   \tau_r \psi = \tau_\psi \\
   \tau_r \theta = \tau_\theta 
   \]

2) At \( r = \delta \) and \( \psi = \psi \)

   \[
   u = U_0 \\
   v = V_0 \\
   w = W_0 \\
   \tau_r \psi = \tau_r \theta = 0 .
   \]

**Viscous Layer: Integral Form.** - The integral method of von Karman-Pohlhausen will be employed to obtain an approximate solution to the viscous layer equations. Before performing the necessary integration of the momentum equations across the viscous layer, it is desirable to undertake simplification by incorporating the continuity equation within each of the momentum equations. Consider for example the \( \psi \) momentum equation, (29). Let

\[
\frac{v}{r} \frac{\partial v}{\partial \psi} = \frac{1}{2r} \frac{\partial (v^2)}{\partial \psi}
\]

and

\[
u \frac{\partial v}{\partial r} = \frac{\partial uv}{\partial r} - v \frac{\partial u}{\partial r} .
\]
By means of the continuity equation, the last equation becomes

\[
\frac{u}{r} \frac{\partial v}{\partial r} = \frac{\partial uv}{\partial r} + \frac{1}{2r} \frac{\partial (v^2)}{\partial \psi} + \frac{R + 2r \sin \psi}{r(R + r \sin \psi)} uv + \frac{\cos \psi}{R + r \sin \psi} v^2.
\]

Combining the first and third term in Equation 30 and substituting into the \( \psi \) momentum yields an improved version of that equation

\[
\frac{1}{r(R + r \sin \psi)} \frac{\partial [r(R + r \sin \psi)uv]}{\partial r} + \frac{1}{r} \frac{\partial (v^2)}{\partial \psi} + \frac{(v^2 - w^2) \cos \psi}{R + r \sin \psi} = \frac{1}{\rho} \left( - \frac{1}{r} \frac{\partial p}{\partial \psi} + \frac{\partial \tau_r \psi}{\partial r} \right)
\]

The \( \theta \) momentum equation likewise becomes

\[
\frac{1}{r(R + r \sin \psi)} \frac{\partial [r(R + r \sin \psi)uw]}{\partial r} + \frac{1}{r} \frac{\partial vw}{\partial \psi} + \frac{2 \cos \psi}{R + r \sin \psi} vw = \frac{1}{\rho} \frac{\partial \tau_r \theta}{\partial r}.
\]

At this point it is important to make a distinction between the "shedding layer" and the "boundary layer" which together comprise the viscous flow region. The boundary layer is defined in the usual sense for the axial component of velocity, \( w \). That is, the thickness of the boundary layer is defined to be the distance from the wall, \( \delta_2 \), at which \( \partial w / \partial r \) becomes zero, or approximately so. Likewise the thickness of the shedding layer, \( \delta_1 \), may be defined as the distance from the wall at which \( \partial v / \partial r \) becomes approximately zero as \( v \) approaches \( V_0 \). These two thicknesses will not be equal (except near \( \psi = 0 \) and 180 degrees) although this is the assumption made in previous shedding
layer type analyses. In the present development however this assumption will not be employed. The thickness of the boundary layer, $\delta_2$, is not completely independent of the shedding layer flow, of course, because of the appearance of the tangential component of velocity in the $\theta$ momentum equation.

Integrating the $\psi$ momentum equation across the shedding layer and employing some of the boundary conditions yields

$$
- U_0 V_0 + \frac{1}{a} \int_{a-\delta_1}^{a} \frac{\partial(v^2)}{\partial \psi} \, dr + \frac{\cos \psi}{R+a \sin \psi} \left( \int_{a-\delta_1}^{a} v^2 \, dr - \int_{a-\delta_1}^{a} w^2 \, dr \right) \\
= \frac{1}{\rho} \left( - \frac{\delta_1}{a} \frac{dp}{d\psi} + \tau \right). \quad (33)
$$

Similarly, by integrating across the boundary layer, the $\theta$ momentum equation becomes

$$
- U_0 W_\theta + \frac{1}{a} \int_{a-\delta_2}^{a} \frac{\partial(v w)}{\partial \psi} \, dr + \frac{2 \cos \psi}{R+a \sin \psi} \int_{a-\delta_2}^{a} v w \, dr = \frac{1}{\rho} \tau \theta. \quad (34)
$$

where it has been assumed that, since the viscous layer is considered to be thin with respect to the pipe radius, $a$, the variable $r$ in the coefficients takes the value of the pipe radius, $a$.

The thicknesses $\delta_1$ and $\delta_2$ have been employed to define the thickness of the shedding and boundary layers, respectively, across which the integrations are performed. Since the viscous layer is considered to be thin, then the difference between $\delta_1$ and $\delta_2$ is assumed to be small so that the error incurred by not employing identical lower limits on the integral will be insignificant.
By applying Leibnitz rule to the second terms in Equations 33 and 34, the preceding equations may be written as

\[- U_0' V_0' + \frac{d\Theta_{VV}}{d\psi} - V_0' \frac{d\delta_1'}{d\psi} + \frac{\cos \psi}{R' + \sin \psi} (\Theta_{VV} - \Theta_{WW}) = - \delta_1' \frac{dp'}{d\psi} + \tau_\psi' \]

and

\[- U_0' W_0' + \frac{d\Theta_{VW}}{d\psi} - V_0' W_0' \frac{d\delta_2'}{d\psi} + 2 \frac{\cos \psi}{R' + \sin \psi} \Theta_{VW} = \tau_\theta' \]

where the momentum integrals are defined as follows:

\[\Theta_{VV} = \int_{1-\delta_1'}^1 \frac{v^2}{W^2} \, dr' \]  

\[\Theta_{WW} = \int_{1-\delta_1'}^1 \frac{w^2}{W^2} \, dr' \]

\[\Theta_{VW} = \int_{1-\delta_2'}^1 \frac{v w}{W^2} \, dr' \]

The remaining dimensionless variables

\[\delta' = \delta/a \]

\[\tau_\psi' = \tau_\psi/\rho W^2 \]

\[\tau_\theta' = \tau_\theta/\rho W^2 \]

have also been introduced.

Viscous Layer: Relationships for Velocity Profiles and Shearing Stress. - Before Equations 35 and 36 can be solved, additional information is needed with which to evaluate the momentum integrals
and the shear stresses. For turbulent flow, fundamental theoretical concepts are insufficiently developed for this purpose, hence, one must resort to experimental data or empirically derived relationships.

To evaluate the integrals (Equations 37), it is necessary to assume a knowledge of a velocity profile that satisfies the imposed boundary conditions. It has been found that a relationship of the form

\[
\frac{|\vec{v}|}{|\vec{v}_0|} = \left( \frac{n}{\delta} \right)^{1/7}
\]

(38)

adequately represents a turbulent velocity profile in the streamwise direction for three-dimensional boundary layers.\(^{28}\) Here \(n\) is defined as the coordinate normal to the wall and \(|\vec{v}|\) is the magnitude of the streamwise velocity in the viscous layer.

Three-dimensional boundary layers are generally based on a streamline coordinate system\(^{29}\) for simplicity, hence, Equation 38 is in a convenient form for that type of analysis. However, a geodesic coordinate system is employed in the present analysis and the use of streamline quantities will require the introduction of a parameter related to the pitch of the twisting core flow. If the other velocity components in the viscous layer are small with respect to the axial component, then the axial velocity profile can be approximately represented by

\[
\frac{w^l}{W_0^l} = \left( \frac{n^l}{\delta^l_2} \right)^{1/7}
\]

(39)
for turbulent flow in a curved pipe. Within the limitations imposed by previous simplifying assumptions, Equation 39 can be expected to approximate the axial velocity profile, at least at points on the wall away from the region of eddying motion. In the region of eddy motion the fluid exhibits a random movement which cannot be illustrated by such a simple mechanism. Moreover the boundary layer approximations are certainly not valid in this region, particularly if separation occurs.

The tangential turbulent velocities in circular pipes have not been subjected to sufficient investigation to obtain a satisfactory empirical representation of the profile. Weske does, however, illustrate a sampling of measurements made by him which indicates the general shape of the profile. Using certain arguments about the nature of the tangential velocity profile, it is possible to propose the following formulation which roughly represents the measured data

$$\frac{v'}{V_0'} = -\left(\frac{n}{\delta_1'}\right)^{4/7} \left[ v \exp\left(-6 \frac{n}{\delta_1'}\right) - 1 \right]$$

(40)

where $v$ is a function of $\psi$. Figure 7 is a graphical presentation of Equation 40 evaluated for $v = 4$ and 15. The data of Weske is also shown for comparison. It may be observed that while Equation 40 only approximates the experimental data, it does in fact contain several features required for a qualitative representation of the tangential velocity. Some of these are:
Figure 7. Assumed Tangential Velocity Profile Compared with Experimental Data.
1) \( v/V_0' \) approaches \(-0.5\) as \( y' \) approaches \( \delta_1' \), and is zero as \( y' \) approaches zero.

2) The derivative of \( v'/V_0' \) approximately approaches zero as \( n' \) approaches \( \delta_1' \).

3) Near the wall, the profile approaches the characteristic 1/7 turbulent profile (Equation 40)

4) The nonsimilar variable \( v \) controls the peak value that \( v'/V_0' \) may attain in the shedding layer.

The proposed profile should be an improvement over the ones previously employed\(^{24}\) for curved pipe flow especially because of feature 2.

It provides for a finite velocity, near the edge of the shedding layer, opposite in sense to the inwardly directed tangential velocity near the wall. In this way the viscous effects can be included outside the location where the tangential velocity is zero in the shedding layer. Furthermore, the normal derivative of the velocity is zero at the edge of the shedding layer, which is a condition that the previous approximations were incapable of meeting.

Like the turbulent velocity profiles, the relationships for the shear stress must also be determined empirically. For this purpose, one may employ the equation of Blasius\(^ {30} \) for the wall shear stress

\[
\tau_w = 0.0228 \rho u_e^2 \left( \frac{v}{u_e \delta} \right) \quad (41)
\]

where \( u_e \) is the velocity at the edge of the viscous layer. This relationship was originally obtained by the measurement of the pressure drop in smooth tubes. To be applicable to the present problem, it must be
assumed that this relationship is valid for rapidly changing pressure gradients, as has been done in previous developments.

For the axial component of the shear stress at the wall, the Blasius equation gives

\[ \tau_0' = 0.0228 \, w_0' \left( \frac{2}{\delta_0} \right) \frac{1}{4} \]  \hspace{1cm} (42)

where \( \text{Re} \) is the Reynolds number based on pipe diameter. In obtaining an expression for the tangential component of stress, there is once again a lack of experimental data; thus, one must again rely on physical reasoning.

On examining Equation 41 it is observed that the velocity in the equation is the maximum velocity in the viscous layer and the length variable \( \delta \) is the normal distance from the wall at which the maximum velocity occurs. Using these observations as a basis, assume that Equation 41 is valid for the tangential shear stress at the wall, but instead of \( u_e \) use, as its equivalent, the maximum velocity in the shedding layer obtained from Equation 40, and instead of \( \delta \) use the value of \( n \) at which the maximum velocity occurs. Therefore, differentiating Equation 40 with respect to \( n' \) and equating the resulting expression to zero yields

\[ n_m = \frac{v \exp (-6 \, n_m) - 1}{42 \, v \exp (-6 \, n_m)} \]  \hspace{1cm} (43)
where

\[ \eta_m = \frac{n_m'}{\delta_z'} \]

and \( n_m' \) is distance from the wall at which the maximum velocity \( v_m' \) occurs. The roots of Equation 43 may easily be found numerically by the process of iteration. \(^{31}\) The relationship for the maximum tangential velocity is

\[ v_m' = \frac{-K_2' \cos \psi \eta_m^{1/7} \left[ v \exp (-6 \eta_m) - 1 \right]}{R' + \sin \psi} \]  \hspace{1em} \text{(44)}

The tangential shear stress is then

\[ \tau_{\psi'} = \pm 0.228 v_m'^{7/4} \left( \frac{2}{\eta_m \delta_z' Re} \right)^{1/4} \]

\[ - \psi = \frac{3}{2} \pi \rightarrow \frac{5\pi}{2} \]

\[ - \psi = \frac{1}{2} \pi \rightarrow \frac{3}{2} \pi \]  \hspace{1em} \text{(45)}

where

\[ Re = \frac{2 a W_p}{\mu} \]

is the Reynolds number.

In order to derive useful forms of the momentum equations for the evaluation of the shedding layer and boundary layer thicknesses, it is necessary to introduce a transformation in one of the coordinates. Let

\[ r' = 1 - n' \]

hence the momentum integrals (Equation 42) will take the general form
\[
\int_{1-\delta'}^{\delta'} f(r) \, dr = \int_0^{\delta'} f(1-n') \, dy'.
\]

Observe also that when the relationships for the axial and tangential velocity profiles (Equations 39 and 40) are substituted into the momentum integrals (Equation 37) the result is a combination of integrals of the general form

\[
I_{a,b} = \frac{1}{\delta'} \int_0^{\delta'} \left( \frac{n'}{\delta'} \right)^{a/\gamma} \left[ \exp \left( -b \frac{6n'}{\delta'} \right) \right] dy'.
\] (46)

Now let

\[ t = b \frac{6n'}{\delta'} \]

or

\[ y = \frac{\delta t}{6b} \]

\[ dy = \frac{\delta}{6b} \, dt \]

Therefore

\[
I_{a,b} = \frac{1}{\delta'} \int_0^{\delta t/6b} \left( \frac{t}{6b} \right)^{a/\gamma} \left[ \exp \left( -t \frac{\delta}{6b} \right) \right] \, dt
\]

which becomes

\[
I_{a,b} = \frac{1}{(6b)^{a/\gamma} + 1} \int_0^{\infty} t^{a/\gamma} e^{-t} \, dt
\] (47)

since the integrand contributes nothing beyond \( y = \delta \). The integral on the right hand side is recognized as the gamma function. Thus,

\[
I_{a,b} = \frac{1}{(6b)^{a/\gamma} + 1} \Gamma \left( \frac{a}{\gamma} + 1 \right)
\] (48)
where as usual $\Gamma(x)$ represents the gamma function of the argument $x$.

The values of $I_{a,b}$ were found to be as follows:

$$I_{1,1} = 0.12027579$$
$$I_{2,1} = 0.089444482$$
$$I_{2,2} = 0.03682574$$

Evaluating the momentum integrals with the aid of the velocity profiles yields

$$\Theta_{VV} = V_0^2 \delta_1 \left( \nu^2 I_{2,2} - 2 \nu I_{2,1} + \frac{7}{9} \right)$$  \hspace{1cm} (49a)
$$\Theta_{WW} = \frac{7}{9} \delta_1 W_0^2$$  \hspace{1cm} (49b)
$$\Theta_{VW} = V_0' W_0' \delta_2 \left( \frac{7}{9} - \nu I_{2,1} \right)$$  \hspace{1cm} (49c)

and for the derivatives

$$\frac{d\Theta_{VV}}{d\psi} = \frac{\Theta_{VV}}{V_0^2 \delta_1} \left( V_0^2 \left( \frac{d}{d\psi} + \delta_1 \frac{dV_0^2}{d\psi} \right) + 2 V_0^2 \delta_1 \left( \nu I_{2,2} - I_{2,1} \right) \frac{dv}{d\psi} \right)$$  \hspace{1cm} (50a)
$$\frac{d\Theta_{WW}}{d\psi} = \frac{7}{9} \left( \delta_1 \frac{dW_0^2}{d\psi} + W_0' \frac{d\delta_1'}{d\psi} \right)$$  \hspace{1cm} (50b)

$$\frac{d\Theta_{VW}}{d\psi} = \left( V_0' W_0' \left( \frac{d\delta_2}{d\psi} + \delta_2 \frac{dV_0'}{d\psi} \right) \right) \frac{\Theta_{VW}}{V_0' W_0' \delta_2} - V_0' W_0' \delta_2 I_{2,1} \frac{dv}{d\psi}$$  \hspace{1cm} (50c)
Additional Restraints

The $\psi$ and $\theta$ momentum equations (35 and 36) constitute a set of two independent equations and three unknown dependent variables, $\delta_1', \delta_2'$, and $\nu$ plus two unknown constants. In order to obtain a solution, additional equations must be found to complete the set. Since the continuity equation was only used to change the original form of the two momentum equations and not to eliminate a variable, the conservation of mass can be employed for one of the remaining relationships.

Consider now the volume flow, $\hat{q}_{SL}$, in the shedding layer. Thus

$$\hat{q}_{SL} = \int_{1-\delta_1'}^{1} \nu' \, dr'$$

which becomes by virtue of the relationship for the variation of $\nu'$ across the shedding layer

$$\hat{q}_{SL} = -\delta_1' \nu_0' \left( \nu \, I_{1,1} - \frac{7}{8} \right).$$

Likewise the volume flow of fluid exchanged between the shedding layer and the central core across the boundary of the shedding layer is

$$\hat{q}_c = \frac{K_2' \, (1-\delta_1')} {R' + (1-\delta_1') \sin \psi} \cos \psi$$

which becomes approximately

$$\hat{q}_c \approx \nu_0' \, (1-\delta_1').$$
if $\delta_1' << 1$. Since mass must be conserved, the flow across the boundary of the shedding layer must be equal to the negative of the flow within the shedding layer. Hence,

$$\dot{q}_c = -\dot{q}_{SL}$$

or

$$\delta_1' = \frac{1}{\nu I_{1,1} + 0.125}$$  \hspace{1cm} (53a)$$

$$\nu = \frac{1}{I_{1,1}} \left( \frac{1}{\delta_1'} - 0.125 \right)$$  \hspace{1cm} (53b)$$

$$\frac{d\nu}{d\psi} = -\frac{1}{\delta_1'^2 I_{1,1}} \frac{d\delta_1'}{d\psi}. \hspace{1cm} (53c)$$

Eliminating $d\nu/d\psi$ from Equations 35 and 36 by means of Equation 53c, the $\psi$ and the $\theta$ momentum equations become

$$\frac{d\delta_1'}{d\psi} = \left\{ -U_0' V_0' + \frac{\cos \psi}{R' + \sin \psi} (\Theta_{VV} - \Theta_{WW}) + \delta_1' \frac{d\psi}{d\psi} + \tau_\psi' \right. + \frac{\Theta_{VV}}{V_0'^2} \frac{dV_0'^2}{d\psi} \right\} / V_0'^2 \left\{ -\nu^2 I_{2,2} \right. + 2 \nu I_{2,1} + \frac{2}{9} + \frac{2}{\delta_1'^2 I_{1,1}} (\nu I_{2,2} - I_{2,1}) \right\} \hspace{1cm} (54a)$$

$$\frac{d\delta_2'}{d\psi} = \left\{ -U_0' W_0' + \frac{\Theta_{VV}}{V_0'^2 W_0'^2} \frac{dV_0'^2}{d\psi} W_0' - V_0' W_0' \delta_2' I_{2,1} \frac{d\nu}{d\psi} \right. \right. + \frac{2 \cos \psi}{R' + \sin \psi} \Theta_{VV} + \tau_\theta \right) / V_0' W_0' \left( \nu I_{2,2} + \frac{2}{9} \right) \hspace{1cm} (54b)$$
where

\[
\frac{dV_0'^2}{d\psi} = -2K_2'^2 \left\{ \frac{\sin \psi \cos \psi}{(R' + \sin \psi)^2} + \frac{2 \cos^2 \psi}{(R' + \sin \psi)^3} \right\} \quad (55a)
\]

\[
\frac{dV_0'W_0'}{d\psi} = -K_1'K_2' \left\{ \frac{\sin \psi}{(R' + \sin \psi)^2} + \frac{2 \cos^2 \psi}{(R' + \sin \psi)^3} \right\} \]

\[
\frac{dp'}{d\psi} = \frac{(K_1'^2 + K_2'^2) \cos \psi}{(R' + \sin \psi)^3} \quad (55b)
\]

The equality of volume of flow through the pipe to the integral of the axial component of velocity over the cross-sectional area is another condition that must be fulfilled. Thus if \( Q \) is the volume flow rate through the pipe then

\[
Q = 2 \int_{-\pi/2}^{\pi/2} \int_0^\alpha w^* r \, d\psi \, dr \quad (56)
\]

or in dimensionless terms

\[
1 = \frac{2}{\pi} \int_0^{\pi/2} \int_{-\pi/2}^{\pi/2} \frac{W}{W} r' \, d\psi \, dr' - \frac{2}{\pi} \int_1^{1-\delta_2'} \int_{-\pi/2}^{\pi/2} \left( \frac{W_0}{W} - \frac{W}{W} \right) r' \, d\psi \, dr' \quad (57)
\]

where \( \delta_2' \) is the average boundary layer thickness. The first integral becomes

\[
\frac{2}{\pi} \int_0^{\pi/2} \int_{-\pi/2}^{\pi/2} \frac{W}{W} r' \, d\psi \, dr' = \frac{2K_1'}{\pi} \int_0^{\pi/2} \int_{-\pi/2}^{\pi/2} \frac{r' \, d\psi \, dr'}{R' + r' \sin \psi} = 2K_1' \left[ R' - (R'^2 - 1)^{1/2} \right] \quad (58)
\]

Now if the flow in the pipe is completely inviscid, the last integral in Equation 57 will be zero, and the first integral should tend
to unity as $R'$ increases in magnitude. Since the right hand side of Equation 58 is indeterminate in its present form, it must be expanded in a binominal series expansion to determine its limiting value. Thus

$$\left( R'^2 - 1 \right)^{\frac{1}{2}} = R' \left( 1 - \frac{1}{2R'} - \frac{1}{8R'^3} - \frac{1}{16R'^5} - \frac{5}{128R'^7} - \ldots \right) \quad (59)$$

Hence

$$\frac{2}{\pi} \int_{0}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{W}{W} r' \, d\psi \, dr' \approx K_1 \left( \frac{1}{R'^2} + \frac{1}{4R'^4} + \ldots \right) \quad (60)$$

neglecting terms of $R'^5$ and higher. Since $R'$ is always greater than unity and $K_1$ must approach $R'$ for large $R'$ to satisfy Equation 15, then it is clear that the value of the first integral approaches unity as $R'$ increases.

Considering now the first part of the second integral of Equation 57

$$\frac{2}{\pi} \int_{1-\delta_2}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{W_0}{W} r' \, d\psi \, dr' = \frac{2K_1}{\pi} \int_{1-\delta_2}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r' \, d\psi \, dr'}{R' + \sin \psi}$$

$$= \frac{K_1 \left( 2 \delta_2^2 - \delta_2^2 \right)}{(R'^2 - 1)^{\frac{1}{2}}} \quad (61)$$

The last part of the second integral is

$$\frac{2}{\pi} \int_{1-\delta_2}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{w}{W} r \, d\psi \, dr = \frac{2K_1}{\pi} \int_{1-\delta_2}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(1-r')^{\frac{1}{2}} \, r' \, d\psi \, dr'}{R' + r' \sin \psi}$$

$$= \frac{K_1}{(R'^2 - 1)^{\frac{1}{2}}} \left( \frac{7}{4} \delta_2^2 - \frac{14}{15} \delta_2^2 \right) \quad (62)$$
Thus from Equations 57, 58, 61 and 62, a relationship for $K_1'$ is obtained:

$$
\frac{1}{K_1'} = 2 \left[ R' - (R'^2 - 1)^{\frac{1}{2}} \right] + \frac{1}{4} \frac{\delta'_{21} - \frac{1}{15} \delta'_{22}}{(R'^2 - 1)^{\frac{1}{2}}} + \delta'.
$$

(63)

The means of determining the precise value of $K_1'$ has now been established. A trial and error procedure is required whereby values of $K_1'$ are assumed until the value computed from Equation 63 is equal to the assumed value. The boundary layer thickness is obtained from the numerical integration of Equation 54b.

Before the numerical solutions to Equation 54 can be initiated, some additional conditions are needed to determine the initial starting values for $\delta'_1$ and $\delta'_2$. Because of the presence of $V_0'$ in the denominators of the right hand side of Equation 54, the derivatives are unbounded at the $\psi = 90$ and 270-degree positions in the pipe, hence the numerical integration procedure cannot be started there. Therefore, to give a starting value of $\delta'_1$, it was assumed that at $\psi = 0$ and 180 degrees

$$
\frac{d\delta'_1}{d\psi} = 0
$$

This assumption can be justified on the basis that, near these values of $\psi$, the mass exchange with the central core will experience a change in sense because of the wall curvature. Also one might expect this condition from the influence of the centrifugal force which is a maximum in the tangential direction at these values of $\psi$. 
This is to say that the shedding layer thickness is a maximum halfway around the inside wall. The starting values of $\delta_1'$ can now be found from Equation 54a by a numerical trial and error process.

Considerable effort was devoted to looking for two additional restraints with which $K_2'$ and a starting value of $\delta_2'$ could be independently determined. One such possibility exists, for instance, in the equation for volume flow in the shedding layer, which is

$$\dot{q}_{SL} = - \delta_1' V_0' \left( \nu I_{1,1} - \frac{7}{8} \right) = - V_0' (1 - \delta_1') .$$

Since the volume flow is zero at $\psi = 90$ and 270 degrees then this condition could be satisfied by letting $\delta_1' = 1$ and $\nu I_{1,1} = 7/8$ at $\psi = 90$ and 270 degrees. Unfortunately, however, the integration cannot be started on the plane of the bend and furthermore the condition that the volume flow in the shedding layer be zero at the stagnation points is identically satisfied by the presence of the $V_0'$ factor. Hence this consideration introduces no additional relationships to be satisfied.

Another example may be provided by the fact that since the flow field is a reflection about the plane of curvature $d\delta_2'/d\psi$ should be zero at $\psi = 90$ and 270 degrees. This condition should provide an independent equation from Equation 54a for $\theta$ momentum which must be satisfied in the plane of the bend. Since the integration of the $\psi$ momentum equation provided a shedding layer thickness that tended to zero as the plane of the bend was approached it was impossible to determine a finite value of $\nu$ at the stagnation points because of the relationship
between \( \nu \) and \( \delta_1' \) (Equation 53). Therefore this approach also fails to yield any new relations. It will be assumed then in the calculations that follow that the starting value of \( \delta_2' \) is the same as the starting value of \( \delta_1' \) and the constant \( K_2' \) must be determined "experimentally".

The numerical evaluation of the present theory was undertaken on the Univac located at the University of Alabama Research Institute. The integration of the \( \psi \) and \( \theta \) momentum equations was accomplished by means of the Runge-Kutta method (see Appendix). The average value of the boundary layer thickness was found as follows:

\[
\bar{\delta}_2' = \frac{1}{\psi_0 - \psi} \int_{\psi_0}^{\psi} \delta_2' \, d\psi.
\]  

(64)

Results

An original objective of this analysis was to obtain a theoretical solution of the flow variables in a pipe bend which is independent of any experimentally obtained data. While realistic from a physical standpoint, the particular relationship employed for the shedding layer velocity profile and the fact that the shedding layer and boundary layer thicknesses are separately identified have introduced an excessive number of unknowns which prevent a complete theoretical determination of the flow field variables. In addition, the form of the relationships resulting from these particular assumptions precluded the determination of additional restraints which could be used to complete the set of
equations relating the various constants and variables. For these reasons it has thus far been impossible to establish an independent procedure for determining $K_2'$ and an initial value for $\delta_2'$. The purpose of this section is to present some of the computation results from the theory as outlined above and to examine in particular the consequences of the necessity of providing an "educated guess" for the value of $K_2'$.

Unfortunately there is only fragmentary data available with which to examine the assumptions regarding curved flow in circular ducts. Weske$^{15}$ has measured the velocities at the exits of 30-degree bends which, for purposes of comparison, will be considered approximately representative of the conditions at the position at which the minimum pressure occurs in bends of larger deflection angles. Figure 8 illustrates the profiles of axial velocity in the plane of the bend and the tangential velocity parallel to the plane of the bend. The experimental value of $W'$ at $r = 0$ will yield the experimental value of $K_1'$ through Equation 15c and the value of $V'$ at $r = 0$ will yield the value of $K_2'$ through Equation 15b. Table 1 presents the resultant theoretical and experimental values of $K_1'$ ($K_2'$ was assumed to be equal to $R'/30$).

One notes from Figure 8 that the experimental velocity in the pipe cross section parallel to the plane of curvature is everywhere directed towards the inside of the bend. Since this condition violates the principle of the conservation of mass, it must be explained through experimental error or in some manifestation of the oscillatory motion of the fluid in the transition region which was predicted by Hawthorne$^{13}$.
Figure 8a. Velocities at the Exit of a 30-Degree Bend - Comparison Between Experiment and Present Analysis (Re = 5.35 × 10^5, R' = 8)
Figure 8b. Velocities at the Exit of a 30-Degree Bend - Comparison Between Experiment and Present Analysis (Re = 5.35 x 10^5, R' = 3)
Figure 8c. Velocities at the Exit of a 30-Degree Bend—Comparison Between Experiment and Present Analysis (Re = 5.35 x 10^5, R_i = 2)
TABLE 1
COMPARISON BETWEEN EXPERIMENTAL DATA\textsuperscript{15} AND PRESENT THEORY

<table>
<thead>
<tr>
<th>$R'$</th>
<th>$K_1'$</th>
<th>$K_2'$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Theory</td>
<td>Experimental</td>
</tr>
<tr>
<td>8.0</td>
<td>8.120</td>
<td>8.07</td>
</tr>
<tr>
<td>3.0</td>
<td>2.960</td>
<td>3.27</td>
</tr>
<tr>
<td>2.0</td>
<td>1.895</td>
<td>2.20</td>
</tr>
</tbody>
</table>

and experimentally observed by others.\textsuperscript{14} Weske does not comment on these unusual observations, however. At any rate they do indicate that the secondary velocities will be exceedingly small compared with the axial velocity and hence $K_2'$ will be small compared with $K_1'$ (or $R'$).

The effect on the pressure resulting from "guessing" the value of $K_2'$ will be examined by comparison with the minimum pressure coefficient for completely potential flow, e.g., for the condition that

$$\delta_2'_{o} = \delta_2' = K_2' = 0$$

Figure 9 illustrates the comparison of $C_{p_{\text{min}}}$ for potential flow with the present theory for $R'/K_2' = 30$ and 15 for Reynolds numbers of $5 \times 10^4$ and $10^6$. Although it is not evident from the graphical presentation, the value of $C_{p_{\text{min}}}$ computed from the theory with a value of $R'/K_2' = 15$ deviates from the potential $C_{p_{\text{min}}}$ by about 5 percent at $R' = 1.5$ to about 3 percent at $R' = 5$. For a value of $R'/K_2' = 30$, the
Figure 9. Effects of Various Choices of $K_2$ on the Minimum Pressure Coefficient.
computed $C_{p_{\text{min}}}$ was indistinguishable from those computed for $R'/K_2' = 15^*$. On the basis of this comparison and the previous discussion concerning the relative magnitudes of the velocity components, it appears that the value of $K_2'$ is quite small with respect to $R'$, and that the minimum pressure coefficient is relatively insensitive to its particular value and the value of the Reynolds number.

As illustrated in Figure 8 and Table 1, the theoretical values of $K_1'$ are at variance with the experiments, especially at the smaller values of $R'$. For the cases examined, the theoretical values of $K_1'$ was approximately equal to the radius of curvature $R'$ while the experimental values of $K_1'$ were slightly larger than the radius of curvature $R'$.

Some of the disagreement between the theoretical and experimental $K_1'$ (or axial velocity profiles) can be easily explained by considering the theoretical variation of the boundary layer thickness about the inside of the pipe (Figure 10). Clearly the theory fails to predict physically reasonable values of $\delta_2'$ in the plane of the bend ($\psi = 90$ and 270 degrees). The experimental values of $\delta_2'$ (from Figure 8) are approximately 0.33 at the inside of the bend and 0.16 at the outside of

---

*Actually because of slight variations resulting from convergence of the numerical iteration procedures, the values of $C_{p_{\text{min}}}$ for $R'/K_2' = 30$, and 15 oscillated about one another at different values of $R'$. $C_{p_{\text{min}}}$ did not change appreciably for variations in Reynolds number.*
Figure 10. Boundary Layer Thickness for Different Values of $R'$
the bend, whereas the theory predicts zero boundary layer thicknesses at both of these locations. For this reason the average value of the theoretical boundary layer thickness \( \delta_2' \) will be considerably less than the actual value of \( \delta_2' \) thus yielding a lower value of \( K_1' \) because of the form of the equation for \( K_1' \) (Equation 63). The failure of the theory to accurately predict the variation of \( \delta_2' \) has particular significance from the standpoint of the primary objective of the analysis since the minimum pressure coefficient is dependent upon the boundary layer thickness at the inside stagnation point and the value of \( K_1' \) squared (Equation 19).

In addition to affecting the magnitude of the minimum pressure in a pipe bend, the boundary layer thickness at the inside of the pipe \( \delta_{2,0}' \) also influences the position of minimum pressure in a bend cross section as Figure 2 illustrates. Pressure measurements by Weske\(^{15} \) indicate that, on approaching the inside wall from the duct center, the pressure decreases as would be expected if the flow were potential in the core. However, approximately at the position corresponding to the interface between the viscous flow and potential flow, as illustrated by velocity measurements (see Figure 8), the pressure reaches a minimum and past that point, toward the duct inside wall, it then experiences a sharp increase attaining a second maximum at or near the inside wall. Weske has designated this viscous layer at the inside wall of thickness \( \delta_{2,0}' \) as the "region of eddy flow".
The effect of the dissimilarity between the theoretical and experimental boundary layer thicknesses and the axial velocity profiles (or $K_1'^*$s) on the minimum pressure coefficient will now be examined. Equation 19 will be used to compute the "experimental" value of $C_{p_{\text{min}}}$ from $K_1'^*$ and $\delta_{2,0}'$ obtained from the experimentally measured axial velocity profile. The experimental value of $K_2'^*$ is assumed to be zero. Figure 11 presents the $C_{p_{\text{min}}}$ derived from the experimental velocity profiles, the potential $C_{p_{\text{min}}}$, and $C_{p_{\text{min}}}$ predicted from the present theory. It is observed that, although the theoretical values of $K_1'^*$ are lower than the experimental values because of a smaller average boundary layer thickness, the theoretical $C_{p_{\text{min}}}$ will be greater than the actual $C_{p_{\text{min}}}$ in the flow field because of its influence of $\delta_{2,0}'$ in the denominator of Equation 19. Thus, there are offsetting factors involved in the theoretical computation of too small a value of boundary layer thickness. One effect is to underpredict the axial velocity magnitude, $K_1'^*$ but conversely to overpredict the $C_{p_{\text{min}}}$ because of the over-canceling influence of $\delta_{2,0}'$ in the denominator of Equation 19.

An explanation as to why the theory fails to predict more reasonable values of the boundary layer thickness could be in the neglect of the $r$ momentum equation (Equation 24). This equation relates the pressure gradient across the viscous layer to the centripetal forces exerted on the fluid as it traverses its circular path through the bend and around the interior of the bend. To estimate the consequences
Figure 11. Comparison of the Experimentally Derived Minimum Pressure Coefficient (Ref. 15) with the Present Theory and Potential Flow.
of neglecting Equation 24, an approximate analysis was performed using this equation and assuming a linear pressure gradient across the shedding layer. The results indicated no appreciable influence on either $K_1$ or the variation of the boundary layer thickness. Including the effects of this equation in a more precise manner will require the simultaneous solution of Equations 54a and 54b.

The inferences and conclusions arising from this presentation must necessarily be regarded as somewhat tentative because of the lack of adequate experimental data with which to make comparisons and to clarify the nature of curved flow in the transition region. The data used herein$^{15}$ does not actually apply to the problem at hand because it was taken at the exit of a bend where the flow is in a transition from curvilinear to rectilinear motion. The author was unable to find any experimental information$^7$ which would yield the minimum pressure or more fundamentally the velocity profiles in the transition portion of the bend. All of the available measurements have been made at the bend exit, in the case of velocities, or along the bend wall rather than in the flow field, in the case of pressures.

As a result of this study, the following statements can be tendered regarding turbulent flow at which the minimum pressure exists at the deflection angle in a pipe bend:
1) The flow variables are essentially independent of Reynolds number.

2) The secondary circulation is small and has little effect upon the primary flow variables.

3) The flow in the central core is potential but the magnitude of the axial velocity is higher than full potential flow because of the constriction of the boundary layer.

4) Simple potential flow considerations will yield a minimum pressure less than the actual minimum pressure.

5) The thickness of the region of eddy motion is an important consideration in predicting the minimum pressure.
CHAPTER III

EXPERIMENTAL INVESTIGATION

Experimentation on curved pipes has in the past emphasized primarily the losses associated with turning the flow since it has traditionally been of major significance in hydraulic designs. Experimental studies of the pressures have been rather limited in scope, although pipe bends have some utility as flow meters, with the pressure differential between the inside and the outside walls as the flow rate indication.31-33

The purpose of the experimental investigation is to obtain the static pressure profiles in the plane of the bend at the walls of 90-degree circular pipe bends. The preceding analysis applied only to the cross-sectional plane at which the pressure was assumed to be stationary but it was not able to specify the position of this plane. Therefore, the primary objective of the experimental investigation is to determine the axial position of the minimum pressure (and its value) at the wall. If the assumption is made that the minimum wall pressure and the minimum flow field pressure lie in the same cross-sectional plane, then these results should provide the foundation for more detailed pressure traverses in the flow field at the cross-sectional plane where the greatest pressure variations across the pipe

60
bend arise. It is expected that the results of these investigations will help in acquiring a better understanding of curved pipe flow in the transition region and will become the basis for future studies of bend cavitation and for establishing the utility of curved pipe analyses in the transition region.

The present investigation was conducted using water as the fluid medium. A range of Reynolds numbers, based on pipe diameter, was studied with $1.8 \times 10^6$ being the maximum and $2.5 \times 10^5$ being the minimum. Five bend radii of curvatures were investigated, namely, 2.00, 1.50, 1.25, and 0.75 times the inside diameter, which was nominally 4 inches.

**Apparatus**

**General Description.** - The facility (Figure 12a) used in this study is a constant diameter closed return loop oriented in the vertical plane and using water as the test medium. The elbow test sections form the second bend in the upper limb of the test loop (Figure 12b) which is primarily fabricated from 4-inch-diameter heavy wall copper tubing (Type L) with standard bronze sweat solder elbows and 150-pound flanges. To insure the attainment of fully developed turbulent flow at the test bend entrance, the elbow test sections are connected on the upstream side to a straight pipe run which is 60 diameters long and is preceded by a flow straightener. To minimize the transmission of vibrations to the test sections, flexible metal hoses form the portion
Figure 12a. Photograph of the Test Facility Used in the Experimental Investigations
Figure 12b. Schematic of the Test Facility Used in the Experimental Investigations
of the loop immediately adjacent to the control valve and the pump. Also the pipe sections are insulated from the supports with rubber padding. A 62-gallon accumulator tank is provided to impose additional hydrostatic head on the system and to absorb spurious pressure surges in the test loop. To partially remove some of the dissolved air, the water from the utility line is used to impose a mild vacuum in the ullage. Control of the flow rate through the system is achieved by regulating the pressure drop with a gate valve mounted to the outlet side of the pump.

The loop is driven by a commercially available close coupled pump which was selected to achieve minimal vibration associated with shaft misalignment. Design rating of the pump is 1100 gal/min with a total head rise of 150 feet of liquid at 1770 rpm. The pump is powered by a 60 horsepower, 440 volt, three-phase motor. The ultimate Reynolds number with the present components is about $1.8 \times 10^6$.

**Test Sections.** - Five different elbow test sections were used in the present investigation. The radii of centerline curvature were 2.00, 1.50, 1.25, 1.00 and 0.75 times the internal diameter.

The test sections were fabricated from transparent acrylic plastic. The internal contour of each elbow was accurately machined into two plastic sheets of appropriate thickness. The two halves were then bonded together with an epoxy cement to form a pipe bend with the seam in the plane of curvature. Flanges were then bonded to the ends
of the curved section. The surfaces were polished to approximately 10 rms finish so that the flow could be visually observed.

Instrumentation of the test bends consisted of a number of static pressure taps as illustrated in Figure 13. The taps were formed by drilling 3/16-inch holes to within 1/4 inch of the inside wall and then drilling through the surface with a 1/32-inch-diameter drill. The taps at the flanges were not drilled but were formed by milling 0.016-inch deep circular grooves into the parts before assembly. Wires of 0.032-inch diameter were coated with a mould release and positioned in the grooves before the ports were cemented together. After the parts were cemented together, these wires were then removed when the cement had set. The openings were carefully hand polished to remove burrs and imperfections which could affect the accuracy of the measured results. Brass connectors with a 3/32-inch straight thread on one end and 1/16-inch hose fitting on the other were cemented into the taps in the bend. Where space was limited at the inside of some of the small curvature bends, a 1/16-inch countersink was used and 1/16-inch-diameter copper tubing was employed as hose connectors.

Instrumentation. - Test section pressures were measured by a commercially available 10-tube manometer with a 60-inch range. All measurements were made with mercury as the indicating fluid and were referenced to a static pressure tap upstream of the test section. Piezometer lines were of transparent plastic tubing which were connected to the test section through a series of valves used to bleed off
Figure 13. Schematic of the Elbow Test Section
trapped air bubbles. Pressure data were recorded on 2 by 3 inch lantern slides with a tripod mounted camera. The slides were then projected onto a screen which permitted readings of the pressure differences to within approximately ±0.02 inches of mercury. The mercury was periodically removed from the manometer and was cleaned by filtering three times through 50 percent dilute nitric acid. The manometer was cleaned with detergent and water followed by an acetone rise; the glass tubes were cleaned with bichromate acid.

Flow rates through the loop are measured with a propeller type volumetric flowmeter located in the 4-inch-diameter line on the suction side of the pump. The nominal linear range is in the interval between 75 gal/min to 1250 gal/min with a stated accuracy of ±0.02 percent in this range. For limited operation the flowmeter can be operated at 1500 gal/min without damage to the pickup sensor. The ac signal from the flowmeter was converted to a voltage output which was recorded on a direct reading oscillograph. The converter has a stated analog output accuracy of ±0.1 percent. A calibration oscillator was used to calibrate the converter. The tuning fork that generates the signal has a stated accuracy of ±0.05 percent.

Additional instrumentation consists of a small pressure-vacuum gage mounted on the surge tank and a well-type thermometer mounted in the 4-inch diameter line downstream of the test bend, which can be read to within about ±1.0°F.
Procedure

**Facility Operation and Techniques.** Untreated tap water was used as the test medium in the present study; however, before it was introduced into the system it was allowed to stand in the accumulator tank while the jet pump created a mild vacuum in the ullage. The length of time it was permitted to remain under vacuum varied from overnight to two or three days. This process sufficiently removed enough dissolved air so that the manometer lines could be bled more easily.

Before the system was filled for each run, it was thoroughly flushed with fresh tap water to remove loose scale and corrosion from the iron pump casing. The deaerated water in the surge tank was then allowed to slowly fill the loop under gravity head while air was vented from the system through the main vent, small vents on the upper leg and the pump casing vent. Thorough venting of the pump casing required rotating the impeller by hand until all trapped air was removed. The pump was then pulsed several times while the bleeding operation was intermittently performed. This procedure resulted in only a very small amount of air remaining in the system.

After filling the system the accumulator tank was partially filled with tap water and a small pressure was applied using the tap water pressure. The piezometer lines were then bled free of air by opening all of the manometer control valves. To bleed the lines between the valves and the manometer, the manometer well was raised to a position
about 20 inches from the top of the manometer. The bleed valve and the piezometer valve of each tube were then alternately opened and closed, thus forcing the air bubbles out of the vent. This operation was repeated in turn for all manometer tubes plus the well. This procedure completely filled the piezometer connecting lines and manometer with water and removed all entrapped air from the pressure instrumentation system. The flowmeter recorder was then calibrated using the 100, 120, 200 and 500 hertz signal generated by the calibrator.

To perform one test run, the pump motor was started with the control valve about 1/4 open and the flow rate was then adjusted to the lowest value desired (for instance, 40 percent of full flow). After the manometer indicated that equilibrium had been reached, a Polaroid picture was taken of the mercury columns and a recording was made of the temperature. The flow rate was then increased to the next highest level and the procedure repeated until the full range of flow rates had been examined for the particular elbow installed in the system. This completed a test run.

The film transparencies of the pressure data were projected on a screen and then recorded. The difference between the measured pressures and the reference pressure were taken and then corrected for the weight of the additional column of water on the lower mercury column. Hence the corrected pressure difference in inches of mercury is
\[(h_{\text{ref} - h})_{\text{corrected}} = (h_{\text{ref} - h})_{\text{measured}} - \frac{(h_{\text{ref} - h})_{\text{measured}}}{13.56} \]

\[= 0.92625 (h_{\text{ref} - h})_{\text{measured}} \] (65)

where 13.56 is taken as the specific gravity of mercury and \(h_{\text{ref}}\) corresponds to \(p_1\) (except for units). The flow rates were obtained by relating the ac signal generated by the flowmeter, to the flow rate. For the particular flowmeter used, this relationship is

\[Q = 2.882 f \] (66)

Mean velocities obtained from the one-dimensional continuity equation

\[\Bar{W} = \frac{Q}{A} \] (67)

and the Reynolds number from

\[Re = \frac{2 \rho \Bar{W} a}{\mu} \] (68)

using the viscosity data of References 27 and 28.

**Analysis.** In the analytical portion of the investigation a relationship for the minimum pressure coefficient was derived (Equation 19)

\[C_{p_{\text{min}}} = \frac{p_1 - p_{\text{min}}}{\rho \Bar{W}^2 / 2} \]

\[= \frac{K_1^{12} + K_2^{12}}{[R^{1} - (1 - 5_{2,1})^2]} - W_1^{12} \]

where \(K_2^{1}\) is approximately zero and \(W_1^{1}\) is approximately unity. \(K_1^{1}\) is determined by using mass flow considerations and is related to the
bend radius of curvature and the average boundary layer thickness as follows (Equation 63)

\[
\frac{1}{K_1} = 2 \left[ R' - R'^2 - 1 \right]^{\frac{1}{2}} + \frac{1}{4} \frac{5 \gamma_1 - 1}{15} \frac{5 \gamma_2}{(R'^2 - 1)^{\frac{1}{2}}} \quad (70)
\]

The preceding equation can be written more conveniently as

\[
\frac{1}{K_1} = 2 \left[ R' - (R'^2 - 1)^{\frac{1}{2}} \right] + \frac{C}{(R'^2 - 1)^{\frac{1}{2}}} \quad (71)
\]

Now from Equations 69 and 71, the minimum pressure coefficient at the wall is

\[
P_{\text{min, wall}} = \frac{p_{1} - p_{\text{min, wall}}}{\rho \bar{W}^2/2}
= \frac{1}{(R'^2 - 1)^{\frac{1}{2}}} \left\{ 2 \left[ R' - (R'^2 - 1)^{\frac{1}{2}} \right] + \frac{C_{\text{wall}}}{(R'^2 - 1)^{\frac{1}{2}}} \right\}^{-2} \bar{W}_1^2
\]

(72)

where \( C_{\text{wall}} \) is an empirically obtained correlation coefficient which is a function of Reynolds number and radius of curvature of the bend. It is interesting to note the different roles of the empirical constant in the present analysis and the analysis of Addison. In the latter's analysis the empirical constant is a discharge coefficient multiplicative on the first term in the brackets whereas in the present analysis it is an additive term.
Results

The measured minimum pressure coefficients ($C_{p_{\text{min,wall}}}$) are shown in Figure 14 as a function of pipe Reynolds number, $Re$.

For the four bends of largest radii of curvature ($R/2 = 2, 1.5, 1.25, 1.00$), the pressure coefficients indicate an independence of Reynolds number over the range investigated. For the bend of smallest radius of curvature ($R/2a = 0.75$) the pressure coefficient varies with Reynolds number (Figures 14 and 15) but appears to become independent of Reynolds number with values of $Re$ greater than about $1.4 \times 10^6$.

The independence of $C_{p_{\text{wall}}}$ with Reynolds number is in accord with the relationship obtained for other bend parameters, namely the flow discharge coefficient $^{31,32}$ and bend excess loss coefficient $^{34}$. These results also support the conclusions reached through analytical means in the previous section. To ascertain if there were other independent variables which would affect the results, a test run was made deviating from the normal procedure. In this run the flow rate was held essentially constant and the Reynolds number was allowed to vary by the changing viscosity of the water as it became heated. These points are indicated in Figure 14 for the small radius bend. For the range of variables examined in this exploratory investigation, the pressure coefficients were indistinguishable from those of other runs.

In Figure 16 the pressure profiles over the outside and the inside of the bends are presented. The bends of largest radii of
Figure 14. Variation of Minimum Wall Pressure Coefficient with Reynolds Number
Figure 15. Variation of the Wall Pressure Coefficient with Reynolds Number (R/2a = 0.75)
curvature yield pressure coefficients at the wall which are essentially in accord with the results of Yarnell and Nagler;\textsuperscript{35} i.e., the minimum pressure occurs at a deflection angle of about 22.5 degrees and the maximum pressure at about 60 degrees. However, as the radius of curvature is reduced (Figures 16c and 16d), the point of minimum pressure shifts to about 30 degrees bend deflection angle and a region of constant pressure, which is first evident for the $R/2a = 1.50$ bend, appears at a deflection angle of 67.5 degrees and continues for the remainder of the bend. This region is interpreted to be the portion of the flow in which the boundary layer has separated; this was observed also by Weske\textsuperscript{15} with velocity traverses at the outlets of bends of various deflection angles.

The pressure variation on the bend of smallest radius of curvature (Figure 16e) deserves special mention. First, it is to be noted that the point of minimum pressure has shifted back to a deflection angle of 22.5 degrees. The region of separation, exemplified by the constant pressures, is seen to be independent of Reynolds number, which is surprising since separation is a viscosity induced phenomenon and should therefore be a function of Reynolds number. Furthermore, in the straight section downstream of the bend the higher pressures occur at the lower Reynolds numbers, whereas the reverse is true for the bend itself. The pressures in the bend would be expected to decrease rather than increase with increasing Reynolds number since a decrease represents an approach towards potential flow.
Figure 16a. Variation of Wall Pressure Coefficient with Distance Along Bend Centerline (R/2a = 2.00)
Figure 16b. Variation of Wall Pressure Coefficient with Distance Along Bend Centerline (R/2a = 1.50)
Figure 16c. Variation of Wall Pressure Coefficient with Distance Along Bend Centerline (R/2a = 1.25)
Figure 16d. Variation of Wall Pressure Coefficient with Distance Along Bend Centerline (R/2a = 1.00)
Figure 16e. Variation of Wall Pressure Coefficient with Distance Along Bend Centerline (R/2a = 0.75)
Figure 17 presents a comparison of the measured minimum pressures with the results from potential flow theory. As illustrated from the static pressure distribution across the bend (Figure 2) the pressure measured at the wall will be greater than the potential wall pressure since the potential pressure is approximately an interpolation of the central core pressure distribution through the point of $C_{p_{\text{min,core}}}$ to the wall. Clearly the pressure jump across the region of eddy motion becomes greater as the bend radius decreases or as the energy of the opposing peripheral flow, which is converted into pressure upon stagnating at the plane of curvature, increases. Comparison with the "experimentally" obtained $C_{p_{\text{min,core}}}$ of Figure 11 shows that this jump is indeed small however.

Finally, Figure 18 presents the relationship between the empirical coefficient $C_{\text{wall}}$ and the radius of curvature. The fact that the data points do not lie on a smoothly faired curve possibly indicates an installation misalignment of one of the bends. It is more likely due to normal scatter of the data and the fact that Equation 72, when solved for $C_w$, is in the form of a difference between two terms of very nearly equal magnitude.

**Conclusions**

Wall pressure measurements in the plane of curvature of 90-degree circular pipe bends have led to the following conclusions:
Figure 17. Comparison of Measured Minimum Wall Pressure Coefficient with Potential Flow Theory
Figure 18. Variation of $C_{\text{wall}}$ with Radius of Curvature, $R'$
1) For a sufficiently high Reynolds number the wall pressure coefficient is independent of Reynolds number. The value at which the transition to a constant pressure coefficient takes place in $\text{Re} = 1.4 \times 10^6$ for the smallest bend radius of curvature tested ($R/2a = 0.75$). For the larger bend curvatures the transition point was not observed for the range of Reynolds numbers tested ($0.5 \times 10^6 < \text{Re} < 1.7 \times 10^6$).

2) The minimum wall pressure occurred at a deflection angle of 22.5 degrees for the two largest curvatures ($R/2a = 2, 1.5$) and the smallest curvature. For the intermediate curvatures ($R/2a = 1.25, 1.00$) the minimum pressure was at 30-degree deflection angle. For all bends tested the maximum pressure on the outside wall occurred at a deflection angle of about 60 degrees.

3) For bends with a radius of curvature ($R/2a$) less than 1.50 a region of constant wall pressure appeared at about 67.5 degrees on the inside wall and persisted to the bend exit. This region is expected to coincide with a region of separated flow.

4) The minimum wall pressures at the inside of the bend are greater than that predicted from potential flow with the difference increasing for smaller bend curvatures.
LIST OF REFERENCES


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APPENDIX

The method of Runge-Kutta is an accurate method of numerically integrating ordinary differential equations and has been found to be quite useful for automatic computing machines. Since the method is treated in detail in textbooks on numerical analysis only the basic equations are presented as they are employed for the present problem.

Let the ordinary differential equation form of the momentum equations be

\[ \frac{d\delta_i}{d\psi} = g(\psi, \delta) \]

Then the following parameters are computed

\[ k_j = g(\psi_{i-1} + C_j \Delta \psi, \delta_{i-1} + C_j k_{j-1}) \Delta \psi \]

where the index \( j \) takes the values 1, 2, 3 and 4 and the index \( i \) is defined by

\[ i = \frac{\psi}{\Delta \psi} + 1 \]

The constant \( \Delta \psi \) is the finite increment of \( \psi \). The incremental value of \( \delta \) occurring over the increment \( \Delta \psi \) is

\[ \Delta \delta = \frac{1}{6} \sum_{j=1}^{4} D_j k_j \]
thus at $\psi = (i-1) \Delta \psi$ the new value of $\delta^i$ is

$$\delta^i_i = \delta^i_{i-1} + \Delta \psi(i-1)$$

The following constants are employed in the preceding equations:

| $C_1$ | $D_1$ |
| 0.0   | 1.0   |
| $C_2$ | $D_2$ |
| 0.5   | 2.0   |
| $C_3$ | $D_3$ |
| 0.5   | 2.0   |
| $C_4$ | $D_4$ |
| 1.0   | 1.0   |