

$P$ -ALGEBRAS AND  $Q$ -ALGEBRAS

by

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# Abstract

We define two classes of algebras  $P$ - and  $Q$ -, which are derived from the definitions of  $BCK$ - and  $BCI$ - algebras. The birth of  $P$ -algebras is based on the symmetric difference in set theory. We prove that the class of  $P$ -algebras is a variety, and the definition of  $P$ -algebras is an alternative definition for groups of exponent 2, which we call  $P$ -groups. The class of  $Q$ -algebras consists of a combination of three axioms of  $BCK$ - and  $P$ -algebras. We study the relationship among  $P$ -,  $Q$ - and  $BCI$ - algebras. The theory of  $P$ - and  $Q$ - algebras is developed parallel to the theory of  $BCK$ - and  $BCI$ - algebras.

*To my daughters Sayumi, Kelly and Amy,  
and my wife Achini.*

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# Chapter 1

## Introduction

The theory of  $P$ - and  $Q$ - algebras is primarily based on the theory of  $BCK$ - and  $BCI$ -algebras.

The notion of  $BCK$ -algebras was first introduced by Kiyoshi Iseki in 1966. The foundation of  $BCK$ -algebras is an axiomatic system based on the properties of the set difference in set theory, and the implication functor in propositional calculi. We introduce the theory of  $BCK$ -algebras in Chapter 2. As a generalization of the theory of  $BCK$ -algebras, Kiyoshi Iseki introduced the notion of  $BCI$ -algebras. We briefly discuss the theory of  $BCI$ -algebras in Chapter 3.

The symmetric difference of sets is an operation in set theory, which had not been defined algebraically. In Chapter 4, We define an algebraic structure, which is based on the symmetric difference in set theory, and name it  $P$ -algebras. The definition of  $P$ -algebras consists of a set of three axioms, which is a derivation of the definition of  $BCK$ -algebras. We construct two sets of conditions that can be used as alternative definitions of  $P$ -algebras. We prove that the class of  $P$ -algebras is a variety, and the definition of  $P$ -algebras is an alternative definition for groups of all of whose elements are of exponent 2, which we define as  $P$ -groups. The relationship between the  $P$ - and  $BCI$ - algebras is also given.

In Chapter 5, we define the other algebraic structure,  $Q$ -algebras. The definition of  $Q$ -algebras consists of a combination of three axioms of  $BCK$ -algebras and  $P$ -algebras. We study the relationship among the  $P$ -,  $Q$ - and  $BCI$ - algebras.

Chapter 6 concludes the study with a discussion of future research possibilities.

# Chapter 2

## *BCK*–Algebras

### 2.1 Introduction

The notion of *BCK*–Algebras was first introduced by Kiyoshi Iseki in 1966 [Ise66]. It originated from two different roots of mathematics. One is based on set theory, and the other is from classical and non-classical propositional calculi.

In set theory, there are three most fundamental and elementary operations: the union, intersection, and set difference. The union and intersection appear as binary operations in many algebraic structures such as Boolean algebras and distributive lattices. But the set difference has not been considered systematically before the introduction of *BCK*–Algebras.

In propositional calculi, there are some systems which contain only the implication functor. The notion of implicative algebras [Ras74] is an example of such systems. There is a close relationship between the implication functor in logical systems and the notion of the set difference in set theory.

Consider the following simple relations in set theory:

$$(A - B) - (A - C) \subseteq C - B,$$

$$A - (A - B) \subseteq B.$$

In propositional calculi, corresponding relations are given by

$$(p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r)),$$

$$p \rightarrow ((p \rightarrow q) \rightarrow q).$$

The foundation of *BCK*–algebras is an axiomatic system based on the properties of the set difference and the implication functor.

## 2.2 Definitions of *BCK*–Algebras

**Definition 2.1.** Let  $X$  be a nonempty set with a binary operation “ $\star$ ” and a constant 0.

Then  $(X; \star, 0)$  is said to be a *BCK*–algebra if it satisfies the following axioms:

$\forall x, y, z \in X,$

1.  $((x \star y) \star (x \star z)) \star (z \star y) = 0,$
2.  $x \star y = 0$  and  $y \star x = 0$  implies  $x = y,$
3.  $x \star x = 0,$
4.  $(x \star (x \star y)) \star y = 0,$
5.  $0 \star x = 0.$

A binary relation “ $\leq$ ” on  $(X; \star, 0)$  can be defined as  $x \leq y$  if and only if  $x \star y = 0$ . Then a set of axioms corresponding to the above can be given as follows:

$\forall x, y, z \in X,$

1.  $(x \star y) \star (x \star z) \leq (z \star y),$
2.  $x \leq y$  and  $y \leq x$  implies  $x = y,$
3.  $x \leq x,$
4.  $x \star (x \star y) \leq y,$
5.  $0 \leq x.$

In a *BCK*–algebra  $(X; \star, 0)$ , “ $\star$ ” and “ $\leq$ ” are called the *BCK*– operation and *BCK*– ordering on  $X$ , respectively.

## 2.3 Examples of *BCK*–Algebras

**Example 2.1.** The difference of sets.

Let  $A$  be any set;  $X = 2^A$ , and  $0 = \phi$ . For any  $x$  and  $y$  in  $X$ , let the binary operation  $\star$  be defined as follows:

$$x \star y = x \cap y^c.$$

Then  $(X; \star, 0)$  is a *BCK*–algebra.

**Example 2.2.** Let  $X$  be a partially ordered set with the minimum element 0. For any  $x$  and  $y$  in  $X$ , let the binary operation  $\star$  be defined as follows:

$$x \star y = \begin{cases} 0 & \text{if } x \leq y, \\ x & \text{otherwise.} \end{cases}$$

Then  $(X; \star, 0)$  is a *BCK*-algebra.

**Example 2.3.** Let  $X$  be the set of all non-negative integers. For any  $x$  and  $y$  in  $X$ , let the binary operation  $\star$  be defined as follows:

$$x \star y = \begin{cases} 0 & \text{if } x \leq y, \\ x - y & \text{otherwise.} \end{cases}$$

Then  $(X; \star, 0)$  is a *BCK*-algebra.

**Example 2.4.** Let  $A$  be a nonempty set, and let  $X$  be the set of all real valued functions defined on  $A$ . For any  $f$  and  $g$  in  $X$ , and any  $x$  in  $A$ , let the binary operation  $\star$  be defined as follows:

$$(f \star g)(x) = \begin{cases} 0 & \text{if } f(x) \leq g(x), \\ f(x) - g(x) & \text{otherwise.} \end{cases}$$

Then  $(X; \star, 0)$  is a *BCK*-algebra.

**Example 2.5.** [Ise82] Let  $X$  be a partially ordered set with the minimum element  $0$ ; a distinguished element  $e$ , such that  $0 \leq x$  for all  $x \in X$ , and  $e \leq x$  for all nonzero  $x \in X$ . For any  $x$  and  $y$  in  $X$ , let the binary operation  $\star$  be defined as follows:

$$x \star y = \begin{cases} 0 & \text{if } x \leq y, \\ x & \text{if } y = 0, \\ e & \text{otherwise.} \end{cases}$$

Then  $(X; \star, 0)$  is a *BCK*-algebra.

## 2.4 Basic Theorems

Proofs of the following theorems can be found in [IT78] or in [MJ94].

**Theorem 2.1.** *Let  $(X; \star, 0)$  be a *BCK*-algebra, and suppose  $x, y$  and  $z$  are in  $X$ . Then,*

1.  $x \leq y$  implies  $z \star y \leq z \star x$ ,
2.  $x \leq y$  implies  $x \star z \leq y \star z$ ,
3.  $x \leq y$  and  $y \leq z$  implies  $x \leq z$ .

The third and fourth axioms of *BCI*-ordering, together with the third part of the above theorem, make  $(X; \leq)$  a partially ordered set.

**Theorem 2.2.** *Let  $(X; \star, 0)$  be a *BCK*-algebra, and suppose  $x, y$  and  $z$  are in  $X$ . Then,*

$$(x \star y) \star z = (x \star z) \star y.$$

**Theorem 2.3.** *Let  $(X; \star, 0)$  be a BCK–algebra, and suppose  $x, y$  and  $z$  are in  $X$ . Then,*

1.  $(x \star y) \star z = 0$  implies  $(x \star z) \star y = 0$ ,

2.  $((x \star z) \star (y \star z)) \star (x \star y) = 0$ ,

3.  $x \star y = 0$  implies  $(x \star z) \star (y \star z) = 0$ ,

4.  $(x \star y) \star x = 0$ ,

5.  $x \star 0 = x$ .

**Theorem 2.4.** *Let  $(X; \star, 0)$  be a BCK–algebra, and suppose  $x, y$  and  $z$  are in  $X$ . Then,*

$$x \star y = z \text{ implies } z \star x = y.$$

**Theorem 2.5.** *Let  $X$  be a nonempty set with a binary operation  $\star$  and a constant  $0$ . Then  $(X; \star, 0)$  is a BCK–algebra if and only if, for all  $x, y, z$  in  $X$ , it satisfies the following conditions:*

1.  $((x \star y) \star (x \star z)) \star (z \star y) = 0$ ,

2.  $x \star (0 \star y) = x$ ,

3.  $x \star y = 0$  and  $y \star x = 0$  implies  $x = y$ .



## 2.5 Sub–Algebras

**Definition 2.2.** Let  $(X; \star, 0)$  be a *BCK*–algebra, and  $X_0$  be a nonempty subset of  $X$ . Then  $X_0$  is said to be a *sub-algebra* of  $X$  if  $X_0$  is closed under the binary operation  $\star$  in  $X$ .

**Theorem 2.6.** Let  $(X; \star, 0)$  be a *BCK*–algebra, and  $X_0$  be a sub-algebra of  $X$ . Then,

1.  $0 \in X_0$ ,
2.  $(X_0; \star, 0)$  is also a *BCK*–algebra,
3.  $X$  is a sub-algebra of  $X$ ,
4.  $(\{0\}; \star, 0)$  is a sub-algebra of  $X$ .

**Theorem 2.7.** Let  $(X; \star, 0)$  be a *BCK*–algebra, and “ $a$ ” be a nonzero element of  $X$ . Then  $(\{0, a\}; \star, 0)$  is a sub-algebra of  $X$ .

The next chapter introduces the notion of *BCI*–algebras.

# Chapter 3

## *BCI*–Algebras

### 3.1 Introduction

Kiyoshi Iseki introduced the notion of *BCI*–Algebras, which is a generalization of *BCK*–Algebras, in 1966 [Ise66]. There are a few different axiomatic definitions of *BCI*–Algebras. In this study, we consider two definitions. The first definition, which consists of four axioms, is derived from the axiom system of *BCK*–algebras. The second definition, which consists of three axioms, is a simplified version of the first definition.

### 3.2 Definitions of *BCI*–Algebras

**Definition 3.1.** Let  $X$  be a nonempty set with a binary operation “ $\star$ ” and a constant  $0$ . Then  $(X; \star, 0)$  is said to be a *BCI*–algebra if it satisfies the following axioms:

$\forall x, y, z \in X,$

1.  $((x \star y) \star (x \star z)) \star (z \star y) = 0,$
2.  $x \star y = 0$  and  $y \star x = 0$  implies  $x = y,$

3.  $x \star x = 0$ ,

4.  $(x \star (x \star y)) \star y = 0$ .

Li Hui Shi published the following definition in 1985 [Shi85], which consists of the first two axioms of the above definition and a third axiom.

**Definition 3.2.** Let  $X$  be a nonempty set with a binary operation “ $\star$ ” and a constant  $0$ . Then  $(X; \star, 0)$  is said to be a *BCI–algebra* if it satisfies the following axioms:

$\forall x, y, z \in X$ ,

1.  $((x \star y) \star (x \star z)) \star (z \star y) = 0$ ,

2.  $x \star y = 0$  and  $y \star x = 0$  implies  $x = y$ ,

3.  $x \star 0 = x$ .

The proof of equivalence of the two definitions can be found in Li Hui Shi’s original paper [Shi85] or in [Yis06].

**Definition 3.3.** A *BCI–algebra*  $(X; \star, 0)$  is said to be *associative* if  $(x \star y) \star z = x \star (y \star z)$  for all  $x, y, z \in X$ .

### 3.3 Examples of *BCI–Algebras*

Since the class of *BCK–algebras* is a sub-class of the class of *BCI–algebras*, the examples of *BCK–algebras* given in Chapter 2 are also examples of *BCI–algebras*. Here are some other examples.

**Example 3.1.** Let  $(G; \cdot, e)$  be an abelian group, and suppose  $x, y \in G$ . Define a binary operation  $\star$  as,

$$x \star y = x \cdot y^{-1}.$$

Then  $(G; \star, e)$  is a *BCI*-algebra.

**Example 3.2.** Let  $X = \{0, 1, 2\}$ , and the binary operation  $\star$  be given by the following Cayley table:

$\star$	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

Then  $(X; \star, 0)$  is a *BCI*-algebra.

### 3.4 Basic Theorems

Proofs of the following propositions and theorems can be found in [Yis06].

**Proposition 3.1.** *Let  $(X; \star, 0)$  be a *BCI*-algebra, and suppose  $x$  and  $y$  are in  $X$ . Define a binary relation “ $\leq$ ” on  $X$  such that,*

$$x \leq y \text{ if and only if } x \star y = 0.$$

*Then  $(X; \leq)$  is a partially ordered set, and 0 is a minimal element.*

The partial ordering defined in proposition 3.1 is called the *BCI*-ordering on  $X$ .

**Theorem 3.2.** *Let  $X$  be a nonempty set with a binary operation  $\star$  and a constant  $0$ . Then  $(X; \star, 0)$  is a BCI–algebra if and only if there is a partial ordering  $\leq$  on  $X$  such that, for all  $x, y, z$  in  $X$ , the following conditions hold:*

1.  $(x \star y) \star (x \star z) \leq (z \star y)$ ,
2.  $x \star y = 0$  if and only if  $x \leq y$ ,
3.  $x \star (x \star y) \leq y$ .

**Theorem 3.3.** *Let  $(X; \star, 0)$  be a BCI–algebra, and suppose  $x, y$  and  $z$  are in  $X$ . Then,*

1.  $x \leq y$  implies  $z \star y \leq z \star x$ ,
2.  $x \leq y$  implies  $x \star z \leq y \star z$ .

**Theorem 3.4.** *Let  $(X; \star, 0)$  be a BCI–algebra, and suppose  $x, y$  and  $z$  are in  $X$ . Then,*

$$(x \star y) \star z = (x \star z) \star y.$$

**Theorem 3.5.** *Let  $(X; \star, 0)$  be a BCI–algebra, and suppose  $x$  and  $y$  are in  $X$ . Then,*

$$0 \star (x \star y) = (0 \star x) \star (0 \star y).$$

## 3.5 Sub–Algebras

**Definition 3.4.** Let  $(X; \star, 0)$  be a BCI–algebra, and  $X_0$  be a nonempty subset of  $X$ . Then  $X_0$  is said to be a *sub-algebra* of  $X$  if  $X_0$  is closed under the binary operation  $\star$  in  $X$ .

**Theorem 3.6.** *Let  $(X; \star, 0)$  be a BCI–algebra, and  $X_0$  be a sub-algebra of  $X$ . Then,*

1.  $0 \in X_0$ ,
2.  $(X_0; \star, 0)$  is also a BCI–algebra,
3.  $X$  is a sub-algebra of  $X$ ,
4.  $(\{0\}; \star, 0)$  is a sub-algebra of  $X$ .

The next chapter develops the theory of  $P$ –algebras.

# Chapter 4

## $P$ –Algebras

### 4.1 Introduction

The foundation of the theory of  $BCK$ –algebras and  $BCI$ –algebras had been partially based on the set difference in set theory. The symmetric difference of sets is another operation in set theory, which had not been defined algebraically. In order to define the symmetric difference as an algebraic operation, let’s first define an algebraic structure similar to that of a  $BCK$ –algebra, which consists of a non-empty set  $X$ , a constant element  $0$ , and a binary operation “ $\star$ .”

Let  $A$  be any set and  $X = 2^A$ . Since the empty set is a natural choice for the constant element, let’s define  $0$  as the empty set. Suppose  $x, y \in X$ , and define a binary operation “ $\star$ ” in  $X$  as,

$$x \star y = (x \cap y^c) \cup (y \cap x^c).$$

Then the binary operation  $\star$  satisfies the following axioms of  $BCK$ –algebras:

1.  $((x \star y) \star (x \star z)) \star (z \star y) = 0$ ,
2.  $x \star y = 0$  and  $y \star x = 0$  implies  $x = y$ ,

$$3. x \star x = 0,$$

$$4. (x \star (x \star y)) \star y = 0.$$

But, it does not satisfy the fifth axiom:

$$5. 0 \star x = 0.$$

In fact, for any  $x \in X$ ,

$$\begin{aligned} 0 \star x &= (\phi \cap x^c) \cup (x \cap \phi^c) \\ &= \phi \cup x \\ &= x. \end{aligned}$$

Therefore, by changing the fifth axiom, a new set of axioms can be obtained as follows:

$$1. ((x \star y) \star (x \star z)) \star (z \star y) = 0$$

$$2. x \star y = 0 \text{ and } y \star x = 0 \text{ implies } x = y$$

$$3. x \star x = 0$$

$$4. (x \star (x \star y)) \star y = 0$$

$$5. 0 \star x = x$$

Due to the symmetry of the set union operation,  $x \star y = 0$  implies  $y \star x = 0$ . Thus,  $y \star x = 0$  in the second axiom is redundant, and hence the second axiom can be simplified as:

$$2. x \star y = 0 \text{ implies } x = y.$$

Furthermore, the third and fourth axioms can be derived from the other three axioms. Therefore, the set of axioms reduces to three axioms. We named the new algebraic structure, *P-algebras*.



## 4.2 The Definition of $P$ -algebras

**Definition 4.1.** Let  $X$  be a nonempty set with a binary operation “ $\star$ ” and a constant  $0$ .

Then  $(X; \star, 0)$  is said to be a  $P$ -algebra if it satisfies the following axioms:

$\forall x, y, z \in X$ ,

1.  $((x \star y) \star (x \star z)) \star (z \star y) = 0$ ,
2.  $x \star y = 0$  implies  $x = y$ ,
3.  $0 \star x = x$ .

Next we consider the independence of the axioms of  $P$ -algebras. The following example shows the independence of the first axiom.

**Example 4.1.** Let  $X = \{0, 1, 2\}$ , and the binary operation  $\star$  be given by the following Cayley table:

$\star$	0	1	2
0	0	1	2
1	1	0	2
2	2	2	0

**Axiom 1:**  $((x \star y) \star (x \star z)) \star (z \star y) = 0$

When  $x = 2$ ,  $y = 0$ , and  $z = 1$ :

$$((x \star y) \star (x \star z)) \star (z \star y) = ((2 \star 0) \star (2 \star 1)) \star (1 \star 0) = (2 \star 2) \star 1 = 0 \star 1 = 1 \neq 0.$$

Thus,  $(X; \star, 0)$  does not satisfy the first axiom.

**Axiom 2:**  $x \star y = 0$  implies  $x = y$

Since  $0$  appears in the Cayley table only on the diagonal from upper left to lower right, the hypothesis is true only when  $x = y = 0$ ,  $x = y = 1$ , or  $x = y = 2$ . In all three cases, the conclusion is also true. Thus,  $(X; \star, 0)$  satisfies the second axiom.

**Axiom 3:**  $0 \star x = x$

The first row of the Cayley table shows that this is the case. Thus,  $(X, \star, 0)$  satisfies the third axiom.

The above example shows that  $(X; \star, 0)$  satisfies the second and third axioms, but not the first. The next example shows the independence of the second axiom.

**Example 4.2.** Let  $X = \{0, 1, 2\}$ , and the binary operation  $\star$  be given by the following Cayley table:

$\star$	0	1	2
0	0	1	2
1	1	0	0
2	2	0	0

**Axiom 1:**  $((x \star y) \star (x \star z)) \star (z \star y) = 0$

For all possible combinations of  $x$ ,  $y$ , and  $z$ , the results of  $((x \star y) \star (x \star z)) \star (z \star y)$  are given in the following table:

$(x, y, z)$	$((x \star y) \star (x \star z)) \star (z \star y)$
$(0, 0, 0)$	$((0 \star 0) \star (0 \star 0)) \star (0 \star 0) = (0 \star 0) \star 0 = 0 \star 0 = 0$
$(0, 0, 1)$	$((0 \star 0) \star (0 \star 1)) \star (1 \star 0) = (0 \star 1) \star 1 = 1 \star 1 = 0$
$(0, 0, 2)$	$((0 \star 0) \star (0 \star 2)) \star (2 \star 0) = (0 \star 2) \star 2 = 2 \star 2 = 0$
$(0, 1, 0)$	$((0 \star 1) \star (0 \star 0)) \star (0 \star 1) = (1 \star 0) \star 1 = 1 \star 1 = 0$
$(0, 1, 1)$	$((0 \star 1) \star (0 \star 1)) \star (1 \star 1) = (1 \star 1) \star 0 = 0 \star 0 = 0$
$(0, 1, 2)$	$((0 \star 1) \star (0 \star 2)) \star (2 \star 1) = (1 \star 2) \star 0 = 0 \star 0 = 0$
$(0, 2, 0)$	$((0 \star 2) \star (0 \star 0)) \star (0 \star 2) = (2 \star 0) \star 2 = 2 \star 2 = 0$
$(0, 2, 1)$	$((0 \star 2) \star (0 \star 1)) \star (1 \star 2) = (2 \star 1) \star 0 = 0 \star 0 = 0$

$$\begin{aligned}
(0, 2, 2) & ((0 \star 2) \star (0 \star 2)) \star (2 \star 2) = (2 \star 2) \star 0 = 0 \star 0 = 0 \\
(1, 0, 0) & ((1 \star 0) \star (1 \star 0)) \star (0 \star 0) = (1 \star 1) \star 0 = 0 \star 0 = 0 \\
(1, 0, 1) & ((1 \star 0) \star (1 \star 1)) \star (1 \star 0) = (1 \star 0) \star 1 = 1 \star 1 = 0 \\
(1, 0, 2) & ((1 \star 0) \star (1 \star 2)) \star (2 \star 0) = (1 \star 0) \star 2 = 1 \star 2 = 0 \\
(1, 1, 0) & ((1 \star 1) \star (1 \star 0)) \star (0 \star 1) = (0 \star 1) \star 1 = 1 \star 1 = 0 \\
(1, 1, 1) & ((1 \star 1) \star (1 \star 1)) \star (1 \star 1) = (0 \star 0) \star 0 = 0 \star 0 = 0 \\
(1, 1, 2) & ((1 \star 1) \star (1 \star 2)) \star (2 \star 1) = (0 \star 0) \star 0 = 0 \star 0 = 0 \\
(1, 2, 0) & ((1 \star 2) \star (1 \star 0)) \star (0 \star 2) = (0 \star 1) \star 2 = 1 \star 2 = 0 \\
(1, 2, 1) & ((1 \star 2) \star (1 \star 1)) \star (1 \star 2) = (0 \star 0) \star 0 = 0 \star 0 = 0 \\
(1, 2, 2) & ((1 \star 2) \star (1 \star 2)) \star (2 \star 2) = (0 \star 0) \star 0 = 0 \star 0 = 0 \\
(2, 0, 0) & ((2 \star 0) \star (2 \star 0)) \star (0 \star 0) = (2 \star 2) \star 0 = 0 \star 0 = 0 \\
(2, 0, 1) & ((2 \star 0) \star (2 \star 1)) \star (1 \star 0) = (2 \star 0) \star 1 = 2 \star 1 = 0 \\
(2, 0, 2) & ((2 \star 0) \star (2 \star 2)) \star (2 \star 0) = (2 \star 0) \star 2 = 2 \star 2 = 0 \\
(2, 1, 0) & ((2 \star 1) \star (2 \star 0)) \star (0 \star 1) = (0 \star 2) \star 1 = 2 \star 1 = 0 \\
(2, 1, 1) & ((2 \star 1) \star (2 \star 1)) \star (1 \star 1) = (0 \star 0) \star 0 = 0 \star 0 = 0 \\
(2, 1, 2) & ((2 \star 1) \star (2 \star 2)) \star (2 \star 1) = (0 \star 0) \star 0 = 0 \star 0 = 0 \\
(2, 2, 0) & ((2 \star 2) \star (2 \star 0)) \star (0 \star 2) = (0 \star 2) \star 2 = 2 \star 2 = 0 \\
(2, 2, 1) & ((2 \star 2) \star (2 \star 1)) \star (1 \star 2) = (0 \star 0) \star 0 = 0 \star 0 = 0 \\
(2, 2, 2) & ((2 \star 2) \star (2 \star 2)) \star (2 \star 2) = (0 \star 0) \star 0 = 0 \star 0 = 0
\end{aligned}$$


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Thus,  $(X; \star, 0)$  satisfies the first axiom.

**Axiom 2:**  $x \star y = 0$  implies  $x = y$

When  $x = 1$  and  $y = 2$ :

$x \star y = 0$ , but  $x \neq y$ .

Thus,  $(X; \star, 0)$  does not satisfy the second axiom.

**Axiom 3:**  $0 \star x = x$

The first row of the Cayley table shows that this is the case. Thus,  $(X, \star, 0)$  satisfies the third axiom.

The above example shows that  $(X; \star, 0)$  satisfies the first and third axioms, but not the second. The next example shows the independence of the third axiom.

**Example 4.3.** Let  $X = \{0, 1, 2\}$ , and the binary operation  $\star$  be given by the following Cayley table:

$\star$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

**Axiom 1:**  $((x \star y) \star (x \star z)) \star (z \star y) = 0$

For all possible combinations of  $x$ ,  $y$ , and  $z$ , the results of  $((x \star y) \star (x \star z)) \star (z \star y)$  are given in the following table:

$(x, y, z)$	$((x \star y) \star (x \star z)) \star (z \star y)$
$(0, 0, 0)$	$((0 \star 0) \star (0 \star 0)) \star (0 \star 0) = (0 \star 0) \star 0 = 0 \star 0 = 0$
$(0, 0, 1)$	$((0 \star 0) \star (0 \star 1)) \star (1 \star 0) = (0 \star 2) \star 1 = 1 \star 1 = 0$
$(0, 0, 2)$	$((0 \star 0) \star (0 \star 2)) \star (2 \star 0) = (0 \star 1) \star 2 = 2 \star 2 = 0$
$(0, 1, 0)$	$((0 \star 1) \star (0 \star 0)) \star (0 \star 1) = (2 \star 0) \star 2 = 2 \star 2 = 0$
$(0, 1, 1)$	$((0 \star 1) \star (0 \star 1)) \star (1 \star 1) = (2 \star 2) \star 0 = 0 \star 0 = 0$
$(0, 1, 2)$	$((0 \star 1) \star (0 \star 2)) \star (2 \star 1) = (2 \star 1) \star 1 = 1 \star 1 = 0$
$(0, 2, 0)$	$((0 \star 2) \star (0 \star 0)) \star (0 \star 2) = (1 \star 0) \star 1 = 1 \star 1 = 0$
$(0, 2, 1)$	$((0 \star 2) \star (0 \star 1)) \star (1 \star 2) = (1 \star 2) \star 2 = 2 \star 2 = 0$

$$\begin{array}{l}
(0, 2, 2) \quad ((0 \star 2) \star (0 \star 2)) \star (2 \star 2) = (1 \star 1) \star 0 = 0 \star 0 = 0 \\
(1, 0, 0) \quad ((1 \star 0) \star (1 \star 0)) \star (0 \star 0) = (1 \star 1) \star 0 = 0 \star 0 = 0 \\
(1, 0, 1) \quad ((1 \star 0) \star (1 \star 1)) \star (1 \star 0) = (1 \star 0) \star 1 = 1 \star 1 = 0 \\
(1, 0, 2) \quad ((1 \star 0) \star (1 \star 2)) \star (2 \star 0) = (1 \star 2) \star 2 = 2 \star 2 = 0 \\
(1, 1, 0) \quad ((1 \star 1) \star (1 \star 0)) \star (0 \star 1) = (0 \star 1) \star 2 = 2 \star 2 = 0 \\
(1, 1, 1) \quad ((1 \star 1) \star (1 \star 1)) \star (1 \star 1) = (0 \star 0) \star 0 = 0 \star 0 = 0 \\
(1, 1, 2) \quad ((1 \star 1) \star (1 \star 2)) \star (2 \star 1) = (0 \star 2) \star 1 = 1 \star 1 = 0 \\
(1, 2, 0) \quad ((1 \star 2) \star (1 \star 0)) \star (0 \star 2) = (2 \star 1) \star 1 = 1 \star 1 = 0 \\
(1, 2, 1) \quad ((1 \star 2) \star (1 \star 1)) \star (1 \star 2) = (2 \star 0) \star 2 = 2 \star 2 = 0 \\
(1, 2, 2) \quad ((1 \star 2) \star (1 \star 2)) \star (2 \star 2) = (2 \star 2) \star 0 = 0 \star 0 = 0 \\
(2, 0, 0) \quad ((2 \star 0) \star (2 \star 0)) \star (0 \star 0) = (2 \star 2) \star 0 = 0 \star 0 = 0 \\
(2, 0, 1) \quad ((2 \star 0) \star (2 \star 1)) \star (1 \star 0) = (2 \star 1) \star 1 = 1 \star 1 = 0 \\
(2, 0, 2) \quad ((2 \star 0) \star (2 \star 2)) \star (2 \star 0) = (2 \star 0) \star 2 = 2 \star 2 = 0 \\
(2, 1, 0) \quad ((2 \star 1) \star (2 \star 0)) \star (0 \star 1) = (1 \star 2) \star 2 = 2 \star 2 = 0 \\
(2, 1, 1) \quad ((2 \star 1) \star (2 \star 1)) \star (1 \star 1) = (1 \star 1) \star 0 = 0 \star 0 = 0 \\
(2, 1, 2) \quad ((2 \star 1) \star (2 \star 2)) \star (2 \star 1) = (1 \star 0) \star 1 = 1 \star 1 = 0 \\
(2, 2, 0) \quad ((2 \star 2) \star (2 \star 0)) \star (0 \star 2) = (0 \star 2) \star 1 = 1 \star 1 = 0 \\
(2, 2, 1) \quad ((2 \star 2) \star (2 \star 1)) \star (1 \star 2) = (0 \star 1) \star 2 = 2 \star 2 = 0 \\
(2, 2, 2) \quad ((2 \star 2) \star (2 \star 2)) \star (2 \star 2) = (0 \star 0) \star 0 = 0 \star 0 = 0
\end{array}$$


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Thus,  $(X; \star, 0)$  satisfies the first axiom.

**Axiom 2:**  $x \star y = 0$  implies  $x = y$

Since 0 appears in the Cayley table only on the diagonal from upper left to lower right, the hypothesis is true only when  $x = y = 0$ ,  $x = y = 1$ , or  $x = y = 2$ . In all three cases, the conclusion is also true. Thus,  $(X; \star, 0)$  satisfies the second axiom.

**Axiom 3:**  $0 \star x = x$

When  $x = 1$ , according to the first row of the Cayley table,  $0 \star 1 = 2 \neq 1$ . Thus,  $(X; \star, 0)$  does not satisfy the third axiom.

The above example shows that  $(X; \star, 0)$  satisfies the first and second axioms, but not the third.

The three examples above show that the axioms are independent of each other. Next we will illustrate some examples of  $P$ -algebras.

### 4.3 Examples of $P$ -Algebras

The premier example of  $P$ -algebras is the symmetric difference of sets.

**Example 4.4.** The symmetric difference of sets.

Let  $A$  be any set;  $X = 2^A$ , and  $0 = \phi$ . For any  $x$  and  $y$  in  $X$ , define the binary operation  $\star$  as,

$$x \star y = (x \cap y^c) \cup (y \cap x^c).$$

Then  $(X; \star, 0)$  is a  $P$ -algebra.

*Proof.* Let  $A$  be a nonempty set, and suppose  $x, y, z \in X = 2^A$ .

**Axiom 1:**  $((x \star y) \star (x \star z)) \star (z \star y) = 0$

$$\begin{aligned}
(x \star y) \star (x \star z) &= (((x \cap y^c) \cup (y \cap x^c)) \cap ((x \cap z^c) \cup (z \cap x^c))^c) \\
&\quad \cup (((x \cap z^c) \cup (z \cap x^c)) \cap ((x \cap y^c) \cup (y \cap x^c))^c) \\
&= (((x \cap y^c) \cup (y \cap x^c)) \cap ((x \cap z^c)^c \cap (z \cap x^c)^c)) \\
&\quad \cup (((x \cap z^c) \cup (z \cap x^c)) \cap ((x \cap y^c)^c \cap (y \cap x^c)^c)) \\
&= (((x \cap y^c) \cup (y \cap x^c)) \cap ((x^c \cup z) \cap (z^c \cup x))) \\
&\quad \cup (((x \cap z^c) \cup (z \cap x^c)) \cap ((x^c \cup y) \cap (y^c \cup x))) \\
&= ((x \cap y^c) \cap ((x^c \cup z) \cap (z^c \cup x))) \cup ((y \cap x^c) \cap ((x^c \cup z) \cap (z^c \cup x))) \\
&\quad \cup ((x \cap z^c) \cap ((x^c \cup y) \cap (y^c \cup x))) \cup ((z \cap x^c) \cap ((x^c \cup y) \cap (y^c \cup x))) \\
&= (((x \cap y^c) \cap (x^c \cup z)) \cap (z^c \cup x)) \cup (((y \cap x^c) \cap (z^c \cup x)) \cap (x^c \cup z)) \\
&\quad \cup (((x \cap z^c) \cap (x^c \cup y)) \cap (y^c \cup x)) \cup (((z \cap x^c) \cap (y^c \cup x)) \cap (x^c \cup y)) \\
&= (((x \cap y^c \cap x^c) \cup (x \cap y^c \cap z)) \cap (z^c \cup x)) \\
&\quad \cup (((y \cap x^c \cap z^c) \cup (y \cap x^c \cap x)) \cap (x^c \cup z)) \\
&\quad \cup (((x \cap z^c \cap x^c) \cup (x \cap z^c \cap y)) \cap (y^c \cup x)) \\
&\quad \cup (((z \cap x^c \cap y^c) \cup (z \cap x^c \cap x)) \cap (x^c \cup y)) \\
&= ((\phi \cup (x \cap y^c \cap z)) \cap (z^c \cup x)) \cup (((y \cap x^c \cap z^c) \cup \phi) \cap (x^c \cup z)) \\
&\quad \cup ((\phi \cup (x \cap z^c \cap y)) \cap (y^c \cup x)) \cup (((z \cap x^c \cap y^c) \cup \phi) \cap (x^c \cup y)) \\
&= ((x \cap y^c \cap z) \cap (z^c \cup x)) \cup ((y \cap x^c \cap z^c) \cap (x^c \cup z)) \\
&\quad \cup ((x \cap z^c \cap y) \cap (y^c \cup x)) \cup ((z \cap x^c \cap y^c) \cap (x^c \cup y))
\end{aligned}$$

$$\begin{aligned}
&= ((x \cap y^c \cap z \cap z^c) \cup (x \cap y^c \cap z \cap x)) \cup ((y \cap x^c \cap z^c \cap x^c) \cup (y \cap x^c \cap z^c \cap z)) \\
&\quad \cup ((x \cap z^c \cap y \cap y^c) \cup (x \cap z^c \cap y \cap x)) \cup ((z \cap x^c \cap y^c \cap x^c) \cup (z \cap x^c \cap y^c \cap y)) \\
&= (\phi \cup (x \cap y^c \cap z)) \cup ((y \cap x^c \cap z^c) \cup \phi) \cup (\phi \cup (x \cap z^c \cap y)) \cup ((z \cap x^c \cap y^c) \cup \phi) \\
&= (x \cap y^c \cap z) \cup (y \cap x^c \cap z^c) \cup (x \cap z^c \cap y) \cup (z \cap x^c \cap y^c) \\
&= ((x \cap y^c \cap z) \cup (x^c \cap y^c \cap z)) \cup ((x^c \cap y \cap z^c) \cup (x \cap y \cap z^c)) \\
&= ((x \cap (y^c \cap z)) \cup (x^c \cap (y^c \cap z))) \cup ((x^c \cap (y \cap z^c)) \cup (x \cap (y \cap z^c))) \\
&= (y^c \cap z) \cup (y \cap z^c) \\
&= (y \cap z^c) \cup (z \cap y^c).
\end{aligned}$$

Therefore,

$$\begin{aligned}
((x \star y) \star (x \star z)) \star (z \star y) &= (((y^c \cap z) \cup (y \cap z^c)) \cap ((z \cap y^c) \cup (y \cap z^c)))^c \\
&\quad \cup (((z \cap y^c) \cup (y \cap z^c)) \cap ((z \cap y^c) \cup (y \cap z^c)))^c \\
&= \phi \cup \phi \\
&= \phi.
\end{aligned}$$

Thus,  $(X; \star, 0)$  satisfies the first axiom.

**Axiom 2:**  $x \star y = 0$  implies  $x = y$

$$\begin{aligned}
x \star y = 0 &\Rightarrow (x \cap y^c) \cup (y \cap x^c) = \phi \\
&\Rightarrow x \cap y^c = \phi \text{ and } y \cap x^c = \phi \\
&\Rightarrow x \subseteq y \text{ and } y \subseteq x \\
&\Rightarrow x = y.
\end{aligned}$$

Thus,  $(X; \star, 0)$  satisfies the second axiom.



**Axiom 3:**  $0 \star x = x$

$$\begin{aligned} 0 \star x &= (\phi \cap x^c) \cup (x \cap \phi^c) \\ &= \phi \cup x \\ &= x. \end{aligned}$$

Thus,  $(X; \star, 0)$  satisfies the third axiom.

Hence  $(X; \star, 0)$  is a  $P$ -algebra. □

The validity of the first axiom in the above example can be illustrated using Venn diagrams as follows.

Let  $A$  be a nonempty set, and suppose  $x, y, z \in X = 2^A$ . Following is a Venn diagram of the sets  $x, y$  and  $z$  (let  $(0)$  represents the empty set).

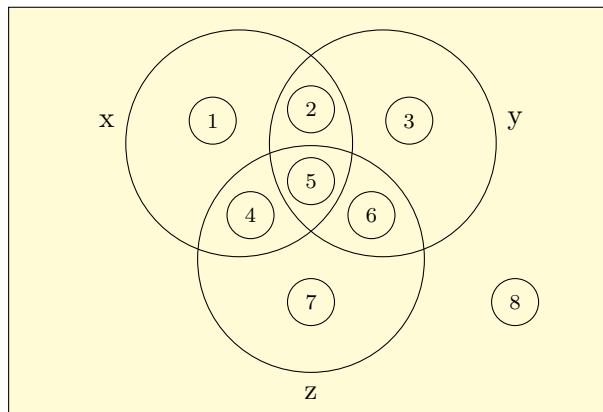


Figure 4.1: Venn diagram of  $x, y$  and  $z$ .

**Axiom 1:**  $((x \star y) \star (x \star z)) \star (z \star y) = 0$

$$(x \star y) : (1), (3), (4), (6)$$

$$(x \star z) : (1), (2), (6), (7)$$

$$(x \star y) \star (x \star z) : (3), (4), (2), (7)$$

$$(z \star y) : (4), (7), (2), (3)$$

$$((x \star y) \star (x \star z)) \star (z \star y) : (0)$$

Therefore  $((x \star y) \star (x \star z)) \star (z \star y) = 0$ .

**Example 4.5.** Let  $X = \{1, -1\}$ ;  $0 = 1$ , and let the binary operation  $\star$  be the ordinary multiplication. Then  $(X; \star, 0)$  is a  $P$ -algebra.

*Proof.*

**Axiom 1:**  $((x \star y) \star (x \star z)) \star (z \star y) = 0$

$$\begin{aligned} ((x \star y) \star (x \star z)) \star (z \star y) &= ((xy)(xz))(zy) \\ &= (xyz)^2 \\ &= 1 \\ &= 0. \end{aligned}$$

Thus,  $(X; \star, 0)$  satisfies the first axiom.

**Axiom 2:**  $x \star y = 0$  implies  $x = y$

$$\begin{aligned}x \star y = 0 &\Rightarrow xy = 1 \\ &\Rightarrow x = y = 1 \text{ or } x = y = -1 \\ &\Rightarrow x = y.\end{aligned}$$

Thus,  $(X; \star, 0)$  satisfies the second axiom.

**Axiom 3:**  $0 \star x = x$

$$\begin{aligned}0 \star x &= 1x \\ &= x.\end{aligned}$$

Thus,  $(X; \star, 0)$  satisfies the third axiom.

Hence  $(X; \star, 0)$  is a  $P$ -algebra. □

**Example 4.6.** Let  $X = \{0, 1\}$ ;  $0 = 0$ , and let the binary operation  $\star$  be the addition modulo 2. Then  $(X; \star, 0)$  is a  $P$ -algebra.

*Proof.*

**Axiom 1:**  $((x \star y) \star (x \star z)) \star (z \star y) = 0$

$$\begin{aligned}((x \star y) \star (x \star z)) \star (z \star y) &= ((x + y)(\text{mod } 2) \star (x + z)(\text{mod } 2)) \star (z + y)(\text{mod } 2) \\ &= (2x + y + z)(\text{mod } 2) \star (z + y)(\text{mod } 2) \\ &= (2x + 2y + 2z)(\text{mod } 2) \\ &= 0.\end{aligned}$$

Thus,  $(X; \star, 0)$  satisfies the first axiom.

**Axiom 2:**  $x \star y = 0$  implies  $x = y$

$$\begin{aligned}x \star y = 0 &\Rightarrow (x + y) \pmod{2} = 0 \\ &\Rightarrow x = y = 0 \text{ or } x = y = 1 \\ &\Rightarrow x = y.\end{aligned}$$

Thus,  $(X; \star, 0)$  satisfies the second axiom.

**Axiom 3:**  $0 \star x = x$

$$\begin{aligned}0 \star x &= (0 + x) \pmod{2} \\ &= x.\end{aligned}$$

Thus,  $(X; \star, 0)$  satisfies the third axiom.

Hence  $(X; \star, 0)$  is a  $P$ -algebra. □

**Example 4.7.** Let  $X = \{0, 1, 2, 3\}$ , and let the binary operation  $\star$  be given by the following Cayley table:

$\star$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then  $(X; \star, 0)$  is a  $P$ -algebra.

*Proof.*

**Axiom 1:**  $((x \star y) \star (x \star z)) \star (z \star y) = 0$

For all possible combinations of  $x$ ,  $y$ , and  $z$ , the results of  $((x \star y) \star (x \star z)) \star (z \star y)$  are given in the following table:

$(x, y, z)$	$((x \star y) \star (x \star z)) \star (z \star y)$
$(0, 0, 0)$	$((0 \star 0) \star (0 \star 0)) \star (0 \star 0) = (0 \star 0) \star 0 = 0 \star 0 = 0$
$(0, 0, 1)$	$((0 \star 0) \star (0 \star 1)) \star (1 \star 0) = (0 \star 1) \star 1 = 1 \star 1 = 0$
$(0, 0, 2)$	$((0 \star 0) \star (0 \star 2)) \star (2 \star 0) = (0 \star 2) \star 2 = 2 \star 2 = 0$
$(0, 0, 3)$	$((0 \star 0) \star (0 \star 3)) \star (3 \star 0) = (0 \star 3) \star 3 = 3 \star 3 = 0$
$(0, 1, 0)$	$((0 \star 1) \star (0 \star 0)) \star (0 \star 1) = (1 \star 0) \star 1 = 1 \star 1 = 0$
$(0, 1, 1)$	$((0 \star 1) \star (0 \star 1)) \star (1 \star 1) = (1 \star 1) \star 0 = 0 \star 0 = 0$
$(0, 1, 2)$	$((0 \star 1) \star (0 \star 2)) \star (2 \star 1) = (1 \star 2) \star 3 = 3 \star 3 = 0$
$(0, 1, 3)$	$((0 \star 1) \star (0 \star 3)) \star (3 \star 1) = (1 \star 3) \star 2 = 2 \star 2 = 0$
$(0, 2, 0)$	$((0 \star 2) \star (0 \star 0)) \star (0 \star 2) = (2 \star 0) \star 2 = 2 \star 2 = 0$
$(0, 2, 1)$	$((0 \star 2) \star (0 \star 1)) \star (1 \star 2) = (2 \star 1) \star 3 = 3 \star 3 = 0$
$(0, 2, 2)$	$((0 \star 2) \star (0 \star 2)) \star (2 \star 2) = (2 \star 2) \star 0 = 0 \star 0 = 0$
$(0, 2, 3)$	$((0 \star 2) \star (0 \star 3)) \star (3 \star 2) = (2 \star 3) \star 1 = 1 \star 1 = 0$
$(0, 3, 0)$	$((0 \star 3) \star (0 \star 0)) \star (0 \star 3) = (3 \star 0) \star 3 = 3 \star 3 = 0$
$(0, 3, 1)$	$((0 \star 3) \star (0 \star 1)) \star (1 \star 3) = (3 \star 1) \star 2 = 2 \star 2 = 0$
$(0, 3, 2)$	$((0 \star 3) \star (0 \star 2)) \star (2 \star 3) = (3 \star 2) \star 1 = 1 \star 1 = 0$
$(0, 3, 3)$	$((0 \star 3) \star (0 \star 3)) \star (3 \star 3) = (3 \star 3) \star 0 = 0 \star 0 = 0$
$(1, 0, 0)$	$((1 \star 0) \star (1 \star 0)) \star (0 \star 0) = (1 \star 1) \star 0 = 0 \star 0 = 0$
$(1, 0, 1)$	$((1 \star 0) \star (1 \star 1)) \star (1 \star 0) = (1 \star 0) \star 1 = 1 \star 1 = 0$
$(1, 0, 2)$	$((1 \star 0) \star (1 \star 2)) \star (2 \star 0) = (1 \star 3) \star 2 = 2 \star 2 = 0$
$(1, 0, 3)$	$((1 \star 0) \star (1 \star 3)) \star (3 \star 0) = (1 \star 2) \star 3 = 3 \star 3 = 0$

$$\begin{aligned}
(1, 1, 0) & ((1 \star 1) \star (1 \star 0)) \star (0 \star 1) = (0 \star 1) \star 1 = 1 \star 1 = 0 \\
(1, 1, 1) & ((1 \star 1) \star (1 \star 1)) \star (1 \star 1) = (0 \star 0) \star 0 = 0 \star 0 = 0 \\
(1, 1, 2) & ((1 \star 1) \star (1 \star 2)) \star (2 \star 1) = (0 \star 3) \star 3 = 3 \star 3 = 0 \\
(1, 1, 3) & ((1 \star 1) \star (1 \star 3)) \star (3 \star 1) = (0 \star 2) \star 2 = 2 \star 2 = 0 \\
(1, 2, 0) & ((1 \star 2) \star (1 \star 0)) \star (0 \star 2) = (3 \star 1) \star 2 = 2 \star 2 = 0 \\
(1, 2, 1) & ((1 \star 2) \star (1 \star 1)) \star (1 \star 2) = (3 \star 0) \star 3 = 3 \star 3 = 0 \\
(1, 2, 2) & ((1 \star 2) \star (1 \star 2)) \star (2 \star 2) = (3 \star 3) \star 0 = 0 \star 0 = 0 \\
(1, 2, 3) & ((1 \star 2) \star (1 \star 3)) \star (3 \star 2) = (3 \star 2) \star 1 = 1 \star 1 = 0 \\
(1, 3, 0) & ((1 \star 3) \star (1 \star 0)) \star (0 \star 3) = (2 \star 1) \star 3 = 3 \star 3 = 0 \\
(1, 3, 1) & ((1 \star 3) \star (1 \star 1)) \star (1 \star 3) = (2 \star 0) \star 2 = 2 \star 2 = 0 \\
(1, 3, 2) & ((1 \star 3) \star (1 \star 2)) \star (2 \star 3) = (2 \star 3) \star 1 = 1 \star 1 = 0 \\
(1, 3, 3) & ((1 \star 3) \star (1 \star 3)) \star (3 \star 3) = (2 \star 2) \star 0 = 0 \star 0 = 0 \\
(2, 0, 0) & ((2 \star 0) \star (2 \star 0)) \star (0 \star 0) = (2 \star 2) \star 0 = 0 \star 0 = 0 \\
(2, 0, 1) & ((2 \star 0) \star (2 \star 1)) \star (1 \star 0) = (2 \star 3) \star 1 = 1 \star 1 = 0 \\
(2, 0, 2) & ((2 \star 0) \star (2 \star 2)) \star (2 \star 0) = (2 \star 0) \star 2 = 2 \star 2 = 0 \\
(2, 0, 3) & ((2 \star 0) \star (2 \star 3)) \star (3 \star 0) = (2 \star 1) \star 3 = 3 \star 3 = 0 \\
(2, 1, 0) & ((2 \star 1) \star (2 \star 0)) \star (0 \star 1) = (3 \star 2) \star 1 = 1 \star 1 = 0 \\
(2, 1, 1) & ((2 \star 1) \star (2 \star 1)) \star (1 \star 1) = (3 \star 3) \star 0 = 0 \star 0 = 0 \\
(2, 1, 2) & ((2 \star 1) \star (2 \star 2)) \star (2 \star 1) = (3 \star 0) \star 3 = 3 \star 3 = 0 \\
(2, 1, 3) & ((2 \star 1) \star (2 \star 3)) \star (3 \star 1) = (3 \star 1) \star 2 = 2 \star 2 = 0 \\
(2, 2, 0) & ((2 \star 2) \star (2 \star 0)) \star (0 \star 2) = (0 \star 2) \star 2 = 2 \star 2 = 0 \\
(2, 2, 1) & ((2 \star 2) \star (2 \star 1)) \star (1 \star 2) = (0 \star 3) \star 3 = 3 \star 3 = 0 \\
(2, 2, 2) & ((2 \star 2) \star (2 \star 2)) \star (2 \star 2) = (0 \star 0) \star 0 = 0 \star 0 = 0 \\
(2, 2, 3) & ((2 \star 2) \star (2 \star 3)) \star (3 \star 2) = (0 \star 1) \star 1 = 1 \star 1 = 0 \\
(2, 3, 0) & ((2 \star 3) \star (2 \star 0)) \star (0 \star 3) = (1 \star 2) \star 3 = 3 \star 3 = 0
\end{aligned}$$

$$\begin{array}{ll}
(2, 3, 1) & ((2 \star 3) \star (2 \star 1)) \star (1 \star 3) = (1 \star 3) \star 2 = 2 \star 2 = 0 \\
(2, 3, 2) & ((2 \star 3) \star (2 \star 2)) \star (2 \star 3) = (1 \star 0) \star 1 = 1 \star 1 = 0 \\
(2, 3, 3) & ((2 \star 3) \star (2 \star 3)) \star (3 \star 3) = (1 \star 1) \star 0 = 0 \star 0 = 0 \\
(3, 0, 0) & ((3 \star 0) \star (3 \star 0)) \star (0 \star 0) = (3 \star 3) \star 0 = 0 \star 0 = 0 \\
(3, 0, 1) & ((3 \star 0) \star (3 \star 1)) \star (1 \star 0) = (3 \star 2) \star 1 = 1 \star 1 = 0 \\
(3, 0, 2) & ((3 \star 0) \star (3 \star 2)) \star (2 \star 0) = (3 \star 1) \star 2 = 2 \star 2 = 0 \\
(3, 0, 3) & ((3 \star 0) \star (3 \star 3)) \star (3 \star 0) = (3 \star 0) \star 3 = 3 \star 3 = 0 \\
(3, 1, 0) & ((3 \star 1) \star (3 \star 0)) \star (0 \star 1) = (2 \star 3) \star 1 = 1 \star 1 = 0 \\
(3, 1, 1) & ((3 \star 1) \star (3 \star 1)) \star (1 \star 1) = (2 \star 2) \star 0 = 0 \star 0 = 0 \\
(3, 1, 2) & ((3 \star 1) \star (3 \star 2)) \star (2 \star 1) = (2 \star 1) \star 3 = 3 \star 3 = 0 \\
(3, 1, 3) & ((3 \star 1) \star (3 \star 3)) \star (3 \star 1) = (2 \star 0) \star 2 = 2 \star 2 = 0 \\
(3, 2, 0) & ((3 \star 2) \star (3 \star 0)) \star (0 \star 2) = (1 \star 3) \star 2 = 2 \star 2 = 0 \\
(3, 2, 1) & ((3 \star 2) \star (3 \star 1)) \star (1 \star 2) = (1 \star 2) \star 3 = 3 \star 3 = 0 \\
(3, 2, 2) & ((3 \star 2) \star (3 \star 2)) \star (2 \star 2) = (1 \star 1) \star 0 = 0 \star 0 = 0 \\
(3, 2, 3) & ((3 \star 2) \star (3 \star 3)) \star (3 \star 2) = (1 \star 0) \star 1 = 1 \star 1 = 0 \\
(3, 3, 0) & ((3 \star 3) \star (3 \star 0)) \star (0 \star 3) = (0 \star 3) \star 3 = 3 \star 3 = 0 \\
(3, 3, 1) & ((3 \star 3) \star (3 \star 1)) \star (1 \star 3) = (0 \star 2) \star 2 = 2 \star 2 = 0 \\
(3, 3, 2) & ((3 \star 3) \star (3 \star 2)) \star (2 \star 3) = (0 \star 1) \star 1 = 1 \star 1 = 0 \\
(3, 3, 3) & ((3 \star 3) \star (3 \star 3)) \star (3 \star 3) = (0 \star 0) \star 0 = 0 \star 0 = 0
\end{array}$$


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Thus,  $(X; \star, 0)$  satisfies the first axiom.

**Axiom 2:**  $x \star y = 0$  implies  $x = y$

Since 0 appears in the Cayley table only on the diagonal from upper left to lower right, the hypothesis is true only when  $x = y = 0$ ,  $x = y = 1$ ,  $x = y = 2$ , or  $x = y = 3$ . In all four cases, the conclusion is also true. Thus,  $(X; \star, 0)$  satisfies the second axiom.

**Axiom 3:**  $0 \star x = x$

The first row of the Cayley table shows that this is the case. Thus,  $(X, \star, 0)$  satisfies the third axiom.

Hence  $(X; \star, 0)$  is a  $P$ -algebra. □

## 4.4 Basic Theorems

**Theorem 4.1.** *Every  $P$ -algebra is commutative.*

*Proof.* Let  $(X; \star, 0)$  be a  $P$ -algebra, and suppose  $x, y \in X$ . Then,

$$\begin{aligned} ((0 \star x) \star (0 \star y)) \star (y \star x) &= 0 && \{\text{by the first axiom}\} \\ (x \star y) \star (y \star x) &= 0 && \{\text{by the third axiom, } 0 \star x = x \text{ and } 0 \star y = y\} \\ x \star y &= y \star x && \{\text{by the second axiom}\}. \end{aligned}$$

Thus, every  $P$ -algebra is commutative. □

**Corollary 4.2.** *Let  $(X; \star, 0)$  be a  $P$ -algebra. Then, for any  $x$  in  $X$ ,  $x \star 0 = x$ .*

*Proof.* Let  $(X; \star, 0)$  be a  $P$ -algebra, and suppose  $x \in X$ . Then,

$$\begin{aligned} x \star 0 &= 0 \star x && \{\text{by the commutative property}\} \\ &= x && \{\text{by the third axiom}\}. \end{aligned}$$

Thus, for any  $x$  in  $X$ ,  $x \star 0 = x$ . □

The following theorem shows the validity of the third axiom of  $BCK$ -algebras in  $P$ -algebras.



**Theorem 4.3.** *Let  $(X; \star, 0)$  be a  $P$ -algebra. Then, for any  $x$  in  $X$ ,  $x \star x = 0$ .*

*Proof.* Let  $(X; \star, 0)$  be a  $P$ -algebra, and suppose  $x \in X$ . Then,

$$\begin{aligned} ((x \star 0) \star (x \star 0)) \star (0 \star 0) &= 0 && \{\text{by the first axiom}\} \\ (x \star x) \star 0 &= 0 && \{\text{by corollary 4.2, } x \star 0 = x\} \\ x \star x &= 0 && \{\text{by the second axiom}\}. \end{aligned}$$

Thus, for any  $x$  in  $X$ ,  $x \star x = 0$ . □

The following theorem shows the validity of the fourth axiom of  $BCK$ -algebras in  $P$ -algebras.

**Theorem 4.4.** *Let  $(X; \star, 0)$  be a  $P$ -algebra. Then, for any  $x$  and  $y$  in  $X$ ,  $(x \star (x \star y)) \star y = 0$ .*

*Proof.* Let  $(X; \star, 0)$  be a  $P$ -algebra, and suppose  $x, y \in X$ . Then,

$$\begin{aligned} ((x \star 0) \star (x \star y)) \star (y \star 0) &= 0 && \{\text{by the first axiom}\} \\ (x \star (x \star y)) \star y &= 0 && \{\text{by corollary 4.2}\}. \end{aligned}$$

Thus, for any  $x$  and  $y$  in  $X$ ,  $(x \star (x \star y)) \star y = 0$ . □

**Corollary 4.5.** *Let  $(X; \star, 0)$  be a  $P$ -algebra. Then, for any  $x$  and  $y$  in  $X$ ,  $x \star (x \star y) = y$ .*

*Proof.* Let  $(X; \star, 0)$  be a  $P$ -algebra, and suppose  $x, y \in X$ . Then,

$$\begin{aligned} (x \star (x \star y)) \star y &= 0 && \{\text{by theorem 4.4}\} \\ x \star (x \star y) &= y && \{\text{by the second axiom}\}. \end{aligned}$$

Thus, for any  $x$  and  $y$  in  $X$ ,  $x \star (x \star y) = y$ . □

**Theorem 4.6.** *Let  $(X; \star, 0)$  be a  $P$ -algebra. Then, for any  $x$  and  $y$  in  $X$ ,  $x \star y = 0$  if and only if  $x = y$ .*

*Proof.* Let  $(X; \star, 0)$  be a  $P$ -algebra. Suppose there exist  $x, y$  in  $X$  such that  $x \star y = 0$ . Then, by the second axiom,  $x = y$ .

Now suppose  $x = y$ . Then  $x \star y = y \star y = 0$ . Thus, for any  $x$  and  $y$  in  $X$ ,  $x \star y = 0$  if and only if  $x = y$ . □

**Theorem 4.7.** *Every  $P$ -algebra is associative.*

*Proof.* Let  $(X; \star, 0)$  be a  $P$ -algebra, and suppose  $x, y, z \in X$ . Then,

$$\begin{aligned}
 ((y \star z) \star (y \star (x \star y))) \star ((x \star y) \star z) &= 0 && \{\text{by the first axiom}\} \\
 ((y \star z) \star (y \star (y \star x))) \star ((x \star y) \star z) &= 0 && \{\text{by the commutative property}\} \\
 ((y \star z) \star x) \star ((x \star y) \star z) &= 0 && \{\text{by corollary 4.5}\} \\
 (x \star (y \star z)) \star ((x \star y) \star z) &= 0 && \{\text{by the commutative property}\} \\
 x \star (y \star z) &= (x \star y) \star z && \{\text{by the second axiom}\}.
 \end{aligned}$$

Thus, every  $P$ -algebra is associative. □

**Theorem 4.8.** *Let  $X$  be a nonempty set with a binary operation  $\star$  and a constant  $0$ . Then  $(X; \star, 0)$  is a  $P$ -algebra if and only if the binary operation  $\star$  satisfies the following three conditions:*

1. *the binary operation  $\star$  is associative,*
2.  *$x \star x = 0$  for any  $x$  in  $X$ ,*
3.  *$x \star (x \star y) = y$  for any  $x$  and  $y$  in  $X$ .*

*Proof.* Suppose  $(X; \star, 0)$  is a  $P$ -algebra. Then, by theorem 4.7, the binary operation  $\star$  is associative; by theorem 4.3,  $x \star x = 0$  for any  $x \in X$ ; and by corollary 4.5,  $x \star (x \star y) = y$  for any  $x$  and  $y$  in  $X$ . Thus, if  $(X; \star, 0)$  is a  $P$ -algebra then the binary operation  $\star$  satisfies the three conditions.

Now suppose the binary operation  $\star$  satisfies the three conditions.

Suppose  $x, y, z \in X$ . Then,

$$\begin{aligned}
 ((x \star y) \star (x \star z)) \star (z \star y) &= (x \star y) \star ((x \star z) \star (z \star y)) && \{\text{by the associative property}\} \\
 &= (x \star y) \star (x \star (z \star (z \star y))) && \{\text{by the associative property}\} \\
 &= (x \star y) \star (x \star y) && \{\text{by the third condition}\} \\
 &= 0 && \{\text{by the second condition}\}.
 \end{aligned}$$

Thus,  $(X; \star, 0)$  satisfies the first axiom of  $P$ -algebras.

Suppose  $x, y \in X$ , and  $x \star y = 0$ . Then,

$$\begin{aligned}
 x &= x \star (x \star x) && \{\text{by the third condition}\} \\
 &= x \star 0 && \{\text{by the second condition}\} \\
 &= x \star (x \star y) && \{\text{by hypothesis, } x \star y = 0\} \\
 &= y && \{\text{by the third condition}\}.
 \end{aligned}$$

Thus,  $(X; \star, 0)$  satisfies the second axiom.

For any  $x \in X$ ,

$$\begin{aligned} 0 \star x &= (0 \star 0) \star x && \{\text{by the second condition}\} \\ &= 0 \star (0 \star x) && \{\text{by the associative property}\} \\ &= x && \{\text{by the third condition}\}. \end{aligned}$$

Thus,  $(X; \star, 0)$  satisfies the third axiom.

Hence  $(X; \star, 0)$  is a  $P$ -algebra. □

**Theorem 4.9.** *Let  $X$  be a nonempty set with a binary operation  $\star$  and a constant  $0$ . Then  $(X; \star, 0)$  is a  $P$ -algebra if and only if the binary operation  $\star$  satisfies the following three conditions:*

1. *the binary operation  $\star$  is associative,*
2.  *$x \star x = 0$  for any  $x$  in  $X$ ,*
3.  *$0 \star x = x$  for any  $x$  in  $X$ .*

*Proof.* Suppose  $(X; \star, 0)$  is a  $P$ -algebra. Then, by theorem 4.7, the binary operation  $\star$  is associative; by theorem 4.3,  $x \star x = 0$  for any  $x \in X$ ; and by definition,  $0 \star x = x$  for any  $x \in X$ . Thus, if  $(X; \star, 0)$  is a  $P$ -algebra then the binary operation  $\star$  satisfies the three conditions.

Now suppose the binary operation  $\star$  satisfies the three conditions. The third condition is identical to the third axiom of  $P$ -algebras. Let us show that  $(X; \star, 0)$  satisfies the first and second axioms.

Suppose  $x, y, z \in X$ . Then,

$$\begin{aligned}
((x \star y) \star (x \star z)) \star (z \star y) &= (x \star y) \star ((x \star z) \star (z \star y)) && \{\text{by the associative property}\} \\
&= (x \star y) \star (((x \star z) \star z) \star y) && \{\text{by the associative property}\} \\
&= (x \star y) \star ((x \star (z \star z)) \star y) && \{\text{by the associative property}\} \\
&= (x \star y) \star ((x \star 0) \star y) && \{\text{by the second condition}\} \\
&= (x \star y) \star (x \star (0 \star y)) && \{\text{by the associative property}\} \\
&= (x \star y) \star (x \star y) && \{\text{by the third condition}\} \\
&= 0 && \{\text{by the second condition}\}.
\end{aligned}$$

Thus,  $(X; \star, 0)$  satisfies the first axiom of  $P$ -algebras.

Suppose  $x, y \in X$ , and  $x \star y = 0$ . Then,

$$\begin{aligned}
x &= 0 \star x && \{\text{by the third condition}\} \\
&= (x \star x) \star x && \{\text{by the second condition}\} \\
&= x \star (x \star x) && \{\text{by the associative property}\} \\
&= x \star 0 && \{\text{by the second condition}\} \\
&= x \star (y \star y) && \{\text{by the second condition}\} \\
&= (x \star y) \star y && \{\text{by the associative property}\} \\
&= 0 \star y && \{\text{by hypothesis, } x \star y = 0\} \\
&= y && \{\text{by the third condition}\}.
\end{aligned}$$

Thus,  $(X; \star, 0)$  satisfies the second axiom.

Hence  $(X; \star, 0)$  is a  $P$ -algebra. □

The two sets of conditions of theorems 4.8 and 4.9 can be used as alternative sets of axioms of  $P$ -algebras.

**Theorem 4.10.** *Every  $P$ -algebra is a  $BCI$ -algebra.*

*Proof.* Let  $(X; \star, 0)$  be a  $P$ -algebra. Suppose  $x, y, z \in X$ . Then, by the definition of a  $P$ -algebra,

$$((x \star y) \star (x \star z)) \star (z \star y) = 0.$$

Thus,  $(X; \star, 0)$  satisfies the first axiom of the definition 3.2 of  $BCI$ -algebras.

Suppose  $x \star y = 0$  and  $y \star x = 0$ . Then  $x \star y = 0$ , and by the definition of a  $P$ -algebra,  $x = y$ . That is,

$$x \star y = 0 \text{ and } y \star x = 0 \text{ implies } x = y.$$

Thus,  $(X; \star, 0)$  satisfies the second axiom of the definition 3.2 of  $BCI$ -algebras.

The third axiom of the definition 3.2 follows from corollary 4.2.

Thus, every  $P$ -algebra is a  $BCI$ -algebra. □

The converse of the above theorem is not true. That is, there exist  $BCI$ -algebras which are not  $P$ -algebras. The following is an example of such a  $BCI$ -algebra.

**Example 4.8.** Let  $X = \{0, 1\}$ , and the binary operation  $\star$  be given by the following Cayley table:

$\star$	0	1
0	0	0
1	1	0

Then  $(X; \star, 0)$  is a  $BCI$ -algebra, but not a  $P$ -algebra.

*Proof.*

Let us first show that  $(X; \star, 0)$  satisfies the three axioms of the definition 3.2 of  $BCI$ -algebras.

**Axiom 1:**  $((x \star y) \star (x \star z)) \star (z \star y) = 0$

For all possible combinations of  $x$ ,  $y$ , and  $z$ , the results of  $((x \star y) \star (x \star z)) \star (z \star y)$  are given in the following table:

$(x, y, z)$	$((x \star y) \star (x \star z)) \star (z \star y)$
$(0, 0, 0)$	$((0 \star 0) \star (0 \star 0)) \star (0 \star 0) = (0 \star 0) \star 0 = 0 \star 0 = 0$
$(0, 0, 1)$	$((0 \star 0) \star (0 \star 1)) \star (1 \star 0) = (0 \star 0) \star 1 = 0 \star 1 = 0$
$(0, 1, 0)$	$((0 \star 1) \star (0 \star 0)) \star (0 \star 1) = (0 \star 0) \star 0 = 0 \star 0 = 0$
$(0, 1, 1)$	$((0 \star 1) \star (0 \star 1)) \star (1 \star 1) = (0 \star 0) \star 0 = 0 \star 0 = 0$
$(1, 0, 0)$	$((1 \star 0) \star (1 \star 0)) \star (0 \star 0) = (1 \star 1) \star 0 = 0 \star 0 = 0$
$(1, 0, 1)$	$((1 \star 0) \star (1 \star 1)) \star (1 \star 0) = (1 \star 0) \star 1 = 1 \star 1 = 0$
$(1, 1, 0)$	$((1 \star 1) \star (1 \star 0)) \star (0 \star 1) = (0 \star 1) \star 0 = 0 \star 0 = 0$
$(1, 1, 1)$	$((1 \star 1) \star (1 \star 1)) \star (1 \star 1) = (0 \star 0) \star 0 = 0 \star 0 = 0$

Thus,  $(X; \star, 0)$  satisfies the first axiom of the definition 3.2.

**Axiom 2:**  $x \star y = 0$  and  $y \star x = 0$  implies  $x = y$

According to the Cayley table, the hypothesis is true only when  $x = y = 0$  or  $x = y = 1$ . In both cases, the conclusion is also true. Thus,  $(X; \star, 0)$  satisfies the second axiom of the definition 3.2.

**Axiom 3:**  $x \star 0 = x$

The first column of the Cayley table shows that this is the case. Thus,  $(X; \star, 0)$  satisfies the third axiom of the definition 3.2.

Hence  $(X; \star, 0)$  is a *BCI*-algebra.

But, when  $x = 1$ , according to the first row of the Cayley table,  $0 \star 1 = 0 \neq 1$ . That is,  $(X; \star, 0)$  does not satisfy the third axiom of the definition of *P*-algebras, and hence  $(X; \star, 0)$  is not a *P*-algebra. Thus, there exist *BCI*-algebras which are not *P*-algebras.  $\square$

**Theorem 4.11.** *Every associative BCI–algebra is a P–algebra.*

*Proof.* Let  $(X; \star, 0)$  be an associative BCI–algebra. Suppose  $x, y$  and  $z$  are in  $X$ . Then, by definition of BCI–algebras,

$$((x \star y) \star (x \star z)) \star (z \star y) = 0.$$

Thus,  $(X; \star, 0)$  satisfies the first axiom of the definition of  $P$ –algebras.

Suppose  $x \star y = 0$ . Then,

$$\begin{aligned} x &= x \star 0 && \{\text{by the third axiom of definition 3.2}\} \\ &= x \star (x \star y) && \{\text{by hypothesis, } x \star y = 0\} \\ &= (x \star x) \star y && \{\text{by the associative property}\} \\ &= 0 \star y && \{\text{by the third axiom of definition 3.1}\} \\ &= (y \star y) \star y && \{\text{by the third axiom of definition 3.1}\} \\ &= y \star (y \star y) && \{\text{by the associative property}\} \\ &= y \star 0 && \{\text{by the third axiom of definition 3.1}\} \\ &= y && \{\text{by the third axiom of definition 3.2}\}. \end{aligned}$$

That is,

$$x \star y = 0 \text{ implies } x = y.$$

Thus,  $(X; \star, 0)$  satisfies the second axiom of the definition of  $P$ –algebras.



$$\begin{aligned}
x &= x \star 0 && \{\text{by the third axiom of definition 3.2}\} \\
&= x \star (x \star x) && \{\text{by the third axiom of definition 3.1}\} \\
&= (x \star x) \star x && \{\text{by the associative property}\} \\
&= 0 \star x && \{\text{by the third axiom of definition 3.1}\}.
\end{aligned}$$

Thus,  $(X; \star, 0)$  satisfies the third axiom of  $P$ -algebras, and hence every associative  $BCI$ -algebra is a  $P$ -algebra.  $\square$

By definition, every  $BCK$ -algebra is a  $BCI$ -algebra. By theorem 4.10, every  $P$ -algebra is a  $BCI$ -algebra. The following theorem states that the intersection of two sub-algebras of a  $BCI$ -algebra, when one sub-algebra is a  $BCK$ -algebra, and the other is a  $P$ -algebra, is the singleton  $\{0\}$ .

**Theorem 4.12.** *Let  $(X; \star, 0)$  be a  $BCI$ -algebra;  $A$  and  $B$  be two non-empty subsets of  $X$  such that,  $(A; \star, 0)$  and  $(B; \star, 0)$  are two sub-algebras of  $(X; \star, 0)$ . If  $(A; \star, 0)$  is a  $BCK$ -algebra and  $(B; \star, 0)$  is a  $P$ -algebra, then  $A \cap B = \{0\}$ .*

*Proof.* Let  $(X; \star, 0)$  be a  $BCI$ -algebra;  $A$  and  $B$  be two non-empty subsets of  $X$  such that,  $(A; \star, 0)$  and  $(B; \star, 0)$  are two sub-algebras of  $(X; \star, 0)$ . Suppose  $(A; \star, 0)$  is a  $BCK$ -algebra and  $(B; \star, 0)$  is a  $P$ -algebra.

Since  $A \neq \emptyset$ , there exists  $x \in X$  such that  $x \in A$ . Since  $(A; \star, 0)$  is a  $BCK$ -algebra,

$$x \star x = 0 \in A.$$

Similarly, since  $B \neq \emptyset$ , there exists  $y \in X$  such that  $y \in B$ , and since  $(B; \star, 0)$  is a  $P$ -algebra,

$$y \star y = 0 \in B.$$

Therefore,

$$0 \in A \cap B.$$

Now suppose there exists  $z \in X$  such that  $z \in A \cap B$ .

$$z \in A \cap B \Rightarrow z \in A \text{ and } z \in B$$

$$z \in A \Rightarrow 0 \star z \in A \text{ and } 0 \star z = 0$$

$$z \in B \Rightarrow 0 \star z \in B \text{ and } 0 \star z = z$$

$$0 \star z \in A \text{ and } 0 \star z \in B \Rightarrow 0 \star z \in A \cap B \text{ and } 0 = 0 \star z = z$$

Thus,  $A \cap B = \{0\}$ . □

**Theorem 4.13.** *Let  $(X; \star, 0)$  be a BCI-algebra, and define the set  $A$  as,*

$$A = \{x | x \in X \text{ and } 0 \star x = x\}.$$

*Then  $(A; \star, 0)$  is a sub-algebra of  $(X; \star, 0)$ . Furthermore,  $(A; \star, 0)$  is a  $P$ -algebra.*

*Proof.* Let  $(X; \star, 0)$  be a BCI-algebra, and let  $A$  be defined as above. Since  $0 \in X$  and  $0 \star 0 = 0$ ,  $0 \in A$ . Therefore,  $A$  is non-empty. Suppose  $x, y \in X$ . Then  $0 \star x = x$  and  $0 \star y = y$ .

$$\begin{aligned} x \star y &= (0 \star x) \star (0 \star y) \\ &= 0 \star (x \star y) && \{\text{by theorem 3.5}\} \\ &\in A. \end{aligned}$$

Thus,  $A$  is closed under the binary operation  $\star$ , and hence  $(A; \star, 0)$  is a sub-algebra of  $(X; \star, 0)$ . Next, let us show that  $(A; \star, 0)$  satisfies the three axioms of the definition of  $P$ -algebras.

Suppose  $x, y, z \in A$ . Then  $x, y, z \in X$ , and since  $(X; \star, 0)$  is a  $BCI$ -algebra,

$$((x \star y) \star (x \star z)) \star (z \star y) = 0.$$

Thus,  $(A; \star, 0)$  satisfies the first axiom of the definition of  $P$ -algebras.

Suppose  $x \star y = 0$ . Then,

$$\begin{aligned} ((0 \star x) \star (0 \star y)) \star (y \star x) &= 0 && \{\text{by the first axiom}\} \\ (x \star y) \star (y \star x) &= 0 && \{\text{by the definition of the set } A, 0 \star x = x \text{ and } 0 \star y = y\}. \end{aligned}$$

Similarly,

$$((0 \star y) \star (0 \star x)) \star (x \star y) = (y \star x) \star (x \star y) = 0.$$

By the second axiom of the definition of  $BCI$ -algebra,

$$((x \star y) \star (y \star x) = 0 \text{ and } (y \star x) \star (x \star y) = 0) \Rightarrow (x \star y = y \star x).$$

By our supposition,  $x \star y = 0$ , and hence  $y \star x = 0$ . Again, by the second axiom of the definition of  $BCI$ -algebra,

$$x \star y = 0 \text{ and } y \star x = 0 \Rightarrow x = y.$$

Thus,  $(A; \star, 0)$  satisfies the second axiom of the definition of  $P$ -algebras.

By definition of the set  $A$ ,  $(A; \star, 0)$  satisfies the third axiom of the definition of  $P$ -algebras. Thus,  $(A; \star, 0)$  is a  $P$ -algebra. □

## 4.5 Sub-Algebras

**Definition 4.2.** Let  $(X; \star, 0)$  be a  $P$ -algebra, and  $X_0$  be a nonempty subset of  $X$ . Then  $X_0$  is said to be a *sub-algebra* of  $X$  if  $X_0$  is closed under the binary operation  $\star$  in  $X$ .

**Theorem 4.14.** Let  $(X; \star, 0)$  be a  $P$ -algebra, and  $X_0$  be a sub-algebra of  $X$ . Then,

1.  $0 \in X_0$ ,
2.  $(X_0; \star, 0)$  is also a  $P$ -algebra,
3.  $X$  is a sub-algebra of  $X$ ,
4.  $(\{0\}; \star, 0)$  is a sub-algebra of  $X$ .

The proof is trivial, and hence omitted.

**Theorem 4.15.** Let  $(X; \star, 0)$  be a  $P$ -algebra, and “ $a$ ” be any nonzero element of  $X$ . Then  $(\{0, a\}; \star, 0)$  is a sub-algebra of  $X$ .

*Proof.* Let  $(X; \star, 0)$  be a  $P$ -algebra, and “ $a$ ” be any nonzero element of  $X$ . The binary operation  $\star$  in  $\{0, a\}$  is given by the following Cayley table:

$\star$	$0$	$a$
$0$	$0$	$a$
$a$	$a$	$0$

Since the binary operation  $\star$  is closed in  $\{0, a\}$ ,  $(\{0, a\}; \star, 0)$  is a  $P$ -algebra. Furthermore,  $\{0, a\}$  is a nonempty subset of  $X$ . Thus,  $(\{0, a\}; \star, 0)$  is a sub-algebra of  $X$ . □

## 4.6 Direct Products

**Definition 4.3.** Let  $\{X_i | i \in I\}$  be an indexed family of  $P$ -algebras. The *Cartesian product* of  $\{X_i | i \in I\}$  is the set

$$\prod_{i \in I} X_i = \{ \{x_i\}_{i \in I} | x_i \in X_i \},$$

where  $\{x_i\}_{i \in I}$  is a sequence in  $\prod_{i \in I} X_i$ . Furthermore,  $\{x_i\}_{i \in I} = \{y_i\}_{i \in I}$  if and only if  $x_i = y_i$  for all  $i \in I$ .

The binary operation  $\star$  on  $\prod_{i \in I} X_i$  is defined as,

$$\{x_i\}_{i \in I} \star \{y_i\}_{i \in I} = \{x_i \star y_i\}_{i \in I}.$$

**Theorem 4.16.** Let  $\{X_i | i \in I\}$  be an indexed family of  $P$ -algebras, and  $\star$  be the binary operation in definition 4.3. Then  $(\prod_{i \in I} X_i; \star, 0)$  is a  $P$ -algebra, where  $0 = \{0_i\}_{i \in I}$ .

The proof is trivial, and hence omitted.

**Definition 4.4.**  $(\prod_{i \in I} X_i; \star, 0)$  is the *direct product* of  $\{X_i | i \in I\}$ , where  $\star$  is the binary operation in definition 4.3, and  $0 = \{0_i\}_{i \in I}$ .

## 4.7 Varieties

**Definition 4.5.** A nonempty class of algebras is called a *variety* if it is closed under sub-algebras, Cartesian products, and homomorphic images.

Neither the class of  $BCK$ -algebras nor the class of  $BCI$ -algebras is a variety. Using a counter example, A. Wroński proved that the class of  $BCK$ -algebras is not a variety [Wro83].

**Definition 4.6.** Let  $\Sigma$  be a set of identities. Define  $M(\Sigma)$  to be the class of algebras  $A$  satisfying  $\Sigma$ . A class  $K$  of algebras is an *equational class* if there is a set of identities  $\Sigma$  such that  $K = M(\Sigma)$  [BS81].

**Theorem 4.17** (Birkhoff). [Bir35] *A class  $K$  of algebras is a variety if and only if  $K$  is an equational class.*

The proof of the above theorem can be found in [BS81].

**Theorem 4.18.** *The class of  $P$ -algebras is a variety.*

*Proof.* By theorem 4.9,  $P$ -algebras can be defined by a set of three identities. Therefore, the class of  $P$ -algebras is an equational class. By the Birkhoff theorem, the class of  $P$ -algebras is a variety. □

## 4.8 $P$ -Groups

Let  $(X; \star, 0)$  be a  $P$ -algebra. Then the binary operation  $\star$  is commutative and satisfies the group axioms. Therefore,  $P$ -algebras can be considered as abelian groups.

**Definition 4.7.** A group of exponent 2 is called a  $P$ -group.

**Theorem 4.19.** *Let  $X$  be a non-empty set,  $\star$  a binary operation, and  $0$  is a constant element in  $X$ . Then  $(X; \star, 0)$  is a  $P$ -algebra if and only if it is a  $P$ -group.*

*Proof.* Suppose  $(X; \star, 0)$  is a  $P$ -algebra. Then, by theorem 4.7, the binary operation  $\star$  is associative. By the third axiom of the definition of  $P$ -algebras, for any  $x \in X$ ,

$$0 \star x = x.$$

Furthermore, by corollary 4.2, for any  $x \in X$ ,

$$x \star 0 = x.$$

That is, for any  $x \in X$ ,

$$0 \star x = x \star 0 = x.$$

Thus, 0 is the identity element of the  $P$ -group. By theorem 4.3, for any  $x \in X$ ,

$$x \star x = 0.$$

That is, every element  $x$  is its inverse, and hence is of order 2. Thus,  $(X; \star, 0)$  is a  $P$ -group.

Now suppose  $(X; \star, 0)$  is a  $P$ -group. Then, for any  $x, y, z \in X$ ,

$$\begin{aligned} ((x \star y) \star (x \star z)) \star (z \star y) &= (x \star y) \star ((x \star z) \star (z \star y)) \\ &= (x \star y) \star (((x \star z) \star z) \star y) \\ &= (x \star y) \star ((x \star (z \star z)) \star y) \\ &= (x \star y) \star ((x \star 0) \star y) \\ &= (x \star y) \star (x \star y) \\ &= 0. \end{aligned}$$

Thus,  $(X; \star, 0)$  satisfies the first axiom of the definition of  $P$ algebras.

If  $x \star y = 0$ , then,

$$x = y^{-1} = y.$$

Thus,  $(X; \star, 0)$  satisfies the second axiom of the definition of  $P$ algebras.

By definition of  $P$ -groups, for any  $x \in X$ ,

$$0 \star x = x.$$

Thus,  $(X; \star, 0)$  satisfies the third axiom of the definition of  $P$ -algebras, and hence  $(X; \star, 0)$  is a  $P$ -algebra. □

Because of the relationship between  $P$ -algebras and  $P$ -groups, properties of  $P$ -algebras can be studied in terms of groups of exponent 2, and vice versa.

The next chapter develops the theory of  $Q$ -algebras.



# Chapter 5

## $Q$ -Algebras

### 5.1 Introduction

Let  $X$  be a nonempty set with a binary operation “ $\star$ ” and a constant  $0$ . Suppose the binary operation  $\star$  satisfies the second and third axioms of the definition of  $P$ -algebras. If  $(X; \star, 0)$  is a  $P$ -algebra, then by theorem 4.4, for any  $x$  and  $y$  in  $X$ ,

$$(x \star (x \star y)) \star y = 0. \tag{5.1}$$

Is the converse of the above statement true? That is, when the binary operation  $\star$  satisfies the condition (5.1), and the second and third axioms, is  $(X; \star, 0)$  a  $P$ -algebra? Or else, can the first axiom of the definition of  $P$ -algebras be replaced with the condition (5.1)? The following example shows that it is not the case.

**Example 5.1.** Let  $X = \{0, 1, 2\}$ , and the binary operation  $\star$  be given by the following Cayley table:

$\star$	0	1	2
0	0	1	2
1	1	0	2
2	2	1	0

Let's first show that  $(X; \star, 0)$  satisfies the condition (5.1), and the second and third axioms of the definition of  $P$ -algebras.

**Condition (5.1):**  $(x \star (x \star y)) \star y = 0$

For all possible combinations of  $x$  and  $y$ , the results of  $(x \star (x \star y)) \star y$  are given in the following table:

$(x, y)$	$(x \star (x \star y)) \star y$
$(0, 0)$	$(0 \star (0 \star 0)) \star 0 = (0 \star 0) \star 0 = 0 \star 0 = 0$
$(0, 1)$	$(0 \star (0 \star 1)) \star 1 = (0 \star 1) \star 1 = 1 \star 1 = 0$
$(0, 2)$	$(0 \star (0 \star 2)) \star 2 = (0 \star 2) \star 2 = 2 \star 2 = 0$
$(1, 0)$	$(1 \star (1 \star 0)) \star 0 = (1 \star 1) \star 0 = 0 \star 0 = 0$
$(1, 1)$	$(1 \star (1 \star 1)) \star 1 = (1 \star 0) \star 1 = 1 \star 1 = 0$
$(1, 2)$	$(1 \star (1 \star 2)) \star 2 = (1 \star 2) \star 2 = 2 \star 2 = 0$
$(2, 0)$	$(2 \star (2 \star 0)) \star 0 = (2 \star 2) \star 0 = 0 \star 0 = 0$
$(2, 1)$	$(2 \star (2 \star 1)) \star 1 = (2 \star 1) \star 1 = 1 \star 1 = 0$
$(2, 2)$	$(2 \star (2 \star 2)) \star 2 = (2 \star 0) \star 2 = 2 \star 2 = 0$

Thus,  $(X; \star, 0)$  satisfies the condition (5.1).

**Axiom 2:**  $x \star y = 0$  implies  $x = y$

Since 0 appears in the Cayley table only on the diagonal from upper left to lower right, the hypothesis is true only when  $x = y = 0$ ,  $x = y = 1$ , or  $x = y = 2$ . In all three cases, the conclusion is also true. Thus,  $(X; \star, 0)$  satisfies the second axiom.

**Axiom 3:**  $0 \star x = x$

The first row of the Cayley table shows that this is the case. Thus,  $(X, \star, 0)$  satisfies the third axiom.

So,  $(X, \star, 0)$  satisfies (5.1), and the second and third axioms of the definition of  $P$ -algebras.

But, when  $x = 2$ ,  $y = 1$ , and  $z = 0$ :

$$\begin{aligned} ((x \star y) \star (x \star z)) \star (z \star y) &= ((2 \star 1) \star (2 \star 0)) \star (0 \star 1) \\ &= (1 \star 2) \star 1 \\ &= 2 \star 1 \\ &= 1 \\ &\neq 0. \end{aligned}$$

Thus,  $(X, \star, 0)$  does not satisfy the first axiom of the definition of  $P$ -algebras. Hence  $(X, \star, 0)$  is not a  $P$ -algebra. Since the first axiom of the definition of  $P$ -algebras is identical to that of  $BCI$ -algebras,  $(X, \star, 0)$  is not a  $BCI$ -algebra either.

Therefore, the condition (5.1), along with the second and third axioms of the definition of  $P$ -algebras, defines a new algebraic structure. We named the new algebraic structure,  $Q$ -algebras.

## 5.2 The Definition of $Q$ -algebras

**Definition 5.1.** Let  $X$  be a nonempty set with a binary operation “ $\star$ ” and a constant 0.

Then  $(X; \star, 0)$  is said to be a  $Q$ -algebra if it satisfies the following axioms:

$\forall x, y, z \in X,$

1.  $(x \star (x \star y)) \star y = 0,$
2.  $x \star y = 0$  implies  $x = y,$
3.  $0 \star x = x.$

Next we consider the independence of the axioms of  $Q$ -algebras. The following example shows the independence of the first axiom.

**Example 5.2.** Let  $X = \{0, 1, 2\}$ , and the binary operation  $\star$  be given by the following Cayley table:

$\star$	0	1	2
0	0	1	2
1	1	0	2
2	2	2	0

**Axiom 1:**  $(x \star (x \star y)) \star y = 0$

When  $x = 2$  and  $y = 1$ :

$$(x \star (x \star y)) \star y = (2 \star (2 \star 1)) \star 1 = (2 \star 2) \star 1 = 0 \star 1 = 1 \neq 0.$$

Thus,  $(X, \star, 0)$  does not satisfy the first axiom.

**Axiom 2:**  $x \star y = 0$  implies  $x = y$

Since 0 appears in the Cayley table only on the diagonal from upper left to lower right, the hypothesis is true only when  $x = y = 0$ ,  $x = y = 1$ , or  $x = y = 2$ . In all three cases, the conclusion is also true. Thus,  $(X; \star, 0)$  satisfies the second axiom.

**Axiom 3:**  $0 \star x = x$

The first row of the Cayley table shows that this is the case. Thus,  $(X, \star, 0)$  satisfies the third axiom.

The above example shows that  $(X; \star, 0)$  satisfies the second and third axioms, but not the first. The next example shows the independence of the second axiom.

**Example 5.3.** Let  $X = \{0, 1, 2\}$ , and the binary operation  $\star$  be given by the following Cayley table:

$\star$	0	1	2
0	0	1	2
1	1	0	0
2	2	0	0

**Axiom 1:**  $(x \star (x \star y)) \star y = 0$

For all possible combinations of  $x$  and  $y$ , the results of  $(x \star (x \star y)) \star y$  are given in the following table:

$(x, y)$	$(x \star (x \star y)) \star y$
$(0, 0)$	$(0 \star (0 \star 0)) \star 0 = (0 \star 0) \star 0 = 0 \star 0 = 0$
$(0, 1)$	$(0 \star (0 \star 1)) \star 1 = (0 \star 1) \star 1 = 1 \star 1 = 0$
$(0, 2)$	$(0 \star (0 \star 2)) \star 2 = (0 \star 2) \star 2 = 2 \star 2 = 0$
$(1, 0)$	$(1 \star (1 \star 0)) \star 0 = (1 \star 1) \star 0 = 0 \star 0 = 0$
$(1, 1)$	$(1 \star (1 \star 1)) \star 1 = (1 \star 0) \star 1 = 1 \star 1 = 0$
$(1, 2)$	$(1 \star (1 \star 2)) \star 2 = (1 \star 0) \star 2 = 1 \star 2 = 0$
$(2, 0)$	$(2 \star (2 \star 0)) \star 0 = (2 \star 2) \star 0 = 0 \star 0 = 0$

$$(2, 1) \quad (2 \star (2 \star 1)) \star 1 = (2 \star 0) \star 1 = 2 \star 1 = 0$$

$$(2, 2) \quad (2 \star (2 \star 2)) \star 2 = (2 \star 0) \star 2 = 2 \star 2 = 0$$


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Thus,  $(X; \star, 0)$  satisfies the first axiom.

**Axiom 2:**  $x \star y = 0$  implies  $x = y$

When  $x = 1$  and  $y = 2$ :

$$x \star y = 0, \text{ but } x \neq y.$$

Thus,  $(X, \star, 0)$  does not satisfy the second axiom.

**Axiom 3:**  $0 \star x = x$

The first row of the Cayley table shows that this is the case. Thus,  $(X, \star, 0)$  satisfies the third axiom.

The above example shows that  $(X; \star, 0)$  satisfies the first and third axioms, but not the second. The next example shows the independence of the third axiom.

**Example 5.4.** Let  $X = \{0, 1, 2\}$ , and the binary operation  $\star$  be given by the following Cayley table:

$\star$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

**Axiom 1:**  $(x \star (x \star y)) \star y = 0$

For all possible combinations of  $x$  and  $y$ , the results of  $(x \star (x \star y)) \star y$  are given in the following table:

$(x, y)$	$(x \star (x \star y)) \star y$
$(0, 0)$	$(0 \star (0 \star 0)) \star 0 = (0 \star 0) \star 0 = 0 \star 0 = 0$
$(0, 1)$	$(0 \star (0 \star 1)) \star 1 = (0 \star 0) \star 1 = 0 \star 1 = 0$
$(0, 2)$	$(0 \star (0 \star 2)) \star 2 = (0 \star 1) \star 2 = 2 \star 2 = 0$
$(1, 0)$	$(1 \star (1 \star 0)) \star 0 = (1 \star 1) \star 0 = 0 \star 0 = 0$
$(1, 1)$	$(1 \star (1 \star 1)) \star 1 = (1 \star 0) \star 1 = 1 \star 1 = 0$
$(1, 2)$	$(1 \star (1 \star 2)) \star 2 = (1 \star 2) \star 2 = 2 \star 2 = 0$
$(2, 0)$	$(2 \star (2 \star 0)) \star 0 = (2 \star 2) \star 0 = 0 \star 0 = 0$
$(2, 1)$	$(2 \star (2 \star 1)) \star 1 = (2 \star 1) \star 1 = 1 \star 1 = 0$
$(2, 2)$	$(2 \star (2 \star 2)) \star 2 = (2 \star 0) \star 2 = 2 \star 2 = 0$

Thus,  $(X, \star, 0)$  satisfies the first axiom.

**Axiom 2:**  $x \star y = 0$  implies  $x = y$

Since 0 appears in the Cayley table only on the diagonal from upper left to lower right, the hypothesis is true only when  $x = y = 0$ ,  $x = y = 1$ , or  $x = y = 2$ . In all three cases, the conclusion is also true. Thus,  $(X; \star, 0)$  satisfies the second axiom.

**Axiom 3:**  $0 \star x = x$

This is not true when  $x = 1$ , since according to the first row of the Cayley table,  $0 \star 1 = 0 \neq 1$ . Thus,  $(X; \star, 0)$  does not satisfy the third axiom.

The above example shows that  $(X; \star, 0)$  satisfies the first and second axioms, but not the third.

The three examples above show that the axioms are independent of each other. Next we will illustrate some examples of  $Q$ -algebras.

### 5.3 Examples of $Q$ -Algebras

The algebra in example 5.1 is a  $Q$ -algebra. Here are some more examples.

**Example 5.5.** Let  $X$  be the set of all non-negative integers. For any  $x$  and  $y$  in  $X$ , let the binary operation  $\star$  be defined as follows:

$$x \star y = \begin{cases} y & \text{if } x < y, \\ x - y & \text{otherwise.} \end{cases}$$

Then  $(X; \star, 0)$  is a  $Q$ -algebra.

*Proof.*

**Axiom 1:**  $(x \star (x \star y)) \star y = 0$

**Case 1:**  $x < y$

$$\begin{aligned} (x \star (x \star y)) \star y &= (x \star y) \star y \\ &= y \star y \\ &= y - y \\ &= 0. \end{aligned}$$

**Case 2:**  $x \geq y$

$$\begin{aligned} (x \star (x \star y)) \star y &= (x \star (x - y)) \star y \\ &= (x - (x - y)) \star y \\ &= y \star y \\ &= y - y \\ &= 0. \end{aligned}$$



Thus,  $(X; \star, 0)$  satisfies the first axiom.

**Axiom 2:**  $x \star y = 0$  implies  $x = y$

$$\begin{aligned} x \star y = 0 &\Rightarrow y = 0 \text{ or } x - y = 0 \\ &\Rightarrow x - y = 0 \\ &\Rightarrow x = y. \end{aligned}$$

Thus,  $(X; \star, 0)$  satisfies the second axiom.

**Axiom 3:**  $0 \star x = x$

Since  $\forall x \in X, x \geq 0$ , by definition,  $0 \star x = x$  for any  $x$  in  $X$ . Thus,  $(X; \star, 0)$  satisfies the third axiom.

Hence  $(X; \star, 0)$  is a  $Q$ -algebra. □

The Cayley table of the above  $Q$ -algebra is given below:

$\star$	0	1	2	3	4	5	...
0	0	1	2	3	4	5	...
1	1	0	2	3	4	5	...
2	2	1	0	3	4	5	...
3	3	2	1	0	4	5	...
4	4	3	2	1	0	5	...
5	5	4	3	2	1	0	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

**Example 5.6.** Let  $X$  be the set of all non-positive integers. For any  $x$  and  $y$  in  $X$ , let the binary operation  $\star$  be defined as follows:

$$x \star y = \begin{cases} y & \text{if } x > y, \\ x - y & \text{otherwise.} \end{cases}$$

Then  $(X; \star, 0)$  is a  $Q$ -algebra.

*Proof.*

**Axiom 1:**  $(x \star (x \star y)) \star y = 0$

**Case 1:**  $x > y$

$$\begin{aligned} (x \star (x \star y)) \star y &= (x \star y) \star y \\ &= y \star y \\ &= y - y \\ &= 0. \end{aligned}$$

**Case 2:**  $x \leq y$

$$\begin{aligned} (x \star (x \star y)) \star y &= (x \star (x - y)) \star y \\ &= (x - (x - y)) \star y \\ &= y \star y \\ &= y - y \\ &= 0. \end{aligned}$$

Thus,  $(X; \star, 0)$  satisfies the first axiom.

**Axiom 2:**  $x \star y = 0$  implies  $x = y$

$$\begin{aligned} x \star y = 0 &\Rightarrow y = 0 \text{ or } x - y = 0 \\ &\Rightarrow x - y = 0 \\ &\Rightarrow x = y. \end{aligned}$$

Thus,  $(X; \star, 0)$  satisfies the second axiom.

**Axiom 3:**  $0 \star x = x$

Since  $\forall x \in X, x \leq 0$ , by definition,  $0 \star x = x$  for any  $x$  in  $X$ . Thus,  $(X; \star, 0)$  satisfies the third axiom.

Hence  $(X; \star, 0)$  is a  $Q$ -algebra. □

The Cayley table of the above  $Q$ -algebra is given below:

$\star$	0	-1	-2	-3	-4	-5	...
0	0	-1	-2	-3	-4	-5	...
-1	-1	0	-2	-3	-4	-5	...
-2	-2	-1	0	-3	-4	-5	...
-3	-3	-2	-1	0	-4	-5	...
-4	-4	-3	-2	-1	0	-5	...
-5	-5	-4	-3	-2	-1	0	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

**Example 5.7.** Let  $A$  be a nonempty set, and let  $X$  be the set of all non-negative real valued functions defined on  $A$ . For any  $f$  and  $g$  in  $X$ , and any  $x$  in  $A$ , let the binary operation  $\star$  be defined as follows:

$$(f \star g)(x) = \begin{cases} g(x) & \text{if } f(x) < g(x), \\ f(x) - g(x) & \text{otherwise.} \end{cases}$$

Then  $(X; \star, 0)$  is a  $Q$ -algebra.

The proof is similar to that of example 5.5, and hence omitted.

## 5.4 Basic Theorems

**Theorem 5.1.** *Let  $(X; \star, 0)$  be a  $Q$ -algebra. Then, for any  $x$  in  $X$ ,  $x \star x = 0$ .*

*Proof.* Let  $(X; \star, 0)$  be a  $Q$ -algebra, and suppose  $x \in X$ . Then,

$$\begin{aligned} (0 \star (0 \star x)) \star x &= 0 && \{\text{by the first axiom}\} \\ (0 \star x) \star x &= 0 && \{\text{by the third axiom, } 0 \star x = x\} \\ x \star x &= 0 && \{\text{by the third axiom, } 0 \star x = x\}. \end{aligned}$$

Thus, for any  $x$  in  $X$ ,  $x \star x = 0$ . □

**Theorem 5.2.** *Let  $(X; \star, 0)$  be a  $Q$ -algebra. Then, for any  $x$  in  $X$ ,  $(x \star 0) \star x = 0$ .*

*Proof.* Let  $(X; \star, 0)$  be a  $Q$ -algebra, and suppose  $x \in X$ . Then,

$$(x \star (x \star x)) \star x = 0 \quad \{\text{by the first axiom}\}$$

$$(x \star 0) \star x = 0 \quad \{\text{by theorem 5.1}\}.$$

Thus, for any  $x$  in  $X$ ,  $(x \star 0) \star x = 0$ . □

**Theorem 5.3.** *Let  $(X; \star, 0)$  be a  $Q$ -algebra. Then, for any  $x$  in  $X$ ,  $x \star (0 \star x) = 0$ .*

*Proof.* Let  $(X; \star, 0)$  be a  $Q$ -algebra, and suppose  $x \in X$ . Then,

$$x \star x = 0 \quad \{\text{by theorem 5.1}\}$$

$$x \star (0 \star x) = 0 \quad \{\text{by the third axiom, } 0 \star x = x\}.$$

Thus, for any  $x$  in  $X$ ,  $x \star (0 \star x) = 0$ . □

**Corollary 5.4.** *Let  $(X; \star, 0)$  be a  $Q$ -algebra. Then, for any  $x$  in  $X$ ,  $(x \star 0) \star x = x \star (0 \star x)$ .*

*Proof.* Let  $(X; \star, 0)$  be a  $Q$ -algebra, and suppose  $x \in X$ . Then, by theorems 5.2 and 5.3,

$$(x \star 0) \star x = 0 = x \star (0 \star x).$$

Thus, for any  $x$  in  $X$ ,  $(x \star 0) \star x = x \star (0 \star x)$ . □

**Corollary 5.5.** *Let  $(X; \star, 0)$  be a  $Q$ -algebra. Then, for any  $x$  in  $X$ ,  $x \star 0 = x$ .*

*Proof.* Let  $(X; \star, 0)$  be a  $Q$ -algebra, and suppose  $x \in X$ . Then,

$$\begin{aligned} (x \star 0) \star x &= 0 && \{\text{by theorem 5.2}\} \\ x \star 0 &= x && \{\text{by the second axiom}\}. \end{aligned}$$

Thus, for any  $x$  in  $X$ ,  $x \star 0 = x$ . □

**Theorem 5.6.** *Let  $(X; \star, 0)$  be a  $Q$ -algebra. Then, for any  $x$  and  $y$  in  $X$ ,  $x \star y = 0$  if and only if  $x = y$ .*

*Proof.* Let  $(X; \star, 0)$  be a  $Q$ -algebra. Suppose there exist  $x, y$  in  $X$  such that  $x \star y = 0$ . Then, by the second axiom,  $x = y$ .

Now suppose  $x = y$ . Then  $x \star y = y \star y = 0$ . Thus, for any  $x$  and  $y$  in  $X$ ,  $x \star y = 0$  if and only if  $x = y$ . □

**Corollary 5.7.** *Let  $(X; \star, 0)$  be a  $Q$ -algebra. Then, for any  $x$  in  $X$ ,  $x \star y = 0$  if and only if  $y \star x = 0$ .*

*Proof.* Let  $(X; \star, 0)$  be a  $Q$ -algebra, and suppose  $x, y \in X$ . Then,

$$\begin{aligned} x \star y = 0 &\Leftrightarrow x = y && \{\text{by theorem 5.6}\} \\ &\Leftrightarrow y = x \\ &\Leftrightarrow y \star x = 0 && \{\text{by theorem 5.6}\}. \end{aligned}$$

Thus, for any  $x$  in  $X$ ,  $x \star y = 0$  if and only if  $y \star x = 0$ . □

**Theorem 5.8.** *Every  $P$ -algebra is a  $Q$ -algebra.*

*Proof.* Let  $(X; \star, 0)$  be a  $P$ -algebra, and suppose  $x, y \in X$ . Then,

$$\begin{aligned} (x \star (x \star y)) \star y &= 0 && \{\text{by theorem 4.4}\} \\ x \star y = 0 &\Rightarrow x = y && \{\text{by the second axiom of } P\text{-algebras}\} \\ 0 \star x &= x && \{\text{by the third axiom of } P\text{-algebras}\}. \end{aligned}$$

Therefore,  $(X; \star, 0)$  satisfies the three axioms of the definition of  $Q$ -algebras, and hence is a  $Q$ -algebra. Thus, every  $P$ -algebra is a  $Q$ -algebra.  $\square$

In general,  $Q$ -algebras are neither commutative nor associative. In example 5.5,  $1 \star 3 = 3$  while  $3 \star 1 = 2$ . Furthermore,  $1 \star (3 \star 2) = 1 \star 1 = 0$  while  $(1 \star 3) \star 2 = 3 \star 2 = 1$ . Thus, the  $Q$ -algebra in example 5.5 is neither commutative nor associative.

On the other hand, every  $P$ -algebra is commutative. Therefore, the converse of theorem 5.8 is not true.

**Theorem 5.9.** *Every associative  $Q$ -algebra is a  $P$ -algebra.*

*Proof.* Let  $(X; \star, 0)$  be an associative  $Q$ -algebra. Then, by theorem 5.1,  $x \star x = 0$  for any  $x \in X$ ; and by definition,  $0 \star x = x$  for any  $x \in X$ . That is, the binary operation  $\star$  satisfies the three conditions of theorem 4.9. Thus,  $(X; \star, 0)$  is a  $P$ -algebra, and hence every associative  $Q$ -algebra is a  $P$ -algebra.  $\square$

**Corollary 5.10.** *Every associative  $BCI$ -algebra is a  $Q$ -algebra.*

*Proof.* By theorem 4.11, every associative  $BCI$ -algebra is a  $P$ -algebra. By theorem 5.8, every  $P$ -algebra is a  $Q$ -algebra. Thus, every associative  $BCI$ -algebra is a  $Q$ -algebra.  $\square$

## 5.5 Sub–Algebras

**Definition 5.2.** Let  $(X; \star, 0)$  be a  $Q$ –algebra, and  $X_0$  be a nonempty subset of  $X$ . Then  $X_0$  is said to be a *sub-algebra* of  $X$  if  $X_0$  is closed under the binary operation  $\star$  in  $X$ .

**Theorem 5.11.** Let  $(X; \star, 0)$  be a  $Q$ –algebra, and  $X_0$  be a sub-algebra of  $X$ . Then,

1.  $0 \in X_0$ ,
2.  $(X_0; \star, 0)$  is also a  $Q$ –algebra,
3.  $X$  is a sub-algebra of  $X$ ,
4.  $(\{0\}; \star, 0)$  is a sub-algebra of  $X$ .

The proof is trivial, and hence omitted.

**Theorem 5.12.** Let  $(X; \star, 0)$  be a  $Q$ –algebra, and “ $a$ ” be any nonzero element of  $X$ . Then  $(\{0, a\}; \star, 0)$  is a sub-algebra of  $X$ .

The proof is similar to that of theorem 4.15, and hence omitted.

**Example 5.8.** Consider the  $Q$ –algebra in example 5.5, where  $X$  is the set of all non-negative integers, and for any  $x$  and  $y$  in  $X$ , the binary operation  $\star$  is defined as,

$$x \star y = \begin{cases} y & \text{if } x < y, \\ x - y & \text{otherwise.} \end{cases}$$

Let  $n$  be any positive integer, and  $X_0$  be the set of all non-negative multiples of  $n$ . That is,

$$X_0 = \{0, n, 2n, 3n, \dots\}.$$

Then  $(X_0; \star, 0)$  is a sub-algebra of  $(X; \star, 0)$ .



*Proof.*

Suppose  $x, y \in X_0$ . Then,  $x = jn$  and  $y = kn$  for some non-negative integers  $j$  and  $k$ . If  $x < y$ , then,

$$x \star y = y = kn \in X_0.$$

If  $x \geq y$ , then,

$$\begin{aligned} x \star y &= x - y \\ &= jn - kn \\ &= (j - k)n, \end{aligned}$$

where  $j - k$  is a non-negative integer. Hence,  $(j - k)n \in X_0$ . Thus, the binary operation  $\star$  is closed in  $X_0$ , and hence  $(X_0; \star, 0)$  is a sub-algebra of  $(X; \star, 0)$ .  $\square$

# Chapter 6

## Conclusion

In this research we defined two classes of algebras, namely  $P$ -algebras and  $Q$ -algebras, by altering the axioms of  $BCK$ -algebras. We developed the theory of  $P$ - and  $Q$ -algebras parallel to the basic theorems of  $BCK$ - and  $BCI$ -algebras. The two sets of conditions of theorems 4.8 and 4.9 can be used as alternative sets of axioms of  $P$ -algebras. While classes of  $BCK$ - and  $BCI$ -algebras are not varieties, we proved that the class of  $P$ -algebras is a variety. The question “Is the class of  $Q$ -algebras a variety?”, remains open.

We proved that  $P$ -algebras are abelian groups of exponent two. It may be possible to derive simplified versions of existing group theorems, which are valid only for abelian groups of exponent two, using the theory of  $P$ -algebras. On the other hand, one can attempt to construct a set of axioms similar to that of  $P$ -algebras, which is valid for any group. Then it can be extended to develop other axiom systems for algebraic structures such as rings, fields, etc.

Another area of interest is the study of the binary operation  $\star$  as a linear, quadratic, or cubic polynomial of two variables over a field, and derive the values of the coefficients.

We studied the relationship between  $BCI$ - and  $P$ - algebras, and proved that every  $P$ -algebra is a  $BCI$ -algebra. Furthermore, every associative  $BCI$ -algebra is a  $P$ -algebra. A similar relationship is evident between  $P$ - and  $Q$ - algebras. That is, every  $P$ -algebra is a  $Q$ - algebra, and every associative  $Q$ -algebra is a  $P$ -algebra.

One application of  $P$ -algebras is the “ $XOR$  function in computer programming. For example, it may be efficient to code the  $XOR$  function in algorithms, using the axioms of  $P$ -algebras.

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